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NON-ABELIAN TENSOR AND EXTERIOR PRODUCTS MODULO q AND UNIVERSAL q-CENTRAL RELATIVE EXTENSION OF LIE ALGEBRAS

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Abstract

The notions of tensor end exterior products modulo q of two crossed P-modules, where q is a positive integer and P is a Lie algebra, are introduced and some properties are established. The condition for the existence of a universal q-central relative extension of a Lie epimorphism is given and this extension is described as an exterior product modulo q.

Introduction

The non-abelian tensor product of groups was introduced by Brown and Loday [3,4] and has applications in homotopy theory and in non-abelian (co)homology theory of groups [11,13,14].

In [5] Conduche and Rodriguez-Fernandez introduce the non-abelian tensor product modulo an integer q of groups, generalizing definitions of Brown [2] and Ellis and Rodriguez [9]. This construction is the mod q version of the non-abelian tensor product of groups of Brown and Loday.

In [6] Ellis developed an analogous theory of non-abelian tensor product for Lie algebras (see also [7]). Using tensor (exterior) product of Lie algebras Ellis describes the universal central extension of Lie algebras. The importance of this product is given by Guin in [12], constructing the non-abelian homology of Lie algebras in low dimensions, which has applications in cyclic homology.

In the present paper we introduce the non-abelian tensor (exterior) product modulo q, $M \otimes^q N$ ($M \wedge^q N$), where M and N are two crossed P-modules, in the context of Lie algebras, as the mod q version of Ellis' tensor (exterior) product of Lie algebras and investigate its properties. The general aim introducing this notion is to describe the universal q-central relative extension of a Lie epimorphism, analogously to Conduche-Rodriguez-Fernandez's result in the group case [5, Theorem 2.11].

In [16] Kassel and Loday give the notion of relative extension of a Lie epimorphism $\alpha : P \to Q$ and prove that a universal central relative extension exists if and only if the relative homology group $H_2(Q, P; \Lambda) = 0$, where Λ is a principal ring. In this paper we introduce the definition of a q-central relative extension of a Lie epimorphism, which is the mod q version of Kassel-Loday's notion and give our main result (Theorem 2.8): for a short exact sequence of Lie algebras

$$0 \to N \to P \xrightarrow{\alpha} Q \to 0$$

there exists a universal q-central relative extension of α if and only if $N = N\#_q P$, where $N\#_q P$ is the submodule of P generated by the elements $[n, p]$ and qn for $n \in N$, $p \in P$.

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In the rest of the paper the following interesting properties of non-abelian tensor (exterior) product modulo q, $M \otimes^q N$ ($M \wedge^q N$), of Lie algebras are given. The existence of a unique isomorphism $M \otimes^q N \to N \otimes^q M$ ($M \wedge^q N \to N \wedge^q M$) (Proposition 1.7) is shown. Compatibility of tensor product modulo q of crossed modules with the direct limit of crossed modules is established (Theorem 1.9). Some examples of crossed squares of Lie algebras are given. Using a slightly generalized version of Whitehead's universal quadratic functor (for the definition see bellow) the relation between the Lie exterior product modulo q and the Lie tensor product modulo q is estabished (Theorem 1.17). Finally, the relation between Ellis' non-abelian tensor product and the non-abelian tensor product modulo q of two Lie algebras with compatible actions on each other is given (Theorem 1.22).

Notation. We shall use the term Lie algebra to mean a Lie algebra over Λ, where Λ is a commutative ring with identity. We denote by $[,]$ the Lie bracket and by q a non-negative integer. For any Lie algebra X, an ideal $Y \subseteq X$ and $x \in X$ we shall write $cl(x)$ to denote the coset $x + Y$.

1. Tensor and exterior products modulo q of Lie algebras

Let P and M be two Lie algebras. By an action of P on M we mean a Λ-bilinear map $P \times M \to M$, $(p, m) \mapsto {}^p m$ satisfying

$$ {}^{[p,p']}m = {}^p({}^{p'}m) - {}^{p'}({}^p m), $$

$$ {}^p[m, m'] = [{}^p m, m'] + [m, {}^p m'] $$

for all $m, m' \in M$, $p, p' \in P$. Note that any Lie algebra acts on its ideals by Lie multiplication.

Recall from [16] (see also [6]) that, in the context of Lie algebras, a crossed P-module is a Lie homomorphism $\mu : M \to P$ together with an action of P on M which satisfies the following conditions:

(i) $\mu({}^p m) = [p, \mu(m)]$,

(ii) ${}^{\mu(m)}m' = [m, m']$

for all $m, m' \in M$, $p \in P$.

A *morphism of crossed modules* $\mu : M \to P$ and $\mu' : M' \to P'$ is a pair ($f : M \to M'$, $\varphi : P \to P'$) of Lie homomorphisms such that $f({}^p m) = {}^{\varphi(p)} f(m)$ for all $m \in M$, $p \in P$ and $\mu' f = \varphi \mu$.

Suppose that $\mu : M \to P$ and $\nu : N \to P$ are two crossed P-modules and consider the pullback

$$ \begin{array}{ccc} M \times_P N & \xrightarrow{\pi_2} & N \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\mu} & P \end{array} $$

Let $K = M \times_P N = \{(m, n) \in M \times N | m \in M, n \in N, \mu(m) = \nu(n)\}$. In this diagram each Lie algebra acts on any other via its image in the Lie algebra P.

Definition 1.1. *The tensor product modulo q, $M \otimes^q N$, of the crossed P-modules μ and ν is the Lie algebra generated by the symbols $m \otimes n$ and $\{k\}$, $m \in M$, $n \in N$, $k \in K$ subject to the following relations:*

$$ \lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n, \tag{1.1} $$

$$ \begin{aligned} (m + m') \otimes n &= m \otimes n + m' \otimes n, \\ m \otimes (n + n') &= m \otimes n + m \otimes n', \end{aligned} \tag{1.2} $$

$$[m,m']\otimes n = m\otimes {}^{m'}n - m'\otimes {}^{m}n,\qquad m\otimes [n,n'] = {}^{n'}m\otimes n - {}^{n}m\otimes n', \tag{1.3}$$

$$[m\otimes n, m'\otimes n'] = -{}^{n}m\otimes {}^{m'}n', \tag{1.4}$$

$$[\{k\}, m\otimes n] = {}^{qk}m\otimes n + m\otimes {}^{qk}n, \tag{1.5}$$

$$\{\lambda k+\lambda' k'\} = \lambda\{k\}+\lambda'\{k'\}, \tag{1.6}$$

$$[\{k\},\{k'\}] = \pi_1(qk)\otimes\pi_2(qk'), \tag{1.7}$$

$$\{(-{}^{n}m, {}^{m}n)\} = q(m\otimes n) \tag{1.8}$$

for all $m,m'\in M,\ \ n,n'\in N,\ \ k,k'\in K,\ \ \lambda,\lambda'\in\Lambda.$

Let $M\Box N$ be the submodule of $M\otimes^q N$ generated by the elements $m\otimes n$ with $\mu(m)=\nu(n)$. Then $M\Box N$ lies in the centre of $M\otimes^q N$ since for any $m\otimes n\in M\Box N$, $m'\otimes n'\in M\otimes N$ we have $[m\otimes n, m'\otimes n']=0$ (see [6]) and for any $\{k\}\in K$ by relations (1.3), (1.5) one has

$$\begin{aligned}
&[\{k\}, m\otimes n] = {}^{qk}m\otimes n + m\otimes {}^{qk}n = [\pi_1(qk), m]\otimes n + m\otimes {}^{qk}n\\
&= \pi_1(qk)\otimes {}^{m}n - m\otimes {}^{qk}n + m\otimes {}^{qk}n = \pi_1(qk)\otimes {}^{\mu(m)}n\\
&= \pi_1(qk)\otimes {}^{\nu(n)}n = \pi_1(qk)\otimes [n,n] = 0.
\end{aligned}$$

In particular, $M\Box N$ is an ideal of $M\otimes^q N$.

Definition 1.2. *The exterior product modulo* q, $M\wedge^q N$, *of the crossed* P*-modules* μ *and* ν *is the quotient*

$$M\otimes^q N/M\Box N.$$

In other words, the Lie algebra $M\wedge^q N$ is the quotient of the Lie algebra $M\otimes^q N$ by the relation

$$\pi_1(k)\otimes\pi_2(k) = 0\ ,\quad k\in K. \tag{1.9}$$

Let us denote by $m\wedge n$ the image of $m\otimes n$ in $M\wedge^q N$.

Proposition 1.3. *There are two Lie homomorphisms* $\xi : M\otimes^q N\to M$ *and* $\xi' : M\otimes^q N\to N$ *defined by*

$$\xi(m\otimes n) = -{}^{n}m\ ,\quad \xi(\{k\}) = \pi_1(qk),$$

$$\xi'(m\otimes n) = {}^{m}n\ ,\quad \xi'(\{k\}) = \pi_2(qk).$$

Moreover, these homomorphisms factor through $M\wedge^q N$.

Proof. [6, Proposition 2] leaves us to show that ξ and ξ' commute with relations (1.5)-(1.9). In effect,

$$\begin{aligned}
&\xi({}^{qk}m\otimes n + m\otimes {}^{qk}n) = -{}^{n}({}^{qk}m) - {}^{({}^{qk}n)}m = -{}^{n}({}^{qk}m) - {}^{\nu({}^{\mu\pi_1(qk)}n)}m\\
&= -{}^{n}({}^{qk}m) - {}^{[\mu\pi_1(qk),\nu(n)]}m = -{}^{n}({}^{qk}m) - {}^{\mu\pi_1(qk)}({}^{n}m) + {}^{n}({}^{qk}m)\\
&= -[\pi_1(qk), {}^{n}m] = \xi([\{k\}, m\otimes n]).
\end{aligned}$$

The proof of the rest is left as an exercise. $\Box$

Remark 1.4. *There is the canonical Lie homomorphism* $\delta : M\otimes^q N\to M\times_P N$ *(resp.* $\delta' : M\wedge^q N\to M\times_P N$*), given, for* $x\in M\otimes^q N$ *(resp.* $x\in M\wedge^q N$*) by* $\delta(x) = (\xi(x),\xi'(x))$ *(resp.* $\delta'(x) = (\xi(x),\xi'(x))$*). In the case* $q=1$ *the map* δ *is an isomorphism of Lie algebras.*

Lemma 1.5. (i) *Let* $m,m',m''\in M$ *and* $n,n',n''\in N$ *be such that* $\mu(m)=\nu(n)=\nu(n'')$ *and* $\mu(m')=\mu(m'')=\nu(n')$*, then*

$$qm''\otimes qn'' = -qm\otimes qn' = qm'\otimes qn.$$

(ii) *Let $k, k' \in K$ and suppose $[k, k'] = 0$, then*

$$q(\pi_1(k) \otimes \pi_2(k')) = 0.$$

Proof. (i) By the relation (1.7) one has

$$\begin{aligned} qm'' \otimes qn'' &= [\{(m'', n')\}, \{(m, n'')\}] = -[\{(m, n'')\}, \{(m'', n')\}] \\ &= -qm \otimes qn' = -[\{(m, n)\}, \{(m', n')\}] = qm' \otimes qn. \end{aligned}$$

(ii) Follows from the relation (1.8) and the fact that $\{0\} = 0$. $\square$

Recall from [6] Ellis' original definition of the non-abelian tensor product of Lie algebras, $M \otimes N$, which is the Lie algebra generated by elements $m \otimes n$, $m \in M, n \in N$ and subject to the relations (1.1)-(1.4). Furthermore, Ellis' exterior product, $M \wedge N$, is the Lie algebra generated by elements $m \wedge n$, $m \in M, n \in N$ and subject to the relations (1.1)-(1.4) and (1.9) (see [6], [7])

Let $[M, N]$ be the submodule of $K = M \times_P N$ generated by the elements $(-{}^n m, {}^m n)$, $m \in M, n \in N$. It is easy to see that $[M, N]$ is an ideal of K. Further, $[M, N]$ contains the commutator $[K, K]$ of K since for $k, k' \in K$ one has

$$[k, k'] = (-{}^{\pi_2(k')}\pi_1(k), {}^{\pi_1(k)}\pi_2(k')).$$

We have the following

Proposition 1.6. *There is a commutative diagram of Lie algebras*

$$\begin{tikzcd} M \otimes N \arrow[r, "\varphi"] \arrow[d] & M \otimes^q N \arrow[r] \arrow[d] & K/[M,N] \arrow[r] \arrow[d, equal] & 0 \\ M \wedge N \arrow[r, "\psi"] & M \wedge^q N \arrow[r] & K/[M,N] \arrow[r] & 0 \end{tikzcd}$$

with exact rows.

Proof. At first note that the Lie algebra $K/[M, N]$ is abelian. The homomorphism φ (resp. ψ) is defined by $\varphi(m \otimes n) = m \otimes n$ (resp. $\psi(m \wedge n) = m \wedge n$). By (1.5) $\operatorname{Im}\varphi$ (resp. $\operatorname{Im}\psi$) is an ideal of $M \otimes^q N$ (resp. $M \wedge^q N$). It is clear that the quotient of $M \otimes^q N$ (resp. $M \wedge^q N$) by $\varphi(M \otimes N)$ (resp. $\psi(M \wedge N)$) is generated by elements $\{k\}$, $k \in K$ with the relations $\{\lambda k + \lambda' k'\} = \lambda\{k\} + \lambda'\{k'\}$, $[\{k\}, \{k'\}] = 0$, $\{(-{}^n m, {}^m n)\} = 0$ and the diagram is commutative. $\square$

The tensor and exterior products of Lie algebras modulo q are symmetric as we shall show now.

Proposition 1.7. *Let (M, μ) and (N, ν) be crossed P-modules. Then there is a unique isomorphism of Lie algebras*

$$s : M \otimes^q N \longrightarrow N \otimes^q M \quad (s : M \wedge^q N \longrightarrow N \wedge^q M)\ ,$$

such that $s(m \otimes n) = -(n \otimes m)$ ($s(m \wedge n) = -(n \wedge m)$), $s(\{k\}) = \{\overline{k}\}$, where $\overline{k} = (\pi_2(k), \pi_1(k))$ for all $m \in M$, $n \in N$ and $k \in K$.

Proof. We have only to show that s commutes with relations (1.5)-(1.9) (for relations (1.1)-(1.4) see [6]). In effect,

$$\begin{aligned} s([\{k\}, m \otimes n]) &= [\{\overline{k}\}, -n \otimes m] = -({}^{q\overline{k}}n \otimes m + n \otimes {}^{q\overline{k}}m) \\ &= -({}^{qk}n \otimes m + n \otimes {}^{qk}m) = s({}^{qk}m \otimes n + m \otimes {}^{qk}n), \end{aligned}$$

$$s(\{\lambda k + \lambda' k'\}) = \{\overline{\lambda k + \lambda' k'}\} = \lambda\{\overline{k}\} + \lambda'\{\overline{k'}\} = s(\lambda\{k\} + \lambda'\{k'\}).$$

By Lemma 1.5(i) one has

$$s([\{k\},\{k'\}]) = [\{\overline{k}\},\{\overline{k'}\}] = \pi_1(q\overline{k}) \otimes \pi_2(q\overline{k'})$$
$$= \pi_2(qk') \otimes \pi_1(qk) = s(\pi_1(qk) \otimes \pi_2(qk)),$$

$$s(\{(-{}^m n, {}^m n)\}) = \{({}^m n, -{}^n m)\} = -q(n \otimes m) = s(q(m \otimes n)).$$

And finally

$$s(\pi_1(k) \wedge \pi_2(k)) = -(\pi_2(k) \wedge \pi_1(k)) = 0.\Box$$

Proposition 1.8. *Let (M,μ), (N,ν) be crossed P-modules and (M',μ'), (N',ν') be crossed P'-modules. Suppose $\alpha = (f,\varphi) : (M,\mu) \to (M',\mu')$, $\beta = (g,\psi) : (N,\nu) \to (N',\nu')$ are crossed module morphisms such that $\varphi = \psi$. Then there are natural homomorphisms of Lie algebras*

$$\alpha \otimes^q \beta : M \otimes^q N \longrightarrow M' \otimes^q N'$$

$$(\alpha \wedge^q \beta : M \wedge^q N \longrightarrow M' \wedge^q N')\ ,$$

such that $(\alpha \otimes \beta)(m \otimes n) = f(m) \otimes g(n)$ $((\alpha \wedge^q \beta)(m \wedge n) = f(m) \wedge g(n))$, $(\alpha \otimes^q \beta)(\{k\}) = \{(f\pi_1(k), g\pi_2(k))\}$ $((\alpha \wedge^q \beta)(\{k\}) = \{(f\pi_1(k), g\pi_2(k))\})$ for all $m \in M$, $n \in N$ and $k \in K$. Furthermore, if α, β are onto, so also is $\alpha \otimes^q \beta$ $(\alpha \wedge^q \beta)$.

Proof. Note that $(f\pi_1(k), g\pi_2(k)) \in M' \times_{P'} N'$ for all $k \in K = M \times_P N$. $\alpha \otimes^q \beta$ plainly commutes with relations (1.1)-(1.9). For instance,

$$(\alpha \otimes^q \beta)([\{k\}, m \otimes n]) = [\{(f\pi_1(k), g\pi_2(k))\}, f(m) \otimes g(n)]$$
$$= {}^{\mu' f\pi_1(qk)} f(m) \otimes g(n) + f(m) \otimes {}^{\nu' g\pi_2(qk)} g(n)$$
$$= {}^{\varphi\mu\pi_1(qk)} f(m) \otimes g(n) + f(m) \otimes {}^{\varphi\nu\pi_2(qk)} g(n)$$
$$= f({}^{\mu\pi_1(qk)} m) \otimes g(n) + f(m) \otimes g({}^{\nu\pi_2(qk)} n)$$
$$= (\alpha \otimes^q \beta)({}^{qk} m \otimes n + m \otimes {}^{qk} n).\Box$$

Now we investigate the compatibility of the tensor product modulo q, $\otimes^q$, with the direct limit of crossed modules. The group-theoretic version of this result is given in [15].

Theorem 1.9. *Let $\{M_\alpha, \Phi_\alpha^\beta, \alpha \leqslant \beta\}$ and $\{P_\alpha, \Psi_\alpha^\beta, \alpha \leqslant \beta\}$ be two directed systems of Lie algebras. Let $\mu_\alpha : M_\alpha \to P_\alpha$ be a crossed P_α-module for each α such that $(\Phi_\alpha^\beta, \Psi_\alpha^\beta) : (M_\alpha, \mu_\alpha) \to (M_\beta, \nu_\beta)$, $\alpha \leqslant \beta$ is a crossed module morphism. Let $\nu_\alpha : N \to P_\alpha$ be a crossed P_α- module for each α such that $(1, \Psi_\alpha^\beta) : (N, \nu_\alpha) \to (N, \nu_\beta)$, $\alpha \leqslant \beta$ is a crossed module morphism. Then there are natural isomorphisms of Lie algebras*

$$(\varinjlim{}_\alpha\{M_\alpha\}) \otimes^q N \approx \varinjlim{}_\alpha\{M_\alpha \otimes^q N\},\quad \varinjlim{}_\alpha\{M_\alpha\}) \wedge^q N \approx \varinjlim{}_\alpha\{M_\alpha \wedge^q N\},$$

where $\varinjlim{}_\alpha\{M_\alpha\} \otimes^q N$ is considered as the tensor product modulo q of crossed $\varinjlim{}_\alpha\{P_\alpha\}$-modules.

Proof. It is easy to check that a homomorphism $\nu : N \to \varinjlim{}_\alpha\{P_\alpha\}$ defined by $\nu(n) = cl(\nu_\alpha(n))$, $n \in N$ with an action ${}^{cl(p_\alpha)}n = {}^{p_\alpha}n$ is a crossed module.

Let $\mu : \varinjlim{}_\alpha\{M_\alpha\} \to \varinjlim{}_\alpha\{P_\alpha\}$ be the Lie homomorphism defined by $\mu(cl(m_\alpha)) = cl(\mu_\alpha(m_\alpha))$. There is an action of $\varinjlim{}_\alpha\{P_\alpha\}$ on $\varinjlim{}_\alpha\{M_\alpha\}$ defined by ${}^{cl(p_\alpha)}cl(m_\beta) = cl({}^{\Psi_\alpha^\gamma(p_\alpha)}\Phi_\beta^\gamma(m_\beta))$, where $\gamma \geqslant \alpha, \beta$ (the existense of such γ follows from the directeness of the system). It is easy to check that everything is well defined here. Then we have

$$\mu({}^{cl(p_\alpha)}cl(m_\beta)) = \mu(cl({}^{\Psi_\alpha^\gamma(p_\alpha)}\Phi_\beta^\gamma(m_\beta))) = cl(\mu_\gamma({}^{\Psi_\alpha^\gamma(p_\alpha)}\Phi_\beta^\gamma(m_\beta)))$$
$$= cl([\Psi_\alpha^\gamma(p_\alpha), \mu_\gamma(\Phi_\beta^\gamma(m_\beta))]) = cl([\Psi_\alpha^\gamma(p_\alpha), \Psi_\beta^\gamma\mu_\beta(m_\beta)])$$
$$= [cl(p_\alpha), \mu_(cl(m_\beta))],\ \text{ where }\ \gamma \geqslant \alpha, \beta;$$

$$\begin{aligned}
&{}^{\mu(cl(m_\alpha))}cl(m_\beta) = {}^{cl(\mu_\alpha(m_\alpha))}cl(m_\beta) = cl({}^{\Psi^\gamma_\alpha\mu_\alpha(m_\alpha)}\Phi^\gamma_\beta(m_\beta)) \\
&= cl({}^{\mu_\gamma\Phi^\gamma_\alpha(m_\alpha)}\Phi^\gamma_\beta(m_\beta)) = cl(\Phi^\gamma_\alpha(m_\alpha),\Phi^\gamma_\beta(m_\beta)]) \\
&= [cl(m_\alpha),cl(m_\beta)], \quad \text{where } \gamma \geqslant \alpha,\beta.
\end{aligned}$$

Hence μ is a crossed $\varinjlim_\alpha\{P_\alpha\}$-module.

Suppose

$$t : (\varinjlim_\alpha\{M_\alpha\}) \otimes^q N \longrightarrow \varinjlim_\alpha\{M_\alpha \otimes^q N\}$$

$$(\text{resp. } t : (\varinjlim_\alpha\{M_\alpha\}) \wedge^q N \longrightarrow \varinjlim_\alpha\{M_\alpha \wedge^q N\})$$

is a homomorphism defined by the formula $t(cl(m_\alpha)\otimes n) = cl(m_\alpha \otimes n)$ (resp. $t(cl(m_\alpha)\wedge n) = cl(m_\alpha \wedge n)$) and $t(\{(cl(m_\alpha),n)\}) = cl(\{(\Phi^\beta_\alpha(m_\alpha),n)\})$. Note that there is a $\beta \geqslant \alpha$ such that $(\Phi^\beta_\alpha(m_\alpha),n) \in M_\beta \times_{P_\beta} N$, when $(cl(m_\alpha),n) \in (\varinjlim_\alpha M_\alpha) \times_{\varinjlim_\alpha P_\alpha} N$. To prove that t is well defined we repeat the corresponding part of the proof of Theorem 1.5 in [15].

It is clear that t commutes with relations (1.1) and (1.2). Let us show the compatibility with relations (1.3)-(1.9).

$$\begin{aligned}
&t([cl(m_\alpha),cl(m_\beta)]\otimes n) = t(cl([\Phi^\gamma_\alpha(m_\alpha),\Phi^\gamma_\beta(m_\beta)])\otimes n) \\
&= cl([\Phi^\gamma_\alpha(m_\alpha),\Phi^\gamma_\beta(m_\beta)]\otimes n) \\
&= cl(\Phi^\gamma_\alpha(m_\alpha)\otimes {}^{\mu_\gamma\Phi^\gamma_\beta(m_\beta)}n - \Phi^\gamma_\beta(m_\beta)\otimes {}^{\mu_\gamma\Phi^\gamma_\alpha(m_\alpha)}n) \\
&= cl((\Phi^\gamma_\alpha \otimes^q 1_N)(m_\alpha \otimes {}^{\Psi^\gamma_\beta\mu_\beta(m_\beta)}n) - (\Phi^\gamma_\beta \otimes^q 1_N)(m_\beta \otimes {}^{\Psi^\gamma_\alpha\mu_\alpha(m_\alpha)}n)) \\
&= cl(m_\alpha \otimes {}^{\mu_\beta(m_\beta)}n - cl(m_\beta \otimes {}^{\mu_\alpha(m_\alpha)}n) \\
&= t(cl(m_\alpha)\otimes {}^{cl(m_\beta)}n - cl(m_\beta)\otimes {}^{cl(m_\alpha)}n), \quad \text{for some } \gamma \geqslant \alpha,\beta.
\end{aligned}$$

Similarly it can be proved that t preserves the second relation of (1.3). Next

$$\begin{aligned}
&t([cl(m_\alpha)\otimes n, cl(m_\beta)\otimes n']) = [cl(m_\alpha\otimes n), cl(m_\beta \otimes n')] \\
&= cl([(\Phi^\gamma_\alpha \otimes^q 1_N)(m_\alpha\otimes n), (\Phi^\gamma_\beta \otimes^q 1_N)(m_\beta\otimes n')]) \\
&= cl([\Phi^\gamma_\alpha(m_\alpha)\otimes n, \Phi^\gamma_\beta(m_\beta)\otimes n']) = -cl({}^n\Phi^\gamma_\alpha(m_\alpha)\otimes {}^{\Phi^\gamma_\beta(m_\beta)}n') \\
&= -t({}^n cl(m_\alpha)\otimes {}^{cl(m_\beta)}n'), \quad \gamma \geqslant \alpha,\beta,
\end{aligned}$$

$$\begin{aligned}
&t([\{(cl(m_\alpha),n)\}, cl(m_\beta)\otimes n']) = [cl(\{(\Phi^\gamma_\alpha(m_\alpha),n)\}), cl(m_\beta \otimes n')] \\
&= cl([(\Phi^{\gamma'}_\gamma \otimes^q 1_N)(\{(\Phi^\gamma_\alpha(m_\alpha),n)\}), \Phi^{\gamma'}_\beta(m_\beta\otimes n')]) \\
&= cl([\{(\Phi^{\gamma'}_\alpha(m_\alpha),n)\}, \Phi^{\gamma'}_\beta(m_\beta)\otimes n']) \\
&= cl({}^{qn}\Phi^{\gamma'}_\beta(m_\beta)\otimes n' + \Phi^{\gamma'}_\beta(m_\beta)\otimes {}^{qn}n') \\
&= t({}^{qn}cl(m_\beta)\otimes n' + cl(m_\beta)\otimes {}^{qn}n'), \quad \gamma' \geqslant \gamma,\beta,
\end{aligned}$$

$$\begin{aligned}
&t(\{\lambda(cl(m_\alpha),n) + \lambda'(cl(m_\beta),n')\}) \\
&= t(\{(cl(\Phi^\gamma_\alpha(\lambda m_\alpha) + \Phi^\gamma_\beta(\lambda' m_\beta)), \lambda n + \lambda' n')\}) \\
&= cl(\{(\Phi^{\gamma'}_\gamma(\Phi^\gamma_\alpha(\lambda m_\alpha) + \Phi^\gamma_\beta(\lambda' m_\beta)), \lambda n + \lambda' n')\}) \\
&= cl(\{(\Phi^{\gamma'}_\alpha(\lambda m_\alpha) + \Phi^{\gamma'}_\beta(\lambda' m_\beta)), \lambda n + \lambda' n')\} \\
&= cl(\lambda\{(\Phi^{\gamma'}_\alpha(m_\alpha),n)\} + \lambda'(\{\Phi^{\gamma'}_\beta(m_\beta),n')\}) \\
&= t(\lambda\{(cl(m_\alpha),n)\} + \lambda'\{(cl(m_\beta),n')\}),
\end{aligned}$$

$$\begin{aligned}&t([\{(cl(m_\alpha),n)\},\{(cl(m_\beta),n')\}])\\&=[cl(\{(\Phi_\alpha^\gamma(m_\alpha),n)\},cl(\{(\Phi_\beta^{\gamma'}(m_\beta),n')\})]\\&=cl([\{(\Phi_\alpha^{\gamma''}(m_\alpha),n)\},\{(\Phi_\beta^{\gamma''}(m_\beta),n')\}])\\&=cl(q\Phi_\alpha^{\gamma''}(m_\alpha)\otimes qn')=t(q.cl(m_\alpha)\otimes qn'),\end{aligned}$$

$$\begin{aligned}&t(\{(-^n cl(m_\alpha),{}^{cl(m_\alpha)}n)\})=t(\{(cl(-^n m_\alpha),{}^{m_\alpha}n)\})\\&=cl(\{(\Phi_\alpha^\beta(-^n m_\alpha),{}^{m_\alpha}n)\})=cl(\{(-^n\Phi_\alpha^\beta(m_\alpha),{}^{\Phi_\alpha^\beta(m_\alpha)}n)\})\\&=cl(q(\Phi_\alpha^\beta(m_\alpha)\otimes n))=t(q(cl(m_\alpha)\otimes n)).\end{aligned}$$

If $cl(\mu_\alpha(m_\alpha))=cl(\nu_\alpha(n))$ then there is $\beta\geqslant\alpha$ such that $\Psi_\alpha^\beta\mu_\alpha(m_\alpha)=\Psi_\alpha^\beta\nu_\alpha(n)$ and hence $\mu_\beta\Phi_\alpha^\beta(m_\alpha)=\nu_\beta(n)$. Then

$$t(cl(m_\alpha)\wedge n)=cl(m_\alpha\wedge n)=cl(\Phi_\alpha^\beta(m_\alpha)\wedge n)=0.$$

On the other hand, the homomorphisms

$$\Phi_\alpha\otimes^q 1_N: M_\alpha\otimes^q N\to(\varinjlim{}_\alpha\{M_\alpha\})\otimes^q N$$

$$(\text{resp. }\Phi_\alpha\wedge^q 1_N: M_\alpha\wedge^q N\to(\varinjlim{}_\alpha\{M_\alpha\})\wedge^q N),$$

where $\Phi_\alpha: M_\alpha\to\varinjlim{}_\alpha\{M_\alpha\}$ are the canonical homomorphisms, induce a homomorphism $t':\varinjlim{}_\alpha\{M_\alpha\otimes^q N\}\to(\varinjlim{}_\alpha\{M_\alpha\})\otimes^q N$ (resp. $t':\varinjlim{}_\alpha\{M_\alpha\wedge^q N\}\to(\varinjlim{}_\alpha\{M_\alpha\})\wedge^q N$). It is easy to see that tt', $t't$ are identity maps. □

One has the following generalization of the homomorphism φ in Proposition 1.6

Theorem 1.10. (i) *Let p be a positive integer and let $q'=pq$. Then there is a Lie homomorphism $\varphi': M\otimes^{q'}N\to M\otimes^q N$ given by*

$$\varphi'(m\otimes n)=m\otimes n,\ \ \varphi'(\{k\})=\{pk\}$$

for $m\in M, n\in N, k\in K$. Furthermore, the Lie homomorphism φ' induces a Lie homomorphism $\psi': M\wedge^{q'}N\to M\wedge^q N$.

(ii) *Let $L=K/[M,N]$ then $coker\varphi'$ and $coker\psi'$ are isomorphic to L/pL.*

Proof. (i) We have to show that φ' commutes with relations (1.1)-(1.9). It is clear for relations (1.1)-(1.4) and (1.9). Now, for $m\in M,\ n\in N,\ k,k'\in K,\ \lambda,\lambda'\in\Lambda$ we have

$$\begin{aligned}&\varphi'([\{k\},m\otimes n])=[\{pk\},m\otimes n]={}^{qpk}m\otimes n+m\otimes{}^{qpk}n\\&={}^{q'k}m\otimes n+m\otimes{}^{q'k}n=\varphi'({}^{q'k}m\otimes n+m\otimes{}^{q'k}n);\end{aligned}$$

$$\varphi'(\{\lambda k+\lambda'k'\})=\{p(\lambda k+\lambda'k')\}=\lambda\{pk\}+\lambda'\{pk'\}=\varphi'(\lambda\{k\}+\lambda'\{k'\});$$

$$\begin{aligned}&\varphi'([\{k\},\{k'\}])=[\{pk\},\{pk'\}]=\pi_1(qpk)\otimes\pi_2(qpk')\\&=\varphi'(\pi_1(q'k)\otimes\pi_2(q'k'));\end{aligned}$$

$$\varphi'(\{(-^n m,{}^m n)\})=\{p(-^n m,{}^m n)\}=qq'(m\otimes n)=\varphi'(q'(m\otimes n)).$$

(ii) Can be proved by analogy with the group theoretic version (see [5, Theorem 1.22]). □

Suppose that $\mu: M\to P$ and $\nu: N\to P$ are two crossed P-modules in the context of Lie algebras. There is an action of P on $M\otimes^q N$ ($M\wedge^q N$) which on generators is given by ${}^p(m\otimes n)={}^pm\otimes n+m\otimes{}^pn$ (${}^p(m\wedge n)={}^pm\wedge n+m\wedge{}^pn$) and ${}^p\{k\}=\{{}^pk\}$ for $m\in M$, $n\in N$, $k\in K$ and $p\in P$. The proof of this fact is left for the reader.

Now we give the definition of the crossed square of Lie algebras which is the crossed 2-cube of Lie algebras (see [8, Definitions 1.3 and 1.4]).

Definition 1.11. *A crossed square is a commutative diagram of Lie algebras*

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow{\nu} & P \end{array}$$

endowed with an action of P on each Lie algebra and a bilinear function $h : M \times N \longrightarrow L$ such that

(i) μ, ν *and* $\alpha = \nu\lambda' = \mu\lambda$ *are crossed modules, and the maps* λ, λ' *preserve the actions of* P;

(ii) $\lambda h(m,n) = -{}^{n}m$, $\lambda' h(m,n) = {}^{m}n$;

(iii) $h(\lambda(l), n) = -{}^{n}l$, $h(m, \lambda'(l)) = {}^{m}l$;

(iv) $h([m,m'],n) = h(m, {}^{m'}n) - h(m', {}^{m}n)$,
$h(m,[n,n']) = h({}^{n'}m, n) - h({}^{n}m, n')$;

(v) ${}^{p}h(m,n) = h({}^{p}m, n) + h(m, {}^{p}n)$

for all $m, m' \in M$, $n, n' \in N$, $p \in P$, $l \in L$.

It is easy to obtain the following property of crossed squares of Lie algebras:

Lemma 1.12. *Consider a crossed square of Lie algebras. Then:*

(i) *with the actions induced by the image in* P *the morphisms* λ, λ' *are crossed modules;*

(ii) *the actions of* M *on* $Ker\lambda'$ *and of* N *on* $Ker\lambda$ *are trivial;*

(iii) $h(\lambda(l), \lambda'(l')) = [l, l']$ *for all* $l, l' \in L$.

Now we list some examples of crossed squares of Lie algebras. Throughout, P is an arbitrary Lie algebra and $\mu : M \to P$ and $\nu : N \to P$ are crossed P-modules.

(1) The square (pull)

$$\begin{array}{ccc} M \times_P N & \xrightarrow{\pi_2} & N \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow{\mu} & P \end{array}$$

with $h(m,n) = (-{}^{n}m, {}^{m}n)$ is a crossed square.

(2) The square

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\xi'} & N \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow{\mu} & P \end{array}$$

with $\xi(m \otimes n) = -{}^{n}m$, $\xi'(m \otimes n) = {}^{m}n$ and $h(m,n) = m \otimes n$ is a crossed square.

(3) The square

$$\begin{array}{ccc} M \wedge N & \xrightarrow{\xi'} & N \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow{\mu} & P \end{array}$$

with $h(m,n) = m \wedge n$ is a crossed square.

One has the following

Lemma 1.13. *For $m \in M$, $n \in N$, $k \in K$ we have the following relations:*

$$^{m}\{k\} = m \otimes \pi_2(qk),$$

$$^{n}\{k\} = -\pi_1(qk) \otimes n.$$

Proof. $^{m}\{k\} = \{([m, \pi_1(k)], {}^{m}\pi_2(k))\} = \{(-{}^{\pi_2(k)}m, {}^{m}\pi_2(k))\} = m \otimes \pi_2(qk)$ by formula (1.8).

The proof of the second formula is similar. □

Now we have a fourth example of a crossed square of Lie algebras

Proposition 1.14. *The square*

$$\begin{tikzcd} M \otimes^q N \arrow[r, "\xi'"] \arrow[d, "\xi"'] & N \arrow[d, "\nu"] \\ M \arrow[r, "\mu"] & P \end{tikzcd}$$

is a crossed square with the function h given by $h(m,n) = m \otimes n$, where ξ and ξ' are the Lie homomorphisms defined in Proposition 1.3.

Proof. We have to check each property of a crossed square.

(i) It is easy to see that ξ and ξ' preserve the actions of P. Consider $\alpha = \mu\xi = \nu\xi'$, then:

$$\begin{aligned}&\alpha({}^{p}(m \otimes n)) = \mu\xi({}^{p}m \otimes n + m \otimes {}^{p}n) = \mu(-{}^{n}({}^{p}m) - {}^{({}^{p}n)}m)\\ &= \mu(-{}^{p}({}^{n}m)) = [p, \mu(-{}^{n}m)] = [p, \alpha(m \otimes n)],\end{aligned}$$

$$\alpha({}^{p}\{k\}) = \mu\xi(\{{}^{p}k\}) = \mu\pi_1({}^{p}(qk)) = [p, \mu\pi_1(qk)] = [p, \alpha(\{k\})],$$

$$\begin{aligned}&{}^{\alpha(m\otimes n)}(m' \otimes n') = {}^{\mu(-{}^{n}m)}(m' \otimes n') = {}^{[\mu(m),\nu(n)]}m' \otimes n' + m' \otimes {}^{[\mu(m),\nu(n)]}n'\\ &= -{}^{n}[m, m'] \otimes n' + [m, {}^{n}m'] \otimes n' - m' \otimes [n, {}^{m}n'] + m' \otimes {}^{m}[n, n']\\ &= -[{}^{n}m, m'] \otimes n' + m' \otimes [{}^{m}n, n']\\ &= -{}^{n}m \otimes {}^{m'}n' - m' \otimes [{}^{m}n, n'] + m' \otimes [{}^{m}n, n'] = [m \otimes n, m' \otimes n']\end{aligned}$$

by formulas (1.3) and (1.4),

$$\begin{aligned}&{}^{\alpha(m\otimes n)}\{k\} = {}^{m}({}^{n}\{k\}) - {}^{n}({}^{m}\{k\}) = {}^{m}(-\pi_1(qk) \otimes n) - {}^{n}(m \otimes \pi_2(qk))\\ &= [\pi_1(qk), m] \otimes n - \pi_1(qk) \otimes {}^{m}n - {}^{n}m \otimes \pi_2(qk) - m \otimes [n, \pi_2(qk)]\\ &= \pi_1(qk) \otimes {}^{m}n - m \otimes {}^{qk}n - \pi_1(qk) \otimes {}^{m}n - {}^{n}m \otimes \pi_2(qk) - {}^{qk}m \otimes n\\ &+{}^{n}m \otimes \pi_2(qk) = [m \otimes n, \{k\}]\end{aligned}$$

by Lemma 1.13 and formulas (1.3) and (1.5),

$${}^{\alpha(\{k\})}(m \otimes n) = {}^{\pi_1(qk)}m \otimes n + m \otimes {}^{\pi_1(qk)}n = [\{k\}, m \otimes n]$$

by formula (1.5),

$${}^{\alpha(\{k\})}\{k'\} = {}^{\pi_1(qk)}\{k'\} = \pi_1(qk) \otimes \pi_2(qk') = [\{k\}, \{k'\}]$$

by Lemma 1.13 and formula (1.7).

Now if $x, y \in M \otimes^q N$ are such that $\alpha({}^{p}x) = [p, \alpha(x)]$ and $\alpha({}^{p}y) = [p, \alpha(y)]$, then

$$\begin{aligned}&\alpha({}^{p}[x, y]) = [\alpha({}^{p}x), \alpha(y)] + [\alpha(x), \alpha({}^{p}y)]\\ &= [[p, \alpha(x)], \alpha(y)] + [\alpha(x), [p, \alpha(y)]] = [p, \alpha([x, y])];\end{aligned}$$

Next, if $x_1,\ y_1,\ z_1 \in M \otimes^q N$ are such that ${}^{\alpha(x_1)}y_1 = [x_1, y_1]$ and ${}^{\alpha(x_1)}z_1 = [x_1, z_1]$, then

$${}^{\alpha(x_1)}[y_1, z_1] = [[x_1, y_1], z_1] + [y_1, [x_1, z_1]] = [x_1, [y_1, z_1]];$$

And finally, if $x_2,\ y_2,\ z_2 \in M \otimes^q N$ are such that ${}^{\alpha(x_2)}z_2 = [x_2, z_2]$ and ${}^{\alpha(y_2)}z_2 = [y_2, z_2]$, then

$$\begin{aligned} {}^{\alpha([x_2,y_2])}z_2 &= {}^{\alpha(x_2)}({}^{\alpha(y_2)}z_2) - {}^{\alpha(y_2)}({}^{\alpha(x_2)}z_2) \\ &= [x_2,[y_2,z_2]] - [y_2,[x_2,z_2]] = [[x_2,y_2],z_2]. \end{aligned}$$

Thus α is a crossed module.

(ii), (iv) and (v) are clear.

(iii) For the proof of the first formula we consider the two cases $l = m \otimes n$ and $l = \{k\}$, then one has

$$\xi(m \otimes n) \otimes n' = -{}^{n}m \otimes n' = -{}^{n'}m \otimes n + m \otimes [n,n'] = -{}^{n'}(m \otimes n)$$

by formula (1.3),

$$\xi(\{k\}) \otimes n = \pi_1(qk) \otimes n = -{}^{n}\{k\}$$

by Lemma 1.13, and observe that if $\xi(l) \otimes n = -{}^{n}l$ and $\xi(l') \otimes n = -{}^{n}l'$, for $l,\ l' \in M \otimes^q N$, then by formula (1.3) and (i) it can be written

$$\begin{aligned} \xi([l,l']) \otimes n &= \xi(l) \otimes {}^{\xi(l')}n - \xi(l') \otimes {}^{\xi(l)}n \\ &= {}^{\xi(l')}(\xi(l) \otimes n) - {}^{\xi(l')}\xi(l) \otimes n - {}^{\xi(l)}(\xi(l') \otimes n) + {}^{\xi(l)}\xi(l') \otimes n \\ &= [l', -{}^{n}l] + \xi([l,l']) \otimes n - [l, -{}^{n}l'] + \xi([l,l']) \otimes n, \end{aligned}$$

so that

$$\xi([l,l']) \otimes n = -([l, {}^{n}l'] + [{}^{n}l, l']) = -{}^{n}[l,l'].$$

The proof of the second formula is similar.□

Corollary 1.15. *The square*

$$\begin{array}{ccc} M \wedge^q N & \xrightarrow{\xi'} & N \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\mu} & P \end{array}$$

with $h(m,n) = m \wedge n$ is a crossed square.

Lemma 1.16. (i) *With the action induced by the image of $K = M \times_P N$ in P the Lie homomorphism $\delta : M \otimes^q N \to K$ ($\delta' : M \wedge^q N \to K$), constructed in Remark 1.4 is a crossed module.*

(ii) *If $x \in M \wedge^q N$ (resp. $x \in M \otimes^q N$), then $\{\delta'(x)\} = qx$ (resp. $\{\delta(x)\} = qx$).*

Proof. (i) immediately follows from the direct calculations.

(ii) For $k \in K$ by formula (1.6)

$$\{\delta'(\{k\})\} = \{(\pi_1(qk), \pi_2(qk))\} = \{qk\} = q\{k\},$$

for $m \in M$ and $n \in N$, using (1.8) one has

$$\{\delta'(m \wedge n)\} = \{(-{}^{n}m, {}^{m}n)\} = q(m \wedge n).$$

Now let $x,\ y \in M \wedge^q N$, $\lambda,\ \lambda' \in \Lambda$, $\{\delta'(x)\} = qx$ and $\{\delta'(y)\} = qy$, then by formula (1.6)

$$\{\delta'(\lambda x + \lambda' y)\} = q(\lambda x + \lambda' y).$$

Next, by relation (1.8), Lemma 1.12(iii) and Proposition 1.14 one has

$$\{\delta'([x,y])\} = \{(-{}^{\pi_2\delta'(y)}\pi_1\delta'(x), {}^{\pi_1\delta'(x)}\pi_2\delta'(y))\} = q(\xi(x) \wedge \xi'(y)) = q[x,y].$$

It is enough to see that $\{\delta'(x)\} = qx$ for any $x \in M \wedge^q N$. The equality $\{\delta(x)\} = qx$ for $x \in M \otimes^q N$ can be proved similarly. □

Now we analyse the kernel of the canonical homomorphism $M \otimes^q N \to M \wedge^q N$. In order to do this we consider the generalized version of Whitehead's universal quadratic functor Γ [19], which is defined by Simson and Tye in [18] (see also [6]) for any Λ-module A as the Λ-module $\Gamma(A)$ generated by the symbols $\gamma(a)$ with $a \in A$, subject to the relations

$$\lambda^2\gamma(a) = \gamma(\lambda a),$$

$$\gamma(a+b+c) + \gamma(a) + \gamma(b) + \gamma(c) = \gamma(a+b) + \gamma(a+c) + \gamma(b+c),$$

$$\gamma(\lambda a + b) + \lambda\gamma(a) + \lambda\gamma(b) = \lambda\gamma(a+b) + \gamma(\lambda a) + \gamma(b)$$

for all $\lambda,\ \lambda' \in \Lambda,\ a,\ b,\ c \in A$.

Let $\mu : M \to P$ and $\nu : N \to P$ be two crossed P-modules. Suppose the image of (ξ, ξ') is written $< M, N >$. It is easy to check that $< M, N >$ is an ideal of $K = M \times_P N$ and the quotient is abelian. One has the following

Theorem 1.17. *There is a natural exact sequence of Lie algebras*

$$\Gamma(K/ < M, N >) \xrightarrow{\psi} M \otimes^q N \xrightarrow{t} M \wedge^q N \longrightarrow 0,$$

where $\psi(\gamma(cl(m,n))) = m \otimes n$ *and* $t(m \otimes n) = m \wedge n$.

Proof. It is easily seen from relations (1.1), (1.4)-(1.7) that any element $x \in M \otimes^q N$ is of the form $x = \sum_i m_i \otimes n_i + \{k\}$, so

$$\xi(x) \otimes \xi'(x) = \xi(\sum_i m_i \otimes n_i) \otimes \xi'(\sum_i m_i \otimes n_i)$$
$$+\xi(\sum_i m_i \otimes n_i) \otimes \xi'(\{k\}) + \xi(\{k\}) \otimes \xi'(\sum_i m_i \otimes n_i)$$
$$+\xi(\{k\}) \otimes \xi'(\{k\}).$$

But from the proof of Proposition 14 in [6] we have

$$\xi(\sum_i m_i \otimes n_i) \otimes \xi'(\sum_i m_i \otimes n_i) = 0.$$

Next, using Lemma 1.13

$$\xi(\sum_i m_i \otimes n_i) \otimes \xi'(\{k\}) + \xi(\{k\}) \otimes \xi'(\sum_i m_i \otimes n_i)$$
$$= \sum_i (-{}^{n_i}m_i \otimes \pi_2(qk) + \pi_1(qk) \otimes {}^{m_i}n_i) = \sum_i (-{}^{({}^{n_i}m_i)}\{k\} - {}^{({}^{m_i}n_i)}\{k\})$$
$$= \sum_i (-{}^{[\nu(n_i),\mu(m_i)]}\{k\} - {}^{[\mu(m_i),\nu(n_i)]}\{k\}) = 0,$$

By relation (1.7)

$$\xi(\{k\}) \otimes \xi'(\{k\}) = \pi_1(qk) \otimes \pi_2(qk) = 0.$$

So $\xi(x) \otimes \xi'(x) = 0$ for every $x \in M \otimes^q N$. Thus if $cl(m,n) = 0$ then $\psi(\gamma(cl(m,n))) = m \otimes n = 0$. Clearly ψ commutes with the defining relations of $\Gamma(-)$ and $\operatorname{Im}\psi = M \square N = \operatorname{Ker} t$ (see Definition 1.2). But $M \square N$ is in the centre of $M \otimes^q N$ and so ψ is a Lie homomorphism. □

Recall the definition of compatible actions of Lie algebras from [6]

Definition 1.18. *Let M and N be two Lie algebras with actions on each other. The actions are compatible if*

$${}^{({}^n m)}n' = [n', {}^m n] \text{ and } {}^{({}^m n)}m' = [m', {}^n m]$$

for all $m, m' \in M$ $n, n' \in N$.

Let M and N be two Lie algebras with compatible actions on each other. We shall denote by ${}^{M}N$ the submodule of N generated by the elements of the form ${}^{m}n$, $m \in M$, $n \in N$. It follows from the compatibility condition that ${}^{M}N$ is an ideal of N.

According to the definition of the Peiffer product of groups (see [19],[10]) we have the following

Definition 1.19. *The Peiffer product, $M \bowtie N$, of two Lie algebras M and N with compatible actions on each other is the quotient of the coproduct $M * N$ by the relations:*

$$[m,n] = {}^{m}n,\ [n,m] = {}^{n}m$$

for all $m \in M$, $n \in N$.

As a consequence of the compatibility condition the actions of $M * N$ on M and on N factor through $M \bowtie N$ and the canonical maps $M \to M \bowtie N$ and $N \to M \bowtie N$ are crossed modules. So we can define an 'absolute' tensor product modulo q of two Lie algebras M and N acting on each other compatibly, by considering them as crossed $M \bowtie N$-modules.

Theorem 1.20. *If M and N act trivially on each other (i.e. ${}^{M}N = \{0\}$ and ${}^{N}M = \{0\}$) then there is an isomorphism*

$$M \otimes^{q} N \approx (M^{ab}/qM^{ab}) \otimes_{\Lambda/q\Lambda} (N^{ab}/qN^{ab}),$$

where $M^{ab} = M/[M,M]$, $N^{ab} = N/[N,N]$ and $[M,M]$, $[N,N]$ are commutants of M and N respectively.

Proof. In the case of trivial actions $[m,n] = 0$ in $M \bowtie N$ for all $m \in M$, $n \in N$ and hence the Peiffer product $M \bowtie N = M \times N$. Clearly $K = M \times_{M \bowtie N} N = 0$. So the Lie homomorphism φ in Proposition 1.6 is surjective. By [6] in the case of trivial actions one has $M \otimes N \approx M^{ab} \otimes_{\Lambda} N^{ab}$. By relation (1.8) every element in $M \otimes^{q} N$ has an order dividing q. Then

$$\begin{aligned} M \otimes^{q} N &\approx M \otimes N/q(M \otimes N) \approx M^{ab} \otimes_{\Lambda} N^{ab}/q(M^{ab} \otimes_{\Lambda} N^{ab}) \\ &\approx (M^{ab}/qM^{ab}) \otimes_{\Lambda/q\Lambda} (N^{ab}/qN^{ab}). \ \square \end{aligned}$$

Now the relation between Ellis' non-abelian tensor product of Lie algebras and the non-abelian tensor product modulo q of Lie algebras with compatible actions on each other will be given, which is the Lie algebra analogue of [15, Theorem 1.9]

First we study the Peiffer product of Lie algebras. Let M and N be two Lie algebras acting compatible on each other and let $\psi : M * N \to M \bowtie N$ be the natural Lie homomorphism. Then modulo $Ker\psi$, $[m,n] \equiv {}^{m}n$, so that every element of $M \bowtie N$ can be written as $\psi(m) + \psi(n)$ for suitable m and n. We denote $\psi(m) + \psi(n)$ by $< m,n >$. It is easy to see that the relations

$$\begin{aligned} &[< m,n >, < m',n' >] = < [m,m'] + {}^{n}m' - {}^{n'}m, [n,n'] > \\ &= < [m,m'], [n,n'] + {}^{m}n' - {}^{m'}n > \end{aligned}$$

are defining relations for $M \bowtie N$ on the generators $< m,n >$ and the Peiffer product is a homomorphic image of the semidirect products $M \ltimes N$ and $M \rtimes N$. Furthermore, $M \bowtie N$ is obtaind from $M \ltimes N$ (resp. $M \rtimes N$) by imposing the relation

$$({}^{n}m, {}^{m}n) = 0$$

for all $m \in M$ and $n \in N$, since if L is an ideal of $M \ltimes N$ (resp. $M \rtimes N$) generated by the set $\{({}^{n}m, {}^{m}n) | m \in M, n \in N\}$, then we have a Lie homomorphism $M \ltimes N/L \xrightarrow{\epsilon} M \bowtie N$ (resp. $M \rtimes N/L \xrightarrow{\epsilon} M \bowtie N$), $\epsilon(cl(m,n)) = < m,n >$. On the other hand, there is a Lie homomorphism $M \bowtie N \xrightarrow{\epsilon'} M \ltimes N/L$ (resp. $M \bowtie N \xrightarrow{\epsilon'} M \rtimes N/L$) induced by the canonical homomorphisms $M \to M \ltimes N$ and $N \to M \ltimes N$ (resp. $M \to M \rtimes N$ and $N \to M \rtimes N$). It is clear that $\epsilon\epsilon'$ and $\epsilon'\epsilon$ are identity maps.

Let $\mu : M \to P$ and $\nu : N \to P$ be two crossed P-modules.The actions of M and N on each other via P are always compatible and it is easy to prove the following

Proposition 1.21. *There is an exact sequence of Lie algebras*

$$0 \longrightarrow K/[M,N] \xrightarrow{j} M \bowtie N \xrightarrow{t} P \tag{1.10}$$

where the map j is induced by the map $K \to M \ltimes N$ given by $(m,n) \mapsto (m,-n)$ and $t(< m,n >) = \mu(m) + \nu(n)$.

Observe that this result is the Lie algebra analogue of Proposition 2.5 [1] (see also [10], [15]).

By (1.10) and Proposition 1.6 one has the following exact sequence of Lie algebras

$$M \otimes N \xrightarrow{\varphi} M \otimes^q N \longrightarrow M \bowtie N \longrightarrow P. \tag{1.11}$$

In the case of the 'absolute' tensor product modulo q of Lie algebras M and N acting compatibly on each other and considered as crossed $M \bowtie N$-modules, the natural homomorphism $M \bowtie N \to P = M \bowtie N$ is the identity map. Thus from (1.11) $\varphi : M \otimes N \to M \otimes^q N$ is an epimorphism and $K = [M,N]$.

Theorem 1.22. *Let M and N be Lie algebras equipped with compatible actions on each other. Then there is a short exact sequence of Lie algebras*

$$0 \longrightarrow q(Ker\lambda \cap Ker\lambda') \longrightarrow M \otimes N \xrightarrow{\varphi} M \otimes^q N \longrightarrow 0,$$

where $\lambda : M \otimes N \to M$, $\lambda' : M \otimes N \to N$ are Lie homomorphisms defined on generators by $\lambda(m \otimes n) = -{}^n m$, $\lambda'(m \otimes n) = {}^m n$ (see [6, Proposition 2]).

Proof. Any element $x \in M \otimes N$ is of the form $x = \sum_i m_i \otimes n_i$. Let $x \in Ker\lambda \cap Ker\lambda'$, then by the formulas (1.8), (1.6) one has

$$\begin{aligned}\varphi(qx) &= \sum_i q(m_i \otimes n_i) = \sum_i \{(-{}^{n_i}m_i, {}^{m_i}n_i)\} \\ &= \{(\sum_i(-{}^{n_i}m_i), \sum_i {}^{m_i}n_i)\} = \{(\lambda(x), \lambda'(x))\} = \{(0,0)\} = 0.\end{aligned}$$

This proves that $\varphi(q(Ker\lambda \cap Ker\lambda')) = 0$. Hence φ induces a natural Lie homomorphism

$$\psi : M \otimes N/q(Ker\lambda \cap Ker\lambda') \longrightarrow M \otimes^q N.$$

Since $K = [M,N]$ (see above), any element $k \in K$ is of the form $k = \sum_i(-{}^{n_i}m_i, {}^{m_i}n_i)$ for suitable $m_i \in M$, $n_i \in N$. Let us define a homomorphism $\psi' : M \otimes^q N \longrightarrow M \otimes N/q(Ker\lambda \cap Ker\lambda')$ as follows: $\psi'(m \otimes n) = cl(m \otimes n)$, $\psi'(\{k\}) = cl(\sum_i q(m_i \otimes n_i))$. It is easy to see that ψ' is correctly defined, it preserves the relations (1.1)-(1.8) and $\psi\psi'$, $\psi'\psi$ are identity maps. □

Note that by [12] if $N \to M$ (resp. $M \to N$) is a crossed M-module (resp. N-module) then $Ker\lambda'$ (resp. $Ker\lambda$) is the first non-abelian homology $H_1(M,N)$ (resp. $H_1(N,M)$) of the Lie algebra M (resp.N) with coefficients in the Lie algebra N (resp. M)

Corollary 1.23. *If M is a perfect Lie algebra (i.e. $M = [M,M]$) then one has the following short exact sequence of Lie algebras*

$$0 \to qH_2(M) \to M \otimes M \to M \otimes^q M \to 0$$

Proof. Follows from Theorem 1.22 and the fact that if M is a perfect Lie algebra then $Ker\lambda = H_2(M)$ [6, Theorem 11]. □

2. The universal q-central relative extension of Lie algebras

Let M and N be two ideals of the Lie algebra P, so that there is a canonical identification $K = M \times_P N = M \cap N$, sending (k,k) onto k. We denote by $M\#_q N$ the image of $M \otimes^q N$ and $M \wedge^q N$ in $K = M \cap N$. Whence $M\#_q N$ is the ideal of K generated by elements $[m,n]$ and qk for $m \in M$, $n \in N$ and $k \in K$.

Proposition 2.1. *Suppose that M and N are two ideals of a Lie algebra P and $M \cap N = M\#_q N$, then we have*

$$M \otimes^q N = M \wedge^q N.$$

Proof. For $k \in K = M \cap N = M\#_q N$ there exists $x \in M \otimes^q N$ such that $k = \xi(x)$, then by Lemme 1.12(iii) we have

$$k \otimes k = \xi(x) \otimes \xi(x) = [x,x] = 0. \ \square$$

Now we give the following definition from [16]

Definition 2.2. (i) *Let $\alpha : P \to Q$ be a Lie epimorphism and A be a Q-module. A relative extension of α by A is an exact sequence of Lie algebras*

$$0 \to A \to E \overset{\mu}{\to} P \overset{\alpha}{\to} Q \to 0$$

such that μ is a crossed P-module.

(ii) *a morphism between the relative extensions*

$$0 \to A \to E \overset{\mu}{\to} P \overset{\alpha}{\to} Q \to 0$$

and

$$0 \to A' \to E' \overset{\mu'}{\to} P \overset{\alpha}{\to} Q \to 0$$

is a P-equivariant Lie homomorphism $\varphi : E \to E'$ (i.e. $\varphi({}^p x) = {}^p\varphi(x)$ for $x \in E$, $p \in P$) such that $\mu'\varphi = \mu$.

(iii) *A relative extension of α by A is called a central relative extension if Q acts trivially on A.*

Following definitions are the Lie algebra analogues of Definitions 2.3-2.5 in [5].

Definition 2.3. *A relative extension of α by A is called a q-central relative extension if Q acts trivially on A and $qa = 0$ for any $a \in A$. Such q-central relative extension is called universal if there exists a unique morphism of relative extensions from it to any q-central relative extension of α.*

Note that if $Q = \{0\}$, then the q-central relative extension of α by A is a q-central extension of P by A, i.e. a central extension of Lie algebras

$$0 \to A \to E \to P \to 0,$$

such that $qa = 0$ for all $a \in A$.

Definition 2.4. *A P-Lie algebra A (P acts on A) is called P-q-perfect if A is generated by elements of the form $[a,a'] - {}^p a'$ and qa, $a, a' \in A$, $p \in P$.*

Note that if P acts trivially on A, then the P-q-perfect Lie algebra A is a q-perfect Lie algebra, i.e. A is generated by elements of the form qa and $[a',a'']$, $a, a', a'' \in A$.

Now we obtain the conditions for the existence of a universal q-central relative extension of a Lie epimorphism and describe this extension using exterior (tensor) product modulo q.

Lemma 2.5. *Let P be a Lie algebra and N be an ideal of P. Then the P-Lie algebra $N \wedge^q P$ ($N \otimes^q P$) is P-q-perfect if and only if $N = N\#_q P$.*

Proof. First suppose $N = N\#_q P$, then for any $n \in N$ there exists $x \in N \wedge^q P$ such that $n = \xi(x)$, thus, by Lemma 1.16(ii) $\{n\} = \{\xi(x)\} = qx$ and by Definition 1.11(iii) and Corollary 1.15 we have $n \wedge p = \xi(x) \wedge p = -{}^p x$. As $N \wedge^q P$ is generated by elements $n \wedge p$ and $\{n\}$ it is P-q-perfect.

Conversely, let $N \wedge^q P$ be P-q-perfect. Consider the surjective Lie homomorphism $\psi : N \wedge^q P \to N/[N,P]$ given by Proposition 1.6. As the elements $[x,x'] - {}^p x'$ and qx generate $N \wedge^q P$, then their images $\psi([x,x'] - {}^p x') = cl([\xi(x'), p - \xi(x)]) = 0$ and $\psi(qx) = cl(\xi(qx))$ generate $N/[N,P]$, so $N = N\#_q P$. □

Lemma 2.6. *A short exact sequence of Lie algebras*

$$0 \to N \to P \xrightarrow{\alpha} Q \to 0$$

gives rise to an exact sequence of Lie algebras

$$N \wedge^q P \xrightarrow{\xi} P \xrightarrow{\alpha} Q \to 0$$

if and only if $N = N\#_q P$.

Proposition 2.7. *Suppose that*

$$0 \to N \to P \xrightarrow{\alpha} Q \to 0$$

is a short exact sequence of Lie algebras and let

$$0 \to A \to E \xrightarrow{\mu} P \xrightarrow{\alpha} Q \to 0$$

be a q-central relative extension of α. If $N \neq N\#_q P$ then the Lie algebra E is not P-q-perfect and this q-central relative extension is not universal.

Proof. If the Lie algebra E is P-q-perfect then by surjectivity of the Lie homomorphism $E \xrightarrow{\mu} N$ we obtain that N is P-q-perfect i.e. $N = N\#_q P$. So E is not P-q-perfect.

Let E_P^q be the submodule of E generated by the elements $[x,x'] - {}^p x'$ and qx, $x, x' \in E$, $p \in P$. It is easy to see that E_P^q is an ideal of E, $E/E_P^q \neq 0$ is abelian and Q acts trivially on E/E_P^q. Then the exact sequence

$$0 \to E/E_P^q \xrightarrow{i} E/E_P^q \times N \xrightarrow{\pi} P \xrightarrow{\alpha} Q \to 0,$$

where $\pi(x,n) = n$, is a q-central relative extension of α.

Let us define Lie homomorphisms $f_1,\ f_2 : E \to E/E_P^q \times N$ as follows: $f_1(x) = (cl(x), \mu(x))$ and $f_2(x) = (0, \mu(x))$ for all $x \in E$. Clearly f_1 and f_2 are morphisms of relative extensions and $f_1 \neq f_2$. Hence the q-central relative extension $0 \to A \to E \xrightarrow{\mu} P \xrightarrow{\alpha} Q \to 0$ is not universal. □

Theorem 2.8. *Let*

$$0 \to N \to P \xrightarrow{\alpha} Q \to 0$$

be a short exact sequence of Lie algebras and $N = N\#_q P$. Then the exact sequence

$$0 \to V \to N \wedge^q P \xrightarrow{\xi} P \xrightarrow{\alpha} Q \to 0$$

is the universal q-central relative extension of α, where $V = Ker\xi$.

Proof. By Lemme 2.6 this sequence is exact. By Lemma 1.12(ii) and Corollary 1.15 the Lie algebra Q acts trivially on $Ker\xi$ and by Lemma 1.16(ii) one has $qx = \{\xi(x)\} = 0$ for $x \in Ker\xi$. So the sequence is a q-central relative extension of α.

Let

$$0 \to A \to E \xrightarrow{\mu} P \xrightarrow{\alpha} Q \to 0$$

be another q-central relative extension of α. Suppose $\vartheta : N \to E$ be a set-theoretic section of μ and let us define a map $k : N \wedge^q P \to E$ as follows: $k(n \wedge p) = -{}^p\vartheta(n)$ and $k(\{n\}) = q\vartheta(n)$. We must show that k commutes with relations (1.1) to (1.9).

Clearly

$$k(n \wedge \lambda p) = -{}^{\lambda p}\vartheta(n) = \lambda k(n \wedge p).$$

Note that if $x, y \in E$ and $x - y \in A$, we have ${}^p x = {}^p y$ for all $p \in P$. Then

$$k(\lambda n \wedge p) = -{}^p\vartheta(\lambda n) = -{}^p(\lambda\vartheta(n)) = \lambda k(n \wedge p);$$

$$k((n + n') \wedge p) = -{}^p(\vartheta(n + n')) = -{}^p(\vartheta(n) + \vartheta(n')) = k(n \wedge p) + k(n' \wedge p).$$

Clearly

$$k(n \wedge (p + p')) = k(n \wedge p) + k(n \wedge p');$$

Using the defining conditions of crossed module

$$\begin{aligned} k([n, n'] \wedge p) &= -{}^p\vartheta([n, n']) = -{}^p[\vartheta(n), \vartheta(n')] \\ &= -[{}^p\vartheta(n), \vartheta(n')] - [\vartheta(n), {}^p\vartheta(n')] = -{}^{({}^p\vartheta(n))}\vartheta(n') + {}^{({}^p\vartheta(n'))}\vartheta(n) \\ &= -{}^{[p,n]}\vartheta(n') + {}^{[p,n']}\vartheta(n) = -k(n' \wedge {}^n p) + k(n \wedge {}^{n'} p); \end{aligned}$$

and

$$\begin{aligned} k(n \wedge [p, p']) &= -{}^{[p,p']}\vartheta(n) = -{}^p({}^{p'}\vartheta(n)) + {}^{p'}({}^p\vartheta(n)) \\ &= -{}^p\vartheta({}^{p'}n) + {}^{p'}\vartheta({}^p n) = k({}^{p'}n \wedge p) - k({}^p n \wedge p'). \end{aligned}$$

The proof of the commutativity of k with relations (1.4) and (1.5) is similar. Next, since $qx = qy$ for all $x, y \in E$ such that $x - y \in A$, we have

$$\begin{aligned} k(\{\lambda n + \lambda' n'\}) &= q\vartheta(\lambda n + \lambda' n') = q(\lambda\vartheta(n) + \lambda'\vartheta(n')) \\ &= \lambda k(\{n\}) + \lambda' k(\{n'\}); \end{aligned}$$

$$\begin{aligned} k([\{n\}, \{n'\}]) &= [q\vartheta(n), q\vartheta(n')] = -{}^{q\vartheta(n')}(q\vartheta(n)) \\ &= -{}^{qn'}\vartheta(qn) = k(qn \wedge qn'); \end{aligned}$$

$$k(\{[n, p]\}) = q\vartheta([n, p]) = -q{}^p\vartheta(n) = k(q(n \wedge p)).$$

Finally

$$k(n \wedge n) = -{}^n\vartheta(n) = -{}^{\mu\vartheta(n)}\vartheta(n) = -[\vartheta(n), \vartheta(n)] = 0.$$

Since $\vartheta({}^p n) - {}^p\vartheta(n) \in A$ one has

$$k({}^p\{n\}) = q\vartheta({}^p n) = q{}^p\vartheta(n) = {}^p k(\{n\});$$

and

$$k({}^p(n \wedge p')) = -{}^{p'}\vartheta({}^p n) - {}^{[p,p']}\vartheta(n) = -{}^p({}^{p'}\vartheta(n)) = {}^p k(n \wedge p').$$

Thus k is P-equivariant.

Suppose $k' : N \wedge^q P \to E$ is an other homomorphism such that $\mu k = \mu k' = \xi$, then $k(y) - k'(y) \in Ker\mu = A$ and $k(qy) = k'(qy)$ for all $y \in N \wedge^q P$. On the other hand for any $x, x' \in N \wedge^q P$ and $p \in P$ we have ${}^{(\mu(x)-p)}(k(x') - k'(x')) = 0$ from which comes

$$k([x, x'] - {}^p x') = k'([x, x'] - {}^p x').$$

Thus $k = k'$ since $N \wedge^q P$ is P-q-perfect and is generated by elements $[x, x'] - {}^p x'$ and qy, for all $x, x', y \in N \wedge^q P$, $p \in P$. □

Remark 2.9. *In Theorem 2.8 $N \wedge^q P$ can be replaced by $N \otimes^q P$.*

Corollary 2.10. *If P is a q-perfect Lie algebra, then the exact sequence of Lie algebras*

$$0 \to V \to P \wedge^q P \xrightarrow{\xi} P \to 0$$

is the universal q-central extension, where $V = Ker\xi$.

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