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Solution to the Presentation Problem for Powers of the Augmentation Ideal of Torsion Free and Torsion Abelian Groups

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Abstract

This paper solves for torsion free and torsion abelian groups G the problem of presenting n-th powers $\Delta^n(G)$ of the augmentation ideal $\Delta(G)$ of an integral group ring $\mathbb{Z}G$, in terms of the standard additive generators of $\Delta^n(G)$. A concrete basis for $\Delta^n(G)$ is obtained when G itself has a basis and is torsion. The results are applied to describe the homology of the sequence $\Delta^n(N)G \rightarrowtail \Delta^n(G) \twoheadrightarrow \Delta^n(G/N)$.

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1 Introduction

Let G denote a group or monoid and $\mathbb{Z}G$ its integral group or monoid ring. The kernel $\Delta(G)$ of the augmentation homomorphism $\mathbb{Z}G \to \mathbb{Z}, \sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g$, is called the **augmentation ideal** of $\mathbb{Z}G$. It is clear that $\Delta(G)$ is the free abelian group on the elements

$$[g] := g - 1, \quad g \in G$$

modulo the relation $[1] = 0$ (cf. [6]). This implies that the n-th power ideal $\Delta^n(G) := (\Delta G)^n$ of the augmentation ideal $\Delta(G)$ is generated as an abelian group by the products

$$[g_1, \ldots, g_n] := [g_1] \cdots [g_n], \quad g_1, \ldots, g_n \in G,$$

called the **standard generators of** $\Delta^n(G)$. It is a classical problem in the theory of group rings to find all relations among the standard generators of $\Delta^n(G)$. The current paper solves this problem for torsion free and torsion abelian groups, as well as free abelian monoids.

There are two obvious relations which hold for any group or monoid G. The first is the

(N) normalizing relation : $[g_1, \ldots, g_n] = 0$, whenever some $g_i = 1 (n \geq 1)$.

The second is a consequence of the fact that the symbol $[g_1, g_2]$ is a 2-cocycle:

$$[g_2, g_3] - [g_1 g_2, g_3] + [g_1, g_2 g_3] - [g_1, g_2] = 0.$$

This implies the

(R) cocycle relation: $[g_1, \ldots, g_n]$ is a 2-cocycle in g_{i-1}, g_i

when the other variables are fixed ($n \geq 2$). Other relations depend essentially on the structure of the group G.

If the group or monoid is abelian then we have the

(S) symmetric relation: $[g_1, \cdots, g_n] = [g_{\sigma(1)}, \cdots, g_{\sigma(n)}]$ for any permutation σ of n letters ($n \geq 1$).

To the above list of relations, we shall add when $n \geq 3$ and G is a torsion abelian group, two more complicated relations called T and U and define a possibly infinite number $n(G) \geq 2$. Our main results are summarized in the following theorem.

THEOREM (1.1) Let G be a torsion free or torsion abelian group. Then the following holds.

(1.1.1) N, R, and S are a defining set of relations for $\Delta^n(G)$ when either $n = 2$ or $n \geq 2$ and G is torsion free or a direct limit of cyclic groups.

(1.1.2) N, R, S, and T are a defining set of relations for $\Delta^n(G)$ when either G is p-elementary or G is torsion and $n \leq n(G)$.

(1.1.3) N, R, S, T, and U are a defining set of relations for $\Delta^n(G)$ when G is torsion.

Since $\Delta^n(G)$ is a subgroup of the free abelian group $\mathbb{Z}G$, it has a $\mathbb{Z}$-basis. The strategy of the proof of the theorem is to find a $\mathbb{Z}$-basis for $\Delta^n(G)$ and enough relations on $\Delta^n(G)$ to allow writing each standard generator as a sum of the $\mathbb{Z}$-basis elements. It follows then that these relations are a defining set of relations for $\Delta^n(G)$.

For $n = 1$, it is obvious that for any group G, the standard generators $[g]$ such that $g \neq 1$ form a basis for $\Delta^1(G)$. Thus $\Delta^1(G)$ is presented by the relation N on the standard generators of $\Delta^1(G)$.

For $n \geq 2$, finding a $\mathbb{Z}$-basis for $\Delta^n(G)$ is easier if G itself has an ordered basis. By definition, an ordered (not necessarily finite) basis for an abelian group G consists of a totally ordered set I and a function $g : I \to G, i \mapsto g_i$, such that if $\langle g_i \rangle$ denotes the cyclic subgroup of G generated by g_i then $G = \coprod_{i \in I} \langle g_i \rangle$. Fortunately we can reduce to the case G has an ordered basis, because Δ^n commutes with direct limits, every abelian group is a direct limit of its finitely generated subgroups, and every finitely generated abelian group has an ordered basis (I, g) (such that I is finite). Since there is essentially no difference between handling groups with an arbitrary ordered basis and groups with a finite ordered basis, we shall treat the general case. Given an abelian group G with ordered basis (I, g), we define relations $T_I(G)$ and $U_I(G)$ and a finite number $n_I(G)$. The relations $N, R, S, T_I(G)$, and $U_I(G)$ will be enough to present $\Delta^n(G)$. The precise theorem will be formulated below. To obtain the relations T and U used in the theorem above, we take respectively the unions $\bigcup_H (\bigcup_I T_I(H))$ and $\bigcup_H (\bigcup_I U_I(H))$ where H ranges over all subgroups of G having an ordered basis and I over all ordered bases of H. The number $n(G)$ in the theorem above is defined as $\sup_H (\sup_I n_I(H, I))$ where H and I range as above. Since

$n(G)$ can be infinite, whereas $n_I(H)$ is always finite, one sees that the limiting process is interesting, and not just a routine procedure to reduce to the case that G has an ordered basis.

Theorem 1.1 is deduced routinely from the theorem below for groups G with an ordered basis (I, g). The latter theorem makes reference to NRS-generators and special NRS-generators for $\Delta^n(G)$, which are defined in the main body of the paper.

THEOREM 1.2 Let G be a free or torsion abelian group with ordered basis (I, g). Then the following holds.

(1.2.1) The assertion of (1.1.1). Furthermore the NRS-generators of $\Delta^n(G)$ are a $\mathbb{Z}$-basis when G is torsion.

(1.2.2) N, R, S, and T_I are a defining set of relations for $\Delta^n(G)$ when G is p-elementary or G is torsion and $n \leq n_I(G)$. Furthermore the special NRS-generators form a $\mathbb{Z}$-basis of $\Delta^n(G)$.

(1.2.3) N, R, S, T_I, and U_I are a defining set of relations for $\Delta^n(G)$ when G is torsion. (Furthermore the proof of this assertion provides a procedure for constructing, but not uniquely, a $\mathbb{Z}$-basis of $\Delta^n(G)$, starting from the special NRS-generators in $\Delta^n(G)$).

Whereas the relations N, R, and S are well known and special cases of T and T_I are found in the literature, the relations U and U_I and the numbers $n(G)$ and $n_I(G)$ are completely new.

Since the relations N, R, and S hold in any abelian group or monoid, it makes sense singling out the universal object they define. Accordingly, we let

$$\hat{\Delta}^n(G)$$

denote the free abelian group on the standard generators of $\Delta^n(G)$, modulo the relations N, R, and S. The strategy of the proof of (1.2.1) is as follows. If G is either torsion or a free abelian monoid with an ordered monoid basis, we show that the NRS-generators generate $\hat{\Delta}^n(G)$ and form a $\mathbb{Z}$-basis of $\Delta^n(G)$. It follows immediately that $\hat{\Delta}^n(G) \cong \Delta^n(G)$ and this gives the presentation of $\Delta^n(G)$. If G is a free abelian group, we use a trick to deduce the presentation of $\Delta^n(G)$ from that in the case G is a free abelian monoid. The strategy of the proof of (1.2.2) is to show that the special NRS-generators generate $\hat{\Delta}^n(G)/\langle T_I \rangle$ and form a $\mathbb{Z}$-basis of $\Delta^n(G)$. It follows immediately that $\hat{\Delta}^n(G)/\langle T_I\rangle \cong \Delta^n(G)$ and this gives the presentation of $\Delta^n(G)$. The strategy of the proof of (1.2.3) is to replace in a systematic way the special NRS-generators by another set of elements which generates $\hat{\Delta}^n(G)/\langle T_I, U_I\rangle$ and is a $\mathbb{Z}$-basis of $\Delta^n(G)$. It follows immediately that $\hat{\Delta}^n(G)/\langle T_I, U_I\rangle \cong \Delta^n(G)$ and this gives the presentation of $\Delta^n(G)$.

Set

$$\omega^n(G) = \text{Ker } (\hat{\Delta}^n(G) \to \Delta^n(G)).$$

The group $\hat{\Delta}^n(G)$ has a right action of G which is compatible under the canonical homomorphism $\hat{\Delta}^n(G) \to \Delta^n(G)$ with the right action of G on $\Delta^n(G)$ by multiplication. Thus $\omega^n(G)$ is a G-module and we call it the **relation module** of $\Delta^n(G)$. According to Theorem 1.1, $\omega^n(G)$ is trivial if G is either torsion free or a direct limit of cyclic groups; in the remaining cases, it is generated by the T and U relations. It is reasonable to expect using these generators to compute $\omega^n(G)$, for example by refining the Main Construction 3.9 and Main Lemma 3.10. This will not be attempted, however, in the current paper.

Relation modules describe the failure of exactness of the well known zero sequence in the theorem below.

THEOREM 1.3 Let G denote an abelian group and $N \subseteq G$ a subgroup. Let $\Delta^n(N)G$ denote the G-submodule of $\Delta^n(G)$ generated by the image $(\Delta^n(N) \to \Delta^n(G))$. Then the homology of the sequence $\Delta^n(N)G \rightarrowtail \Delta^n(G) \twoheadrightarrow \Delta^n(G/N)$ is computed by $\omega^n(G/N)/\omega^n(G)$. Moreover if G is finite (resp. torsion) then so is $\omega^n(G/N)/\omega^n(G)$.

There is evidence suggesting that the relation module functor ω^n on abelian groups is the fundamental group functor $\pi_1\underline{\underline{\Delta}}^n$ of a suitably defined functor $\underline{\underline{\Delta}}^n$: ((abelian groups)) $\to$ ((pointed topological spaces)). Here π_1 denotes as usual the fundamental group functor on pointed topological spaces. We construct at the end of the paper a functor $\underline{\underline{\Delta}}^n$ and a surjective natural transformation $\omega^n \to \pi_1\underline{\underline{\Delta}}^n$, whose kernel we describe by certain generators. It is an open question whether or not these generators are zero. If they are zero then the higher homotopy groups of $\underline{\underline{\Delta}}^n(G)$ can be thought of as higher relation modules of $\Delta^n(G)$.

We summarize now some of the previous literature related to our results. For the moment, let G denote an arbitrary group. The relation $[f,g] = [fg] - [f] - [g]$ in $\Delta^2(G)$ induces the relation

$$[g_1, \cdots, g_k, f, g, g_{k+3}, \cdots, g_{n+1}] = [g_1, \cdots, g_k]([fg] - [f] - [g])[g_{k+3} \cdots, g_{n+1}]$$

in $\Delta^{n+1}(G)$, whose right hand side represents naturally an element of $\Delta^n(G)$. Factoring out of $\Delta^n(G)$, the

(B) bilinearizing relation: $[g_1, \cdots, g_k]([fg] - [f] - [g])[g_{k+3}, \cdots, g_{n+1}] = 0$,

we get the group $\Delta^n(G)/B = \Delta^n(G)/\Delta^{n+1}(G)$.

This group has been intensively studied in the literature and in the case G is finite abelian, a presentation in terms of standard generators was provided over two decades ago in the celebrated paper [7] of I.R.S. Passi and L.P.V. Vermani. A survey of this and important related results is found in Passi's survey article [5]. A significant impulse for the presentation of Passi and Vermani was an earlier result of F. Bachmann and L. Gruenenfelder [1]. It says that for a finite abelian group, the quotients $\Delta^n(G)/\Delta^{n+1}(G)$ stabilize for large n, i.e. there exists an N such that for all $n \geq N, \Delta^N(G)/\Delta^{N+1}(G) \cong \Delta^n(G)/\Delta^{n+1}(G)$. A number of papers were written concerned with the problem of determining N and computing the isomorphism class of $\Delta^N(G)/\Delta^{N+1}(G)$. Definitive results are found in A.W. Hales' paper [3]. This paper contains also another presentation of $\Delta^n(G)/\Delta^{n+1}(G)$ for any n. For $n < N$ and G an elementary group, the (nonstable) quotients above were computed recently in G. Tang [10].

The first steps in finding a presentation of $\Delta^n(G)$ for G torsion abelian were taken in A.Bak-N.Vavilov [2], in connection with the generalized Milnor conjecture for quadratic forms over a field F. The conjecture predicts a certain presentation of the n'th power ideal $I^n(F)$ of the fundamental ideal $I(F)$ of the Witt ring $W(F)$. Setting $G = F^\bullet/F^{\bullet 2}$, we get a canonical surjective ring homomorphism $\mathbb{Z}G \to W(F)$ taking each $\Delta^n(G)$ onto $I^n(F)$. So if the conjecture for $I^n(F)$ is true then one ought to be able to find a presentation of $\Delta^n(G)$ which is compatible with that conjectured for $I^n(F)$. Such a presentation is given in [2] and coincides with that in (1.1.1) and (1.1.2). This is the first appearance of T relations. A presentation of $\Delta^n(G)$ for any elementary p-group is provided by the combined work of M.M. Parmenter [4] and G. Tang [9] and coincides with that given in (1.1.1) and (1.1.2).

The rest of the paper is organized as follows. In section 2, we define NRS-generators. Then we prove Theorems 1.1.1 and 1.2.1. The presentation part of the results is translated into the language of Rees rings of augmentation ideals. We do this because the Rees ring of an augmentation ideal maps canonically onto the associated graded ring of the augmentation ideal, and the associated graded ring is the construction used in [7], [5], [6] and [3] in formulating presentation results. In section 3, we define special NRS-generators, and show that they are linearly independent in $\hat{\Delta}^n(G)$ and $\Delta^n(G)$. Then we define the numbers $n_I(G)$ and $n(G)$ and prove Theorems 1.1.2 and 1.2.2. The presentation part of the results is translated into the language of Rees rings. In section 4, we prove Theorems (1.1.3) and (1.2.3) and translate the presentation there into the language of Rees rings. In section 5, we develop results concerning the relation module $\omega^n(G)$, prove Theorem 1.3, and consider the problem of higher relation modules.

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2 $N, R,$ and S relations

The goal of this section is to prove Theorems 1.1.1 and 1.2.1. Let G denote a torsion free or torsion abelian group. Theorem 1.1.1 says the relations N, R and S on the standard generators of $\Delta^n(G)$, present the group $\Delta^n(G)$ when $n = 2$ or $n \geq 3$ and G is either torsion free or a direct limit of cyclic groups, i.e. a subgroup of $\mathbb{Q}/\mathbb{Z}$. Theorem 1.2.1 supposes additionally that G has an ordered basis. It makes the same assertion as (1.1.1) and the additional one that a certain subset of the standard generators, namely the NRS-generators, is a basis for $\Delta^n(G)$ when G is torsion. By a straightforward direct limit argument, one shows that (1.1.1) follows from (1.2.1).

To prove (1.2.1), we begin with some general preliminaries which will play a role in the entire paper. Let G denote a torsion abelian group with an ordered basis or a free abelian monoid with an ordered monoid basis. Our first task is to reduce the number of standard generators required to generate $\Delta^n(G)$. Since only the $N, R,$ and S relations are used in the reduction process, the set of generators we arrive at will also generate $\hat{\Delta}^n(G)$. This set will be called the NRS-generators of $\Delta^n(G)$ and $\hat{\Delta}^n(G)$. In the special case either G is a torsion abelian group and the hypotheses of (1.2.1) is satisfied or G is a free abelian monoid, we show that the NRS-generators are a basis for $\Delta^n(G)$. It follows immediately that the canonical map $\hat{\Delta}^n(G) \to \Delta^n(G)$ is an isomorphism and $\Delta^n(G)$ is presented by the N, R and S relations on the standard generators of $\Delta^n(G)$. The presentation of $\Delta^n(G)$ when G is a free abelian group is deduced by a trick from the case G is a free abelian monoid.

We begin by studying the influence of a product decomposition $G = H \times \langle g \rangle$ on reducing the number of symbols required for generating $\hat{\Delta}^n(G)$.

LEMMA 2.1 Let G denote a torsion abelian group and $G = H \times \langle g \rangle$ a direct product decomposition of G such that $\langle g \rangle$ is a finite cyclic group generated by g. Let ε denote the exponent of g. Let

$$\begin{aligned}
\mathcal{G}^n(g) &= \{[g^i, \underbrace{g, \cdots, g}_{n-1}] & &\mid 1 \leq i \leq \varepsilon - 1\} \\
\mathcal{G}^n(H, g) &= \{[h, g^i, \underbrace{g, \cdots, g}_{n-2}] & &\mid 1 \leq i \leq \varepsilon - 1, h \in H \backslash \langle 1 \rangle\} \\
\mathcal{G}^n(H, *, g) &= \{[h_1, \cdots h_{n-j}, \underbrace{g, \cdots, g}_{j}] & &\mid 0 \leq j \leq n-2, h_1, \cdots, h_{n-j} \in H \backslash \langle 1 \rangle\}.
\end{aligned}$$

(Obviously if $H = \langle 1 \rangle$ then $\mathcal{G}^n(H, g) = \mathcal{G}^n(H, *, g) = \emptyset$.) The conclusion is that $\mathcal{G}^n(g) \cup \mathcal{G}^n(H, g) \cup \mathcal{G}^n(H, *, g)$ generates $\hat{\Delta}^n(G)$.

PROOF The result is trivially true for $n = 1$. So we can assume $n \geq 2$.

We show first that any generator $[g_1, \cdots, g_n]$ of $\hat{\Delta}^n(G)$ can be written as a sum of generators $[g'_1, \cdots, g'_n]$ such that each g'_i is either in $\langle g \rangle$ or in H. Let $\#[g_1, \cdots, g_n]$ denote the number of $g'_i s$ such that g_i is neither in $\langle g \rangle$ nor in H. If $\#[g_1, \cdots, g_n] = 0$ then we are done. We proceed by induction on the value of $\#$. Suppose $\#[g_1, \cdots, g_n] > 0$. By (S), we can assume that $g_1 = h_1 g^i$ where $g^i \neq 1$ and $h_1 \in H \backslash \langle 1 \rangle$. Let $g_2 = h_2 g^j$ where $h_2 \in H$. Let $[g_1, g_2, --] = [g_1, g_2, g_3, \cdots, g_n]$. Suppose $j = 0$. Then $[g_1, g_2, --] = [h_1 g^i, h_2, --] =$ (by (R)) $[g^i, h_1 h_2, --] + [h_1, h_2, --] - [g^i, h_1, --]$. Since the value of $\#$ on each generator on the right hand side of the equation above is strictly less than $\#[g_1, g_2, --]$, we can write by induction on the value of $\#$, each generator on the right hand side as a sum of generators of the desired kind. Suppose $h_2 = 1$. Then $[g_1, g_2, --] = [h_1 g^i, g^j, --] =$ (by (R)) $[h_1, g^{i+j}, --] + [g^i, g^j, --] - [h_1, g^i, --]$ and we can finish again by induction on the value of $\#$. Suppose now that $g^j \neq 1$ and $h_2 \in H \backslash \langle 1 \rangle$. Then $[g_1, g_2, --] = [h_1 g^i, h_2 g^j, --] =$ (by (R)) $[h_1 g^{i+j}, h_2, --] + [h_1 g^i, g^j, --] - [g^j, h_2, --]$ and we can finish by induction on the value of $\#$.

By the above and (S), it is clear that any generator of $\hat{\Delta}^n(G)$ is a sum of generators of the kind $[h_1, \cdots, h_{n-j}, g^{i_1}, \cdots g^{i_j}]$ where $0 \leq j \leq n$ and $h_1, \cdots, h_{n-j} \in H$. Let $[h_1, \cdots, h_{n-j}, g^{i_1}, \cdots, g^{i_j}] = [--, g^{i_1}, \cdots, g^{i_j}]$. If $j \geq 2$, we show next that $[--, g^{i_1}, \cdots, g^{i_j}]$ is a sum of generators of the kind $[--, g^i, g, \cdots, g]$ where $1 \leq i \leq \varepsilon - 1$. By (N), we can assume that $1 \leq i_k \leq \varepsilon - 1$ for each k such that $1 \leq k \leq j$. We proceed by induction on $i_2 + \cdots + i_j$. If $i_2 + \cdots + i_j = j - 1$ then $i_2 = \cdots = i_j = 1$ and we are done. Suppose $i_2 + \cdots + i_j > j-1$. By (S), we can assume $i_2 > 1$. By (R), we can write $[--, g^{i_1}, g^{i_2}, \cdots, g^{i_j}] = [--, g^{i_1+i_2-1}, g, g^{i_3}, \cdots, g^{i_j}] + [--, g^{i_1}, g^{i_2-1}, g^{i_3}, \cdots, g^{i_j}] - [--, g^{i_2-1}, g, g^{i_3}, \cdots, g^{i_j}]$. By induction, we can write each of the generators on the right hand side of the equation above as sums of generators of the kind $[--, g^i, g, \cdots, g]$ where $1 \le i \le \varepsilon - 1$. This completes the demonstration.

Suppose $n \ge 3$ and $n - j \ge 2$. Let $h_1, \cdots, h_{n-j-2} \in H$ and for any $g_{n-j-1}, \cdots, g_n \in G$, let $[--, g_{n-j-1}, \cdots, g_n] = [h_1, \cdots, h_{n-j-2}, g_{n-j-1}, \cdots, g_n]$. To complete the proof it suffices to show that a generator of the kind $[--, h', h, \underbrace{g^i, g, \cdots, g}_{j}]$ where $h', h \in H$ and $i > 1$ is a sum of generators of the kind $[--, h^{(2)}, \underbrace{g^{i'}, g, \cdots, g}_{j+1}]$ and $[--, h^{(3)}, h^{(4)}, \underbrace{g, \cdots, g}_{j}]$ where $h^{(2)}, h^{(3)}, h^{(4)} \in H$. In fact, the proof will show that $i > i' \ge 1$. Clearly it suffices to check the case $n = 3$. The proof is by induction on i. By (R), $[h', h, g^i] = [h', hg^{i-1}, g] + [h', h, g^{i-1}] - [h', g^{i-1}, g] =$ (by(R)) $[h'h, g^{i-1}, g] + [h', h, g] - [h, g^{i-1}, g] + [h', h, g^{i-1}] - [h', g^{i-1}, g]$. All the generators on the right hand side of the equation above, except $[h', h, g^{i-1}]$ are of the kind we want. If $i = 2$ then $[h', h, g^{i-1}]$ is also of the kind we want. If $i > 2$ then by induction on $i, [h', h, g^{i-1}]$ is a sum of generators of the kind we want. This finishes the proof. $\square$

Definition 2.2 Let $G = \coprod_{i \in I} G_i$ be a direct sum of nontrivial finite cyclic groups $G_i = \langle g_i \rangle$. Give I a total ordering and if $i \in I$ is not the smallest element of I, set $G_{<i} = \coprod_{j<i} G_j$. A standard generator $[x(1), x(2), \cdots, x(n)]$ of $\hat{\Delta}^n(G)$ or $\Delta^n(G)$ is called an **NRS-generator** with respect to I, of degree n, if either

(2.2.1)

$$[x(1), x(2), \cdots, x(n)] = [g_{i_1}^e, g_{i_2}, \cdots, g_{i_n}] \text{ where } i_1 = i_2 \le i_3 \le \cdots, \le i_n \in I, e \ne 0$$

or

(2.2.2)
$$[x(1), x(2), \cdots, x(n)] = [h, g_{i_2}^e, g_{i_3}, \cdots, g_{i_n}] \text{ where } i_2 \le i_3 \le \cdots, \le i_n \in I,$$
$$i_2 \text{ is not the smallest element of } I, e \ne 0, \text{ and } h \in G_{<i_2} \backslash \langle 1 \rangle.$$

Corollary 2.3 Let $G = \coprod_{i \in I} G_i$ be as in (2.2). Then $\hat{\Delta}^n(G)$ is generated by NRS-generators.

PROOF If $J \subseteq I$, let $G_J = \coprod_{i \in J} G_i$. Since $\hat{\Delta}^n(G)$ is canonically isomorphic to the direct limit $\lim_{\longrightarrow J} \hat{\Delta}^n(G_J)$ where J ranges over all nonempty finite subsets of I with the induced ordering, we can reduce to the case I is finite. We proceed now by induction on the order $|I|$ of I. If $|I| = 1$ then by Lemma 2.1, $\hat{\Delta}^n(G)$ is generated by $\mathcal{G}(g)$. Obviously each element of $\mathcal{G}(g)$ is an NRS-generator. Suppose $|I| > 1$. Let $I = \{1, \cdots, \ell\}$. Let $g = g_\ell$. For $1 \leq m \leq n$ and $2 \leq k \leq \ell$, let $\bar{\Delta}^m(G_{<k}) = \text{image } \hat{\Delta}^m(G_{<k}) \to \hat{\Delta}^n(G), [x(1), \cdots, x(m)] \mapsto [x(1), \cdots, x(m), \underbrace{g, \cdots, g}_{n-m}]$. Decompose $G = G_{<\ell} \times \langle g \rangle$. By Lemma 2.1, $\hat{\Delta}^n(G)$ is generated by $\mathcal{G}^n(g), \mathcal{G}^n(G_{<\ell}, g)$, and $\sum_{m=2}^{n} \bar{\Delta}^m(G_{<\ell})$. By definition, the elements of $\mathcal{G}^n(g)$ and $\mathcal{G}^n(G_{<\ell}, g)$ are NRS-generators. By induction on $|I|$, each group $\hat{\Delta}^m(G_{<\ell})$ is generated by NRS-generators and clearly the map $\hat{\Delta}^m(G_{<\ell}) \to \hat{\Delta}^n(G)$ preserves NRS-generators. This completes the proof. $\square$

COROLLARY 2.4 Let $G = \coprod_{i \in I} G_i$ be as in (2.2). Let i_0 denote the smallest element of I, which might not exist. Then the set of all NRS-generators of $\hat{\Delta}^2(G)$, i.e. $\cup_{i \in I} \mathcal{G}^2(g_i) \cup \cup_{i \in I \setminus \{i_0\}} \mathcal{G}^2(G_{<i}, g_i)$, is a $\mathbb{Z}$-basis for $\hat{\Delta}^2(G)$ and $\Delta^2(G)$. In particular the canonical homomorphism $\hat{\Delta}^2(G) \to \bar{\Delta}^2(G)$ is an isomorphism.

PROOF Let $\mathcal{B} = \cup_{i \in I} \mathcal{G}^2(g_i) \cup \cup_{i \in I \setminus \{i_0\}} \mathcal{G}^2(G_{<i}, g_i)$. It suffices to show that $\mathcal{B}$ generates $\hat{\Delta}^2(G)$ and the elements of $\mathcal{B}$ are $\mathbb{Z}$-linearly independent in $\Delta^2(G)$. Since G is canonically isomorphic to the direct limit $\lim_{\longrightarrow J} \left(\coprod_{j \in J} G_j \right)$ where J ranges over all nonempty finite subsets of I with the induced ordering and since $\hat{\Delta}^2$ and Δ^2 commute with direct limits, it suffices to prove the above when I is finite. If I has one element then by Lemma 2.1, $\mathcal{B}$ generates $\hat{\Delta}^2(G)$. If I has more than one element then using induction on the number of elements of I and Lemma 2.1, one concludes that $\mathcal{B}$ generates $\hat{\Delta}^2(G)$. Let $|G|$ denote the order of G. Clearly rank $\Delta(G) = |G| - 1$. It is well known (cf. [6], [8]) that the abelian group $\Delta(G)/\Delta^2(G)$ is annihilated by some power of $|G|$. Thus rank $\Delta^2(G) = \text{rank } \Delta(G) = |G| - 1$. But $\mathcal{B}$ generates $\Delta^2(G)$ and has precisely $|G| - 1$ elements. It follows that the elements of $\mathcal{B}$ must be $\mathbb{Z}$-linearly independent. $\square$

THEOREM 2.5 Let G denote a torsion abelian group. Then the canonical homomorphism $\hat{\Delta}^2(G) \to \Delta^2(G)$ is an isomorphism.

PROOF Since G is a direct limit of finite subgroups and since $\hat{\Delta}^2$ and Δ^2 commute with direct limits, we can reduce to the case G is finite. In this case the result follows from (2.4). $\square$

The next result is also a corollary of Lemma 2.1.

COROLLARY 2.6 Let $G = \langle g \rangle$ denote a finite cyclic group. Then for any n, the set $\mathcal{G}^n(g)$ is a basis for $\hat{\Delta}^n(G)$ and $\Delta^n(G)$. Thus the canonical homomorphism $\hat{\Delta}^n(G) \to \Delta^n(G)$ is an isomorphism.

PROOF By Lemma 2.1, the elements of $\mathcal{G}^n(g)$ generate $\hat{\Delta}^n(G)$ and therefore also $\Delta^n(G)$. It is well known (cf. [6], [8]) that the abelian group $\Delta(G)/\Delta^n(G)$ is annihilated by a power of $|G|$. Thus rank $\Delta^n(G) =$ rank $\Delta(G) = |G| - 1$. Since $\mathcal{G}^n(g)$ generates $\Delta^n(G)$ and has precisely $|G|-1$ elements, it must be a basis of $\Delta^n(G)$. Thus $\mathcal{G}^n(g)$ is also a basis of $\hat{\Delta}^n(G)$. The last assertion of the lemma is now trivial. $\square$

COROLLARY 2.7 If G is a direct limit of finite cyclic groups, i.e. a subgroup of $\mathbb{Q}/\mathbb{Z}$, then the canonical homomorphism $\hat{\Delta}^n(G) \to \Delta^n(G)$ is an isomorphism.

PROOF This follows directly from the previous corollary and the fact that $\hat{\Delta}^n$ and Δ^n commute with direct limits. $\square$

The next lemma is an analog of (2.1), for abelian monoids. Its proof is the same as that of (2.1), but simpler, and will be omitted.

LEMMA 2.8 Let M denote an abelian monoid and $M = H \times \langle g \rangle$ a direct product decomposition of M such that $\langle g \rangle$ is the free monoid generated by g. Let

$$\begin{array}{ll} \mathcal{G}^n(g) & = \{[g^i, g, \cdots, g] \mid i > 0\} \\ \mathcal{G}^n(H, g) & = \{[h, g^i, g, \cdots, g] \mid i > 0, h \in H \backslash \langle 1 \rangle\} \\ \mathcal{G}^n(H, *, g) & = \{[h_1, \cdots, h_{n-j}, \underbrace{g, \cdots, g}_{j}] \mid 0 \le j \le n-2, h_1, \cdots, h_{n-j} \in H \backslash \langle 1 \rangle\}. \end{array}$$

Then the union of the sets of generators above generates $\hat{\Delta}^n(M)$.

DEFINITION 2.9 Let $M = \coprod_{i \in I} G_i$ be a direct sum of free monoids $G_i = \langle g_i \rangle$. Give I a total ordering and if $i \in I$ is not the smallest element of I, set $G_{<i} = \coprod_{j<i} G_j$. A standard generator $[x(1), x(2), \cdots, x(n)]$ of $\hat{\Delta}^n(M)$ or $\Delta^n(M)$ is called an **NRS-generator** with respect to I, of degree n, if either

(2.9.1)
$$[x(1), x(2), \cdots, x(n)] = [g_{i_1}^e, g_{i_2}, \cdots, g_{i_n}] \text{ where } i_1 = i_2 \le i_3 \le \cdots, \le i_n \in I, e > 0$$

or

$$[x(1), x(2), \cdots, x(n)] = [h, g_{i_2}^e, g_{i_3}, \cdots, g_{i_n}] \text{ where } i_2 \leq i_3 \leq \cdots, \leq i_n \in I, \tag{2.9.2}$$
$$i_2 \text{ is not the smallest element of } I, e > 0, \text{ and } h \in G_{<i_2} \backslash \langle 1 \rangle.$$

LEMMA 2.10 Let $M = \coprod_{i \in I} G_i$ be as in (2.9). Then the set of all NRS-generators in (2.9) generates $\hat{\Delta}^n(M)$ and is a $\mathbb{Z}$-basis for $\Delta^n(M)$. Consequently the canonical homomorphism $\hat{\Delta}^n(M) \to \Delta^n(M)$ is an isomorphism.

PROOF The proof that NRS-generators generate $\hat{\Delta}^n(M)$ is the same as that of (2.3). Next we show that they are $\mathbb{Z}$-linearly independent in $\Delta^n(M)$. This will complete the proof of the lemma.

$\mathbb{Z}G$ is a polynomial ring over $\mathbb{Z}$ in the indeterminates $\{g_i \mid i \in I\}$. It suffices to show that any finite set $[x(1)_1, \cdots, x(n)_1], \cdots, [x(1)_k, \cdots, x(n)_k]$ of distinct NRS-generators of the same total degree is linearly independent. Suppose $\sum_{i=1}^{k} a_i[x(1)_i, \cdots, x(n)_i] = 0$. Then clearly $\sum_{i=1}^{k} a_i x(1)_i \cdots x(n)_i = 0$. The key observation now is that if $[x(1), \cdots, x(n)]$ and $[x(1)', \cdots, x(n)']$ are NRS-generators of the same total degree then $[x(1), \cdots, x(n)] = [x(1)', \cdots x(n)'] \Leftrightarrow x(1) \cdots x(n) = x(1)' \cdots x(n)'$. Thus the monomials in $\{x(1)_i \cdots x(n)_i \mid 1 \leq i \leq k\}$ are distinct and therefore linearly independent. Thus $a_i = 0$ for $1 \leq i \leq k$. □

THEOREM 2.11 Let M denote an abelian monoid such that if $x, y \in M$ then $xy = 1 \Leftrightarrow x = y = 1$. Then the canonical homomorphism $\hat{\Delta}^n(M) \to \Delta^n(M)$ is an isomorphism.

PROOF Since the functors $\hat{\Delta}^n$ and Δ^n commute with direct limits and since M is a direct limit of its (finitely generated) free submonoids, we can reduce to the case covered in Lemma 2.10. □

DEFINITION-LEMMA 2.12 Let G denote an abelian group or monoid. Let $[x(1), \cdots, x(n)]$ denote an arbitrary standard generator for $\hat{\Delta}^n(G)$ and let g denote an arbitrary element of G. Then the rule $[x(1), \cdots, x(n)]g = [x(1), \cdots, x(n-1), x(n)g] - [x(1), \cdots, x(n-1), g]$ defines an action of G on $\hat{\Delta}^n(G)$ such that the canonical homomorphism $\hat{\Delta}^n(G) \to \Delta^n(G)$ is one of G-modules.

PROOF Since $\Delta^n(G)$ is an ideal of $\mathbb{Z}G$, it has a natural G-action given by multiplication and this action satisfies the rule above. Thus it suffices to show that the rule defines a G-action on $\hat{\Delta}^n(G)$. This follows from the equations $[ef, g]h - [e, fg]h + [e, f]h - [f, g]h = 0$ and $[f, g]h - [g, f]h = 0$ in $\hat{\Delta}^2(G)$, for arbitrary $e, f, g, h \in G$. These equations are straightforward to verify. □

THEOREM 2.13 Let G denote a torsion free abelian group. Then the canonical homomorphism $\hat{\Delta}^n(G) \to \Delta^n(G)$ is an isomorphism.

PROOF Since the functors $\hat{\Delta}^n$ and Δ^n commute with direct limits and since every torsion free abelian group is a direct limit of (finitely generated) free subgroups, we can reduce to the case G is free abelian. Pick a basis for G and let G^+ denote the (free) submonoid generated by the basis elements. From Theorem 2.11, it follows that the canonical map $\hat{\Delta}^n(G^+) \to \Delta^n(G)$ is injective and thus we can identify $\hat{\Delta}^n(G^+)$ with its image in $\hat{\Delta}^n(G)$. If $[x(1), \cdots, x(n)]$ is a standard generator of $\hat{\Delta}^n(G)$ then there is an element $g \in G^+$ such that $[x(1), \cdots, x(n)]g \in \hat{\Delta}^n(G^+)$. Since $\hat{\Delta}^n(G^+)G^+ \subseteq \hat{\Delta}^n(G^+)$, it follows that given $x \in \hat{\Delta}^n(G)$, there is an element $g \in G^+$ such that $xg \in \hat{\Delta}^n(G^+)$. Thus Theorem 2.13 follows from Theorem 2.11 or even better from Lemma 2.10. $\square$

Corollary 2.7, Theorem 2.11, and Theorem 2.13 have the following immediate consequence for the Rees ring of $\mathbb{Z}G$.

COROLLARY 2.14 Let G denote an abelian group. Let $\mathbb{Z} \oplus \Delta(G) \oplus \Delta^2(G) \oplus \cdots$ denote the Rees ring of $\Delta(G)$ in $\mathbb{Z}G$. Let $X(G)$ denote the free abelian group on all symbols X_g such that $g \in G$ and let $TX(G)$ denote the tensor algebra of $X(G)$ over $\mathbb{Z}$. We introduce the following relations into $TX(G)$.

(N') $X_1 = 0$.

(R') $X_g X_h - X_{fg} X_h + X_f X_{gh} - X_f X_g = 0$ for all $f, g, h \in G$.

(S') $X_f X_g = X_g X_f$ for all $f, g \in G$.

If G is torsion free or a direct limit of cyclic groups then the canonical graded surjective ring homomorphism $TX(G) \to \mathbb{Z} \oplus \Delta(G) \oplus \Delta^2(G) \oplus \cdots, X_g \mapsto (g-1)$, has kernel the 2-sided ideal generated by N', R', and S'.

PROOF The tensor ring $TX(G)$ is a graded ring. Let $TX(G)_n$ denote its n-th homogeneous component. The Rees ring $R(G)$ of the augmentation ideal $\Delta(G)$ is also a graded ring whose n-th homogeneous component is $\Delta^n(G)$. The ring homomorphism $TX(G) \to R(G), X_g \mapsto [g]$, is a surjective graded ring homomorphism, taking $TX(G)_n$ onto $\Delta^n(G)$. The group $TX(G)_n$ $(n \geq 1)$ is free abelian on the products $X_{g_1} \cdots X_{g_n}$ where $g_1, \cdots, g_n$ ranges over all elements of G. The map $\varphi_n = \varphi|_{TX(G)_n} : TX(G)_n \to \Delta^n(G)$ takes the generator $X_{g_1} \cdots X_{g_n}$ to the standard generator $[g_1, \cdots, g_n]$ of $\Delta^n(G)$. Obviously φ_1 kills the relation N' and φ_2 the relations R' and S'. Thus the ideal $\mathfrak{q}$ generated by these relations is contained in Ker φ. Since the relations N', R' and S' are homogeneous, $\mathfrak{q}$ is a direct sum $\mathfrak{q} = \bigoplus_{n\geq 1} \mathfrak{q}_n$ of its homogeneous components $\mathfrak{q}_n$. The component $\mathfrak{q}_1$ is obviously additively generated by X_1, the component $\mathfrak{q}_2$ additively by all $X_{f,g,h} := X_gX_h - X_{fg}X_h + X_fX_{gh} - X_fX_g$ and all $X_{f,g} := X_fX_g - X_gX_f$, and the component $\mathfrak{q}_n$ ($n \geq 3$) additively by all $(X_{f,g,h})(X_{g_3}\cdots X_{g_n}), (X_{f,g})(X_{g_3}\cdots X_{g_n}), (X_{g_3}\cdots X_{g_i})(X_{f,g,h})(X_{g_{i+1}}\cdots X_{g_n})$, and $(X_{g_3}\cdots X_{g_i})(X_{f,g})(X_{g_{i+1}}\cdots X_{g_n})$ where $g_3, \cdots, g_n$ ranges over G and i over $\{3, \cdots, n\}$. But the additive generators above of the subgroup $\mathfrak{q}_n$ of the free abelian group $TX(G)_n$ on the elements $X_{g_1}\cdots X_{g_n}$ are the N, R, and S relations for $\Delta^n(G)$. Thus by Theorem 2.13 and Corollary 2.7, $\mathfrak{q}_n =$ Ker φ_n. $\square$

3 T relations and presenting Δ^n for $n \leq n(G)$ or G p-elementary

The goal of this section is to prove Theorems 1.1.2 and 1.2.2. Let G denote a torsion abelian group. Theorem 1.1.2 says that the relations N, R, S and T on the standard generators of $\Delta^n(G)$, present the group $\Delta^n(G)$ when $n \leq n(G)$ or G is p-elementary. Theorem 1.2.2 supposes additionally that G has an ordered basis I, which holds of course if G is p-elementary. The theorem says that the relations N, R, S and T_I on the standard generators of $\Delta^n(G)$, present $\Delta^n(G)$ when $n \leq n_I(G)$ or G is p-elementary. Furthermore it says that the special NRS-generators are a basis of $\Delta^n(G)$. One deduces by a straightforward direct limit argument that (1.1.2) follows from (1.2.2), using the fact that relations $T \supseteq$ relations T_I and $n(G) = \sup_H(\sup_I n_I(H))$ where H ranges over all subgroups of G with an ordered basis and I ranges over all ordered bases of H.

The first task of the section is defining the T relations $T^n(G)$ for $\Delta^n(G)$. This will take several pages and it is recommended that at first reading, the reader just glance at the result obtained in Definition 3.5. After this definition, we assume G has an ordered basis and define special NRS-generators. They are used in turn to define in Definition 3.8 the T_I relations $T_I^n(G)$ for $\Delta^n(G)$. At the conclusion of the definition, the strategy of the proof of Theorem 1.2.2 is elucidated.

Let p denote a natural prime and t a natural number. Let $\langle h\rangle$ denote a group of order p^t generated by h. By (2.6), we know for any n the free $\mathbb{Z}$-module $\Delta^n(\langle h\rangle)$ has the $\mathbb{Z}$-basis

$$\{(h^j - 1)(h-1)^{n-1} \mid 1 \leqslant j \leqslant p^t - 1\}.$$

From the binomial formula applied to the right hand side of the equation $1 = h^{p^t} = ((h-1)+1)^{p^t}$, we obtain that

$$p^t(h-1) = -\sum_{i=2}^{p^t} \binom{p^t}{i} (h-1)^i.$$

From this equation, it follows that for any n there exists an l such that $p^l(h-1) \in \Delta^n(\langle h\rangle)$, e.g. $l = t(n-1)$. The next lemma describes precisely how large l must be.

LEMMA 3.1 Let $\langle h\rangle$ denote a group of order p^t generated by h. Then

$$\begin{array}{ll} p^l(h-1) \in \Delta(\langle h\rangle) \smallsetminus \Delta^2(\langle h\rangle) & \text{if } 0 \leqslant l < t. \\ p^l(h-1) \in \Delta^{(p-1)(l-t+1)+1}(\langle h\rangle) \smallsetminus \Delta^{(p-1)(l-t+1)+2}(\langle h\rangle) & \text{if } l \geqslant t. \end{array}$$

PROOF Suppose $0 \leqslant l < t$. It is obvious that $p^l(h-1) \in \Delta(\langle h\rangle)$. If $p^l(h-1) \in \Delta^2(\langle h\rangle)$ then there are integers $c_1, \cdots, c_{p^t-1}$ such that

$$p^l(h-1) = \sum_{i=1}^{p^t-1} c_i(h^i-1)(h-1).$$

So

$$p^l - \sum_{i=1}^{p^t-1} c_i(h^i-1) = d\sum_{i=0}^{p^t-1} h^i$$

for some $d \in \mathbb{Z}$, since the annihilator of $(h-1)$ in $\mathbb{Z}(\langle h\rangle)$ is $\mathbb{Z}(\sum_{i=0}^{p^t-1} h^i)$. Multiplying both sides of the identity above by $\sum_{i=0}^{p^t-1} h^i$, we get the equation

$$p^l\sum_{i=0}^{p^t-1} h^i = d(\sum_{i=0}^{p^t-1} h^i)^2 = dp^t\sum_{i=0}^{p^t-1} h^i,$$

which implies that $p^l = dp^t$, contrary to the assumption that $l < t$.

Suppose $l \geqslant t$. We use induction on $l-t$ to prove first that $p^l(h-1) \in \Delta^{(p-1)(l-t+1)+1}(\langle h\rangle)$.

When $l = t$, then

$$p^t(h-1) = -\sum_{i=2}^{p^t} \binom{p^t}{i} (h-1)^i.$$

If $p=2$, it is obvious that $p^t(h-1)\in\Delta^2(\langle h\rangle)$. So we assume that $p\geqslant 3$. Since $(h-1)^i\in\Delta^p(\langle h\rangle)$ for all $i\geqslant p$, it follows that

$$p^t(h-1)\equiv -\sum_{i=2}^{p-1}\binom{p^t}{i}(h-1)^i \qquad (\bmod \Delta^p(\langle h\rangle)).$$

Since each of $\binom{p^t}{2},\cdots,\binom{p^t}{p-1}$ is divisible by p^t, we can iterate the formula above and obtain that $p^t(h-1)\in\Delta^p(\langle h\rangle)$.

Suppose $l>t$. Consider the equation

$$p^l(h-1)=-p^{l-t}\sum_{i=2}^{p^t}\binom{p^t}{i}(h-1)^i.$$

We shall use induction on $l-t$ to show that each summand $p^{l-t}\binom{p^t}{i}(h-1)^i$ on the right such that $p|i$ or $p\nmid i$ and $i>p$, lies in $\Delta^{(p-1)(l-t+1)+1}(\langle h\rangle)$.

Suppose $p|i$. Choose k such that $p^k\|i$ (p^k divides i but p^{k+1} does not divide i). Then $p^{t-k}\|\binom{p^t}{i}$ and thus $p^{l-k}\|p^{l-t}\binom{p^t}{i}$. For the moment assume that $l-k\geqslant t$. Since $l-k-t<l-t$, we can conclude by the induction assumption on $l-t$ that

$$p^{l-t}\binom{p^t}{i}(h-1)^i\in\Delta^{(p-1)(l-k-t+1)+1}(\langle h\rangle)\Delta^{i-1}(\langle h\rangle)\subseteq\Delta^{(p-1)(l-k-t+1)+i}(\langle h\rangle).$$

Since $p^k\|i$ ($k\geqslant 1$), we have

$$(p-1)(l-k-t+1)+i\geqslant(p-1)(l-k-t+1)+p^k\geqslant(p-1)(l-t+1)+1.$$

Thus

$$p^{l-t}\binom{p^t}{i}(h-1)^i\in\Delta^{(p-1)(l-t+1)+1}(\langle h\rangle).$$

Assume now $l-k<t$. Then $l-t<k$, from which it follows that $i\geqslant(p-1)(l-t+1)+1$. This concludes the proof when $p|i$.

Suppose $p\nmid i$ and $i>p$. Then $p^{t-1}|\binom{p^t}{i}$ and thus $p^{l-1}|p^{l-t}\binom{p^t}{i}$. Since $l-1-t<l-t$, we conclude by the induction assumption on $l-t$ that

$$p^{l-t}\binom{p^t}{i}(h-1)^i\in\Delta^{(p-1)(l-t)+1}(\langle h\rangle)\Delta^{(i-1)}(\langle h\rangle)\subseteq\Delta^{(p-1)(l-t+1)+1}(\langle h\rangle).$$

The conclusions above that the summands $p^{l-t}\binom{p^t}{i}(h-1)^i$ lie in $\Delta^{(p-1)(l-t+1)+1}(\langle h\rangle)$ when $p|i$ or $p\nmid i$ and $i>p$, imply that

$$p^l(h-1)\equiv -p^{l-t}\sum_{i=2}^{p-1}\binom{p^t}{i}(h-1)^i \qquad (\bmod\ \Delta^{(p-1)(l-t+1)+1}(\langle h\rangle)).$$

Since $p^l\|p^{l-t}\binom{p^t}{i}$ when $2\leqslant i\leqslant p-1$, we can iterate the formula above and obtain that $p^l(h-1)\in\Delta^{(p-1)(l-t+1)+1}(\langle h\rangle)$.

Next we prove that if $l\geqslant t$ then

$$p^l(h-1)\notin\Delta^{(p-1)(l-t+1)+2}(\langle h\rangle).$$

From the proof above that $p^l(h-1)\in\Delta^{(p-1)(l-t+1)+1}(\langle h\rangle)$, it is easy to see that

$$p^{l-t}\binom{p^t}{i}(h-1)^i\in\Delta^{(p-1)(l-t+1)+2}(\langle h\rangle)$$

for all $i\neq 1,p$. So one has

$$p^l(h-1)\equiv -p^{l-t}\binom{p^t}{p}(h-1)^p \qquad (\bmod\ \Delta^{(p-1)(l-t+1)+2}(\langle h\rangle)).$$

Suppose $p^l(h-1)\in\Delta^{(p-1)(l-t+1)+2}(\langle h\rangle)$. We shall obtain a contradiction. From the congruence above, we obtain that $p^{l-t}\binom{p^t}{p}(h-1)^p\in\Delta^{(p-1)(l-t+1)+2}(\langle h\rangle)$. Thus there are integers $c_1,\cdots,c_{p^t-1}$ such that

$$p^{l-t}\binom{p^t}{p}(h-1)^p=\sum_{i=1}^{p^t-1}c_i(h^i-1)(h-1)^{(p-1)(l-t+1)+1}.$$

Since $\mathbb{Z}(\langle h\rangle)$ has no nonzero nilpotent elements([8], or by an elementary inductive argument), it follows that

$$p^{l-t}\binom{p^t}{p}(h-1)=\sum_{i=1}^{p^t-1}c_i(h^i-1)(h-1)^{(p-1)(l-t)+1}\in\Delta^{(p-1)(l-t)+2}(\langle h\rangle).$$

Since $p^{t-1}\|\binom{p^t}{p}$, one can write $\binom{p^t}{p}=p^{t-1}u$ for some $u\in\mathbb{Z}$ such that $(u,p)=1$. Let $v,w\in\mathbb{Z}$ such that $uv+pw=1$. Since $p^{l-1}u(h-1)=p^{l-t}\binom{p^t}{p}(h-1)\in\Delta^{(p-1)(l-t)+2}(\langle h\rangle)$, it follows that

$$p^{l-1}(h-1)=p^{l-1}uv(h-1)+p^lw(h-1)\in\Delta^{(p-1)(l-t)+2}(\langle h\rangle).$$

Thus we have proved that if

$$p^l(h-1) \in \Delta^{(p-1)(l-t+1)+2}(\langle h \rangle) \qquad \text{then} \qquad p^{l-1}(h-1) \in \Delta^{(p-1)(l-t)+2}(\langle h \rangle).$$

Applying the implication above consecutively for $l, l-1, l-2, \ldots, t$, we get that $p^{t-1}(h-1) \in \Delta^2(\langle h \rangle)$, which contradicts our result in the first paragraph of the proof. $\square$

We shall need the following notation. If z is an integer or rational number, let

$$\{z\}_{\geq 0} = \text{ smallest nonnegative integer } \geq z.$$

If p is a prime number, let

$$v_p : \mathbb{Q} \to \mathbb{Z}, z \mapsto v_p(z)$$

denote the discrete p-adic valuation on $\mathbb{Q}$. If $r \geq s$ are natural numbers and $\binom{r}{s}$ the binomial coefficient they determine then it will be useful to keep in mind that $v_p\binom{r}{s} = v_p(r) - v_p(s)$.

Proposition 3.2 Let $\langle h \rangle$ denote a group of finite order $|h|$ generated by h. Factor $h = h_1 \cdots h_\kappa$ such that each $|h_\alpha|$ $(1 \leq \alpha \leq \kappa)$ is a prime power $p_\alpha^{t_\alpha}$ and $p_\alpha \neq p_\beta$ for $\alpha \neq \beta$. If z is an integer, let

$$c(h,z) = \text{ infimum } \{\{(p_\alpha - 1)(v_{p_\alpha}(z) - t_\alpha + 1)\}_{\geq 0} + 1 \mid 1 \leq \alpha \leq \kappa\}.$$

Then

$$z(h-1) \in \Delta^{c(h,z)}(\langle h \rangle) \backslash \Delta^{c(h,z)+1}(\langle h \rangle).$$

Proof Suppose that $|h| = p^t$ is a prime power. From the definition of v_p, it follows that we can factor $z = ap^{v_p(z)}$ where a is relatively prime to p. By Lemma 3.1, $p^{v_p(z)}(h-1) \in \Delta^{c(h,z)}(\langle h \rangle) \backslash \Delta^{c(h,z)+1}(\langle h \rangle)$. We also know by Lemma 3.1 that the quotient group is annihilated by some power of p. Thus the action of the element a on $\Delta^{c(h,z)}(\langle h \rangle) / \Delta^{c(h,z)+1}(\langle h \rangle)$ by multiplication is a group isomorphism. Thus $z(h-1) \not\equiv 0$ mod $\Delta^{c(h,z)+1}(\langle h \rangle)$, i.e. $z(h-1) \in \Delta^{c(h,z)}(\langle h \rangle) \backslash \Delta^{c(h,z)+1}(\langle h \rangle)$.

Suppose now that $\kappa \geq 2$. Choose β $(2 \leq \beta \leq \kappa)$ such that $c(h,z) = \{(p_\beta - 1)(v_{p_\beta}(z) - t_\beta + 1)\}_{\geq 0} + 1$. (In general β is not unique.) Give $\{1, \cdots, \kappa\}$ its natural ordering $1 < 2 < \cdots < \kappa$. If P denotes a subset of $\{1, \cdots, \kappa\}$, let $(h-1)_P = \prod\limits_{\alpha \in P} (h_\alpha - 1)$. An easy induction argument on κ shows that $(h-1) = \sum_P (h-1)_P$ where P sums over all nonempty subsets of $\{1, \cdots, \kappa\}$. If $\alpha \in P$ then $z(h-1)_P \in \Delta^{\{(p_\alpha - 1)(v_{p_\alpha}(z) - t_\alpha + 1)\}_{\geq 0} + 1}(\langle h \rangle)$ by the paragraph above. Thus $z(h-1) = \sum_P z(h-1)_P \in \Delta^{c(h,z)}(\langle h \rangle)$. Suppose $z(h-1) \in \Delta^{c(h,z)+1}(\langle h \rangle)$. Let $\varphi : \langle h \rangle \to \langle h_\beta \rangle, \prod\limits_{\alpha \neq \beta} h_\alpha \mapsto 1$. The homomorphism φ induces a homomorphism $\Delta^{c(h,z)}(\langle h \rangle) / \Delta^{c(h,z)+1}(\langle h \rangle) \to \Delta^{c(h,z)}(\langle h_\beta \rangle) / \Delta^{c(h,z)+1}(\langle h_\beta \rangle)$ taking $z(h-1)$ to $z(h_\beta - 1)$ and our supposition above implies that $z(h_\beta - 1) \equiv 0 \mod \Delta^{c(h,z)+1}(\langle h_\beta \rangle)$. But since $c(h,z) = c(h_\beta, z)$ and $|h_\beta|$ is a prime power, the last conclusion above contradicts the conclusion of the first paragraph of the proof. $\square$

We are now prepared to begin defining the T relations $T^m(h,g)$ for pairs of elements h and g in a torsion abelian group G such that $3 \leq m \leq |g|+1$. These will be used in turn to define all T relations. The whole process will take several pages.

Let g and m be as above. Let X denote an indeterminate and $\mathbb{Z}[X]$ the integral polynomial ring in X. It is easy to see that for each m, the free $\mathbb{Z}$-module of all polynomials $f(X) \in \mathbb{Z}[X]$ such that degree $(f(X)) \leqslant |g|-1$ and $f(1) = 0$ has a $\mathbb{Z}$-basis consisting of the set

$$\{(X-1)^i \mid 1 \leqslant i \leqslant m-2\} \cup \{(X^i - 1)(X-1)^{m-2} \mid 1 \leqslant i \leqslant |g| - m + 1\}.$$

Let $F_{|g|}^{(m)}(X)$ denote the polynomial

$$F_{|g|}^{(m)}(X) = (X^{|g|-m+2} - 1)(X-1)^{m-2} - X^{|g|} + 1.$$

This polynomial has degree $|g|-1$. Using the basis above, we define the integers $a_i^{(m)}$ $(1 \leqslant i \leqslant m-2)$ and $b_i^{(m)}$ $(1 \leqslant i \leqslant |g|-m+1)$ such that

$$F_{|g|}^{(m)}(X) = \sum_{i=1}^{m-2} a_i^{(m)} (X-1)^i + \sum_{i=1}^{|g|-m+1} b_i^{(m)} (X^i - 1)(X-1)^{m-2}. \tag{3.3}$$

Let $\partial^i F_{|g|}^{(m)}(X)$ denote the i-th derivative of $F_{|g|}^{(m)}(X)$. For $1 \leqslant i \leqslant m-2$, one computes that $\partial^i F_{|g|}^{(m)}(1) = -|g|(|g|-1) \cdots (|g|-i+1) = a_i^{(m)}(i!)$. Thus

$$a_i^{(m)} = -\binom{|g|}{i}.$$

We use the computation above of the $a_i^{(m)}$'s to define further sequences $a_{i,1}^{(m)}, \cdots, a_{i,|h|-1}^{(m)}$ of integers. The next proposition provides the tools for doing this. The sequences will be needed in defining the T relation $T^m(h,g)$.

PROPOSITION 3.4 Let h generate a finite group $\langle h\rangle$ of order $|h|$. Factor $h = h_1 \cdots h_\kappa$ such that each $|h_\alpha| \quad (1 \le \alpha \le \kappa)$ is a prime power $p_\alpha^{t_\alpha}$ and $p_\alpha \ne p_\beta$ for $\alpha \ne \beta$. Let g generate a finite group $\langle g\rangle$ of order $|g|$. Let m be a natural number such that $3 \le m \le |g|+1$.

(3.4.1) For each $\alpha \quad (1 \le \alpha \le \kappa)$ and each $i \quad (1 \le i \le m-2)$, define

$$e_{\alpha,i}^{(m)}(h,g) = \left\{ \frac{m-(i+1)}{p_\alpha - 1} + t_\alpha - 1 - v_{p_\alpha}\binom{|g|}{i} \right\}_{\ge 0}.$$

For each α, define

$$c_i^{(m)}(h,g) = \prod_\alpha p_\alpha^{e_{\alpha,i}^{(m)}(h,g)}.$$

Then, in the notation of (3.2), $c(h, c_i^{(m)}(h,g)\binom{|g|}{i}) \ge m-i$. Furthermore if z is an integer such that $c(h, z\binom{|g|}{i}) \ge m-i$ then $c_i^{(m)}(h,g) \mid z$.

(3.4.2) For a finite set S of natural numbers, let l.c.m(S) denote the least common multiple of the numbers in S. Define

$$c^{(m)}(h,g) = \text{ l.c.m } \{c_i^{(m)}(h,g) \mid 1 \le i \le m-2\}.$$

For each α $(1 \le \alpha \le \kappa)$, define

$$e_\alpha^{(m)}(h,g) = \text{ supremum } \{e_{\alpha,i}^{(m)}(h,g) \mid 1 \le i \le m-2\}.$$

Clearly

$$c^{(m)}(h,g) = \prod_\alpha p_\alpha^{e_\alpha^{(m)}(h,g)}.$$

Our conclusions are: $c(h, c^{(m)}(h,g)\binom{|g|}{i}) \ge m-i$ for all $1 \le i \le m-2$. Furthermore $c^{(m)}(h,g)$ is best possible for this result, i.e. if z is an integer such that $c(h, z\binom{|g|}{i}) \ge m-i$ for all $1 \le i \le m-2$ then $c^{(m)}(h,g) \mid z$.

PROOF (3.4.1) By definition

$$c(h, z\binom{|g|}{i}) = \text{ infimum } \{\{(p_\alpha - 1)(v_{p_\alpha}(z) + v_{p_\alpha}\binom{|g|}{i} - t_\alpha + 1)\}_{\geq 0} + 1 \mid 1 \leq \alpha \leq \kappa\}.$$

Thus $c(h, z\binom{|g|}{i}) \geq m - i \Leftrightarrow$ for each α $(1 \leq \alpha \leq \kappa)$

$$v_{p_\alpha}(z) \geq \frac{m-(i+1)}{p_\alpha - 1} + t_\alpha - 1 - v_{p_\alpha}\binom{|g|}{i}.$$

The assertions of (3.4.1) are an immediate consequence of the above.

(3.4.2) The proof of (3.4.2) is similar to that of (3.4.1). Details are left to the reader. □

DEFINITION 3.5 (**T relations**) Let G be a torsion abelian group. Let $h, g \in G$ and let m be a natural number such that $3 \leq m \leq |g| + 1$. Substituting g for X in the definition of $F^{(m)}_{|g|}(X)$ and using equation (3.3), we get the identity

$$(g^{|g|-m+2} - 1)(g-1)^{m-2} = \sum_{i=1}^{m-2} \binom{|g|}{i} (g-1)^i + \sum_{i=1}^{|g|-m+1} b_i^{(m)} (g^i - 1)(g-1)^{m-2}$$

in $\Delta(G)$. By (3.4.2) and (2.6), there are unique integers $a^{(m)}_{i,1}, \cdots, a^{(m)}_{i,|h|-1}$ $(1 \leq i \leq m-2)$ such that

$$c^{(m)}(h, g)\binom{|g|}{i}(h-1) = \sum_{j=1}^{|h|-1} a^{(m)}_{i,j}(h^j - 1)(h-1)^{m-i-1}.$$

Multiplying both sides of the first identity above by $c^{(m)}(h, g)(h-1)$ and then applying the second identity above, we get the equation

$$\begin{aligned} c^{(m)}(h, g)(h-1)(g^{|g|-m+2} - 1)(g-1)^{m-2} = & \sum_{i=1}^{m-2}\sum_{j=1}^{|h|-1} a^{(m)}_{i,j}(h^j - 1)(h-1)^{m-i-1}(g-1)^i + \\ & \sum_{i=1}^{|g|-m+1} c^{(m)}(h, g) b^{(m)}_i (h-1)(g^i - 1)(g-1)^{m-2} \end{aligned}$$

in $\Delta^m(G)$, i.e.

$$c^{(m)}(h,g)[h, g^{|g|-m+2}, \underbrace{g, \cdots, g}_{m-2}] = \sum_{i=1}^{m-2} \sum_{j=1}^{|h|-1} a_{i,j}^{(m)} [h^j, \underbrace{h, \cdots, h}_{m-i-1}, \underbrace{g, \cdots, g}_{i}] +$$
$$\sum_{i=1}^{|g|-m+1} c^{(m)}(h,g) b_i^{(m)} [h, g^i, \underbrace{g, \cdots, g}_{m-2}].$$

This is by definition the T relation

$$T^m(h,g).$$

Let $n \geq m$. Let $k \geq 0$ be any nonnegative integer such that $n \geq k+m$. Let $f_1, \cdots, f_k$ and $f_{k+m+1}, \cdots, f_n$ be sequences of elements in G. Multiplying the relation $T^m(h,g)$ above on the left by $[f_1, \cdots, f_k]$ and on the right by $[f_{k+m+1}, \cdots, f_n]$, we get the T relation denoted by

$$[f_1, \cdots, f_k] T^m(h,g) [f_{k+m+1}, \cdots, f_n]. \tag{3.5.1}$$

Whereas $T^m(h,g)$ is a relation in $\Delta^m(G)$, $[f_1, \cdots, f_k]\ T^m(h,g)\ [f_{k+m+1}, \cdots, f_n]$ is a relation in $\Delta^n(G)$. For $n \geq 3$, set

$$\begin{aligned} T^n(G) = &\{[f_1, \cdots, f_k] T^m(h,g) [f_{k+m+1}, \cdots, f_n] \mid n \geq m \geq 3, k \geq 0, \\ &n \geq m+k, f_1, \cdots, f_k, h, g, f_{k+m+1}, \cdots, f_n \text{ ranges over} \\ &\text{all sequences of elements in } G \text{ such that } |g|+1 \geq m\}. \end{aligned}$$

Setting $m = 3$ and supposing in our relation $T^m(h,g)$ that $|h| = |g| = 2$, we get the relation (T) in [2], in the equivalent form found in Lemma 3.1 of [2]. Setting $m = 3$ and supposing in our relation $T^m(h,g)$ that $|h| = |g| = p$ is a prime number, we get the relation denoted by (T_p) in [4]. Our relation $T^m(h,g)$ where $3 \leq m \leq p+1, p$ a prime number, and $|h| = |g| = p$ is just the relation (T_m) in [9].

It turns out that the set $T^n(G)$ contains many more relations than are actually required for a presentation of $\Delta^n(G)$. The definitions which follow describe what is actually needed.

A certain subset $T^n_I(G) \subseteq T^n(G)$ depending on an ordered basis I of G will suffice. We recall next the notion of an ordered basis for torsion abelian groups. It has been used implicitly already in the definition of an NRS-generator in (2.2).

DEFINITION 3.6 Let G be a nontrivial torsion abelian group. An **ordered basis** for G consists by definition of a totally ordered set I and a function $g : I \to G \backslash \langle 1 \rangle, i \mapsto g_i$, such that $G = \coprod_{i \in I} \langle g_i \rangle$. Obviously any nontrivial finite torsion abelian group possesses an ordered basis. A torsion abelian group G with a specified ordered basis $(I, g : I \to G\backslash\langle 1 \rangle)$ will be called an **(I, g)- group** or simply **I-group**. If G is an I-group and $i \in I$, define

$$G_{\leq i} = \langle g_j \mid j \leq i \rangle,$$

i.e. $G_{\leq i}$ is the subgroup of G generated by all g_j such that $j \leq i$. If $i \in I$ is not the smallest element of I, define

$$G_{<i} = \langle g_j \mid j < i \rangle.$$

In order to define T_I relations, we need the concept of special NRS-generator. The Main Lemma will show that the special NRS-generators in $\Delta^n(G)$ are linearly in dependent. Special NRS-generators are definded next.

DEFINITION 3.7 Let I denote the natural numbers. Let G be a torsion abelian group and (I, g) an ordered basis for G. If $k \in I$, let $I(k) = \{1, \cdots, k\}$. Thus $(I(k), g|_{I(k)})$ is an ordered basis for $G_{\leqq k}$. For each pair of natural numbers k and n, we define by double induction certain sets $\mathcal{G}^n(k)$ of NRS-generators (see (2.2)) with respect to $I(k)$, of degree n as follows. If $f \in G$, set

$$\mathcal{G}^n(f) = \{[\underbrace{f^e, f, \cdots, f}_{n}] \mid 1 \leq e \leq |f| - 1\}.$$

Define

$$\mathcal{G}^n(1) = \mathcal{G}^n(g_1).$$

Suppose $k > 1$ and $\mathcal{G}^n(k')$ has been defined for all n and all $k' < k$. We define now $\mathcal{G}^n(k)$. Define

$$\mathcal{G}^1(k) = \{[f] \mid f \in G_{\leqq k}\}.$$

Suppose $n > 1$ and $\mathcal{G}^{n'}(k)$ has been defined for all $n' < n$. If $n \leq |g_k|$, let

$$\mathcal{G}^n(j,k) = \begin{cases} \{[x(1), \cdots, x(n-j), \underbrace{g_k, \cdots, g_k}_{j}] \mid [x(1), \cdots, x(n-j)] \in \mathcal{G}^{n-j}(k-1)\}, & \\ & \text{if } 1 \leq j \leq n-2 \\ \{[h, \underbrace{g_k^e, g_k, \cdots, g_k}_{n-1}] \mid h \in G_{\leq (k-1)}, 1 \leq e \leq |g_k| - n + 1\}, & \text{if } j = n-1. \end{cases}$$

If $|g_k| < n$ and $1 \leq j \leq |g_k| - 1$, let

$$\mathcal{G}^n(j,k) = \{[x(1), \cdots, x(n-j), \underbrace{g_k, \cdots, g_k}_{j}] \mid [x(1), \cdots, x(n-j)] \in \mathcal{G}^{n-j}(k-1)\}.$$

Define

$$\mathcal{G}^n(k) = \begin{cases} \mathcal{G}^n(g_k) \cup \mathcal{G}^n(k-1) \cup \bigcup_{j=1}^{n-1} \mathcal{G}^n(j,k), & \text{if } n \leq |g_k| \\ \mathcal{G}^n(g_k) \cup \mathcal{G}^n(k-1) \cup \bigcup_{j=1}^{|g_k|-1} \mathcal{G}^n(j,k), & \text{if } |g_k| < n. \end{cases}$$

It is easy to check that each element of $\mathcal{G}^n(k)$ is an NRS-generator (2.2) for $I(k)$, of degree n. The elements of $\mathcal{G}^n(k)$ will be called **special NRS-generators** for $I(k)$, of degree n.

Let I denote now an arbitrary totally ordered set. Let G be an I-group. An NRS-generator $[x(1), \cdots, x(n)]$ with respect to I, of degree n is called **special** if it is special for a finite subset $J \subseteq I$, under the ordering induced from I. The subgroup of $\Delta^n(G)$ generated by all special NRS-generators for I, of degree n is denoted by

$$S_I \Delta^n(G).$$

Definition 3.8 (**$\mathbf{T_I}$ relations**) Let G denote an (I, g)-group. For $n \geq 3$, let

$$T_I^n(G) \subseteq T^n(G)$$

denote the subset of all relations $[f_1, \cdots, f_k]T^m(h, g')[f_{k+m+1}, \cdots, f_n]$ such that the following holds.

(3.8.1) Suppose $k \neq 0$. Then $h = g_{i_{k+1}}, g' = g_{i_{k+2}}, i_{k+1} < i_{k+2}, m = |g_{i_{k+2}}|$ (thus the left hand side of the relation $T^m(g_{i_{k+1}}, g_{i_{k+2}})$ is $c^{(m)}(g_{i_{k+1}}, g_{i_{k+2}})[g_{i_{k+1}}, \underbrace{g_{i_{k+2}}, \cdots, g_{i_{k+2}}}_{m-1}]$),

$[f_1, \cdots, f_k] = [h', g^e_{i_2}, g_{i_3}, \cdots, g_{i_n}]$ is an NRS-generator of kind (2.2.2), $i_k \leq i_{k+1}$, and $[f_{k+m+1}, \cdots, f_n] = [g_{i_{k+m+1}}, g_{i_{k+m+2}}, \cdots, g_{i_n}]$ is a generator satisfying (i) and (ii) below.

(i) If $i_{k+2} < i_{k+m+1}$ then $[g_{i_{k+m+1}}, g_{i_{k+m+2}}, \cdots, g_{i_n}]$ is a special NRS-generator.

(ii) If $i_{k+2} = i_{k+m+2}$ and $r \leq n$ is the smallest number such that $i_{k+2} < i_r$ then $[g_{i_r}, \cdots, g_{i_n}]$ is a special NRS-generator.

(3.8.2) Suppose $k = 0$. Then $g' = g^e_{i_2}$ for some $i_2 \neq 1 \in I, h \in G_{<i_2}, m = |g_{i_2}| - e + 2$ (thus the left hand side of the relation $T^m(h, g_{i_2})$ is $c^{(m)}(h, g_{i_2})[h, g^e_{i_2}, \underbrace{g_{i_2}, \cdots, g_{i_2}}_{m-2}]$), and

$[f_{m+1}, \cdots, f_n] = [g_{i_{m+1}}, g_{i_{m+2}}, \cdots, g_{i_n}]$ is a generator satisfying (i) and (ii) above.

For fixed h and g', a relation of the form $[f_1, \cdots, f_k]T^m(h, g')[f_{k+m+1}, \cdots, f_n] \in T^n_I(G)$ will be called a

$$\boldsymbol{T^{m,n}(h, g')}\textbf{-relation}.$$

For $n \geq 1$, define

$$\tilde{\Delta}^n_I(G) = \begin{cases} \hat{\Delta}_n(G) \quad , & \text{if } n = 1, 2 \\ \hat{\Delta}_n(G)/T^n_I(G) \quad , & \text{if } n \geq 3. \end{cases}$$

The subgroup of $\tilde{\Delta}^n_I(G)$ generated by all special NRS-generators for I, of degree n is denoted by

$$S\tilde{\Delta}^n_I(G) \text{ or } S_I\tilde{\Delta}^n(G).$$

The purpose of the Main Construction below is to provide a filtration of the quotient group $\tilde{\Delta}^n(G)/S_I\tilde{\Delta}^n(G)$. This filtration is then used in the Main Lemma to show that the group is a torsion group. From this it follows of course that the quotient group $\Delta^n(G)/S_I\Delta^n(G)$ is also torsion. The Main Lemma will also show, as noted above already, that the special NRS-generators in $\Delta^n(G)$ are linearly independent, from which it follows trivially that the map $S_I\tilde{\Delta}^n(G) \to S_I\Delta^n(G)$ is bijective. Thus if $\tilde{\Delta}^n(G) = S_I\tilde{\Delta}^n(G)$ then the special NRS-generators in $\Delta^n(G)$ are a $\mathbb{Z}$-basis for $\Delta^n(G)$ and $\Delta^n(G)$ is presented by the N, R, S and T_I^n relations. The number $n_I(G)$ is defined in (3.11) and Theorem 3.15 will show that if $n \leq n_I(G)$ then $\tilde{\Delta}^n(G) = S_I\tilde{\Delta}^n(G)$. This will complete the proof of Theorem (1.2.2).

Main Construction 3.9 Let (I, g) be an ordered basis for the torsion abelian group G. Let $x = [x(1), \cdots, x(n)]$ be an NRS-generator (2.2) with respect to I, of degree n. Define

$$MI(x) = \text{ infimum } \{i \in I \mid x(1) \in G_{\leq i}\}.$$

$MI(x)$ is called the **minimal index** of x.

For $i \in I$, define

$$L_i(x) = |\{j \mid 1 \leq j \leq n, x(j) \in \langle g_i \rangle\}|.$$

$L_i(x)$ is called the **length of x at i**. There are 2 kinds of NRS-generators according to (2.2.1) and (2.2.2). If x is of kind (2.2.1), write $x = [g_{i_1}^e, g_{i_2}, \cdots, g_{i_n}]$ where $1 \leq e \leq |g_{i_1}| - 1$, and if x is of kind (2.2.2), write $x = [h, g_{i_2}^e, g_{i_3}, \cdots, g_{i_n}]$ where $h \in G_{<i_2}$ and $1 \leq e \leq |g_{i_2}| - 1$. Define

$$E(x) = e.$$

$E(x)$ is called the **exponent** of x. For $i \in I$, define

$$\mathcal{L}_i(x) = \begin{cases} \left.\begin{matrix} 0 \\ \{L_i(x) - |g_i| + 1\}_{\geq 0} \end{matrix}\right\} & \text{if } x \text{ of kind (2.2.1) and } \left\{\begin{matrix} i \leq MI(x) \\ i > MI(x) \end{matrix}\right. \\ \\ \left.\begin{matrix} 0 \\ \{E(x) + L_i(x) - |g_i|\}_{\geq 0} \\ \{L_i(X) - |g_i| + 1\}_{\geq 0} \end{matrix}\right\} & \text{if } x \text{ of kind (2.2.2) and } \left\{\begin{matrix} i \leq MI(x) \\ i = i_2 \\ i > i_2 \end{matrix}\right. \end{cases}.$$

$\mathcal{L}_i(x)$ is called the **reduced length of x at i**. One checks straightforward that $\mathcal{L}_i(x) = 0$ for all $i \in I \Leftrightarrow x$ is a special NRS-generator with respect to I. Moreover for any $x, \mathcal{L}_1(x) = 0$ and for any i and any $x, \mathcal{L}_i(x) \leq n - 1$.

Suppose now that $|I| = k$ is finite and identify $I = \{1, \cdots, k\}$. Let

$$\mathcal{G}^n(k)$$
$$\text{NRS}-\mathcal{G}^n(k)$$

denote respectively the set of all special NRS-generators, respectively NRS-generators, for I of degree n. We construct a filtration bridging the gap between $\mathcal{G}^n(k)$ and NRS-$\mathcal{G}^n(k)$. For any pair $(i, \ell) \in I \times [0, n-1]$ (where $[0, n-1] = \{0, 1, \cdots, n-1\}$), define

$$\mathcal{G}^n_{i,\ell}(k) = \{x \in \text{NRS} - \mathcal{G}^n(k) \mid \mathcal{L}_{i'}(x) = 0 \ \forall\ i' > i, \mathcal{L}_i(x) \leq \ell\}.$$

Defining a total ordering on $I \times [0, n-1]$ by

$$(i, \ell) < (i', \ell') \Leftrightarrow i < i' \text{ or } i = i' \text{ and } \ell < \ell',$$

we obtain a filtration

$$\mathcal{G}^n(k) = \mathcal{G}^n_{2,0}(k) \subseteq \mathcal{G}^n_{2,1}(k) \subseteq \cdots \subseteq \mathcal{G}^n_{k,n-1}(k) = \text{NRS} - \mathcal{G}^n(k). \tag{3.9.1}$$

This in turn determines a filtration

$$S_I\tilde{\Delta}^n(G) = S_{2,0}\tilde{\Delta}^n(G) \subseteq S_{2,1}\tilde{\Delta}^n(G) \subseteq \cdots \subseteq S_{k,n-1}\tilde{\Delta}^n(G) = \tilde{\Delta}^n_I(G) \tag{3.9.2}$$

where we define

$$S_{i,\ell}\tilde{\Delta}^n(G) = \langle \mathcal{G}^n_{i,\ell}(k) \rangle_{\tilde{\Delta}^n_I(G)}$$

and $\langle \mathcal{G}^n_{i,\ell}(k) \rangle_{\tilde{\Delta}^n_I(G)}$ denotes the subgroup of $\tilde{\Delta}^n_I(G)$ generated by $\mathcal{G}^n_{i,\ell}(k)$.

Main Lemma 3.10 Let (I, g) denote an ordered basis for the torsion abelian group G. Let $\mathcal{G}^n(I)$ denote the set of all special NRS-generators for I, of degree n. Then the elements of $\mathcal{G}^n(I)$ are $\mathbb{Z}$-linearly independent in $\tilde{\Delta}^n_I(G)$ and $\Delta^n(G)$ and the quotients $\tilde{\Delta}^n_I(G)/S_I\tilde{\Delta}^n(G)$ and $\Delta^n(G)/S_I\Delta^n(G)$ are torsion. Moreover if I is finite then $|\mathcal{G}^n(I)| = |G| - 1$.

Proof One reduces straigthforward as in the proof of Lemma 2.3 to the case I is finite. Identify I with an interval $\{1, 2, \cdots, u\}$ and adopt the notation developed in (3.7) and (3.9).

Let $k \in I$. We show first by double induction on k and n that $|\mathcal{G}^n(k)| = |G_{\leq k}| - 1$. Suppose $k = 1$. By definition, $\mathcal{G}^n(1) = \mathcal{G}^n(g_1)$ and clearly $|\mathcal{G}^n(g_1)| = |G_{\leq 1}| - 1$. Suppose $k > 1$ and the result has been proved for all $1 \leq k' < k$. Suppose $n = 1$. Trivially $\mathcal{G}^1(k) = \{[f] \mid f \in G_{\leq k} \backslash \langle 1 \rangle\}$. Thus $|\mathcal{G}^1(k)| = |G_{\leq k}| - 1$. Suppose $n > 1$ and the result is true for all $1 \leq n' < n$. According to the definition of $\mathcal{G}^n(k)$, we can divide the proof into 2 cases, namely $n \leq |g_k|$ and $|g_k| < n$.

Suppose $n \leq |g_k|$. Then $|\mathcal{G}^n(k)| = |\mathcal{G}^n(g_k)| + |\mathcal{G}^n(k-1)| + \sum_{j=1}^{n-1} |\mathcal{G}^n(j,k)|$. Clearly $|\mathcal{G}^n(g_k)| = |g_k| - 1$ and by our induction assumption, $|\mathcal{G}^n(k-1)| = |G_{\leq (k-1)}| - 1$, $|\mathcal{G}^n(j,k)| = |G_{\leq (k-1)}| - 1$ $(1 \leq j \leq n-2)$, and $|\mathcal{G}^n(n-1,k)| = (|G_{\leq (k-1)}| - 1)(|g_k| - n + 1)$. Thus $|G_{\leq k}| = (|g_k| - 1) + (|G_{\leq (k-1)}| - 1)|g_k| = |G_{\leq (k-1)}||g_k| - 1 = |G_{\leq k}| - 1$.

Suppose $|g_k| < n$. Then $|\mathcal{G}^n(k)| = |\mathcal{G}^n(g_k)| + |\mathcal{G}^n(k-1)| + \sum_{j=1}^{|g_k|-1} |\mathcal{G}^n(j,k)| = (|g_k| - 1) + (|G_{\leq (k-1)}| - 1)|g_k| = |G_{\leq (k-1)}||g_k| - 1 = |G_{\leq k}| - 1$.

Since we can take above $k = u$, it follows that $|\mathcal{G}^n(I)| = |G| - 1$.

The group $\Delta(G)/\Delta^n(G)$ is torsion by Proposition 3.2. Below it will be shown that the group $\Delta^n(G)/S_I\Delta^n(G)$ is torsion. Thus $|G| - 1 = \text{rank } \Delta(G) = \text{rank } S_I\Delta^n(G)$. Since $\mathcal{G}^n(I)$ has precisely $|G| - 1$ elements and generates $S_I\Delta^n(G)$, it follows that the elements of $\mathcal{G}^n(I)$ are $\mathbb{Z}$-linearly independent in $\Delta^n(G)$. Thus they are obviously $\mathbb{Z}$-linearly independent in $\tilde{\Delta}^n_I(G)$.

To show that $\Delta^n(G)/S_I\Delta^n(G)$ is torsion, it suffices to show that $\tilde{\Delta}^n_I(G)/S_I\tilde{\Delta}^n(G)$ is torsion. We show this next.

It is enough to show that the quotient of any 2 consecutive members of the filtration (3.9.2) is torsion. From the definition of the members $S_{i,\ell}\tilde{\Delta}^n(G)$ of the filtration, it follows that if $i > 2$ then $S_{i,0}\tilde{\Delta}^n(G) = S_{i-1,n-1}\tilde{\Delta}^n(G)$. Thus it is enough to show that for any (i,ℓ) such that $\ell > 0$, the quotient $S_{i,\ell}\tilde{\Delta}^n(G)/S_{i,\ell-1}\tilde{\Delta}^n(G)$ is torsion.

Let $x \in \mathcal{G}^n_{i,\ell}(|I|)$. By definition, $\mathcal{L}_i(x) \leq \ell$. If there are no x's such that $\mathcal{L}_i(x) = \ell$ then $\mathcal{G}^n_{i,\ell}(|I|) = \mathcal{G}^n_{i,\ell-1}(|I|)$ and we are done. So we assume $\mathcal{L}_i(x) = \ell$. We must show that some integral multiple $zx \in S_{i,\ell-1}\tilde{\Delta}^n(G)$.

Suppose x is of kind (2.2.1). Thus x has the form $x = [g^e_{i_1}, g_{i_2}, \cdots, g_{i_n}]$. Since $\mathcal{L}_i(x) = \ell > 0$, it follows that $i = i_r$ for some $2 < r \leq n$. In general, r is not unique. If r is the smallest number such that $i = i_r$ then by a $T^{|g_{i_r}|+1,n}(g_{i_{r-1}}, g_{i_r})$-relation, $c^{(|g_{i_r}|+1)}(g_{i_{r-1}}, g_{i_r})x$ is a sum of generators each of which can be written using (2.3) as a sum of generators in $\mathcal{G}^n_{i,\ell-1}(|I|)$.

Suppose x is of kind (2.2.2). Thus x has the form $x = [h, g^e_{i_2}, g_{i_3}, \cdots, g_{i_n}]$. Since $\mathcal{L}_i(x) = \ell > 0$, it follows that $i = i_r$ for some $2 \leq r \leq n$. Suppose r is the smallest number such that $i = i_r$. If $r = 2$ then by a $T^{m,n}(h, g^e_{i_2})$-relation where $m = |g_{i_2}| - e + 2$, $c^{(m)}(h, g^e_{i_2})x$ is a sum of generators each of which can be written using (2.3) as a sum of generators in $\mathcal{G}^n_{i,\ell-1}(|I|)$. If $r = 3$ then by a $T^{|g_{i_3}|+1,n}(g^e_{i_2}, g_{i_3})$-relation, $c^{(|g_{i_3}|+1)}(g^e_{i_2}, g_{i_3})x$ is a sum of generators each of which can be written using (2.3) as a sum of generators in $\mathcal{G}^n_{i,\ell-1}(|I|)$. Finally if $r > 3$ then by a $T^{|g_{i_r}|+1,n}(g_{i_{r-1}}, g_{i_r})$-relation, $c^{(|g_{i_r}|+1)}(g_{i_{r-1}}, g_{i_r})x$ is a sum of generators each of which can be written using (2.3) as a sum of generators in $\mathcal{G}^n_{i,\ell-1}(|I|)$. $\square$

DEFINITION 3.11 Let (I, g) denote an ordered basis for the torsion abelian group G.

Let i_0 denote the smallest element of I, which might not exist. Let

$$\begin{aligned}\mathcal{M}_I(G) = \ & \{m \in \mathbb{N} \mid \text{ if } 1 \leq k \leq \text{ infimum } \{m-2, |g_i| - 1 \mid i \in I\backslash\{i_0\}\} \\ & \text{then } \forall\ i \in I\backslash\{i_0\},\ \forall\ h \in G_{<i}\backslash\langle 1\rangle, \text{ and } \forall \text{ natural primes } p_\alpha \\ & \text{such that } p_\alpha |\ |h|, m \leq (k+1) + (p_\alpha - 1)(v_{p_\alpha}\tbinom{|g_i|}{k} - v_{p_\alpha}(|h|) + 1)\}.\end{aligned}$$

It is a logical triviality that $1, 2 \in \mathcal{M}_I(G)$, because the inequality $1 \leq k \leq$ infimum $\{m - 2, |g_i| - 1\}$ is never satisfied for $m = 1$ and 2. Define

$$n_I(G) = \text{ supremum } \mathcal{M}_I(G).$$

Suppose now that G is an arbitrary torsion abelian group. Define

$$\begin{aligned}n(G) = \ & \text{supremum } \{n' \mid \text{ given a finite subgroup} \\ & H \subseteq G, \exists \text{ a finite subgroup } H' \supseteq H \text{ and} \\ & \text{an ordered basis } (I', g) \text{ of } H' \text{ such that} \\ & n_{I'}(H') \geq n'\}.\end{aligned}$$

In view of Theorems 3.14 and 3.15 below, it is very useful knowing when the numbers $n_I(G)$ and $n(G)$ are large. The next two lemmas tell us when they are infinite.

LEMMA 3.12 Let (I, g) be an ordered basis for the torsion abelian group G. Let i_0 denote the smallest element of I, which might not exist. If $|I| = 1$ then $n_I(G) = \infty$ and if $|I| \geq 2$ then $2 \leq n_I(G) \leq$ infimum $\{2 + (p_\alpha - 1)v_{p_\alpha}(|g_i|) \mid i \in I\backslash\{i_0\}, h \in G_{<i}\backslash\langle 1\rangle, p_\alpha$ prime number, $p_\alpha \mid |h|\}$.

PROOF It is a logical triviality that $\mathbb{N} \subseteq \mathcal{M}_I(G)$ whenever $|I| = 1$, because $I \backslash \{i_0\} = \emptyset$. Thus if $|I| = 1$ then $n_I(G) = \infty$. Suppose $|I| \geq 2$. It follows directly from (3.11) that $2 \leq n_I(G)$. Suppose $3+(p_\alpha - 1)v_{p_\alpha}(|g_i|) \in \mathcal{M}_I(G)$ for some $i \in I \backslash \{i_0\}, h \in G_{<i} \backslash \langle 1 \rangle$, and prime number p_α such that $p_\alpha \mid |h|$. We shall arrive at a contradiction. By definition, $3 + (p_\alpha - 1)v_{p_\alpha}(|g_i|) \leq (k+1) + (p_\alpha - 1)(v_{p_\alpha}\binom{|g_i|}{k} - v_{p_\alpha}(|h|) + 1)$ for all natural numbers k satisfying $1 \leq k \leq$ infimum $\{1+(p_\alpha - 1)v_{p_\alpha}(|g_i|), |g_i| - 1\}$. Choosing $k = 1$ and keeping in mind that $v_{p_\alpha}\binom{|g_i|}{k} = v_{p_\alpha}(|g_i|) - v_{p_\alpha}(k)$, we deduce from the inequality above that $3 \leq 2$, an obvious contradiction. $\square$

Whereas the lemma above shows that $n_I(G)$ can be infinite only for finite cyclic groups G, the next lemma shows that $n(G)$ is infinite for a much larger class of groups.

LEMMA 3.13 Let $\mathcal{F}$ denote the family of all groups of the form $S_p^{-1}\mathbb{Z}/\mathbb{Z}$ where p denotes an arbitrary natural prime number, S_p the multiplicative set $\{p^i \mid i \geq 0\}$ in $\mathbb{Z}$, and $S_p^{-1}\mathbb{Z}$ the ring of S_p-fractions of $\mathbb{Z}$. If G is an arbitrary direct sum of members of $\mathcal{F}$ then $n(G) = \infty$.

The proof of the lemma is left as an exercise.

THEOREM 3.14 If G is a torsion abelian group and $n \leq n(G)$ or G is p-elementary then the canonical homomorphism $\hat{\Delta}^n(G)/T^n(G) \to \Delta^n(G)$ is an isomorphism.

Theorem 3.14 is an immediate consequence of the following theorem.

THEOREM 3.15 If G is a torsion abelian group with an ordered basis (I, g) and $n \leq n_I(G)$ or G is p-elementary then the canonical homomorphism $\tilde{\Delta}^n_I(G) \to \Delta^n(G)$ is an isomorphism and the set $\mathcal{G}^n(I)$ of all special NRS-generators of degree n is a basis for $\Delta^n(G)$.

PROOF By the Main Lemma 3.10, $\mathcal{G}^n(I)$ is a basis for $S_I\tilde{\Delta}^n(G)$ and $S_I\Delta^n(G)$. Thus the canonical homomorphism $S_I\tilde{\Delta}^n(G) \to S_I\Delta^n(G)$ is an isomorphism. To complete the proof of the theorem, it suffices to show that $\tilde{\Delta}^n_I(G) = S_I\tilde{\Delta}^n(G)$. Reduce as in the proof of Lemma 2.3 to the case I is finite. Let 1 denote the smallest element of I. It is enough now to show that the quotient of any 2 consecutive members of the filtration (3.9.2) is trivial. The proof of the Main Lemma 3.10 shows that this will be true, if the natural numbers

$$\begin{aligned} &c^{(|g_{i_r}|+1)}(g_{i_{r-1}}, g_{i_r}), \\ &c^{(m)}(h, g_{i_2}) \text{ where } m = |g_{i_2}| - e + 2, \\ &c^{(|g_{i_3}|+1)}(g_{i_2}^e, g_{i_3}), \\ &c^{(|g_{i_r}|+1)}(g_{i_{r-1}}, g_{i_r}) \end{aligned} \tag{3.15.1}$$

occurring in the last 2 paragraphs of the proof are all 1.

Suppose G is p-elementary. A natural number $c^{(\ )}(\ ,\ )$ in (3.15.1) is 1 if and only if the integers $e^{(\ )}_{\alpha,i}(\ ,\ )$ of (3.4.1), which are used to define $c^{(\ )}(\ ,\ )$, are all 0. It is easy to check that the integers $e^{(\ )}_{\alpha,i}(\ ,\ ) = 0$ when G is p-elementary.

Suppose that $n \le n_I(G)$. By Theorem 3.5, we can assume that $3 \le n$. Consider the natural numbers in (3.15.1). For the moment, leave aside the second number. Since the indices i_r, i_3, and i_r occurring respectively in the first, third, and fourth numbers, lie in $I\backslash\{1\}$, it follows respectively that $n \ge |g_{i_r}| + 2, n \ge |g_{i_3}| + 2$, and $n \ge |g_{i_r}| + 2$. But $n \le n_I(G)$ and by Lemma 3.12, we know that $n_I(G) \le 2 + (p_\alpha - 1)v_{p_\alpha}(|g_i|)$ for any $i \in I\backslash\{1\}$ and any prime number p_α such that $p_\alpha \mid |h|$ for some $h \in G_{<i}\backslash\langle 1\rangle$. This leads immediately when $i = i_r, i_3$, and i_r to the inequality $p_\alpha^{v_{p_\alpha}(|g_i|)} \le |g_i| \le n \le (p_\alpha - 1)v_{p_\alpha}(|g_i|)$ which is false for any prime p_α. Thus the first, third, and fourth cases in (3.15.1) do not occur under the assumption $n \le n_I(G)$. It remains to show that $c^{(m)}(h, g_{i_2}) = 1$ where $m = |g_{i_2}| - e + 2$.

By (3.4.2)

$$c^{(m)}(h, g_{i_2}) = \prod_\alpha p_\alpha^{e_\alpha^{(m)}(h, g_{i_2})}$$

where p_α runs through all prime numbers such that $p_\alpha \mid |h|$ and

$$e_\alpha^{(m)}(h, g_{i_2}) = \text{ supremum } \{e^{(m)}_{\alpha,k}(h, g_{i_2}) \mid 1 \le k \le m-2\}$$

where

$$e^{(m)}_{\alpha,k}(h, g_{i_2}) = \{\frac{m-(k+1)}{p_\alpha - 1} - v_{p_\alpha}\binom{|g_{i_2}|}{k} + v_{p_\alpha}(|h|) - 1\}_{\ge 0}.$$

We must show for each α and each k $(1 \le k \le m-2)$ that $e^{(m)}_{\alpha,k}(h, g_{i_2}) = 0$, i.e. that

$$m \le (k+1) + (p_\alpha - 1)(v_{p_\alpha}\binom{|g_{i_2}|}{k} - v_{p_\alpha}(|h|) + 1).$$

But $m \le n \le n_I(G)$. Thus $m \in \mathcal{M}_I(G)$ and therefore satisfies the inequality above for any k such that $1 \le k \le$ infimum $\{m-2, |g_{i_2}| - 1\} =$ (because $m = |g_{i_2}| - e + 2$) $m - 2$. $\square$

Theorem 3.14 has the following immediate consequence for the Rees ring of $\mathbb{Z}G$.

Corollary 3.16 Let G denote a torsion abelian group. Let $\mathbb{Z} \oplus \Delta(G) \oplus \Delta^2(G) \oplus \cdots$ denote the Rees ring of $\Delta(G)$ in $\mathbb{Z}G$. Let $TX(G)$ denote the tensor algebra defined in (2.14). Let N', R' and S' denote the relations defined in (2.14). Define the following relation in $TX(G)$.

$$(T') \quad c^m(h,g)X_h X_{g^{|g|-m+2}} \underbrace{X_g \cdots X_g}_{m-2} =$$
$$\sum_{i=1}^{m-2} \sum_{j=1}^{|h|-1} a_{ij}^{(m)} X_{h^j} \underbrace{X_h \cdots X_h}_{m-i-1} \underbrace{X_g \cdots X_g}_{i} +$$
$$\sum_{i=1}^{|g|-m+1} c^{(m)}(h,g) b_i^{(m)} X_h X_{g^i} \underbrace{X_g \cdots X_g}_{m-2}$$

for all $h, g \in G \backslash \langle 1 \rangle$ such that $|h| \neq |g|$ and $|h| \mid |g|$, and all $3 \leq m \leq |g| + 1$ where $c^{(m)}(h,g)$ is defined as in (3.4) and $a_{ij}^{(m)}$ and $b_i^{(m)}$ as in (3.5) and (3.3).

If $n(G) = \infty$ or G is p-elementary then the canonical graded surjective ring homomorphism $TX(G) \to \mathbb{Z} \oplus \Delta(G) \oplus \Delta^2(G) \oplus \cdots, X_g \mapsto (g-1)$, has kernel the 2-sided ideal generated by N', R', S', and T'.

The proof of Corollary 3.16 is similar to that of Corollary 2.14 and will be omitted.

4 U relations and presenting Δ^n

The goal of this section is to prove Theorems 1.1.3 and 1.2.3. Theorem 1.1.3 presents $\Delta^n(G)$ for an arbitary torsion abelian group by adding to the relations N, R, S and T used in Theorem 1.1.2, the new relation U. Theorem 1.2.3 presents $\Delta^n(G)$ for an arbitray torsion abelian group with ordered basis I, by adding to the relations N, R, S and T_I used in Theorem 1.2.2, the new relation U_I. Since relations $T \supseteq$ relations T_I and relations $U \supseteq$ relations U_I, one deduces by a straightforward direct limit argument that (1.1.3) follows from (1.2.3).

We begin by defining the U_I relations when I and therefore G is finite. Let $\tilde{\Delta}_I^n(G) = \hat{\Delta}^n(G)/T_I^n(G)$. Let $S_I\tilde{\Delta}^n(G)$ and $S_I\Delta^n(G)$ denote respectively the subgroups of $\tilde{\Delta}_I^n(G)$ and $\Delta^n(G)$ generated by the special NRS-generators. According to the Main Lemma 3.15, the special NRS-generators are a basis of $S_I\tilde{\Delta}^n(G)$ and $S_I\Delta^n(G)$ and the quotients $\tilde{\Delta}_I^n(G)/S_I\tilde{\Delta}^n(G)$ and $\Delta^n(G)/S_I\Delta^n(G)$ are torsion. The U_I relations are chosen so that they can be used in replacing the special NRS-generators by a set of elements which generates $\tilde{\Delta}^n_I(g)/U^n_I(G)$ and is a basis of $\Delta^n(G)$. This will be enough to prove Theorem 1.2.3.

Let G be a finite abelian group and (I, g) an ordered basis for G. Let $NRS - \mathcal{G}^n(I)$ denote the set of all NRS-generators of degree n for I and let $\mathcal{G}^n(I) \subseteq NRS - \mathcal{G}^n(I)$ denote the subset of all special NRS generators of degree n. Elements of $\mathcal{G}^n(I)$ will be denoted by x's, elements of $NRS - \mathcal{G}^n(I)\backslash\mathcal{G}^n(I)$ by y's, and a general element of $NRS - \mathcal{G}^n(I)$ by z. Totally order the elements of NRS-$\mathcal{G}^n(I)$, which are only finite in number, such that every element of $NRS - \mathcal{G}^n(I)\backslash\mathcal{G}^n(I)$ is greater then every element of $\mathcal{G}^n(I)$. If $z \in NRS - \mathcal{G}(I)$, let $z+1$(resp. $z-1$) denote the next bigger (resp. next smaller) element of $NRS - \mathcal{G}^n(I)$, provided such exists. For $y \in NRS - \mathcal{G}^n(I)\backslash\mathcal{G}^n(I)$, let $\Delta^n_y(G)$ denote the subgroup of $\Delta^n(G)$ generated by all $z \in NRS - \mathcal{G}^n(I)$ such that $z \leq y$. Letting $x_{\max}$ denote the largest element of $\mathcal{G}^n(I)$ and setting $\Delta^n_{x_{\max}}(G) = S_I\Delta^n(G)$, we get a filtration $\Delta^n_{x_{\max}}(G) \subseteq \Delta^n_{x_{\max}+1}(G) \subseteq \cdots \subseteq \Delta^n_y(G) \subseteq \Delta^n_{y+1}(G) \subseteq \cdots \subseteq \Delta^n(G)$ such that $\bigcup_{y \geq x_{\max}} \Delta^n_y(G) = \Delta^n(G)$ and $\Delta^n_y(G) = \Delta^n_{y-1}(G) + \mathbb{Z}y$ for each $y > x_{\max}$. By the Main Lemma 3.10, the quotient $\Delta^n_y(G)/\Delta^n_{y-1}(G)$ is torsion. Let

$$c_y = \text{ smallest natural number such that } c_y y \in \Delta^n_{y-1}(G).$$

Obviously $c_y y$ can be written, not usually in a unique way, as an integral linear combination of elements z such that $z < y$. We shall develop below a systematic procedure to write $c_y y$ in an unambiguous way as an integral linear combination of elements z as above. Suppose this has been done. Write $c_y y$ as this integral linear combination

$$c_y y = \sum_{z<y} d_{y,z} z.$$

Denote this relation by

$$u_y.$$

It is called ***y*-th derived relation** of $(NRS - \mathcal{G}^n(I), \mathcal{G}^n(I))$ with respect to the given total ordering of $NRS - \mathcal{G}^n(I)$. Let

$$U^n_I(G) = \{u_y \mid y > x_{\max}\}.$$

Since $\tilde{\Delta}^n_I(G)$ and $\Delta^n(G)$ have the same generators, the relations in $U^n_I(G)$ make sense in $\tilde{\Delta}^n_I(G)$, although they are not in general satisfied in $\tilde{\Delta}^n_I(G)$. We have the following theorem.

THEOREM 4.1 Let G denote a finite abelian group and (I, g) an ordered basis for G. Then $U^n_I(G)$ generates Ker $(\tilde{\Delta}^n_I(G) \to \Delta^n(G))$. Thus $\hat{\Delta}^n_I(G)/\langle T^n_I(G), U^n_I(G)\rangle \cong \Delta^n(G)$.

The theorem will be proved towards the end of the section, along with extensions to arbitrary torsion abelian groups with and without ordered bases, and to the Rees ring of $\mathbb{Z}G$.

The key to writing $c_y y$ unambiguously as an integral linear combination is the notion of derived basis. This is a general concept which is not restricted to the current context and will be developed below in steps. It should be useful also in the future in handling nonabelian groups G.

DEFINITION 4.2 Let A denote an abelian group and $B \subseteq A$ a subgroup. An element $y \in A$, which represents a torsion element in A/B, is called **good over** B, if the smallest natural number c_y such that $c_y y \in B$ has the property that $c_y y \notin cB$ for any natural number $c > 1$. It follows trivially that y is free.

Suppose that $y \in A$ is good over B. Suppose B is finitely generated and torsion free and let $\mathcal{G}$ be an ordered basis for B. Suppose that $c_y = p^\ell$ is a power of a prime number p. Write $p^\ell y$ as a $\mathbb{Z}$-linear combination $p^\ell y = \sum\limits_{x\in\mathcal{G}} d_x x$ of elements $x \in \mathcal{G}$. Let x_0 denote the smallest element of $\mathcal{G}$ such that p^ℓ and d_{x_0} are relatively prime. Choose integers u and v such that the absolute value $|u|$ of u is as small as possible with respect to the property $up^\ell + vd_{x_0} = 1$. The integers u and v are unique and the set $\mathcal{G}_1 = \{x \in \mathcal{G} \mid x \neq x_0\} \cup \{ux_0 + vy\}$ is a basis for $B_1 = B + \mathbb{Z}y$. Give $\mathcal{G}_1$ the total ordering such that the bijective map $\mathcal{G} \to \mathcal{G}_1$ which sends every element $x \neq x_0$ to itself and $x_0 \mapsto ux_0 + vy$ is order preserving. The ordered basis $\mathcal{G}_1$ is called the **derived basis** of $\mathcal{G}$ and y. It has the property that each of its elements is expressed in a prescribed way as a $\mathbb{Z}$-linear combination of elements of $\mathcal{G} \cup \{y\}$, namely if $x \neq x_0$ then $x = x$ and the remaining element $ux_0 + vy$ has the decomposition $ux_0 + vy = ux_0 + vy$. Note that if $\ell = 0$ then $u = 1$ and $v = 0$; thus $\mathcal{G} = \mathcal{G}_1$.

In the case of a general c_y, we factor $c_y = \prod\limits_{i=1}^{k} p_i^{\ell_i}$ as a product of prime powers $p_i^{\ell_i}$ such that $p_1 < p_2 < \cdots < p_\ell$. Let $\mathcal{G}_0 = \mathcal{G}$ and for $1 \leq i \leq k$, define recursively $\mathcal{G}_i =$ derived basis of $\mathcal{G}_{i-1}$ and $(p_{i+1}^{\ell_{i+1}} \cdots p_k^{\ell_k})y$. Obviously each element of $\mathcal{G}_k$ is expressed in a prescribed way as a $\mathbb{Z}$-linear combination of elements of $\mathcal{G} \cup \{y\}$. $\mathcal{G}_k$ is called the **derived basis** of $\mathcal{G}$ and $\{y\}$.

Let $\mathcal{G}en$ be a totally ordered finite set and let $\mathcal{G} \subseteq \mathcal{G}en$ be a nonempty subset such every element of $\mathcal{G}en\backslash\mathcal{G}$ is greater then every element of $\mathcal{G}$. Elements of $\mathcal{G}$ will be denoted by x's, elements of $\mathcal{G}en\backslash\mathcal{G}$ by y's, and a general element of $\mathcal{G}en$ by z. If $z \in \mathcal{G}en$, let $z+1$(resp. $z-1$) denote the next bigger (resp. smaller) element of $\mathcal{G}$en, provided it exists. Let A be an abelian group. Fix an injective map $\mathcal{G}en \to A$. For each $y \in \mathcal{G}en\backslash\mathcal{G}$ let B_y denote the subgroup of A generated by $\{z \in \mathcal{G}en \mid z \leq y\}$. Let $x_{\max}$ denote the largest element of $\mathcal{G}$ and let $B = B_{x_{\max}}$ denote the subgroup of A generated by $\{x \in \mathcal{G}\} = \{z \in \mathcal{G}en \mid z \leq x_{\max}\}$. The pair $(\mathcal{G}en, \mathcal{G})$ is called a **good set of ordered generators** for A or simply **good** in A, if $\mathcal{G}en$ generates $A, \mathcal{G}$ is a $\mathbb{Z}$-basis for B, and for each $y \in \mathcal{G}en\backslash\mathcal{G}, y$ is good over B_{y-1}. Set $\mathcal{G}_{x_{\max}} = \mathcal{G}$ and define recursively $\mathcal{G}_y =$ derived basis of $\mathcal{G}_{y-1}$ and y. By definition, each element of $\mathcal{G}_y$ is a prescribed $\mathbb{Z}$-linear combination of the element y and elements of $\mathcal{G}_{y-1}$. It follows that each element of $\mathcal{G}_y$ is a prescribed $\mathbb{Z}$-linear combination of elements in $\{z \mid z \leq y\}$. Let c_y denote the smallest natural number such that $c_y y \in B_{y-1}$. Then we can write $c_y y$ in a prescribed way as a $\mathbb{Z}$-linear combination $c_y y = \sum_{z \leq y-1} d_z z$. The difference

$$c_y y - \sum_{z \leq y-1} d_z z \text{ is denoted by } u_y$$

and is called the ***y* -th derived relation of** $(\mathcal{G}en, \mathcal{G})$. Let

$$U(\mathcal{G}en, \mathcal{G}) = \{u_y \mid y \in \mathcal{G}\text{en}\backslash\mathcal{G}\}.$$

This is called the **set of derived relations of** $(\mathcal{G}\text{en}, \mathcal{G})$.

Let $\tilde{A} \to A$ denote a homomorphism of abelian groups. Let $\mathcal{G}en \to \tilde{A}$ denote an injective map such that its image generates $\tilde{A}$ and the diagram

$$\begin{tikzcd} & \mathcal{G}en \arrow[dl] \arrow[dr] & \\ \tilde{A} \arrow[rr] & & A \end{tikzcd}$$

commutes. Because $\mathcal{G}en$ represents generators for both $\tilde{A}$ and A, each derived relation in A makes sense in $\tilde{A}$, although it is not necessarily satisfied in $\tilde{A}$. We have the following lemma.

Lemma 4.3 In the context above, $U(\mathcal{G}en, \mathcal{G})$ generates Ker $(\tilde{A} \to A)$. Thus $\tilde{A}/\langle U(\mathcal{G}en, \mathcal{G})\rangle \cong A$.

Proof Let $\bar{A} = \tilde{A}/\langle U(\mathcal{G}\text{en}, \mathcal{G})\rangle$. Define $\bar{B} = \bar{B}_{x_{\max}}$ and $\bar{B}_y$ analogously to B and B_y, respectively. Clearly the canonical homomorphism $\bar{B} \to B$ is an isomorphism because $\mathcal{G}$ is a basis for both $\bar{B}$ and B. Let $y \in \mathcal{G}en\backslash\mathcal{G}$ and suppose that the canonical homomorphism $\bar{B}_{y-1} \to B_{y-1}$ is an isomorphism. Since the element $y \in A$ is good over B_{y-1}, it follows that the element $y \in \bar{A}$ is good over $\bar{B}_{y-1}$. Let $\mathcal{G}_{y-1}$ denote the derived basis for B_{y-1}. Since $\bar{B}_{y-1} \to B_{y-1}$ is an isomorphism, we can declare $\mathcal{G}_{y-1}$ an ordered basis for $\bar{B}_{y-1}$. Trivially the canonical homomorphism $\bar{B}_y \to B_y$ maps bijectively the derived basis of $\mathcal{G}_{y-1}$ and y for $\bar{B}_y$ onto the derived basis of $\mathcal{G}_{y-1}$ and y for B_y. Thus the map $\bar{B}_y \to B_y$ is an isomorphism. Since $\bigcup\limits_{y\in\mathcal{G}en\backslash\mathcal{G}} \bar{B}_y = \bar{A}$ and $\bigcup\limits_{y\in\mathcal{G}\text{en}\backslash\mathcal{G}} B_y = A$, it follows that $\bar{A} \to A$ is an isomorphism. $\square$

Proof of Theorem 4.1 By definition, $U_I^n(G) = U(NRS - \mathcal{G}^n(I), \mathcal{G}^n(I))$. Thus the theorem follows immediately from Lemma 4.3. $\square$

Theorem 4.1 has the following consequence for the Rees ring of $\mathbb{Z}G$.

Corollary 4.4 Let G denote a finite abelian group with ordered basis (I, g). Let $\mathbb{Z} \oplus \Delta(G) \oplus \Delta^2(G) \oplus \cdots$ denote the Rees ring of $\Delta(G)$ in $\mathbb{Z}G$. Let $TX(G)$ denote the tensor algebra defined in (2.14). Let N', R', and S' denote the relations defined in (2.14). Define the following relations in $TX(G)$.

(T') All relations

$$c^m(h,g)X_hX_{g^{|g|-m+2}}\underbrace{X_g\cdots X_g}_{m-2} =$$
$$\sum_{i=1}^{m-2}\sum_{j=1}^{|h|-1} a_{ij}^{(m)}X_{h^j}\underbrace{X_h\cdots X_h}_{m-i-1}\ \underbrace{X_g\cdots X_g}_{i} +$$
$$\sum_{i=1}^{|g|-m+1} c^{(m)}(h,g)b_i^{(m)}X_hX_{g^i}\underbrace{X_g\cdots X_g}_{m-2}$$

such that

$$c^{(m)}(h,g)[h,g^{|g|-m+2},\underbrace{g,\cdots,g}_{m-2}] = \sum_{i=1}^{m-2}\sum_{j=1}^{|h|-1} a_{i,j}^{(m)}[h^j,\underbrace{h,\cdots,h}_{m-i-1},\underbrace{g,\cdots,g}_{i}] +$$
$$\sum_{i=1}^{|g|-m+1} c^{(m)}(h,g)b_i^{(m)}[h,g^i,\underbrace{g,\cdots,g}_{m-2}]$$

is a relation in $T_I^m(G)$ and $3 \leq m \leq |g|+1$.

(U') All relations

$$\sum_{[g_1,\cdots,g_m]\in NRS-\mathcal{G}^m(I)} d_{[g_1,\cdots,g_m]} X_{g_1}\cdots X_{g_m}$$

such that

$$\sum_{[g_1,\cdots,g_m]\in NRS-\mathcal{G}^m(I)} d_{[g_1,\cdots,g_m]}[g_1,\cdots,g_m] \in U_I^m(G)$$

$$\text{and } 3 \leq m \leq \text{ supremum } \{|g_i|+1 \mid i \in I\}.$$

Then the canonical graded surjective ring homomorphism $TX(G) \to \mathbb{Z}\oplus\Delta(G)\oplus\Delta^n(G)\oplus\cdots, X_g \mapsto (g-1)$, has kernel the 2-sided ideal generated by N', R', S', T', and U'.

The proof of Corollary 4.4 is similar to that of Corollary 2.14 and will be omitted.

The next theorem is an immediate consequence of Theorem 4.1.

THEOREM 4.5 Suppose G is a torsion abelian group. Let $\mathcal{H}$ denote a directed system of finite subgroups $H \subseteq G$ such that the direct limit $\varinjlim_{\mathcal{H}} H = G$. Suppose further that each H comes equipped with an ordered basis (I_H, h). Let $T_{\mathcal{H}}^n(G) = \bigcup_{H\in\mathcal{H}} T_{I_H}^n(H)$ and $U_{\mathcal{H}}^n(G) = \bigcup_{H\in\mathcal{H}} U_{I_H}^n(H)$. Then the canonical homomorphism $\hat{\Delta}^n(G)/\langle T_{\mathcal{H}}^n(G), U_{\mathcal{H}}^n(G)\rangle \to \Delta^n(G)$ is an isomorphism.

There is a result analogous to Theorem 4.5, for Rees rings. Its formulation is left to the reader.

5 Relation modules and $Ker(\Delta^n(G) \to \Delta^n(G/N))$

Let G denote an abelian group or monoid and let $\hat{\Delta}^n(G)$ be defined as in §1. By (2.12), there is an action of G on $\hat{\Delta}^n(G)$ such that the canonical homomorphism $\hat{\Delta}^n(G) \to \Delta^n(G)$ is a G-homomorphism. The $Ker(\hat{\Delta}^n(G) \to \Delta^n(G))$ is denoted by $\omega^n(G)$ and will be called the **relation module** of G of degree n. It is obviously a G-module. According to (2.5), (2.7), (2.11), and (2.13), $\omega^n(G) = 0$ whenever one of the following holds: $n = 2$ and G

is either torsion free or torsion; $n \geq 2$ and G is either torsion free or a direct limit of cyclic groups. Furthermore according to (3.14) and (4.5), if G is torsion then we know generators for it namely the elements of $\omega^n(G)$ defined by the T and U relations.

The theorem below shows that whereas the functor $\hat{\Delta}^n$ is "exact" in an appropriate sense, the funtor Δ^n is not, but its failure to be exact is measured by the functor ω^n.

THEOREM 5.1 Let G denote an abelian group and $N \subseteq G$ a subgroup. Let $\Delta^n(N)G$ (resp. $\hat{\Delta}^n(N)G$) denote the G-submodule of $\Delta^n(G)$ generated by image($\Delta^n(N) \to \Delta^n(G)$) (resp. image ($\hat{\Delta}^n(N) \to \hat{\Delta}^n(G)$)). Then the sequence

$$\hat{\Delta}^n(N)G \rightarrowtail \hat{\Delta}^n(G) \twoheadrightarrow \hat{\Delta}^n(G/N)$$

of G-modules is short exact and the homology of the sequence $\Delta^n(N)G \rightarrowtail \Delta^n(G) \twoheadrightarrow \Delta^n(G/N)$ of G- modules is computed by

$$H(\Delta^n(N)G \rightarrowtail \Delta^n(G) \twoheadrightarrow \Delta^n(G/N)) \cong \omega^n(G/N)/\omega^n(G).$$

Moreover if G is finite, resp. torsion, then so is $\omega^n(G/N)/\omega^n(G)$.

PROOF There is a canonical homomorphism $\hat{\Delta}^n(G)/\hat{\Delta}^n(N)G \to \hat{\Delta}^n(G/N)$ and the rule $\hat{\Delta}^n(G/N) \to \hat{\Delta}^n(G)/\hat{\Delta}^n(N)G, [g_1, \cdots, g_n] \mapsto [\tilde{g}_1, \cdots, \tilde{g}_n] + \hat{\Delta}^n(N)G$ where $\tilde{g}_i \in G$ is any lifting of $g_i \in G/N$ $(1 \leq i \leq n)$, is well defined and yields a homomorphism which is inverse to the one above.

From the sequence of canonical homomorphisms $\omega^n(G) \to \omega^n(G/N) \to \hat{\Delta}^n(G/N) \to \hat{\Delta}^n(G)/\hat{\Delta}^n(N)G \to \Delta^n(G)/\Delta^n(N)G$ and the observation that $\omega^n(G)$ vanishes in $\Delta^n(G)/\Delta^n(N)G$, we obtain an induced homomorphism $\hat{\Delta}^n(G/N)/\omega^n(G) \to \Delta^n(G)/\Delta^n(N)G$. On the other hand the canonical homomorphism $\hat{\Delta}^n(G)/\hat{\Delta}^n(N)G \to \hat{\Delta}^n(G/N)$ induces a homomorphism $\Delta^n(G)/\Delta^n(N)G \to \hat{\Delta}^n(G/N)/\omega^n(G)$ which is mutually inverse to the one above. It follows that $Ker(\Delta^n(G)/\Delta^n(N)G \to \Delta^n(G/N)) \cong Ker(\hat{\Delta}^n(G/N)/\omega^n(G) \to \Delta^n(G/N)) = \omega^n(G/N)/\omega^n(G)$.

To prove the last assertion of the theorem, it suffices to consider the case G is finite. Since $\omega^n(G/N)$ is finitely generated, so is $\omega^n(G/N)/\omega^n(G)$. Thus it suffices to show that $\mathbb{Q} \otimes (\omega^n(G/N)/\omega^n(G)) = 0$. But this follows from the fact that the functor $\mathbb{Q}\otimes\_\_$ commutes with taking homology and by Mashke's theorem, the sequence $\mathbb{Q} \otimes \Delta^n(N)G \rightarrowtail \mathbb{Q}\otimes\hat{\Delta}^n G \twoheadrightarrow \mathbb{Q}\otimes\hat{\Delta}^n(G/N)$ is short exact. $\square$

We draw some easy consequences of the theorem above, using results in previous sections. Further consequences could be drawn by analyzing and computing the quotients $\omega^n(G/N)/\omega^n(G)$, but this is a topic in itself which is not undertaken in the current paper.

Corollary 5.2 Let G denote an abelian group and $N \subseteq G$ a subgroup. If $n = 2$ and G/N is either torsion free or torsion or if $n \geq 2$ and G/N is either torsion free or a direct limit of cyclic groups then the sequence

$$\Delta^n(N)G \rightarrowtail \Delta^n(G) \twoheadrightarrow \Delta^n(G/N)$$

of G-modules is short exact.

Proof The result is an immediate consequence of (5.1), (2.5), (2.7), and (2.13). □

Corollary 5.3 Let G denote a torsion free or torsion abelian group and $N \subseteq G$ a subgroup. If $n = 2$ or if $n \geq 2$ and G is either torsion free or a direct limit of cyclic groups then the homology $H(\Delta^n(N)G \rightarrowtail \Delta^n(G) \twoheadrightarrow \Delta^n(G/N)) \cong \omega^n(G/N)$.

Proof The result is an immediate consequence of (5.1), (2.5), (2.7), and (2.13). □

Corollary 5.4 Let G denote an abelian group and $N \subseteq G$ a subgroup. Then any set theoretic section $s : \underbrace{(G/N) \times \cdots \times (G/N)}_{n} \to \underbrace{G \times \cdots \times G}_{n}$ to the canonical homomorphism $G \times \cdots \times G \to (G/N) \times \cdots \times (G/N)$ defines an isomorphism

$$\hat{\Delta}^n(G/N)/\omega^n(G) \overset{\cong}{\to} \Delta^n(G)/\Delta^n(N)G$$

of G-modules. In particular, if $n = 2$ and G is torsion or if $n \geq 2$ and G is either torsion free or direct limit of cyclic groups then $\hat{\Delta}^n(G/N) \overset{\cong}{\to} \Delta^n(G)/\Delta^n(N)G$.

Proof The first isomorphism was established in the proof of (5.1) and the second isomorphism follows now from (2.5), (2.7), and (2.13). □

Definition-Lemma 5.5 Let G denote an arbitrary group or monoid. Let $\bar{\Delta}^n(G)$ denote the free abelian group on the set of all standard generators $[x(1), \cdots, x(n)]$ of $\Delta^n(G)$ modulo the relations N and R in §1. Let $g \in G$. Then the rule $[x(1), \cdots, x(n)]g = [x(1), \cdots, x(n)g] - [x(1), \cdots, x(n-1), g]$ defines an action of G on $\bar{\Delta}^n(G)$ such that the canonical homomorphism $\bar{\Delta}^n(G) \to \Delta^n(G)$ is one of G-modules.

The proof is similar to that of (2.12) and will be omitted.

For an arbitrary group G, let $\Omega^n(G) = ker(\bar{\Delta}^n(G) \to \Delta^n(G))$. $\Omega^n(G)$ will be called the **nonabelian relation module** of G of degree n. It is obviously a G-module.

The following result is the analog of (5.1) for arbitrary groups. Its proof is similar to that of (5.1) and will be omitted.

THEOREM 5.6 Let G denote an arbitrary group and $N \lhd G$ a normal subgroup. Let $\Delta^n(N)G$ (resp. $\bar{\Delta}^n(N)G$) denote the G-submodule of $\Delta^n(G)$ generated by image($\Delta^n(N) \to \Delta^n(G)$) (resp. image ($\bar{\Delta}^n(N) \to \bar{\Delta}^n(G)$)). Then the sequence

$$\bar{\Delta}^n(N)G \rightarrowtail \bar{\Delta}^n(G) \twoheadrightarrow \bar{\Delta}^n(G/N)$$

of G-modules is short exact and the homology of the sequence $\Delta^n(N)G \rightarrowtail \Delta^n(G) \twoheadrightarrow \Delta^n(G/N)$ of G-modules is computed by

$$H(\Delta^n(N)G \rightarrowtail \Delta^n(G) \twoheadrightarrow \Delta^n(G/N)) \cong \Omega^n(G/N)/\Omega^n(G).$$

Moreover if G is finite then so is $\Omega^n(G/N)/\Omega^n(G)$.

For an arbitrary monoid G, the relation S in §1 makes sense in $\bar{\Delta}^n(G)$, i.e. defines elements of $\bar{\Delta}^n(G)$, such that if G is abelian, the elements live in $\Omega^n(G)$. The next lemma records an obvious relationship between Ω^n and ω^n. Its proof is omitted.

LEMMA 5.7 Let G denote an abelian monoid. Let $\langle S\rangle_{\Omega^n(G)}$ denote the G-submodule of $\Omega^n(G)$ generated by the elements of $\Omega^n(G)$, which are defined by the relation S in §1. Then the canonical G-homomorphism $\Omega^n(G) \to \omega^n(G)$ induces an isomorphism $\Omega^n(G)/\langle S\rangle_{\Omega^n(G)} \overset{\cong}{\to} \omega^n(G)$.

The lemma above allows one to translate computations above for $\omega^n(G)$ into ones for $\Omega^n(G)$, when G is abelian. For arbitrary groups G, sets of generators for $\Omega^n(G)$ will be strongly tied to presentations of G. General results are not available here, although easy examples can be worked out by hand. If F is free, we anticipate that $\Omega^n(F) = 0$. This would show analogously to (5.4), that if G is an arbitrary group and $N \rightarrowtail F \twoheadrightarrow G$ an exact sequence presenting G where F is free then $\bar{\Delta}^n(G) \cong \Delta^n(F)/\Delta^n(N)F$.

We conclude the paper by investigating the notion of higher relation module. Specifically we construct a functor $\underline{\underline{\Delta}}^n$: ((abelian groups)) $\to$ ((pointed topological spaces)), a surjective natural transformation $\tau : \omega^n \to \pi_1\underline{\underline{\Delta}}^n$, and generators for $Ker(\tau)$. It is unclear whether or not all the generators are zero.

Let G denote an abelian group. For each $(n-2)$-tuple $f_1, \cdots, f_{n-2} \in G$, there are G-homomorphisms $\hat{\delta}_{f_1,\cdots,f_{n-2}} : \hat{\Delta}^2(G) \to \hat{\Delta}^n(G), [g_1, g_2] \mapsto [f_1, \cdots, f_{n-2}, g_1, g_2]$, and $\delta_{f_1,\cdots,f_{n-2}} : \Delta^2(G) \to \Delta^n(G), [g_1, g_2] \mapsto [f_1, \cdots, f_{n-2}, g_1, g_2]$. Let $\hat{\Delta}^2_{f_1,\cdots,f_{n-2}}(G) =$ image $(\hat{\delta}_{f_1,\cdots,f_{n-2}})$ and $\Delta^2_{f_1,\cdots,f_{n-2}}(G) =$ image $(\delta_{f_1,\cdots,f_{n-2}})$. Define an abstract simplicial complex $V^n(G)$ as follows. As usual, if $p \in \mathbb{Z}^{\geq 0}$ then $V^n(G)_p$ will denote the p-simplices of $V^n(G)$. Let $V^n(G)_0 = \Delta^n(G)$. If $p \neq 0, 2$, define $V^n(G)_p = \{\{x_0, \cdots, x_p\} \subseteq \Delta^n(G) \mid x_i - x_j \in \Delta^2_{(f_{(i,j)})_1, \cdots, (f_{(i,j)})_{n-2}}(G)$ for some $(n-2)$-tuple $(f_{(i,j)})_1, \cdots, (f_{i,j})_{n-2} \in G\}$. Define $V^n(G)_2 = \{\{x_0, x_1, x_2\} \subseteq \Delta^n(G) \mid \exists\ 0 \leq i \leq 2$ such that for any $k \neq i$ $(0 \leq k \leq 2), x_i - x_k \in \Delta^2_{(f_k)_1, \cdots, (f_k)_{n-2}}(G)$ for some $(n-2)$-tuple $(f_k)_1, \cdots, (f_k)_{n-2} \in G\}$. Let $\underline{\underline{\Delta}}^n(G)$ denote the geometric realization of $V^n(G)$.

Call an element $a \in \hat{\Delta}^n(G)$, a $\hat{\boldsymbol{\Delta}}^{\mathbf{2}}$**- element**, if $a \in \hat{\Delta}^2_{f_1, \cdots, f_{n-2}}(G)$ for some $(n-2)$-tuple $f_1, \cdots, f_{n-2} \in G$. Let $\theta^n(G) = \{a + b + c + d \in \hat{\Delta}^n(G) \mid a, b, c, d$ are $\hat{\Delta}^2$-*elements*$, a + b + c + d \in \omega^n(G)\}$. Let $\hat{\underline{\underline{\Delta}}}^n(G)$ denote the universal connected covering of $\underline{\underline{\Delta}}^n(G)$. One establishes straightforward a functorial one to one correspondence between the elements of $\hat{\Delta}^n(G)/\theta^n(G)$ and the fixed end point contiguity classes of edge paths in $V^n(G)$ starting at the origin, such that the fundamental group $\pi_1(\underline{\underline{\Delta}}^n(G)) = Ker(\hat{\Delta}^n(G)/\theta^n(G) \rightarrow \Delta^n(G)) = \omega^n(G)/\theta^n(G)$. This raises the question: When is $\theta^n(G) = 0$? One could also ask: When is the canonical homomorphism $\hat{\Delta}^2_{f_1, \cdots, f_{n-2}}(G) \rightarrow \Delta^2_{f_1, \cdots, f_{n-2}}(G)$ an isomorphism? Obviously a positive answer to the former implies a positive answer to the latter. It is also interesting to ask: When is the canonical homomorphism $\omega^n(G/N)/\omega^n(G) \rightarrow \omega^n(G/N)/\theta^n(G/N)\omega^n(G)$ an isomorphism?

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