ALGEBRAIC K-THEORY OF NORMED ALGEBRAS

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Abstract. Quillen's algebraic K-theory of discrete rings is extended to the category of normed algebras over a commutative Banach ring k with unit and its relationship with topological K-theory is established. Sufficient conditions for the isomorphism of algebraic and topological K-groups on the category of real normed algebras are given. The isomorphism of algebraic and topological K-functors on the category of polynomial extensions of stable C^*-algebras is proved.

INTRODUCTION AND SUMMARY

In the first section Quillen's Q-construction [13] for exact categories is in fact realized for topological exact categories, namely for the category of finitely generated projective modules over a normed k-algebra A, where k is a commutative Banach ring with unit, and their bounded A-homomorphisms. This construction allows us to extend Quillen's K-theory of discrete rings to the category of normed A-algebras (Definition 1).

The extension of an algebraic K-theory of discrete rings to the category of Banach A-algebras at first was given by Karoubi-Villamayor [7] the polynomial ring $A[x_1, \dots, x_n]$ over A replaced by the ring $A < x_1, \dots, x_n >$ of formal series $S(x_1, \dots, x_n) = \sum a_{i_1 \dots i_n} {x_1}^{i_1} \cdots {x_n}^{i_n}$ such that $\| S \| = \sum \| a_{i_1 \dots i_n} \| < \infty$. A similar extension of Swan's algebraic K-theory to the category of Banach A-algebras has been obtained in [11, 12] free rings replaced by free Banach algebras. In both cases it

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has been shown that for real or complex Banach algebras one gets K-functors isomorphic to topological K-functors. The same fact remains true for Quillen's K-theory. It is proved that Quillen's K-theory on the category of special normed real algebras is isomorphic to topological K-theory (Theorem 2).

The second section deals with the relation between algebraic and topological K-functors on the category of real normed algebras. The natural relationship between algebraic and topological K-functors at Grothendieck functor K_0 level was established by Swan [15]. If X is a compact Hausdorff space and $k(X)$ is the ring of continuous real or complex valued functions on X, then Swan has proved there is a natural equivalence (via the functor Gamma of global sections) between the category of vector bundles on X and the category of finitely generated projective modules over $k(X)$, implying the equality $K_0(k(X)) = K_k^0(X)$. Swan's result has been generalized to the case when k is a C^* algebra with unit (see [12, Chapter V]).

In [8] Karoubi has proved that for any real Banach algebra A with unit one has the following exact sequence

$$A_0^* \xrightarrow{\sigma_0} K_1(A) \xrightarrow{\beta_1} K_1^{\mathrm{top}}(A) \longrightarrow 0 ,$$

where σ_0 and β_1 are natural homomorphisms, σ_0 being injective if A is commutative, K_1 is Whitehead-Bass K-functor and A_0^* is the component of the unit of the topological group A^* of invertible elements of A. This result has been generalized to special normed real algebras with unit and some estimation of the cokernel of the homomorphism $\beta_2 : K_2(A) \to K_2^{\mathrm{top}}(A)$ has been obtained [12, Chapter V, Theorem 2.15].

In connection with this problem Karoubi's Conjecture is well known [8] which predicts the equality of algebraic and topological K-functors on the category of stable C^*-algebras. Karoubi has proved his conjecture for $i \leq 0$ [8]. It was proved for $i = 1$ by de la Harpe and Skandalis [4] and for $i = 2$ by Higson [6] and by Karoubi [9]. Karoubi's Conjecture for Karoubi-Villamayor's algebraic K-functors has been proved by Higson [6]. Recently Suslin and Wodzicki have confirmed Karoubi's Conjecture (for Quillen's algebraic K-functors) [14].

Natural sufficient conditions are obtained for the isomorphism of algebraic and topological K-functors on the category of real normed algebras (Theorems 10 and 22). These theorems allow us to introduce the important notion of K_*-stability of real normed algebras (Definition 12) which is closely related with stability of C^* algebras and with K_*-regularity of discrete rings (Corollary 23). It is shown that a wide class of complex normed algebras including polynomial algebras $A[x_1, \dots, x_n]$ over stable C^*-algebras A, Laurent polynomial algebras $A[t, t^{-1}]$ over stable C^*-algebras A, C^*-algebras of continuous maps from compact spaces to the Calkin algebra, retracts of C^* algebras, are K_i stable for $i \geq 1$. Most of them are not complete and on the category of these complex normed algebras one has the equality of algebraic and topological K-functors (Theorems 14 and 18, Remark 19, Corollary 13, Propositions 20 and 21).

Finally it should be noted that the triple factorization property $(TF)_{\text{right}}$ of a ring introduced by Suslin and Wodzicki play a key role in their work [14]. They have proved that with respect to rings having $(TF)_{\text{right}}$ property Quillen's K-funcotrs have excision property, are Morita invariant and that any C^* algebra has the $(TF)_{\text{right}}$ property. We essentially use these assertions to prove Theorems 14 and 18. The definition of $(TF)_{\text{right}}$ property is given just before Lemma 16.

Now recall the definition of a normed algebra over a Banach ring with unit.

A normed ring A is a ring with non-negative valued map $\| - \|: A \to \mathbb{R}$, called the norme of A, satisfying the following relations

$$\begin{aligned}
&\| x \| = 0 \Longleftrightarrow x = 0 \ , \\
&\| -x \| = \| x \| \ , \\
&\| x + y \| \leq \| x \| + \| y \| \ , \\
&\| xy \| \leq \| x \| \cdot \| y \| \ ,
\end{aligned}$$

for any $x, y \in A$.

A Banach ring is a complete normed ring.

A normed left module M over a normed ring A is a left module over A with a non-negative map $\| - \|: M \to \mathbb{R}$ such that

$$\begin{aligned}
&\| x \| = 0 \Longleftrightarrow x = 0 \ , \\
&\| x + y \| \leq \| x \| + \| y \| \ , \\
&\| ax \| \leq \| a \| \cdot \| x \| \ ,
\end{aligned}$$

for any $x, y \in M$, $a \in A$.

A-module always will mean left A-module. A bounded A homomorphism $f : M \to M'$ of normed A-modules is a A-homomorphism such that there is a constant C satisfying the relation

$$\| f(x) \| \leq C \cdot \| x \| \ , \quad x \in M \ .$$

Since special normed k-algebras and in particular local Banach algebras will play an important role, we recall the definition [2, 12].

A special normed ring A is a normed ring for which there exists a number t , $0 < t \leq 1$, such that for every $a \in A$ with $\| a \| < t$ the element a is quasi-invertible, that is there is a unique $a' \in A$ such that $a + a' + aa' = 0$ and $a' + a + a'a = 0$. A special normed ring A with unit is a normed ring such that the group of its invertible elements forms a open topological subgroup of A. Evidently any Banach k-algebra is a special normed k-algebra. There are many important examples of special normed k-algebras which are not complete (see [11, 12]). Local Banach algebras are special normed

$\mathbb{C}$-algebras [2]. Important examples of local $\mathbb{C}^*$- algebras arise in many applications of K-theory to noncommutative differential geometry.

A local Banach algebra is a normed $\mathbb{C}$-algebra which is closed under holomorphic functional calculus, that is if for any analytic function f on a neighborhood of the spetrum $\mathrm{Sp}_{\widehat{A}}(x)$ of an element $x \in A$ in the completion $\widehat{A}$ of A (with $f(0) = 0$ if A is without unit) we have $f(0) \in A$.

A C^*-algebra A is a complete complex involutive normed algebra with involution $*$ satisfying the relations $\| x \| = \| x^* \|$ and $\| xx^* \| = \| x \|^2$ for any $x \in A$. A (local) C^*-algebra B is called stable if B has the form $A \otimes \mathcal{K}$ for some (local) C^*-algebra A, where $\mathcal{K}$ is the C^*-algebra of compact operators on the infinite dimensional Hilbert space $\mathcal{H}$. The stable algebra $A \otimes \mathcal{K}$ is called the (spatial) C^*-tensor product of A and $\mathcal{K}$ and is defined as follows:

take the algebra $M_\infty(A) = \varinjlim_n M_n(A)$ of infinite matrices over A with only finitely many non-zero entries and with natural inductive limit topology induced by the norm of A. Then $A \otimes \mathcal{K}$ is the completion of $M_\infty(A)$ under this norm.

§1. Quillen's algebraic K-theory of normed algebras

Let A be a normed k-algebra with unit. Let $\underline{P}(A)$ be the category of finitely generated projective A-modules and let $\mathcal{E}$ be the family of short exact sequences in $\underline{P}(A)$. For any finitely generated projective A-module P using an epimorphism $q : \overset{n}{\oplus} A \to P$ one determines a norm on P induced by the norm of A such that q becomes a bounded A-homomorphism. This can be done as follows.

We have the following commutative diagrams:

$$\begin{array}{ccc} P & \longrightarrow & P \otimes_A \widehat{A} \\ \downarrow & & \downarrow \\ \overset{n}{\oplus} A & \longrightarrow & \overset{n}{\oplus} A \otimes_A \widehat{A} \end{array} , \qquad \begin{array}{ccc} \overset{n}{\oplus} A & \longrightarrow & \overset{n}{\oplus} A \otimes \widehat{A} \\ \downarrow & & \downarrow \\ P & \longrightarrow & P \otimes_A \widehat{A} \end{array} ,$$

where $\widehat{A}$ is the completion of A, the vertical map $\sigma : P \to \overset{n}{\oplus} A$ is a splitting for q, i.e. $q\sigma = 1_P$, the horizontal maps are given by $x \mapsto x \otimes e$, e being the unit of A. It follows that the map $P \to P \otimes_A \widehat{A}$ of the right diagram is injective and one gets an epimorphism $q \otimes \widehat{A} : \overset{n}{\oplus} \widehat{A} \to P \otimes_A \widehat{A}$ on a finitely generated projective module $P \otimes_A \widehat{A}$ over a Banach k-algebra $\widehat{A}$. On the other hand it is known that if B is a Banach k-algebra with unit and Q is a finitely generated projective B-module then any epimorphism

$$l : \overset{m}{\oplus} B \longrightarrow Q$$

induces a norm on Q defined by

$$\| x \| = \inf_{y \in l^{-1}(x)} \| y \|, \ \ x \in Q.$$

Therefore , $q \otimes \widehat{A}$ defines a norm on $P \otimes_A \widehat{A}$ which induces a norm on P , since the map $P \to P \otimes_A \widehat{A}$ is injective. Evidently the map $q : \overset{n}{\oplus} A \to P$ is bounded under this norm.

Let $\underline{P'}(A)$ be the category whose objects are normed finitely generated projective A-modules and morphisms are bounded A-homomorphisms. Let P, Q be objects of $\underline{P'}(A)$. One defines on the set $\mathrm{Hom}(P, Q)$ of morphisms of $\underline{P'}(A)$ a norm given by

$$\| f \| = \sup_{x \neq 0} \frac{\| f(x) \|}{\| x \|}, \ \ x \in P.$$

Then the sets $\mathrm{Hom}(P, Q)$ of the category $\underline{P'}(A)$ become normed abelian groups and therefore, are topological abelian groups under the topology induced by this norm.

We deduce that the category $\underline{P'}(A)$ can be viewed as a topological additive category in the sense that the sets of morphisms are topological abelian groups, the composition and the sum of morphisms are continuous under this topology and there exists finite direct sums of objects.

Let $\mathcal{E}'$ be the family of short exact sequences

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{j} M'' \longrightarrow 0$$

in the category $\underline{P'}(A)$.

The homomorphisms i and j satisfy Quillen's conditions of admissible monomorphisms and admissible epimorphisms respectively [13] for discrete rings,real (or complex) Banach algebras and for local Banach algebras.But in general i may be not the k-ernel of j nevertheless all other Quillen's conditions are satisfied. Apply now Quillen's Q-construction [13] to the category $\underline{P'}(A)$ using admissible monomorphimsms and epimorphisms definedby $\mathcal{E}'$. One gets a new topological category $Q\underline{P'}(A)$ defined as follows:

The category $Q\underline{P'}(A)$ has the same objects as the category $P'(A)$. A morphism from an object M to an object M' of the category $Q\underline{P'}(A)$ is an isomorphism class of diagrams

$$M \overset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} M' \ ,$$

where j is an admissible epimorphism and i is an admissible monomorphism. The composite of such morphisms is given by using fiber products. A topology on the sets $\mathrm{Hom}(M, M')$ of morphisms of the category $Q\underline{P'}(A)$ will be introduced by the norm of A. To this end consider the topological product $\mathrm{Hom}(N, M) \times \mathrm{Hom}(N, M')$ of the topological abelian groups $\mathrm{Hom}(N, M)$ and $\mathrm{Hom}(N, M')$ of bounded

A-homomorphisms in the category $P'(A)$ mentioned above. Then take the topological sum

$$L_{M,M'} = \oplus_N \ \text{of topological spaces} \ \ \mathrm{Hom}(N,M) \times \mathrm{Hom}(N,M')\ ,$$

N running over the set of isomorphism classes of objects of the category $P'(A)$. Introduce in $L_{M,M'}$ an equivalence $\sim$ by $(f,g) \sim (f',g')$ if there exists an A-isomorphism $\theta : N \to N'$ such that the diagram

$$\begin{array}{ccccc} M & \overset{f}{\leftarrow} & N & \overset{g}{\rightarrow} & M' \\ || & & \downarrow \theta & & || \\ M & \overset{f'}{\leftarrow} & N' & \overset{g'}{\rightarrow} & M' \end{array}$$

is commutative.

Let $\overline{L}_{M,M'} = L_{M,M'}/\sim$ be the quotient topological space of L. Clearly $\mathrm{Hom}(M,M')$ of the category $Q\underline{P'}(A)$ is a subset of the topological space $\overline{L}_{M,M'}$. Define on $\mathrm{Hom}(M,M')$ the topology induced from $\overline{L}_{M,M'}$.

By this way the category $Q\underline{P'}(A)$ becomes a well defined topological category which coincides with Quillen's Q-contruction when A is a discrete ring.

Consider the nerv $NQ\underline{P'}(A)$ of the category $Q\underline{P'}(A)$ and define on the set of n-simplices, $n \geq 0$, the topology induced by the topology of the category $Q\underline{P'}(A)$. Then $NQ\underline{P'}(A)$ is a simplicial topological space and take its geometric realization which we denote $BQ\underline{P'}(A)$.

Definition 1. *For any normed k-algebra A with unit define*

$$K_n^Q(A) = \pi_{n+1}(BQ\underline{P'}(A), 0), \quad n \geq 0.$$

It is clear that if A is a discrete $\mathbb{Z}$-algebra with unit we recover Quillen's K-groups $K_n(A)$ of the ring A [13]. Let $f : A \to C$ be a bounded homomorphism of normed k-algebras with unit. Let $P \in \mathrm{Ob}\,\underline{P'}(A)$. If the norm on P is induced by the epimorphism $q : \overset{n}{\oplus} A \to P$ (by the aforementioned way) then the norm on $P \otimes_A C$ is induced by the epimorphism $q \otimes C : \overset{n}{\oplus} A \otimes_A C \to P \otimes_A C$. One gets a functor

$$- \otimes_A C : \underline{P'}(A) \longrightarrow \underline{P'}(C),$$

which is exact and continuous.

Thus f induces a continuous map

$$B(f) : BQ\underline{P'}(A) \longrightarrow BQ\underline{P'}(C)$$

and therefore, a homomorphism

$$K_n^Q(f) : K_n^Q(A) \longrightarrow K_n^Q(C), \quad n \geq 0.$$

If A and C are normed k-algebras with unit then the natural continuous map

$$BQ\underline{P'}(A \times C) \longrightarrow BQ\underline{P'}(A) \times BQ\underline{P'}(C)$$

is a weak homotopy equivalence implying an isomorphism

$$K_n^Q(A \times C) \approx K_n^Q(A) \times K_n^Q(C), \quad n \geq 0.$$

Let A be an arbitrary normed k-algebra (not necessarily with unit). Denote

$$A^+ = A + k = \{(a, \lambda),\ a \in A,\ \lambda \in k\}$$

with usual sum, with product given by $(a,\lambda)(a',\lambda') = (aa' + a\lambda' + \lambda a' + \lambda\lambda')$ and with norm given by $\| (a,\lambda) \| = \| a \| + \| \lambda \|$. Take the projection map $p : A^+ \to k$. Then p induces a fibration

$$F \longrightarrow BQ\underline{P'}(A + k) \xrightarrow{B(p)} BQ\underline{P'}(k)$$

with fibre F which will be denoted by $BQ\underline{P'}(A)$.

Define

$$K_n^Q(A) = \pi_{n+1}(F), \quad n \geq 0.$$

Evidently this definition is correct and we have

$$K_n^Q(A + k) = K_n^Q(A) \oplus K_n^Q(k), \quad n \geq 0.$$

Consider now the category $\underline{A}_{\mathbb{R}}$ of real normed algebras. Recall the definition of topological K-functors K_n^{top}, $n \geq 0$, for normed $\mathbb{R}$-algebras [8, 12] .

Let A be a normed $\mathbb{R}$-algebra . Then the groups $K_n^{\mathrm{top}}(A) = \pi_{n-1}\,\mathrm{GL}(A)$, $n \geq 1$, and $K_0^{\mathrm{top}}(A) = K_0(A)$ are called the topological K-groups of A where the topological linear group $GL(A)$ has the inductive limit topology induced by the norm of A.

Theorem 2. *Let A be a special normed $\mathbb{R}$-algebra. Then there is a functorial isomorphism*

$$K_n^Q(A) \approx K_n^{\mathrm{top}}(A), \quad n \geq 0.$$

Proof. It is sufficient to prove the theorem when A is with unit. First it will be shown that the functors K_n^Q, $n \geq 0$, are homotopy functors (in topological sense).

Consider the bounded A-homomorphisms φ_0 , $\varphi_1 : A^I \to A^I$ given by $\varphi_0(f)(x) = f(0)$ and $\varphi_1(f)(x) = f(x)$ where $x \in I = [0,1]$ and $f : I \to A$ is a continuous map. Then the induced continuous maps

$$B(\varphi_0), B(\varphi_1) : BQ\underline{P'}(A^I) \longrightarrow BQ\underline{P'}(A^I)$$

are homotopic, the homotopy being given by $B(\varphi_t)$ where $\varphi_t(f)(x) = f(tx)$, $f \in A^I$ and $x, t \in I$. Therefore, one gets $K_n^Q(\varphi_0) = K_n^Q(\varphi_1)$ for all $n \geq 0$. It follows that $K_n^Q(A) \xrightarrow{\approx} K_n^Q(A^I)$, $n \geq 0$, and this implies that the functors K_n^Q are homotopy functors (see below Proposition 4).

Further we shall prove that $K_0(\underline{P'}(A))$ is isomorphic to $K_0(A)$. Let P be a finitely generated projective A-module and consider two norms on P induced by $q_1 : \overset{n}{\oplus} A \to P$ and $q_2 : \overset{n}{\oplus} A \to P$ respectively. Denote by P_1 and P_2 the A-module P with the first and the second norm respectively. We will show that $[P_1] = [P_2]$ in the group $K_0(\underline{P'}(A))$.

Let $q' : \overset{n+m}{\oplus} A \to P$ be the homomorphism induced by q_1 and q_2, i.e. $q' = q_1 + q_2$. Then there is a commutative diagram

$$\begin{array}{ccccc} \overset{n}{\oplus} A & \longrightarrow & \overset{n+m}{\oplus} A & \longleftarrow & \overset{m}{\oplus} A \\ \downarrow q_1 & & \downarrow q' & & \downarrow q_2 \\ P_1 & \xrightarrow{1_1} & P' & \xleftarrow{1_2} & P_2 \end{array}$$

with bounded A-homomorphisms $1_1 : P_1 \to P'$, $1_2 : P_2 \to P'$ where P' is the A-module P with norm induced by q' and 1_1,1_2 are identity maps.

Therefore, the exact sequences

$$O \longrightarrow P_1 \xrightarrow{1_1} P' \longrightarrow O \longrightarrow O,$$

$$O \longrightarrow P_2 \xrightarrow{1_2} P' \longrightarrow O \longrightarrow O$$

belongs to the family E' and we have $[P_1] = [P] = [P_2]$.

The following step is to show that $K_0^Q(A) = \pi_1(BQ\underline{P'}(A), O)$ is isomorphic to $K_0(A)$. We will follow Quillen's prove of the similar statement for the discrete case ([13] , Theorem 1).

By the same way it can be proved that the category of covering spaces of $BQ\underline{P'}(A)$ is equivalent to the category $\underline{F}$ of morphism inverting functors $F : Ht(Q\underline{P'}(A)) \to Sets$ where $Ht(Q\underline{P'}(A))$ is a category whose objects are the objects of the category $Q\underline{P'}(A)$ and morphisms are equivalence classes of homotopic morphisms of the topological category $Q\underline{P'}(A)$.

We need the following well-known presentation of the Grothendieck group $K_0(A)$ [8]. Let $\mathrm{Proj}(A^n)$ be the set of $n \times n$ matrices p over A satisfying $p^2 = p$ and called projections. Let $\overline{\mathrm{Proj}(A^n)}$ be the quotient set under the equivalence $p \sim p'$ if there exists $\gamma \in \mathrm{GL}_n(A)$ such that $\gamma^{-1} p \gamma = p'$. Take the inductive limit of

$$\overline{\mathrm{Proj}(A^2)} \xrightarrow{i_1} \overline{\mathrm{Proj}(A^4)} \xrightarrow{i_2} \cdots \longrightarrow \overline{\mathrm{Proj}(A^{2n})} \xrightarrow{i_n} \cdots$$

where i_n is induced by $p \mapsto p \oplus p_0$, p_0 being the projection $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ of A. Then $K_0(A)$ is isomorphic to this inductive limit via the map

$$p \longmapsto [\operatorname{Im} p] - [\operatorname{Im} \overbrace{p_0 \oplus \cdots \oplus p_0}^{n}].$$

It is easy to see that using the isomorphism $K_0(\underline{P'}(A)) \approx K_0(A)$ and the aforementioned presentation of $K_0(A)$ we can realize in our case Quillen's proof of the isomorphism $\pi_1(BQ\underline{P'}(A), O) \approx K_0(A)$ if it will be shown that for any continuous map $f : I \longrightarrow \mathrm{Proj}(A^n)$ the elements $f(0)$ and $f(1)$ give the same element in the inductive limit $K_0(A)$. But it is known that if A is a special normed $\mathbb{R}$-algebra with unit then there exists an element $\gamma \in \mathrm{GL}_n(A)$ such that $f(1) = \gamma^{-1} f(0)\gamma$ (see [11] or [12], Theorem 2.10). This means $f(0)$ and $f(1)$ lie in the same equivalence class $[f(0)] = [f(1)] \in \overline{\mathrm{Proj}(A^n)}$. The proof of the isomorphism $K_0(A) \approx \pi_1(BQ\underline{P'}(A), O)$ is complete. We conclude that $K_0(A) \approx K_0^Q(A)$ for any special normed $\mathbb{R}$-algebra A.

Let $\mathfrak{J} = (J, \tau, \delta)$ be the path cotriple in the category $\underline{A_R}$ of normed $\mathbb{R}$-algebras . Let $J_*^+(A) = J_*(A) \xrightarrow{\tau} A$ be the induced augmented simplicial set where $J_*(A) = \{J_n(A), \partial_i^n, s_i^n\}$ with $J_0(A) = J(A) = A^{(I,0)}$, $J_n(A) = J^{n+1}(A) = J(J^n(A))$, $\partial_i^n = J^i \tau J^{n-i}$, $s_i^n = J^i \delta J^{n-i}$, $n \geq 0$. Then $K_n^{top}(A) = \pi_{n-2}(J_*^+(A))$, $n \geq 1$.

Since $J(A)$ is a contractible special normed $\mathbb{R}$-algebra , we have $K_n^Q(J(A)) = 0$ for all $n \geq 0$. The short exact sequence of special normed $\mathbb{R}$-algebras

$$0 \longrightarrow \Omega(A) \xrightarrow{\sigma} J(A) \xrightarrow{\tau} A \longrightarrow 0,$$

where $(\Omega(A), \sigma)$ denotes the kernel of τ, is a GL-fibration relative to the path cotriple $\mathfrak{J}$ and therefore implies $K_{n+1}^{\mathrm{top}}(A) = K_n^{\mathrm{top}}(\Omega(A))$,$n \geq 0$, for any special normed $\mathbb{R}$-algebra A, since $K_n^{\mathrm{top}}(J(A)) = 0$ for $n \geq 1$ and the Grothendieck functor $K_0 = K_0^{top}$ is a homotopy functor in the category of special normed $\mathbb{R}$-algebras [11, 12]. Thus one gets $K_n^{\mathrm{top}}(A) = K_0(\Omega^n(A))$, $n \geq 0$, where $\Omega^{n+1}(A) = \Omega(\Omega^n(A))$, $\Omega^0(A) = A$.

Let A be a special normed $\mathbb{R}$-algebra with unit. There is a short exact sequence

$$0 \longrightarrow \Omega(A) \xrightarrow{\sigma} A^I \xrightarrow{\epsilon} A \times A \longrightarrow 0, \tag{1}$$

where $\epsilon = \epsilon_0 \times \epsilon_1$ with $\epsilon_0(f) = f(0)$, $\epsilon_1(f) = f(1)$. Then the following commutative diagram

$$\begin{array}{ccc} \Omega(A)^+ & \xrightarrow{p} & \mathbb{R} \\ \downarrow \sigma^+ & & \downarrow \Delta \\ A^I & \xrightarrow{\epsilon} & A \times A \end{array}$$

where $\Delta(r) = (r, r)$, $r \in \mathbb{R}$, induces a commutative diagram of fibrations

$$\begin{array}{ccccc} F & \longrightarrow & BQ\underline{P'}(\Omega(A)^+) & \longrightarrow & BQ\underline{P'}(\mathbb{R}) \\ \downarrow \eta & & \downarrow & & \downarrow \\ L & \longrightarrow & BQ\underline{P'}(A^I) & \longrightarrow & BQ\underline{P'}(A \times A) \end{array}$$

where L is the homotopy fibre of the map $BQ\underline{P'}(A^I) \to BQ\underline{P'}(A \times A)$. Later it will be shown (see Proposition 5) that the sequence (1) is a GL-fibration with respect to the path cotriple.

Since the functors K_n^Q are homotopy functors, we have $\mathrm{Coker}\, K_n^Q(\epsilon) \approx K_n^Q(A)$, $n \geq 0$, and the map $\eta : F \to L$ induces an isomorphism of homotopy groups. Therefore, one gets

$$K_{n+1}^Q(A) \approx \mathrm{Coker}\, K_{n+1}^Q(\epsilon) \approx \pi_{n+1}(L) \approx K_n^Q(\Omega(A)) \text{ for all } n \geq 0.$$

It is clear that these isomorphisms hold for arbitrary (not necessarily with unit) special normed $\mathbb{R}$-algebras. The required isomorphism is given by combining

$$K_n^Q(A) \approx K_0^Q(\Omega^n(A)) \approx K_0(\Omega^n(A)) \approx K_n^{\text{top}}(A), \quad n \geq 0.$$

Done with Theorem 2. ■

We see that the algebraic and topological K-functors can be obtained by the same construction, namely by using Quillen's Q-construction (analogical fact was observed in [12] for Swan's K-theory of Banach $\mathbb{R}$-algebras). We deduce there is only one K-theory, the K-theory of normed algebras, the algebraic K-theory being the K-theory of discrete algebras.

§2. Relationship between algebraic and topological K-functors

The natural relation between Quillen's algebraic K-functors and topological K-functors is investigated on the category of real normed algebras.

Definition 3. *A subcategory $\underline{A}$ of the category $\underline{A_{\mathbb{R}}}$ of all normed $\mathbb{R}$-algebras and their bounded (i.e. continuous with respect to the topology induced by the norm) $\mathbb{R}$-homomorphisms will be called homotopy category if the following conditions are satisfied:*

1. *If $A \in \operatorname{Ob}\underline{A}$ then $A^I \in \operatorname{Ob}\underline{A}$; the inclusion $i : A \to A^I$, $i(a) = f_a$ with $f_a(x) = a$, $x \in I$, and the canonical homomorphisms $\epsilon_0, \epsilon_1 : A^I \to A$, $\epsilon_0(f) = f(0)$, $\epsilon_1(f) = f(1)$ are morphisms of $\underline{A}$.*
2. *If $A \in \operatorname{Ob}\underline{A}$ then the homomorphism*

$$\delta : A^I \longrightarrow (A^I)^I = A^{I^2},$$

given by $t \in I$ sending to tx for $x \in I$,

$$\delta(f)(t)(x) = f(tx), \quad t, x \in I,$$

is a morphism of $\underline{A}$.

Recall the definition of a homotopy functor. Let $f, g : A \to B$ be bounded $\mathbb{R}$-homomorphisms of a homotopy category $\underline{A}$. We say that f is homotopic to g, $f \sim g$, if there is a bounded $\mathbb{R}$-homomorphism $h : A \to B^I$ of the category $\underline{A}$ such that $\epsilon_0 h = f$, $\epsilon_1 h = g$. The morphism h is called the homotopy between f and g. A functor $T : \underline{A} \to \underline{\mathfrak{Gr}}$ is called homotopy functor if $T(f) = T(g)$ for $f \sim g$.

Proposition 4. *A functor $T : \underline{A} \to \underline{\mathfrak{Gr}}$ is a homotopy functor if and only if the inclusion $i : A \to A^I$ induces an isomorphism $T(i) : T(A) \to T(A^I)$ for all $A \in \operatorname{Ob}\underline{A}$.*

Proof. If $T(i)$ is an isomorphism then from $T(\epsilon_0)T(i) = T(\epsilon_1)T(i)$ one gets $T(\epsilon_0) = T(\epsilon_1)$ and therefore T is a homotopy functor. Conversely, let T be a homotopy functor. Then it is easy to see that the bounded $\mathbb{R}$-homomorphism $i\epsilon_0 : A^I \to A^I$ is homotopic to the identity map, where $\delta : A^I \to A^{I^2}$ (see Definition 3, 2) is a homotopy between $i\epsilon_0$ and 1_{A^I}. Thus $T(i)T(\epsilon_0) = id = T(\epsilon_0)T(i)$. Done with Proposition 4. ■

The topological K-functors $K_i^{\text{top}} : \underline{A_{\mathbb{R}}} \to \underline{\mathfrak{Ab}}$, $i \geq 1$ to the category of abelian groups are homotopy functors. In effect the short split exact sequence

$$0 \longrightarrow J(A) \longrightarrow A^I \xrightarrow{\epsilon_0} A \longrightarrow 0$$

is a GL-fibration with respect to the path cotriple and gives a split exact sequence

$$0 \longrightarrow K_i^{\text{top}}(J(A)) \longrightarrow K_i^{\text{top}}(A^I) \xrightarrow{K_i^{\text{top}}(\epsilon_0)} K_i^{\text{top}}(A) \longrightarrow 0.$$

But $J(A)$ is an object of the projective class induced by the path cotriple. So $K_i^{\text{top}}(J(A)) = 0$ for all $i \geq 1$, implying an isomorphism $K_i^{\text{top}}(i) : K_i^{\text{top}}(A) \to K_i^{\text{top}}(A^I)$.

Let $T : \underline{A} \to \underline{\mathfrak{Gr}}$ be a functor from a homotopy category of normed $\mathbb{R}$-algebras to the category of groups. As defined in [12] (Chapter V , Definition 2.7]) the homotopization of T is a functor $hT : \underline{A} \to \underline{\mathfrak{Gr}}$ given by

$$hT(A) = \operatorname{Coker}(T(A^I) \underset{T(\epsilon_1)}{\overset{T(\epsilon_0)}{\rightrightarrows}} T(A))$$

for any normed $\mathbb{R}$-algebra A.The homomorphism $hT(f) : hT(A) \to hT(A')$ is defined in a natural way for any homomorphism $f : A \to A'$ of normed $\mathbb{R}$-algebras.

It is evident that $hT = T$ if and only if T is a homotopy functor. Note that if the inclusion $i : A \to A^I$ induces an isomorphism $T(i)$, then $hT(A) = T(A)$.

Proposition 5. *The sequence (1) is a* GL*-fibration with respect to the path cotriple.*

Proof. A similar fact was proved by Swan [16] for discrete rings, A^I replaced in (1) by $A[x]$ and also for Banach algebras over a commutative Banach ring A^I replaced by $A\langle x\rangle$ (see [12]).

Since $J(\epsilon_1)$ is a split homomorphism, the map ϵ_1 induces a surjection

$$\mathrm{GL}(J_*(JA)) \longrightarrow \mathrm{GL}(J_*(A)).$$

Clearly the map $\mathrm{GL}(J_*^+(JA)) \to \mathrm{GL}(J_*^+(A))$ induced by $\epsilon_0 : A^I \to A$, $x \mapsto 0$, is trivial. Therefore, the image of the composites

$$\mathrm{GL}(J_*(A)) \longrightarrow \mathrm{GL}(J_*(A^I)) \longrightarrow \mathrm{GL}(J_*(A)) \times \mathrm{GL}(J_*(A))$$

$$\mathrm{GL}(J_*(JA)) \longrightarrow \mathrm{GL}(J_*(A^I)) \longrightarrow \mathrm{GL}(J_*(A)) \times \mathrm{GL}(J_*(A))$$

is respectively the diagonal subgroup and the subgroup $1 \times \mathrm{GL}(J_*(A))$ where the map $\mathrm{GL}(J_*(A)) \to \mathrm{GL}(J_*(A^I))$ is induced by the canonical inclusion $i : A \to A^I$ and the map $\mathrm{GL}(J_*(A^I)) \to \mathrm{GL}(J_*(A)) \times \mathrm{GL}(J_*(A))$ is induced by $\epsilon_0 \times \epsilon_1$. Since

$1 \times \mathrm{GL}(J_*(A))$ and the diagonal subgroup generate the group $\mathrm{GL}(J_*(A)) \times \mathrm{GL}(J_*(A))$ it follows that $\mathrm{GL}\, J_*(\epsilon_0 \times \epsilon_1)$ is surjective. Done with Proposition 5. ■

Since (1) is a GL-fibration with respect to the cotriple $\mathfrak{J}$, one gets the following short exact sequence

$$0 \longrightarrow K_i^{top}(A^I) \longrightarrow K_i^{top}(A \times A) \longrightarrow K_{i-1}^{top}(\Omega(A)) \longrightarrow 0$$

for all $i \geq 1$.

Proposition 6. *Let* $T : \underline{A_{\mathbb{R}}} \to \underline{\mathfrak{Gr}}$ *be a functor and let* $\mathrm{Im}(T(\epsilon_0) \times T(\epsilon_1))$ *be a normal subgroup of* $T(A) \times T(A)$, $A \in \mathrm{Ob}\, \underline{A_{\mathbb{R}}}$*. Then there is an isomorphism*

$$\psi : hT(A) \xrightarrow{\approx} \mathrm{Coker}(T(\epsilon_0) \times T(\epsilon_1)).$$

Proof. Consider the diagram

$$T(A^I) \underset{T(\epsilon_1)}{\overset{T(\epsilon_0)}{\rightrightarrows}} T(A) \xrightarrow{\eta} hT(A) \longrightarrow 1$$

and let $\tau : T(A) \times T(A) \to \mathrm{Coker}(T(\epsilon_0) \times T(\epsilon_1))$ be the canonical map. Define

$$\psi([x]) = \tau(1, x) \quad , [x] \in hT(A).$$

The map ψ is correctly defined. In effect, if $\eta(x) = \eta(y)$ for $x, y \in T(A)$ then there is an element $z \in T(A^I)$ such that $T(\epsilon_0)(z) = 1$ and $T(\epsilon_1)(z) = xy^{-1}$. Thus the equality $(T(\epsilon_0) \times T(\epsilon_1))(z) = (1, xy^{-1}) = (1, x)(1, y^{-1})$ implies $\tau(1, x) = \tau(1, y)$.

Injectivity of ψ. Suppose $\psi([x]) = 1$. Then $\tau(1, x) = 1$ and there is an element $z \in T(A^I)$ such that $T(\epsilon_0)(z) = 1$ and $T(\epsilon_1)(z) = x$. It follows that $\eta(x) = 1$.

Surjectivity of ψ. For any $(a, b) \in T(A) \times T(A)$ the equality $(a, b)(1, b^{-1}a) = (a, a)$ implies $(T(\epsilon_0) \times T(\epsilon_1))(i(a)) = (a, b)(1, b^{-1}a)$. Hence $\tau(a, b) = \tau(1, ab^{-1})$. Therefore, $\psi([a^{-1}b]) = \tau(1, a^{-1}b) = \tau(a, b)$. Done with Proposition 6. ■

Corollary 7. *Let* $T : \underline{A_{\mathbb{R}}} \to \underline{\mathfrak{Gr}}$ *be a functor as above and such that* $T(0) = \{1\}$*. Then there is a surjection*

$$\mathrm{Coker}\, T(\epsilon_0 \times \epsilon_1) \longrightarrow hT(A), \quad A \in \mathrm{Ob}\, \underline{A_{\mathbb{R}}}.$$

Proof. It suffices to show the surjectivity of the map $\beta : T(A \times A) \to T(A) \times T(A)$ induced by $T(\epsilon_i)$, $i = 0, 1$. Use the map $\gamma : A \to A \times A$ given by $a \mapsto (0, a)$. Then we have $\beta(T(\gamma)(x)) = (1, x)$ for any $x \in T(A)$. For $(x, y) \in T(A) \times T(A)$ take the element $T(\Delta)(x) \cdot T(\gamma)(x^{-1}y)$ of $T(A \times A)$ where $\Delta : A \to A \times A$ is given by $a \mapsto (a, a)$. It is clear that $\beta(z) = (x, y)$. Done with Corollary 7. ■

The canonical map $\eta : T \to hT$ is universal for maps of T into homotopy functors. It follows that the homomorphism $\varphi_i(A) : K_i(A) \to K_i^{top}(A)$, $i \geq 1$, induced by the canonical map $B\,\mathrm{GL}(A)^+ \to B\,\mathrm{GL}^{top}(A)$ yields a homomorphism

$$h\varphi_i(A) : hK_i(A) \longrightarrow K_i^{top}(A), \quad i \geq 1.$$

Proposition 8. *Let A be a normed $\mathbb{R}$-algebra. Then one has an isomorphism*

$$h\varphi_1(A) : hK_1(A) \overset{\approx}{\to} K_1^{top}(A).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc} \mathrm{GL}(J(A)) & \longrightarrow & \mathrm{GL}(A) & \longrightarrow & K_1^{top}(A) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \| & & \\ hK_1(J(A)) & \longrightarrow & hK_1(A) & \xrightarrow{\varphi_1(A)} & K_1^{top}(A) & \longrightarrow & 0, \end{array}$$

where the top row is exact and the vertical homomorphisms are surjective. It follows that the bottom row is also exact. Since hK_1 is a homotopy functor, $hK_1(J(A)) = 0$ implying the isomorphism of $\varphi_1(A)$. Done with Proposition 8. ■

Corollary 9. *If $K_1(A) \overset{\approx}{\to} K_1(A^I)$ then $\varphi_1(A) : K_1(A) \to K_1^{top}(A)$ is an isomorphism.*

Below Proposition 8 and Corollary 9 will be extended to higher K-functors.

Theorem 10. *Let A be a normed $\mathbb{R}$-algebra (not necessarily with unit). Then for fixed $i \geq 1$*

a) *there is an epimorphism*

$$\varphi_{i+1}(A) : K_{i+1}(A) \longrightarrow K_{i+1}^{top}(A)$$

if $K_{j+1}(B^{I^{i-j-1}}) = K_{j+1}(B^{I^{i-j}})$ with $B = A^{2^j}$ for all j, $0 \leq j \leq i-1$,

b) *there is an isomorphism*

$$h\varphi_{i+1}(A) : K_{i+1}(A) \overset{\approx}{\to} K_{i+1}^{top}(A), \quad i \geq 0,$$

if $K_{j+1}(B^{I^{i-j-1}}) = K_{j+1}(B^{I^{i-j+1}})$ with $B = A^{2^j}$ for all j, $0 \leq j \leq i-1$.

Proof. There is a commutative diagram of functors defined on the category of normed $\mathbb{R}$-algebras

$$\begin{array}{ccc} K_i(-) & \underset{\theta_i}{\overset{\approx}{\to}} & K_i^S(-) \\ \downarrow \varphi_i & & \downarrow \alpha_i \\ K_i^{top}(-) & \overset{=}{\to} & K_i^{top}(-) \end{array}$$

where K_i^S is the Swan's K-functor [16], the top row is an equivalence [1] and $\alpha_i(A) : K_i^S(A) \to K_i^{top}(A)$ is induced by the morphism of cotriples $(F, \tau, \delta) \to (J, \tau, \delta)$ given by $|a| \mapsto (t \mapsto at)$ for $a \in A$, $t \in I$.

So Theorem 10 will be proved for Swan's K-functors, i.e. in Theorem 10 Quillen's K-functors replaced by Swan's K-functors and then use the equivalence of Quillen and Swan K-theories.

We need auxiliary K-functors constructed as follows. A similar construction was given for Banach algebras over a commutative Banach ring with unit (see [12]).

Let $f : A \to C$ be a bounded $\mathbb{R}$-homomorphism. Then the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_1F_*^+(A) & \longrightarrow & F_*^+(A^I) & \xrightarrow{F_*^+(\epsilon_0\times\epsilon_1)} & F_*^+(A\times A) \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_1F_*^+(C) & \longrightarrow & F_*^+(C^I) & \xrightarrow{F_*^+(\epsilon_0\times\epsilon_1)} & F_*^+(C\times C) \end{array}$$

with exact rows induces a uniquely defined morphism

$$M_1F_*^+(f) : M_1F_*^+(A) \longrightarrow M_1F_*^+(C)$$

and one gets a functor $M_1F_*^+$ from the category $\underline{A_{\mathbb{R}}}$ to the category of augmented simplicial rings.

Let $M_jF_*^+$ be constructed for $0 \leq j < k$. Then define

$$M_kF_*^+(A) = \operatorname{Ker} M_{k-1}F_*^+(\epsilon_0 \times \epsilon_1)$$

for $A \in \operatorname{Ob}\underline{A_R}$ and $M_kF_*^+(f)$ is induced by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_kF_*^+(A) & \longrightarrow & M_{k-1}F_*^+(A^I) & \xrightarrow{M_{k-1}F_*^+(\epsilon_0\times\epsilon_1)} & M_{k-1}F_*^+(A\times A) \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_kF_*^+(C) & \longrightarrow & M_{k-1}F_*^+(C^I) & \xrightarrow{M_{k-1}F_*^+(\epsilon_0\times\epsilon_1)} & M_{k-1}F_*^+(C\times C) \end{array}$$

In this way one gets for each $k \geq 0$ a functor $M_kF_*^+$ from $\underline{A_{\mathbb{R}}}$ to the category of augmented simplicial rings and define

$$M_kK_i^S(A) = \pi_{i-2}\,\mathrm{GL}(M_kF_*^+(A)) \ \text{ for } \ i \geq 1,\ k \geq 0,$$

$M_kK_0(A) = K_0(S^k(A))$ for $k \geq 0$ where $M_0F_*^+(A) = F_*^+(A)$, $A \in \operatorname{Ob}\underline{A_{\mathbb{R}}}$, and $M_kK_i^S(f)$ is induced by $M_kF_*^+(f)$.

Let $f : A \to B$ be a continuous map with $A, B \in \operatorname{Ob}\underline{A_{\mathbb{R}}}$. Then f induces in a natural way for all $i \geq 1$ a functor a homomorphism $F_i(f) : F_i(A) \to F_i(B)$ and clearly F_i becomes a functor from the category $\underline{A'_{\mathbb{R}}}$ of normed $\mathbb{R}$-algebras with morphisms continuous maps to the category of rings

Any continuous map $f : A \to B$ induces a commutative diagram

$$\begin{array}{ccc} A^I & \xrightarrow{\epsilon_0\times\epsilon_1} & A\times A \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f\times f} \\ B^I & \xrightarrow{\epsilon_0\times\epsilon_1} & B\times B \end{array} \tag{2}$$

where $f^*(\varphi) = f\varphi$ for $\varphi : I \to A$, and f^*, $f \times f$ are continuous maps. Thus we have homomorphisms $F_i(f^*) : F_i(A^I) \to F_i(C^I)$, $F_i(f \times f) : F_i(A \times A) \to F_i(C \times C)$ induced by f^* and $f \times f$ respectively such that $F_i(\epsilon_0 \times \epsilon_1)F_i(f^*) = F_i(f \times f)F_i(\epsilon_0 \times \epsilon_1)$ for $i \geq 1$.

Therefore, any continuous map $f : A \to C$ by the homomorphism $F_i(f^*)$ induces a uniquely defined homomorphism

$$M_1F_i(f) : M_1F_i(A) \longrightarrow M_1F_i(C), \quad i \geq 1.$$

It is evident if $f : A \to C$, $g : C \to D$ are two continuous maps then

$$M_1F_i(gf) = M_1F_i(g) \cdot M_1F_i(f), \ \ ; i \geq 1.$$

Let $f : A \to C$ be a continuous map with $A, B \in \mathrm{Ob}\,\underline{A'_{\mathbb{R}}}$. Then f induces a continuous map $f_1 : \Omega(A) \to \Omega(C)$. Since $\Omega(A \times A) \approx \Omega(A) \times \Omega(A)$ and $\Omega(A^I) \approx \Omega(A)^I$, using the fact M_1F_i is a functor from $\underline{A'_{\mathbb{R}}}$ to the category of rings it can be proved as for the map f and for M_0F_i that the continuous map f_1 induces a uniquely defined homomorphism

$$M_1(M_1F_i)(f_1) : M_1(M_1F_i)(A) \longrightarrow M_1(M_1F_i)(C)$$

for all $i \geq 1$.

Therefore the continuous map f using diagram (2) induces a uniquely defined homomorphism $M_1(M_1F_i)(f_1)$ which we denote by

$$M_2F_i(f) : M_2F_i(A) \longrightarrow M_2F_i(C)$$

and M_2F_i is a functor from $\underline{A'_{\mathbb{R}}}$ to the category of rings for all $i \geq 1$.

Clearly by induction on j similarly it can be shown that any continuous map $f : A \to C$ of normed $\mathbb{R}$-algebras induces a uniquely defined homomorphism

$$M_jF_i(f) : M_jF_i(A) \longrightarrow M_jF_i(C)$$

for all $i \geq 1$, $j \geq 0$, assuming that $M_{j-1}F_i$ is a functor constructed for all $i \geq 1$ by the continuous map $f_{i-1} : S^{i-1}(A) \to S^{i-1}(C)$ induced by f using diagram (2).

Let $f : A \to C$ be a continuous map of normed $\mathbb{R}$-algebras such that there exists a continuous map $f' : C \to A$ with $ff' = 1_C$. Then we have

$$M_jF_i(f) \cdot M_jF_i(f') = 1_{M_jF_i(C)}$$

for all $i \geq 1$, $j \geq 0$. Therefore the induced map

$$\mathrm{GL}(M_jF_*(A)) \longrightarrow \mathrm{GL}(M_jF_*(C))$$

is surjective for all $j \geq 0$ and one gets for every $j \geq 0$ an exact sequence

$$\cdots \longrightarrow M_jK^S_{i+1}(C) \longrightarrow M_jK^S_i(A, \mathrm{Ker}\, f) \longrightarrow M_jK^S_i(A) \longrightarrow$$
$$\longrightarrow M_jK^S_i(C) \longrightarrow M_jK^S_{i-1}(A, \mathrm{Ker}) \longrightarrow \cdots$$

where $M_jK^S_i(A, \mathrm{Ker}) = \pi_{i-2}\,\mathrm{GL}(\mathrm{Ker}\, M_jF^+_*(f))$, $i \geq 1$.

We can apply this to the sequence (1). In effect for the map $\epsilon_0 \times \epsilon_1 : A^I \to A \times A$ there is a continuous map

$$\epsilon' : A \times A \longrightarrow A^I$$

given by $(a,a') \mapsto (t \mapsto (1-t)a + ta')$ such that $(\epsilon_0 \times \epsilon_1)\epsilon' = 1_{A\times A}$. So the sequence (1) gives the following long exact sequence for all $j \geq 0$:

$$(3_j) \quad \cdots \longrightarrow M_{j+1}K^S_{i+1}(A) \longrightarrow M_jK^S_{i+1}(A^I) \longrightarrow M_jK^S_{i+1}(A\times A) \longrightarrow \\ \longrightarrow M_{j+1}K^S_i(A) \longrightarrow M_jK^S_i(A^I) \longrightarrow M_jK^S_i(A\times A) \longrightarrow \cdots \longrightarrow \\ \longrightarrow M_jK^S_1(A\times A) \longrightarrow M_jK_0(\Omega(A)) \longrightarrow M_jK_0(A^I) \longrightarrow M_jK_0(A\times A).$$

It is easy to see that every augmented simplicial ring $M_jF^+_*(A)$, $j \geq 0$, is a simplicial resolution of $\Omega^j(A) = \Omega(\Omega^{j-1}(A))$ in the category of discrete rings.

Clearly the abovementioned morphism $(F,\tau,\delta) \to (J,\tau,\delta)$ induces a morphism $F_*(A) \to J_*(A)$ which gives a morphism

$$M_jF_*(A) \longrightarrow J_*(\Omega^j(A))$$

for all $j \geq 0$.

It follows that for every $j \geq 0$ and any normed $\mathbb{R}$-algebra we have isomorphisms

$$h(M_jK^S_1)(A) \approx K^{top}_1(\Omega^j(A)) \approx K^{top}_{j+1}(A).$$

Let

$$M_jK^S_i(p_0),\ M_jK^S_i(p_1) : M_jK^S_i(A\times A) \longrightarrow M_jK^S_i(A)$$

be homomorphisms induced by the natural projection maps $p_0, p_1 : A \times A \to A$ respectively for $i \geq 1$, $j \geq 0$.

One gets the following commutative diagram

(4_j)

$$\begin{array}{ccccccc} M_jK^S_i(A^I) & \longrightarrow & M_jK^S_i(A\times A) & \longrightarrow & \operatorname{Coker} M_jK^S_i(\epsilon_0\times\epsilon_1) & \longrightarrow & 0 \\ \| & & \downarrow \beta_{j,i} & & \downarrow \overline{\beta_{j,i}} & & \\ M_jK^S_i(A^I) & \longrightarrow & M_jK^S_i(A)\times M_jK^S_i(A) & \longrightarrow & h(M_jK^S_i)(A) & \longrightarrow & 0 \end{array}$$

for all $j \geq 0$, $i \geq 1$, where $\beta_{j,i}$ is induced by the homomorphisms $M_j\underline{K^S_i}(p_0)$, $M_jK^S_i(p_1)$ and the bottom row is exact by Proposition 6. It follows that $\overline{\beta_{j,i}}$ is a surjection.

It is evident that we have an isomorphism $K^S_1(A) \xrightarrow{\approx} K^{top}_1(A)$ if $K^S_1(A) \xrightarrow{\approx} K^S_1(A^I)$, $A \in \operatorname{Ob} \underline{A_{\mathbb{R}}}$.

Therefore, using (3_j) and (4_j) one gets the following commutative diagrams

$$\begin{array}{ccc} M_1K^S_i(A) & \longrightarrow & h(M_0K^S_{i+1})(A) \\ \downarrow & & \downarrow \\ K^{top}_i(\Omega(A)) & \xrightarrow{\approx} & K^{top}_{i+1}(A) \end{array}$$

if $M_0K_i^S(A) \overset{\approx}{\to} M_0K_i^S(A^I)$,

$$\begin{array}{ccc} M_2K_{i-1}^S(A) & \longrightarrow & h(M_1K_i^S)(A) \\ \downarrow & & \downarrow \\ K_{i-1}^{top}(\Omega^2(A)) & \overset{\approx}{\to} & K_{i+1}^{top}(A) \end{array}$$

if $M_1K_{i-1}^S(A) \overset{\approx}{\to} M_1K_{i-1}^S(A^I)$,

....

$$\begin{array}{ccc} M_{i-1}K_2^S(A) & \longrightarrow & h(M_{i-2}K_3^S)(A) \\ \downarrow & & \downarrow \\ K_2^{top}(\Omega^{i-1}(A)) & \overset{\approx}{\to} & K_{i+1}^{top}(A) \end{array}$$

if $M_{i-2}K_2^S(A) \overset{\approx}{\to} M_{i-2}K_2^S(A^I)$,

$$\begin{array}{ccc} M_iK_1^S(A) & \longrightarrow & h(M_{i-1}K_2^S)(A) \\ \downarrow & & \downarrow \\ K_1^{top}(\Omega^i(A)) & \overset{\approx}{\to} & K_{i+1}^{top}(A) \end{array}$$

if $M_{i-1}K_1^S(A) \overset{\approx}{\to} M_{i-1}K_1^S(A^I)$, where the top rows are surjective.

Since the homomorphism

$$M_jK_1^S(A) \longrightarrow K_1^{top}(\Omega^j(A)) \approx K_{j+1}^{top}(A)$$

is surjective for all $j \geq 0$, the homomorphism $M_{i-1}K_2^S(A) \to K_2^{top}(\Omega^{i-1}(A))$ of the last diagram is surjective if $M_{i-1}K_1^S(A) \overset{\approx}{\to} M_{i-1}K_1^S(A^I)$. Thus by the preceeding commutative diagram we conclude that the homomorphism $M_{i-2}K_3^S(A) \to K_3^{top}(\Omega^{i-2}(A))$ is surjective if $M_{i-1}K_1^S(A) \overset{\approx}{\to} M_{i-1}K_1^S(A^I)$ and $M_{i-2}K_2^S(A) \overset{\approx}{\to} M_{i-2}K_2^S(A^I)$.

Continuing in this manner step by step we arrive to the first commutative diagram

$$\begin{array}{ccc} M_1K_i^S(A) & \longrightarrow & h(M_0K_{i+1}^S)(A) \\ \downarrow & & \downarrow \\ K_i^{top}(\Omega(A)) & \overset{\approx}{\to} & K_{i+1}^{top}(A) \end{array}$$

which holds when $K_i^S(A) \overset{\approx}{\to} K_i^S(A^I)$. We deduce that the homomorphism $\alpha_{i+1}(A)$: $K_{i+1}^S(A) \to K_{i+1}^{top}(A)$ is surjective if $M_jK_{i-j}^S(A) \overset{\approx}{\to} M_jK_{i-j}^S(A^I)$ for all j, $0 \leq j \leq i-1$.

If the aforementioned diagrams commute also for A^I then the isomorphism $h(M_iK_1^S)(A) \approx K_1^{top}(\Omega^i(A))$ implies the isomorphism $h\alpha_{i+1}(A) : hK_{i+1}^S(A) \overset{\approx}{\to} K_{i+1}^{top}(A)$ if $M_jK_{i-j}^S(A) \overset{\approx}{\to} M_jK_{i-j}^S(A^I)$ for all j, $0 \leq j \leq i-1$.

The sequence (3_j) gives the following commutative diagram with exact rows

(3'$_j$)

$$\begin{array}{ccccccc} M_jK_{i+1}^S(A\times A) & \longrightarrow & M_{j+1}K_i^S(A) & \longrightarrow & M_jK_i^S(A^I) & \longrightarrow & M_jK_i^S(A\times A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_jK_{i+1}^S((A\times A)^I) & \longrightarrow & M_{j+1}K_i^S(A^I) & \longrightarrow & M_jK_i^S(A^{I^2}) & \longrightarrow & M_jK_i^S((A\times A)^I) \end{array}$$

Thus from the diagrm (3'$_0$) we have $M_1K_i^S(A) \overset{\approx}{\to} M_1K_i^S(A^I)$ if $K_i^S(A^I) \overset{\approx}{\to} K_i^S(A^{I^2})$ and $K_{i+1}^S(A\times A) \overset{\approx}{\to} K_{i+1}^S((A\times A)^I))$. Similarly from the diagram (3'$_1$) one gets $M_2K_i^S(A) \overset{\approx}{\to} M_2K_i^S(A^I)$ if $M_1K_i^S(A^I) \overset{\approx}{\to} M_1K_i^S(A^{I^2})$ and $M_1K_{i+1}^S(A\times A) \overset{\approx}{\to} M_1K_{i+1}^S((A\times A)^I)$. It follows that $M_2K_i^S(A) \overset{\approx}{\to} M_2K_i^S(A^I)$ if $K_i^S(A^{I^2}) \overset{\approx}{\to} K_i^S(A^{I^3})$, $K_{i+1}^S((A\times A)^I) \overset{\approx}{\to} K_{i+1}^S((A\times A)^{I^2})$ and $K_{i+2}^S(A^4) \overset{\approx}{\to} K_{i+2}^S((A^4)^I)$.

Now use induction on j. Assume that the following assertion holds:

the isomorphisms $M_jK_i^S(A) \overset{\approx}{\to} M_jK_i^S(A^I)$ are satisfied for all $j\geq 0$, $i\geq 1$, if we have

$$\begin{gathered} K_i((A)^{I^j}) \overset{\approx}{\to} K_i^S((A)^{I^{j+1}}) \\ K_{i+1}^S((A^2)^{I^{j-1}}) \overset{\approx}{\to} K_{i+1}^S((A^2)^{I^j}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ K_{i+j}^S((A)^{2^j}) \overset{\approx}{\to} K_{i+j}^S((A^{2^j})^I) \end{gathered} \tag{5}$$

and we will prove that

$$M_{j+1}K_i^S(A) \overset{\approx}{\to} M_{j+1}K_i^S(A^{I)})$$

if $K_{i+1}^S((A^{2^l})^{I^{j+1-l}} \overset{\approx}{\to} K_{i+1}^S((A^{2^l})^{I^{j+2-l}}$ for all l, $0\leq l\leq j+1$.

Using the diagram (3'$_j$) one gets $M_{j+1}K_i^S(A) \overset{\approx}{\to} M_{j+1}K_i^S(A^I)$ if $M_jK_i^S(A^I) \overset{\approx}{\to} M_jK_i^S((A)^{I^2})$ and $M_jK_{i+1}^S(A^2) \overset{\approx}{\to} M_jK_{i+1}^S((A^2)^I)$. It is easy to see this holds if we have $K_{i+1}^S((A^{2^l})^{I^{j+1-l}} \overset{\approx}{\to} K_{i+1}^S((A^{2^l})^{j+2-l}$ for all l, $0\leq l\leq j+1$.

Therefore we have proved that for an arbitrary normed $\mathbb{R}$-algebra we have $M_jK_i^S(A) \overset{\approx}{\to} M_jK_i^S(A^I)$, $j\geq 0$, $i\geq 1$, if the isomorphisms (5) hold. We conclude that the isomorphisms $K_{i-l}^S((A^{2^{i-1-l}})^{I^l} \overset{\approx}{\to} K_{i-l}^S(A^{2^{i--1-l}})^{I^{l+1}}$, $0\leq l\leq i-1$, imply not only the isomorphism $M_{i-1}K_1^S(A) \overset{\approx}{\to} M_{i-1}K_1^S(A^I)$ but all the isomorphisms $M_jK_{i-j}^S(A) \overset{\approx}{\to} M_jK_{i-j}^S(A^I)$, $0\leq j\leq i-1$. Evidently to get the isomorphism $h\alpha_{i+1}(A) : hK_{i+1}^S(A) \overset{\approx}{\to} K_{i+1}^{top}(A)$ it is sufficient for isomorphisms (5) to hold for A^I too. Done with Theorem 10. ■

The reason for which such strange normed $\mathbb{R}$-algebra B appears in Theorem 10 is caused by $K_*(A^2)\neq K_*(A)\times K_*(A)$ when A is without unit.

Corollary 11. *Let A be a normed $\mathbb{R}$-algebra. Then for fixed $i\geq 0$*

a) *there is an isomorphism*

$$\varphi_{i+1}(A) : K_{i+1}(A) \overset{\approx}{\to} K_{i+1}^{top}(A), \ \ i\geq 0,$$

if $K_{i+1}(A) \overset{\approx}{\to} K_{i+1}(A^I)$ and $K_{j+1}((A^{2^j})^{I^{i-j-1}} \overset{\approx}{\to} K_{j+1}((A^{2^j})^{I^{i-j+1}}$ for all j, $0\leq j\leq i-1$.

b) *if A is with unit, there is an isomorphism*

$$\varphi_*(A) : K_*(A) \longrightarrow K_*^{top}(A)$$

if $K_*(A) \overset{\approx}{\to} K_*(A^{I^n})$ *for all* $n > 0$.

Similar sufficient conditions have been obtained by Swan for the isomorphism of K_2 and Karoubi-Villamayor's algebraic K-functor k_2 on the category of discrete rings, the rings of continuous functions A^I and A^{I^2} replaced respectively by the polynomial rings $A[x]$ and $A[x,y]$ (see [16, Corollary 4.6]).

Definition 12. *A normed* $\mathbb{R}$*-algebra B will be called K_i-stable if*

$$K_i(B) \overset{\approx}{\to} K_i((B^{2^j})^{I^n})$$

for $j \geq 0$, $n > 0$.

Clearly any special normed $\mathbb{R}$-algebra is K_0-stable, since K_0 is a homotopy functor on the category of special normed $\mathbb{R}$-algebras. Any $\mathbb{C}^*$-algebra B is K_*-stable if $K_*(B) \overset{\approx}{\to} K_*(B^{I^n})$ for all $n > 0$, since $K_*(B^2) \overset{\approx}{\to} K_*(B) \times K_*(B)$.

Corollary 13. *If a normed* $\mathbb{R}$*-algebra B is K_j-stable for all j,* $0 \leq j \leq i \geq 1$, *then the homomorphism*

$$\varphi_i(B) : K_i(B) \longrightarrow K_i^{top}(B)$$

is an isomorphism.

Proof. Follows immediately from Corollary 11 a). Done with Corollary 13. ■

We see that Corollaries 11 and 13 are analogical to the well-known isomorphism of Quillen's and Karoubi-Villamayor's K-groups, K_*-regularity of discrete rings replaced by K_*-stability of normed $\mathbb{R}$-algebras. In fact the stability of $\mathbb{C}_*$-algebras is closely related with these two properties (see below Corollaries 15 and 23).

Let $\underline{D}$ be the category of $\mathbb{C}^*$-algebras. Let $\mathcal{K}$ be the $\mathbb{C}^*$-algebra of compact operators on the infinite dimensional Hilbert space $\mathcal{H}$ and let $\mathcal{K} \otimes A$ be the spatial tensor product of $\mathcal{K}$ and a $\mathbb{C}^*$-algebra A. There is a natural inclusion $\sigma : A \to \mathcal{K} \otimes A$ given by $a \mapsto p \otimes a$ where p is the orthogonal projection on the first basis vector of $\mathcal{H}$. A $\mathbb{C}^*$-algebra X is called stable if it is isomorphic to some $\mathcal{K} \otimes A$.

Recently Suslin and Wodzicki have proved [14] that if A is a stable $\mathbb{C}^*$-algebra then the homomorphism $\varphi_{i+1}(A) : K_{i+1}(A) \to K_{i+1}^{top}(A)$, $i \geq 1$, is an isomorphism for all $i \geq 0$. So was confirmed Karoubi's Conjecture [8] on the equality of algebraic and topological K-theories on the category of stable $\mathbb{C}^*$-algebras.

It is natural to show that stable $\mathbb{C}^*$-algebras satisfy conditions of Corollary 11. In fact we have the following assertion.

Theorem 14. *Let* $T : \underline{D} \to \underline{D}$ *be an endofunctor satisfying the following conditions:*

1. $T(A^I) = T(A)^I$, $A \in \operatorname{Ob} \underline{D}$.

2. *T is left exact with respect to split exact sequences of $\mathbb{C}^*$-algebras, i.e. if*
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$
is a split exact sequence of $\mathbb{C}^*$-algebras then the induced sequence
$$0 \longrightarrow T(A) \longrightarrow T(B) \longrightarrow T(C) \longrightarrow 0$$
is exact.
3. *The homomorphism*
$$K_j(T(\sigma)) : K_j(T(A)) \longrightarrow K_j(T(K \otimes A))$$
is an isomorphism for all j, $0 \leq j \leq i+1$ with fixed $i \geq 0$.

Then for any $\mathbb{C}^*$-algebra A the homomorphism
$$\varphi_{i+1}(A) : K_{i+1}(T(A)) \longrightarrow K_{i+1}^{top}(T(A))$$
is an isomorphism .

Proof. By Corollary 10.4 [14] the sequence
$$0 \longrightarrow K_j(A) \longrightarrow K_j(B) \longrightarrow K_j(C) \longrightarrow 0$$
is exact for any split extension of $\mathbb{C}^*$-algebras
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$
for all $j \geq 0$.

Thus, using Kasparov's results [10,6] on the homotopy invariance of functors defined on the category $\underline{D}$ of $\mathbb{C}^*$-algebras, since Quillen's K-functors are exact on the category $\underline{D}$ (Proposition 1.12, Corollary 10.4 [14]) and condition 3 is satisfied, we conclude that the functor $\underline{D} \to \underline{\mathfrak{Ab}}$ given by $A \mapsto K_j(T(A))$, $0 \leq j \leq i+1$, is a homotopy functor. By Proposition 4 one gets $K_j(T(A)) \overset{\approx}{\to} K_j(T(A^I)) = K_j(T(A)^I)$ and by the exactness of K_* one has an isomorphism $K_j(T(A) \times T(A)) \overset{\approx}{\to} K_j(T(A)) \times K_j(T(A))$ induced by projections for all $0 \leq j \leq i+1$. So all the conditions of Corollary 11 a) are satisfied for any $\mathbb{C}^*$-algebra $T(A)$. Done with Theorem 14. ■

Corollary 15. *Any stable $\mathbb{C}^*$-algebra is K_*-stable.*

Proof. Since Quillen's algebraic K-functors K_i are Morita invariant for $\mathbb{C}^*$-algebras ([14], Corollaries 1.20 and 10.4), the endofunctor $K \otimes - : \underline{D} \to \underline{D}$ sends short exact sequences of $\mathbb{C}^*$- algebras to short exact sequences and one has an isomorphism $A^I \otimes \mathcal{K} \approx (A \otimes \mathcal{K})^I$, it follows that all conditions of Theorem 14 are satisfied and any $\mathbb{C}^*$-algebra $A \otimes \mathcal{K}$ is K_*-stable. Done with Corollary 15. ■

Let A be a normed $\mathbb{R}$-algebra ($\mathbb{C}$-algebra), $A[x]$ the polynomial algebra and $A[t, t^{-1}]$ the Laurent polynomial ring over A. It is well-known that $A[x]$ and $A[t, t^{-1}]$ become normed $\mathbb{R}$-algebras ($\mathbb{C}$-algebras) by the norm $\parallel \sum_i a_i x^i \parallel = \sum_i \parallel a_i \parallel$ and one has natural isomorphisms $(A[x])^{I^n} \approx (A^{I^n})[x]$, $(A[t, t^{-1}])^{I^n} \approx (A^{I^n})[t, t^{-1}]$, $n > 0$.

Now Karoubi's Conjecture on stable $\mathbb{C}^*$-algebras will be extended to polynomial rings over stable $\mathbb{C}^*$algebras (see Theorem 18 and Remark 19).

To this end we need the following notion of triple factorization property introduced in [14] :

It is said that a ring A possesses the property $(TF)_{\text{right}}$ if for any finite collection $a_1, \dots, a_m \in A$ there exist $b_1, \dots, b_m, c, d \in A$ such that $a_i = b_i cd$ for $1 \leq i \leq m$ and the left annihilators in A of c and cd are equal.

Lemma 16. *If a ring A has the property $(TF)_{\text{right}}$ then the polynomial rings $A[x]$ and $A[t, t^{-1}]$ have also the property $(TF)_{\text{right}}$.*

Proof. Let

$$p_1(x) = a_{01}x^{n_1} + a_{11}x^{n_1-1} + \cdots + a_{n_1-11}x + a_{n_11},$$
$$p_2(x) = a_{02}x^{n_2} + a_{12}x^{n_2-1} + \cdots + a_{n_2-12}x + a_{n_22},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$p_k(x) = a_{0k}x^{n_k} + a_{1k}x^{n_k-1} + \cdots + a_{n_k-1k}x + a_{n_kk},$$

be polynomials over A. Since A has the property $(TF)_{\text{right}}$, for the elements

$$a_{01}, \dots, a_{n_11}, a_{02}, \dots, a_{n_22}, \dots, a_{0k}, \dots, a_{n_kk},$$

of the ring A there exist elements

$$b_{01}, \dots, b_{n_11}, b_{02}, \dots, b_{n_22}, \dots, b_{0k}, \dots, b_{n_kk}$$

and c, d of the ring A such that

$$a_{j1} = b_{j1}cd, \quad j = 0, \dots, n_1,$$
$$a_{j2} = b_{j2}cd, \quad j = 0, \dots, n_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{jk} = b_{jk}cd, \quad j = 0, \dots, n_k,$$

and if $ycd = 0$ then $yc = 0$ for $y \in A$. Take the following polynomials

$$q_1(x) = b_{01}x^{n_1} + b_{11}x^{n_1-1} + \cdots + b_{n_11}x + b_{n_11},$$
$$q_2(x) = b_{02}x^{n_2} + b_{12}x^{n_2-1} + \cdots + b_{n_22}x + b_{n_22},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$q_k(x) = b_{0k}x^{n_k} + b_{1k}x^{n_k-1} + \cdots + b_{n_kk}x + b_{n_kk}.$$

Then one has $p_i(x) = q_i(x)cd$ for $1 \leq i \leq k$. If $p(x)cd = 0$ for some $p(x) \in A[x]$, where $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, then $a_jcd = 0$, $0 \leq j \leq n$. Therefore one gets $a_jc = 0$, $0 \leq j \leq n$, implying $p(x)c = 0$. It follows that $A[x]$ had the property

$(TF)_{\rm right}$. We omit the proof for the Laurent polynomial $A[t,t^{-1}]$, since it is similar. Done with Lemma 16. ■

Corollary 17. *If A is a $\mathbb{C}^*$-algebra then the rings $A[x_1,\dots,x_k]$ and $A[t,t^{-1}]$ have the property $(TF)_{\rm right}$.*

Proof. By Proposition 10.2 [14] the ring A has the property $(TF)_{\rm right}$. Hence by Lemma 16 the rings $A[x]$ and $A[t,t^{-1}]$ have also the property $(TF)_{\rm right}$, for the ring $A[x_1,\dots,x_k]$ use induction on k. Done with Corollary 17. ■

By Theorem 3.11 , Corollary 1.20, Corollary 3.12 [14] and Corollary 17 we deduce that for any $\mathbb{C}^*$-algebra A the polynomial rings $A[x_1,\dots,x_k]$ and $A[t,t^{-1}]$ satisfy excision in algebraic K-theory, i.e. if R is a ring with unit containing $A[x_1,\dots,x_k]$ or $A[t,t^{-1}]$ as a two-sided ideal then the canonical maps

$$K_*(A[x_1,\dots,x_k]) \to K_*(R, A[x_1,\dots,x_k]) \ ,$$
$$K_*(A[t,t^{-1}]) \to K_*(R, A[t,t^{-1}])$$

respectively are isomorphisms, and they are Morita invariant, i.e. the natural embeddings

$$A[x_1,\dots,x_k] \longrightarrow M_n(A[x_1,\dots,x_k]),$$
$$A[t,t^{-1}] \longrightarrow M_n(A[t,t^{-1}]),$$

induce respectively isomorphisms

$$K_*(A[x_1,\dots,x_k]) \overset{\approx}{\to} K_*(M_n(A[x_1,\dots,x_k])),$$
$$K_*(A[t,t^{-1}]) \overset{\approx}{\to} K_*(M_n(A[t,t^{-1}])).$$

Let $B = A\otimes\mathcal{K}$ be a stable C^*. Introduce in the polynomial algebra $B[x_1,\dots,x_k]$ a norm given by

$$\parallel p \parallel = \sum \parallel b_{i_1\dots i_k} \parallel \ ,$$

where $p(x) = \sum b_{i_1\dots i_k} x_1^{i_1}\cdots x_k^{i_k}$, $b_{i_1,\dots i_k} \in B$. One gets a complex normed algebra with involution which is not complete. The general linear group $\mathrm{GL}((A\otimes\mathcal{K})[x_1,\dots,x_k])$ becomes a topological group with natural inductive limit topology induced by the norm of $A\otimes\mathcal{K}$ and $K_i^{\rm top}((A\otimes\mathcal{K})[x_1,\dots,x_k]) = \pi_{i-1}(\mathrm{GL}((A\otimes\mathcal{K})[x_1,\dots,x_k]))^{\rm top}$, $i \geq 1$.

Theorem 18. *For any $\mathbb{C}^*$-algebra A the natural homomorphism*

$$\varphi_i : K_i((A\otimes\mathcal{K})[x_1,\dots,x_k]) \longrightarrow K_i^{top}((A\otimes\mathcal{K})[x_1,\dots,x_k])$$

is an isomorphism for all $i \geq 1$, $k \geq 0$.

Proof. The case $k=0$ is Karoubi's Conjecture proved by Suslin and Wodzicki [14]. Consider the functor

$$A \mapsto K_i((A\otimes\mathcal{K})[x_1,\dots,x_k]), \quad i\geq 0,$$

from the category of $\mathbb{C}^*$-algebras to the category of abelian groups. Similarly to the case of stable $\mathbb{C}^*$-algebras (see the proof of Proposition 10.6 [14]) the map $A[x_1,\dots,x_k]\to(A\otimes\mathcal{K})[x_1,\dots,x_k]$ induced by the inclusion $A\to A\otimes\mathcal{K}$ applying to $A\otimes\mathcal{K}$ becomes isomorphic to the natural embedding

$$\begin{gathered}(A\otimes\mathcal{K})[x_1,\dots,x_k]\longrightarrow(A\otimes\mathcal{K}\otimes\mathcal{K})[x_1,\dots,x_k]=(A\otimes M_2(\mathcal{K}))[x_1,\dots,x_k]=\\ =(M_2(A\otimes\mathcal{K}))[x_1,\dots,x_k]=M_2((A\otimes\mathcal{K})[x_1,\dots,x_k]).\end{gathered}$$

Therefore we obtain an isomorphism

$$K_*((A\otimes\mathcal{K})[x_1,\dots,x_k])\longrightarrow K_*((A\otimes\mathcal{K}\otimes\mathcal{K})[x_1,\dots,x_k])$$

induced by the inclusion $A\to A\otimes\mathcal{K}$.

Any short split exact sequence of $\mathbb{C}^*$-algebras

$$0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$$

induces a short split exact sequence of $\mathbb{C}$-algebras

$$0\longrightarrow A_k\longrightarrow B_k\longrightarrow C_k\longrightarrow 0$$

where $A_k=(A\otimes\mathcal{K})[x_1,\dots,x_k]$, $B_k=(B\otimes\mathcal{K})[x_1,\dots,x_k]$, $C_k=(C\otimes\mathcal{K})[x_1,\dots,x_k]$.

The commutative diagram with exact split rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_k & \longrightarrow & B_k & \longrightarrow & C_k & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_k & \longrightarrow & B_k+\mathbb{C} & \longrightarrow & C_k+\mathbb{C} & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{C} & = & \mathbb{C} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

induces the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_*(A_k) & \longrightarrow & K_*(B_k) & \longrightarrow & K_*(C_k) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_*(A_k) & \longrightarrow & K_*(B_k+\mathbb{C}) & \longrightarrow & K_*(C_k+\mathbb{C}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_*(\mathbb{C}) & = & K_*(\mathbb{C}) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the bottom row and the two columns are split exact sequences. It follows that the top row is also a split exact sequence. Therefore the functors $K_i((-\otimes\mathcal{K})[x_1,\dots,x_k])$ are homotopy functors for all $i\geq 0$, $k\geq 0$ and by Proposition 4 the inclusion $A\to A^{I^n}$ induces an isomorphism

$$K_*((A\otimes\mathcal{K})[x_1,\dots,x_k]) \overset{\approx}{\to} K_*((A^{I^n}\otimes\mathcal{K})[x_1,\dots,x_k]) \approx K_*(((A\otimes\mathcal{K})[x_1,\dots,x_k])^{I^n}).$$

Since one has the canonical isomorphism

$$K_*((A\otimes\mathcal{K})[x_1,\dots,x_k])\times(A\otimes\mathcal{K})[x_1,\dots,x_k]) \approx K_*((A\otimes\mathcal{K})[x_1,\dots,x_k])\times \\ \times K_*((A\otimes\mathcal{K})[x_1,\dots,x_k]),$$

all conditions of Corollary 11 are satified. Done with Theorem 18. ■

Remark 19. *The proof of the isomorphism*

$$\varphi_i : K_i((A\otimes\mathcal{K})[t,t^{-1}]) \overset{\approx}{\to} K_i^{top}((A\otimes\mathcal{K})[t,t^{-}]),\quad i\geq 1,$$

where A is a $\mathbb{C}^*$-algebra, is similar to the proof of Theorem 18.

Consider the canonical short exact sequence of $\mathbb{C}^*$-algebras

$$0\longrightarrow\mathcal{K}\longrightarrow\mathcal{B}\longrightarrow\mathcal{A}\longrightarrow 0,$$

where $\mathcal{B}=\mathcal{B}(\mathcal{H})$ is the $\mathbb{C}^*$-algebra of bounded operators on the Hilbert space $\mathcal{H}$ and $\mathcal{A}=\mathcal{B}/\mathcal{H}$ is the Calkin algebra.

Let X be a compact space and $\mathcal{A}^X=\{f : X\to\mathcal{A}\}$ the $\mathbb{C}^*$-algebra of continuous maps from X to the Calkin algebra $\mathcal{A}$.

Proposition 20. *$\mathcal{A}^X$ is a K_*-stable $\mathbb{C}^*$-algebra.*

Proof. For $n\geq 0$ one has an exact sequence of $\mathbb{C}^*$-algebras

$$0\longrightarrow\mathcal{K}^{X\times I^n}\longrightarrow\mathcal{B}^{X\times I^n}\longrightarrow\mathcal{A}^{X\times I^n}\longrightarrow 0.$$

Since $\mathcal{B}$ is an infinite sum ring [17], it is easy to check that $\mathcal{B}^{X\times I^n}$ is also an infinite sum ring. It follows that $K_*(\mathcal{B}^{X\times I^n})=0$ for $n\geq 0$ [17] (see also [12], Chapter II, Corollary 2.50). Thus the exactness of K_* on the category of $\mathbb{C}^*$-algebras implies an isomorphism

$$K_{i+1}(\mathcal{A}^{X\times I^n}) \xrightarrow{\approx} K_i(\mathcal{K}^{X\times I^n}) \text{ for } i,n \geq 0.$$

Clearly $\mathcal{K}^X$ is K_*-stable, i.e. $K_i(\mathcal{K}^X) \xrightarrow{\approx} K_i((\mathcal{K}^X)^{I^n})$ for all $i,n \geq 0$. So one gets isomorphisms

$$K_i(\mathcal{A}^X) \xrightarrow{\approx} K_{i-1}(\mathcal{K}^X) \xrightarrow{\approx} K_{i-1}((\mathcal{K}^X)^{I^n}) \xrightarrow{\approx} K_i((\mathcal{A}^X)^{I^n})$$

for $i > 0$, $n \geq 0$, and their composite coincides with the homomorphism $K_i(\mathcal{A}^X) \to K_i((\mathcal{A}^X)^{I^n})$ induced by the inclusion $\mathcal{A}^X \to (\mathcal{A}^X)^{I^n}$. Done with Proposition 20. ■

Proposition 21. *Let $f : A \to B$ ($g : B \to A$) be a homomorphism of normed $\mathbb{R}$-algebras. If B is K_i-stable and the homomorphism $K_i((A^{2^j})^{I^n}) \to K_i((B^{2^j})^{I^n})$ ($K_i((B^{2^j})^{I^n}) \to K_i((A^{2^j})^{I^n})$) induced by f (by g) is injective (surjective), then A is a K_i-stable normed $\mathbb{R}$-algebra. In particular any retract of a stable $\mathbb{C}^*$-algebra is K_*-stable.*

Proof. The following commutative diagram

$$\begin{array}{ccc} K_i((A^{2^j})^{I^n}) & \longrightarrow & K_i(A) \\ \downarrow & & \downarrow \\ K_i((B^{2^j})^{I^n}) & \longrightarrow & K_i(B) \end{array},$$

where the vertical homomorphisms are induced by f, implies that the homomorphism $K_i((A^{2^j})^{I^n}) \to K_i(A)$ induced by the canonical projection is injective for all $j \geq 0$, $n > 0$. So one has $K_i(A) \xrightarrow{\approx} K_i((A^{2^j})^{I^n})$ for $n,\ j,\ i \geq 0$, i.e. A is K_*-stable. The proof of the dual case is similar. Done with Proposition 21. ■

Let A be a local $\mathbb{C}^*$-algebra. According to ([2], 5.1) the completion of $M_\infty(A) = \varinjlim_n M_n(A)$ is called the stable algebra of A and denoted $A \otimes \mathcal{K}$. Under the norm induced by the norm of A the algebraic limit $M_\infty(A)$ becomes a local $\mathbb{C}^*$-algebra. Since the completion of a local $\mathbb{C}^*$- algebra is a $\mathbb{C}^*$-algebra, one gets an isomorphism of $\mathbb{C}^*$ -algebras $A \otimes \mathcal{K} \xrightarrow{\approx} \overline{A} \otimes \mathcal{K}$, where $\overline{A}$ is the completion of A. It follows that the stable algebra $A \otimes \mathcal{K}$ of a local $\mathbb{C}^*$-algebra A is K_*-stable.

Now it will be shown that conditions analogical to those of Theorem 10 are sufficient for the isomorphism of Karoubi–Villamayor's algebraic K-groups [7] and topological K-groups on the category of real normed algebras.

Recall the definition of Karoubi–Villamayor's algebraic K-functors. On the category of rings consider the polynomial cotriple $\mathfrak{P} = (P, \tau, \delta)$ defined as follows. For any ring A let $A[x]$ be the ring of polynomials over A and $q_0,\ q_1 : A[x] \to A$ the canonical homomorphisms given by $q_0(\sum_{i=0}^n a_i x^i) = a_0,\ q_1(\sum_{i=0}^n a_i x^i) = \sum_{i=0}^n a_i$. Then $P(A)$ is the kernel of q_0 and $\tau_A : P(A) \to A$ is equal to $q_1|_{P(A)}$. We have also a natural homomorphism $\delta_A : P(A) \to P(P(A)) = uvA[u,v]$ given by $x \mapsto uv$. The polynomial cotriple $\mathfrak{P}$ induces in a natural way an augmented simplical ring

$(P_*(A), \tau_A, A)$. By definition $k_i(A) = \pi_{i-2}\,\mathrm{GL}(P_*(A))$ for $i \geq 3$ and the groups $k_1(A)$ and $k_2(A)$ are given by the exact sequence

$$0 \longrightarrow k_2(A) \longrightarrow \pi_0\,\mathrm{GL}(P_*(A)) \longrightarrow \mathrm{GL}(A) \longrightarrow k_1(A) \longrightarrow 0.$$

For any normed $\mathbb{R}$-algebra A one has a canonical homomorphism

$$\beta_i(A) : k_i(A) \longrightarrow K_i^{top}(A), \quad i \geq 1,$$

induced by the natural morphism $\eta : \mathfrak{P} \to \mathfrak{J}$ given by

$$\eta_A(\textstyle\sum_{i=1}^n a_i x^i) = \sum_{i=1}^n a_i t^i, \quad t \in I.$$

Theorem 22. *For any normed $\mathbb{R}$-algebra A the homomorphism*

$$\beta_*(A) : k_*(A) \longrightarrow K_*^{top}(A)$$

is an isomorphism if the inclusion $i : A \to A^{I^n}$ induces an isomorphism $k_*(A) \overset{\approx}{\to} k_*(A^{I^n})$ for all $n > 0$.

Proof. It is similar to the proof of Theorem 10 and is simpler. In effect the sequence (1) is a GL-fibration with respect to the polynomial cotriple $\mathfrak{P}$. This assertion follows from the functorial commutative diagram

$$\begin{array}{ccc} A[x] & \xrightarrow{q_0 \times q_1} & A \times A \\ \downarrow \psi & & \| \\ A^I & \xrightarrow{\epsilon_0 \times \epsilon_1} & A \times A \end{array},$$

where $\psi(\sum_{i=0}^n a_i x^i)(t) = \sum_{i=0}^n a_i t^i$, $t \in I$, using the fact that the map $q_0 \times q_1$ is a GL-fibration with respect to the polynomial cotriple ([16], Lemma 4.5). Replace the auxiliary K-functors $M_k K_i^S(-)$ of Theorem 10 by the K-functors $k_i \Omega^k(-)$, use the natural isomorphism $k_*(A \times A) \overset{\approx}{\to} k_*(A) \times k_*(A)$, the canonical isomorphism of functors $hk_1 \overset{\approx}{\to} K_1^{top}$ and note that in this case $M_j P_*^+(A) = \operatorname{Ker} M_{j-1} P_*^+(\epsilon_0 \times \epsilon_1)$ (with $M_0 P_*^+(A) = P_*^+(A)$) is the $\mathfrak{P}$- projective cotriple resolution of $\Omega^j(A)$ for all $j \geq 1$.

Nevertheless we will show how in this case the proof is simpler for $i = 2$. Let A be a normed $\mathbb{R}$-algebra such that

$$k_j(A) = k_j(A^{I^2}) \ \text{ for } \ j = 1, 2.$$

We have to prove that under these conditions we have isomorphisms $k_1(A) \overset{\approx}{\to} K_1^{\mathrm{top}}(A)$ and $k_2(A) \overset{\approx}{\to} K_2^{\mathrm{top}}(A)$. Since $k_1(A) = k_1(A^I)$ by assumption and $hk_1(A) \overset{\approx}{\to} K_1^{\mathrm{top}}(A)$ for any normed $\mathbb{R}$-algebra A, one gets the first isomorphism $k_1(A) \overset{\approx}{\to}$

$K_1^{\text{top}}(A)$. The short exact sequence (1) yields the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & k_2(A^I) & \longrightarrow & k_2(A\times A) & \longrightarrow & k_1(\Omega A) & \longrightarrow & k_1(A^I) & \longrightarrow & k_1(A\times A) \\
& \downarrow\approx & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \downarrow\psi \\
\cdots \longrightarrow & k_2(A^{I^2}) & \longrightarrow & k_2(A^I\times A^I) & \longrightarrow & k_1(\Omega A^I) & \longrightarrow & k_1(A^{I^2}) & \longrightarrow & k_1(A^I\times A^I)
\end{array}$$

which implies the isomorphism $k_1(\Omega A) \overset{\approx}{\to} k_1(\Omega A)^I$. Thus $k_1(\Omega A) = hk_1(\Omega A)$. But $hk_1(\Omega A) \overset{\approx}{\to} K_1^{\text{top}}(\Omega A) \approx K_2^{\text{top}}(A)$. Therefore one gets $k_1(\Omega A) \approx K_2^{\text{top}}(A)$.

Consider the long exact sequence

$$\cdots \longrightarrow k_2(A^I) \longrightarrow k_2(A\times A) \longrightarrow k_1(\Omega A) \xrightarrow{k_1(\sigma)} k_1(A^I) \xrightarrow{k_1(\epsilon)} k_1(A\times A)\ .$$

Since $k_1(A) = k_1(A^I)$, the homomorphism $k_1(\epsilon)$ is injective and it follows that $hk_2(A) \overset{\approx}{\to} k_1(\Omega A)$. Finally one gets the required second isomorphism

$$k_2(A) = hk_2(A) \overset{\approx}{\to} k_1(\Omega A) \overset{\approx}{\to} K_2^{\text{top}}(A)\ .$$

The proof for $i > 2$ is similar. Done with Theorem 22. ■

Corollary 23. *Any stable $\mathbb{C}^*$ algebra is K_i-regular for all $i \geq 1$.*

Proof. One has the following commutative diagram induced by the inclusion $A\otimes\mathcal{K} \to (A\otimes\mathcal{K})^{I^n}$

$$\begin{array}{ccc}
k_i(A\otimes\mathcal{K}) & \longrightarrow & k_i((A\otimes\mathcal{K})^{I^n}) \\
\downarrow\approx & & \downarrow\approx \\
k_i((A\otimes\mathcal{K})[x_1,\ldots,x_k]) & \longrightarrow & k_i((A\otimes\mathcal{K})^{I^n}[x_1,\ldots,x_k])
\end{array}$$

for $i \geq 1$, $k,\ n > 0$, where the top map is an isomorphism by Proposition 4, since the functor $k_i(-\otimes\mathcal{K})$ is a homotopy functor by Corollary 6.3.10 [6]. Therefore the bottom map is also an isomorphism. Thus by Theorem 22 the natural homomorphism

$$\beta_* : k_*((A\otimes\mathcal{K})[x_1,\ldots,x_k]) \longrightarrow K_*^{top}((A\otimes\mathcal{K})[x_1,\ldots,x_k])$$

is an isomorphism. The case $k = 0$ was proved by Higson ([6], Theorem 6.3.14).

For any normed $\mathbb{R}$-algebra B we have natural homomorphisms

$$K_i(B) \xrightarrow{\vartheta_i} K_i^S(B) \xrightarrow{\psi_i} k_i(B) \xrightarrow{\beta_i} K_i^{top}(B),$$

where $\beta_i\psi_i = \alpha_i$ and $\beta_{ii}\psi_i\vartheta_i = \varphi_i$, $i \geq 1$.

Denote $\overline{\psi_i} = \psi_i\vartheta_{i:K_i(B)\to k_i(B)}$, $i \geq 1$. If $B = (A\otimes\mathcal{K})[x_1,\ldots,x_k]$ where A is a $\mathbb{C}^*$-algebra, then by Theorem 18 the homomorphism $\beta_i\overline{\psi_i}$ is an isomorphism for $i \geq 1$, $k > 0$. It follows that the natural homomorphism

$$\overline{\psi_i} : K_i((A\otimes\mathcal{K})[x_1,\ldots,x_k]) \longrightarrow k_i((A\otimes\mathcal{K})[x_1,\ldots,x_k])$$

is an isomorphism for $i \geq 1$, $k \geq 0$.

The canonical inclusion

$$A \otimes \mathcal{K} \longrightarrow (A \otimes \mathcal{K})[x_1, \ldots, x_k]$$

induces a commutative diagram

$$\begin{array}{ccc} K_i(A \otimes \mathcal{K}) & \longrightarrow & K_i((A \otimes \mathcal{K})[x_1, \ldots, x_k]) \\ \downarrow \overline{\psi_*} & & \downarrow \overline{\psi_*} \\ k_i(A \otimes \mathcal{K}) & \longrightarrow & k_i((A \otimes \mathcal{K})[x_1, \ldots, x_k]) \end{array},$$

where the vertical maps and the bottom map are isomorphisms for $i \geq 1$. Therefore the top map of this diagram is also an isomorphism for $i \geq 1$. Done with Corollary 23. ■

For the functor K_1 Corollary 23 was proved in ([6], Lemma 6.3.12).

We see that any stable $\mathbb{C}^*$-algebra $A \otimes \mathcal{K}$ is K_*-stable and K_i-regular for $i \geq 1$. This means that the natural map

$$\gamma_k : (A \otimes \mathcal{K})[x_1, \ldots, x_k] \longrightarrow (A \otimes \mathcal{K})^{I^k}$$

given by $\lambda x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \mapsto \lambda t_1^{n_1} t_2^{n_2} \cdots t_k^{n_k}$, where $\lambda \in A \otimes \mathcal{K}$, $(t_1, t_2, \cdots, t_k) \in I^k$, $k > 0$, induces a commutative diagram

$$\begin{array}{ccc} K_i(A \otimes \mathcal{K}) & = & K_i(A \otimes \mathcal{K}) \\ \downarrow \approx & & \downarrow \approx \\ K_i((A \otimes \mathcal{K})[x_1, \ldots, x_k]) & \longrightarrow & K_i((A \otimes \mathcal{K})^{I^k}) \end{array}$$

and we conclude that for any stable $\mathbb{C}^*$-algebra $A \otimes \mathcal{K}$ the map γ_k induces an isomorphism

$$K_i((A \otimes \mathcal{K})[x_1, \ldots, x_k]) \xrightarrow{\approx} K_i((A \otimes \mathcal{K})^{I^k})$$

for $i \geq 1$ and $k > 0$.

Corollary 24. *For any stable $\mathbb{C}^*$-algebra $A \otimes \mathcal{K}$ one has an isomorphism*

$$K_i((A \otimes \mathcal{K})[t, t^{-1}]) \approx K_i(A \otimes \mathcal{K}) \oplus K_{i-1}(A \otimes \mathcal{K}), \quad i \geq 1.$$

Proof. The fundamental theorem in algebraic K-theory yields the following isomorphism

$$\begin{gathered} K_i((A \otimes \mathcal{K} + \mathbb{C})[t, t^{-1}]) \approx K_i(A \otimes \mathcal{K} + \mathbb{C}) \oplus K_{i-1}(A \otimes \mathcal{K} + \mathbb{C}) \\ \oplus N K_i(A \otimes \mathcal{K} + \mathbb{C}) \oplus N K_i(A \otimes \mathcal{K} + \mathbb{C}), \quad i \geq 1, \end{gathered}$$

where $NK_i(A \otimes \mathcal{K} + \mathbb{C}) = \mathrm{Coker}(K_i(A \otimes \mathcal{K} + \mathbb{C}) \to K_i((A \otimes \mathcal{K} + \mathbb{C})[t]))$.

By Corollary 23 and using the fact that the rings $A \otimes \mathcal{K}$, $(A \otimes \mathcal{K})[t]$, $(A \otimes \mathcal{K})[t, t^{-}]$ have the property $(TF)_{\text{right}}$ it is easy to check that one has isomorphisms

$$\begin{gathered} K_i((A \otimes \mathcal{K}) + \mathbb{C})[t, t^{-1}]) \approx K_i((A \otimes \mathcal{K})[t, t^{-1}]) \oplus K_i(\mathbb{C}[t, t^{-1}]), \\ K_i((A \otimes \mathcal{K}) + \mathbb{C}) \approx K_i(A \otimes \mathcal{K}) \oplus K_i(\mathbb{C}), \end{gathered}$$

and $NK_i(A\otimes\mathcal{K}+\mathbb{C})$ is trivial for $i\geq 1$, giving the required isomorphism. Done with Corollary 24. ■

In what follows the same is true for normed for $\mathbb{C}$-algebras.

Let $\mathfrak{F}=(F,\tau,\delta)$ be the free cotriple on the category of normed $\mathbb{R}$-algebras. For a normed $\mathbb{R}$-algebra A consider the free $\mathbb{R}$-algebra $F(A)$ generated by A. Define a map $\rho: F(A)\to\mathbb{R}^+$ by

$$\rho(\textstyle\sum_i \lambda_i w_i)=\sum_i \parallel\lambda_i\parallel\cdot\parallel w_i\parallel,$$

where $\lambda_i\in R$, $w_i=|a_{i_1}||a_{i_2}|\cdots|a_{i_k}|$, $a_{i_1},a_{i_2},\dots,a_{i_k}\in A$, and $\parallel w_i\parallel=\parallel a_{i_1}\parallel\cdot\parallel a_{i_2}\parallel\cdots\parallel a_{i_k}\parallel$. Let L be the ideal of $F(A)$ generated by $|0|$ and denote $\widetilde{F}(A)=F(A)/L$. Then the map ρ induces a norm on the $\mathbb{R}$-algebra $\widetilde{F}(A)$ which will be called the free normed $\mathbb{R}$-algebra generated by A. If A is a Banach $\mathbb{R}$-algebra, then the completion $F[A]$ of $\widetilde{F}(A)$ is called the free Banach $\mathbb{R}$-algebra generated by A and $K_i^{top}(F[A])=0$ for all $i\geq 1$ (see [12], Chapter II,3 and Chapter V, Corollary 3.3).

It is clear that any map $\varphi: A\to B$ of a normed $\mathbb{R}$-algebra A to a normed $\mathbb{R}$-algebra B such that $\varphi(0)=0$ and $\parallel\varphi(a)\parallel\leq\parallel a\parallel$ uniquely can be extended to a bounded $\mathbb{R}$-homomorphism $\widetilde{\varphi}:\widetilde{F}(A)\to B$. Therefore the identity map 1_A and the natural injection $\sigma_A: A\to\widetilde{F}(A)$ induce respectively bounded $\mathbb{R}$-homomorphisms $\widetilde{\tau}_A:\widetilde{F}(A)\to A$ and $\widetilde{\delta}_A:\widetilde{F}(A)\to\widetilde{F}(\widetilde{F}(A))=\widetilde{F}^2(A)$. One gets a cotriple $\widetilde{F}=(\widetilde{F},\widetilde{\tau},\widetilde{\delta})$ called the free cotriple on the category of normed $\mathbb{R}$-algebras. Let $\mu(A)=\operatorname{Ker}\widetilde{\tau}_A$ and define $\mu^k(A)=\mu(\mu^{k-1}(A))$, $k\geq 1$, with $\mu^0(A)=A$.

Theorem 25. *For any $\mathbb{C}^*$-algebra A the natural homomorphism*

$$\beta_i: k_i(\mu^k(A\otimes\mathcal{K}))\longrightarrow K_i^{top}(\mu^k(A\otimes\mathcal{K}))$$

is an isomorphism for all $i\geq 1$ and $k=0,\ 1$.

Proof. As noted above the case $k=0$, $\mu^0(A\otimes\mathcal{K})=A\otimes\mathcal{K}$, was proved in [6]. We have only to show that

$$\beta_*: k_*(\mu(A\otimes\mathcal{K}))\longrightarrow K_*^{top}(\mu(A\otimes\mathcal{K}))$$

is an isomorphism.

According to Theorem 6.3.13 [6] the exact sequence of normed $\mathbb{C}$-algebras

$$0\longrightarrow(\mu(A\otimes\mathcal{K}))^{I^m}\longrightarrow(\widetilde{F}(A\otimes\mathcal{K}))^{I^m}\xrightarrow{(\widetilde{\tau})_{I^m}}(A\otimes\mathcal{K})^{I^m}\longrightarrow 0 \tag{6}$$

is a GL-fibration with respect to the polynomial cotriple $\mathfrak{P}$ for all $m\geq 0$, since the map $(\widetilde{\tau})_{I^m}$ is surjective. Now it will be proved that $k_i((\widetilde{F}(A\otimes\mathcal{K}))^{I^m})=0$, $i\geq 1$. Take the $\mathfrak{P}$-projective cotriple resolution $P_*((\widetilde{F}(A\otimes\mathcal{K}))^{I^m})$ of $(\widetilde{F}(A\otimes\mathcal{K}))^{I^m}$. Then the induced augmented simplicial ring

$$P_*((\widetilde{F}(A\otimes\mathcal{K}))^{I^m})\longrightarrow(\widetilde{F}(A\otimes\mathcal{K}))^{I^m} \tag{7}$$

is right contractible ([12], Chapter II,Definition 3.10). In effect the map

$$(\widetilde{\tau})_{I^m} : P((\widetilde{F}(A \otimes \mathcal{K}))^{I^m}) \longrightarrow (\widetilde{F}(A \otimes \mathcal{K}))^{I^m}$$

is splitting, since the injection

$$A \otimes \mathcal{K} \longrightarrow P\widetilde{F}(A \otimes \mathcal{K})$$

given by $l \otimes a \mapsto |l \otimes a|x$ induces a homomorphism

$$g : \widetilde{F}(A \otimes \mathcal{K}) \longrightarrow P\widetilde{F}(A \otimes \mathcal{K})$$

of normed $\mathbb{C}$-algebras and therefore a homomorphism

$$g_{I^m} : (\widetilde{F}(A \otimes \mathcal{K}))^{I^m} \longrightarrow (P\widetilde{F}(A \otimes \mathcal{K}))^{I^m}$$

such that its composite with $(\widetilde{\tau})_{I^m}$ is equal to the identity map of $(\widetilde{F}(A \otimes \mathcal{K}))^{I^m}$. It remains to note that for any normed $\mathbb{C}$-algebra B one has a natural isomorphism $(P(B))^{I^m} \approx P(B^{I^m})$. For the right contractibility of (7) we have to show that there exist homomorphisms of normed $\mathbb{C}$-algebras $\{f_n,\ \ n \geq -1\}$, $f_n : P_n(B) \to P_{n+1}(B)$, where $B = (\widetilde{F}(A \otimes \mathcal{K}))^{I^m}$, such that $\tau_B f_{-1} = 1_B(P_{-1}(B) = P^0(B) = B)$, $\theta_{n+1} f_n = 1_{Pn(B)}$ for $n \geq 0$, $\theta_o f_0 = f_{-1}\tau_B$ and $\theta_i f_n = f_{n-1}\theta_i$ for $n \geq 1$, $i \leq n$. Take $f_{-1} = g_{I^m}$ and $f_n = P_n(g_{I^m}) : P_n(B) = P^{n+1}(B) \to P^{n+2}(B) = P_{n+1}(B)$, $n \geq 0$. Such defined $\{f_n\}_{n\geq -1}$ are contraction homomorphisms for (7). This can be proved by induction assuming contractibility relations for the case $n-1$. One has

$$\begin{aligned}\theta_{n+1} f_n &= (P^{n+1}\tau_B)P^{n+1}(f_{-1}) = P^{n+1}(\tau_B f_{-1}) = P^{n+1}(1_B) = 1_{P^{n+1}(B)},\\ \theta_i f_n &= (P^i \tau_B P^{n+1-i})(P(f_{n-1}) = P(P^{i-1}\tau_B P^{n+1-i}(f_{n-1})) = P(\theta_{i-1} f_{n-1})\\ &= P(f_{n-2}\theta_{i-1}) = P(f_{n-2})P(\theta_{i-1}) = f_{n-1}\theta_i \ \text{ for } \ 0 \leq i \leq n.\end{aligned}$$

It follows that $k_i(B)^{I^m} = 0$ for all $i \geq 1$, $m \geq 0$. Therefore the GL-fibration (6) gives the following commutative diagram

$$\begin{array}{ccc} k_{i+1}(A \otimes \mathcal{K}) & \overset{\approx}{\to} & k_i(\mu(A \otimes \mathcal{K})) \\ \downarrow & & \downarrow \\ k_{i+1}((A \otimes \mathcal{K})^{I^m}) & \overset{\approx}{\to} & k_i((\mu(A \otimes \mathcal{K}))^{I^m}) \end{array}$$

for $i \geq 1$, $m \geq 0$, where the vertival homomorphisms are induced by the inclusion $A \otimes \mathcal{K} \to (A \otimes \mathcal{K})^{I^m}$. Since the functor $k_i(- \otimes K)$, $i \geq 1$, is a homotopy functor, by Proposition 4 one has an isomorphism

$$k_*(A \otimes \mathcal{K}) \overset{\approx}{\to} k_*((A \otimes \mathcal{K})^{I^m})$$

implying the isomorphism

$$k_i(\mu(A \otimes \mathcal{K})) \overset{\approx}{\to} k_i((\mu(A \otimes \mathcal{K}))^{I^m})$$

for all $i \geq 1$, $m \geq 0$. It remains to apply Theorem 22. Done with Theorem 25. ■

Question: Is the homomorphism $\beta_* : k_*(\mu^k(A \otimes \mathcal{K})) \to K_*^{top}(\mu^k(A \otimes \mathcal{K}))$ an isomorphism for $k > 1$?

Remark 26. *Waldhausen's algebraic K-theory [18,19] can be probably extended to topological categories with cofibrations, using Waldhausen's S-construction, which will be isomorphic to Quillen's algebraic K-theory for topological exact categories.*

Remark 27. *Karoubi-Villamayor's algebraic K-theory can also be defined for normed k-algebras generalizing K-theory of Banach k-algebras [7]. For a normed k-algebra A take the subalgebra $A\langle x\rangle$ of $A[[x]]$ consisting of formal series $S = \sum_{i=0}^{\infty} a_i x^i$ such that $\sum_{i=0}^{\infty} |a_i| < \infty$ and $\sum_{i=0}^{\infty} a_i \in A$ with norm $\| S \| = \sum_{i=0}^{\infty} \| a_i \|$. Let $P(A)$ be the kernel of $\epsilon_0 : A\langle x\rangle \to A$ given by $\sum_{i=0}^{\infty} a_i x^i \mapsto a_0$. Then the polynomial cotriple $\mathfrak{P} = (P, \tau, \delta)$ is defined by the usual way [5] and put $k_n(A) = \pi_{n-2}\,\mathrm{GL}(P_*^+(A))$, $n \geq 1$, where $P_*^+(A) = P_*(A) \to A$ is the augmented simplicial normed k-algebra with $P_*(A) = \{P_n(A),\ \partial_i^n,\ s_i^n,\ n \geq 0\}$, $P_n(A) = P^{n+1}(A) = P(P^n(A))$, $\partial_i^n = P^i \tau P^{n-i}$, $s_i^n = P^i \delta P^{n-i}$. It can can be proved that any short exact sequence of normed k-algebras being a GL-fibration with respect to the polynomial cotriple and such that its completion is exact induces a long exact sequence of the K-functors k_*. It can also be proved that for special normed $\mathbb{R}$-algebras the algebraic K-theory k_* is isomorphic to the topological K-theory K_*^{top}.*

References

1. D.W.Anderson, *Relationship among K-theories*, Springer-Verlag Lecture Notes in Math. No. **341** (1973), 52-67.
2. B.Blackadar, *K-theory for operator algebras*, M.S.R.I. Publ. vol. **5**, Springer-Verlag, 1986.
3. B. H. Dayton, *SK_1 of commutative normed algebras*, Springer-Verlag Lecture Notes in Math. No. **551** (1976), 30-43.
4. P. de la Harpe and G. Skandalis, *Sur la simplicite essentielle du groupe des inversibles et du groupe unitaire dans une C^*-algebre simple*, J. Funct. Analysis **62** (1985), 354-378.
5. S. M. Gersten, *On Mayer-Vietoris functors and algebraic K-theory*, J. Algebra, vol. **18**, N **1** (1971), 51-88.
6. N. Higson, *Algebraic K-theory of stable C^*-algebras*, Adv. Math. **67** (1988), 1-140.
7. M. Karoubi and O. Villamayor, *K-theorie algebrique et K-theorie topologique I*, Math. Scand. vol.**28**, fasc. **2** (1971), 265-307.
8. M. Karoubi, *K-theorie algebrique de certaines algebres d'operateurs*, Springer-Verlag Lecture Notes in Math. No. **725** (1979), 254-290.
9. M. Karoubi, *Homologie des groupes discrets associes a des algebres d'operateurs*, J. Operator Theory **15** (1986), 109-162.
10. G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Izv. Akad. Nauk SSSR , Ser. Math. **44** (1980), 571-636 (in Russian).
11. H. Inassaridze, *K-theory of special normed rings*, Springer-Verlag Lecture Notes in Math. No. **1437** (1990), 95-158.
12. H. Inassaridze, *Algebraic K-theory*, Kluwer Academic Publishers, 1995.

13. D. Quillen, *Higher algebraic K-theory*, Springer-Verlag Lecture Notes in Math. No **341** (1973), 77-139.
14. A. Suslin and M. Wodzicki, *Excision in algebraic K-theory*, Annals of Math. vol. **136**, N **1** (1992), 51-122.
15. R. Swan, *Algebraic K-theory*, Springer-Verlag, Lecture Notes in Math. No **76**, 1968.
16. R. Swan, *Some relations between higher K-functors*, J. Algebra, vol. **21**, N**1** (1972), 113-136.
17. J.B.Wagoner, *Delooping classifying spaces in algebraic K-theory*, Topology, vol.**11**, No **4** (1972), 349-370.
18. F. Waldhausen, *Algebraic K-theory of topological spaces I*, Pro. Symp. Pure Math. vol. **32**, part **I**, A.M.S. (1978), 35-60.
19. F. Waldhausen, *Algebraic K-theory of topological spaces II*, Springer-Verlag Lecture Notes in Math. No. **763** (1979), 356-394.
20. X. Wang, *KK-Theories for topological algebras*, K-Theory J., vol. **5**, N**2** (1991), 97-150.