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Topological Methods in Algebra

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1 Introduction

This article discusses the foundations for recent developments introducing a general concept of deformation and homotopy theory into pure algebra. Homotopy theory has a long tradition in analysis and geometry and is grounded in the concepts of topological space, continuous map, path, and deformation of continuous maps. Wherever these notions occur in a natural and meaningful way, one can endevour to apply homotopical concepts and methodology to formulate and solve problems.

The algebraic counterpart of a topological space is a global action. It has a very natural and intuitive notion of path which is easy to formalize and it has a good concept of deformation for morphisms, based on the notion of path. In terms of these concepts, one can develop in a purely algebraic setting the entire spectrum of homotopy theory. This article describes the notion of a global action, several kinds of morphisms between global actions, the notion of path in a global action and the notion of deformation of a morphism.

The body of the paper is organized as follows. We begin by defining a global action and giving a few examples including that of the line L. Next various concepts of morphism between global actions are introduced. A natural, intuitive notion of path in a global action A is described and then formalized as a certain kind of morphism from L to A. The next goal is defining a global action structure on the set of all morphisms from one global action to another and establishing the exponential law. The global action on morphism spaces plays the role of the compact open topology on function spaces in topology. Finally we define when two morphisms are homotopic.

2 Global actions

A group action consists by definition of a group G, a set X and a map $G \times X \to X, (\sigma, \mathrm{x}) \mapsto \sigma\mathrm{x}$, of the Cartesian product $G \times X$ to X such that for all elements $\sigma, \rho \in G$ and $\mathrm{x} \in X, (\sigma\rho)\mathrm{x} = \sigma(\rho\mathrm{x})$ and $1\mathrm{x} = \mathrm{x}$. If G acts on X then we write $G \curvearrowright X$. A morphism $G \curvearrowright X \to H \curvearrowright Y$ of group actions conists of a group homomorphism $\mathrm{f} : G \to H$ and a function $f\,' : X \to Y$ such that for all $\sigma \in G$ and $\mathrm{x} \in X, \mathrm{f}\,'(\sigma\mathrm{x}) = \mathrm{f}(\sigma)\mathrm{f}\,'(\mathrm{x})$.

A global action is formed by fitting together a set of group actions, according to a few simple principles. Each of the group actions making up the global action is then called a local action.

Definition 2.1 A **global action** A consists of a set X_A together with a set $\{(G_A)_\alpha \curvearrowright X_A \mid \alpha \in \Phi_A\}$ of group actions $(G_A)_\alpha \curvearrowright (X_A)_\alpha$ such that each $(X_A)_\alpha \subseteqq X_A$. The set of group actions is structured by equipping the index set Φ_A with a reflexive relation $\leqq$ and imposing the condition that if $\alpha \leqq \beta$ then $(G_A)_\alpha$ leaves $(X_A)_\alpha \cap (X_A)_\beta$ invariant and there is a group homomorphism $(G_A)_{\alpha \leqq \beta} : (G_A)_\alpha \to (G_A)_\beta$ such that if $\sigma \in (G_A)_\alpha$ and $\mathrm{x} \in (X_A)_\alpha \cap (X_A)_\beta$ then $\sigma\mathrm{x} = (G_A)_{\alpha \leqq \beta}(\sigma)\mathrm{x}$.

The index set Φ_A of a global action A is called the **coordinate system** of A and each element of Φ_A is called a **coordinate**. It is possible that for distinct coordinates α and $\beta, (X_A)_\alpha = (X_A)_\beta$, but $(G_A)_\alpha \neq (G_A)_\beta$. This allows one to have distinct groups acting on the same subset of X_A. The function $G_A : \Phi_A \to$ ((groups)), $\alpha \mapsto (G_A)_\alpha$, is called the **global group** of A and each group $(G_A)_\alpha$ is called a **local group** of A. Thus a global group is a group valued function on a coordinate system. Each set $(X_A)_\alpha$ is called a **local set** of A and each group action $(G_A)_\alpha \curvearrowright (X_A)_\alpha$ is called a **local action** of A. The set X_A is called the **enveloping set** of A. The function $\Phi_A \to$ subsets $(X_A), \alpha \mapsto (X_A)_\alpha$, is denoted also by $X_A : \Phi_A \to$ subsets (X_A). When confusion might arise, we write $|X_A|$ for the enveloping set X_A. The notation $|A|$ will also be used for the enveloping set X_A.

A global action A is called **covariant** if the relation $\leqq$ on Φ_A is transitive and if the global group $G_A : \Phi_A \to$ ((groups)) is a covariant functor. In this case, the global group is called **covariant** also. A global action A is called **contravariant** if the relation $\leqq$ on Φ_A is transitive and if $\alpha \leqq \beta \Rightarrow (X_A)_\alpha \supseteqq (X_A)_\beta$. This condition is equivalent to requiring that the function $X_A : \Phi_A \to$ subsets $(X_A), \alpha \mapsto (X_A)_\alpha$, is a contravariant functor. A global action is called **bivariant** of **functorial** if it is both covariant and contravariant.

An important example of a functorial action, which arises in several contexts, is the following.

EXAMPLE 2.2 Let $G \curvearrowright X$ be a group action. Let Φ be a set which indexes a set $\{G_\alpha | \alpha \in \Phi\}$ of subgroups G_α of G. Assume that $G_\alpha = G_\beta \Leftrightarrow \alpha = \beta$. Partially order the set$\{G_\alpha | \alpha \in \Phi\}$ by inclusion and give Φ the induced partial ordering. Clearly the rule $\alpha \mapsto G_\alpha$ defines a functor $G : \Phi \to$ ((groups)). Set $|X| = X$ and define the function $X : \Phi \to$ subsets $|X|$, $\alpha \mapsto X_\alpha$, by $X_\alpha = |X|$ for all $\alpha \in \Phi$. Then one obtains a functorial global action (Φ, G, X).

If U is a set, let

$$\begin{aligned} \mathrm{Perm}(U) =& \text{Group of all bijections of U onto itself,} \\ f\mathrm{Perm}(U) =& \{\sigma \in \mathrm{Perm}(U) | \sigma \text{ fixes all but a finite number of elements of } U\}. \end{aligned}$$

If U is a well ordered nonempty finite set, let

$$\begin{aligned} \mathrm{cPerm(U)} = & \text{ cyclic subgroups of } \mathrm{Perm}(U) \text{ generated by the cyclic} \\ & \text{ permutation which sends each element of } U, \\ & \text{ except for the last, to its successor and} \\ & \text{ sends the last element to the first.} \end{aligned}$$

The next example is important for the homotopy theory of global actions.

EXAMPLE 2.3 This example is called the **line action** and is denoted by L. Let $\Phi_L = \mathbb{Z} \cup \{*\}$. Give Φ_L the partial ordering such that there is no relation between elements of $\mathbb{Z}$ and such that $* \leqq n$ for all $n \in \mathbb{Z}$. Let $|X_L| = \mathbb{Z}$ and define the function $X_L : \Phi \to$ subsets $|X_L|, \alpha = n \mapsto \{n, n+1\}$ and $\alpha = * \mapsto |X_L|$. Define $G_L : \Phi_L \mapsto$ ((groups)),$\alpha = n \mapsto (G_L)_\alpha = \mathrm{Perm}(\{n, n+1\})$ and $\alpha = * \mapsto (G_L)_\alpha = \{1\}$. Then the triple $L = (\Phi_L, G_L, X_L)$ is a functorial action.

The next example extends in several ways the one above, to arbitrary abstract simplicial complexes.

EXAMPLE 2.4 Let S denote an abstract simplicial complex and let $|X_S|$ denote the set of vertices of S. If α is a subcomplex of S, let $(X_S)_\alpha$ denote the set of its vertices. Call a subcomplex α simple, if $(X_S)_\alpha$ has a partition into subsets U such that any finite subset of U is a simplex in α and such that any simplex of α is a subset of some U. Clearly if α

is simple then the partition above of $(X_S)_\alpha$ is unique; let $\mathrm{Part}(X_S)_\alpha$ denote this partition. Let Φ_S denote the set of all simple subcomplexes of S. Partially order Φ_S by defining $\alpha \leqq \beta \Leftrightarrow (X_S)_\alpha \supseteqq (X_S)_\beta$ and every member of $\mathrm{Part}(X_S)_\beta$ is a union of members of the $\mathrm{Part}(X_S)_\alpha$. Clearly the subcomplex whose vertices are $|X_S|$ and whose simplices are the singleton subsets of $|X_S|$ is the smallest element of Φ_S. For $\alpha \in \Phi_S$, define

$$
\begin{aligned}
(G_S)_\alpha &= \prod_{U \in \mathrm{Part}(X_S)_\alpha} \mathrm{Perm}(U) \\
(fG_S)_\alpha &= \prod_{U \in \mathrm{Part}(X_S)_\alpha} f\mathrm{Perm}(U).
\end{aligned}
$$

There is a canonical action of $(G_S)_\alpha$ (resp. $(fG_S)_\alpha$) on $(X_S)_\alpha$ defined by the action of each permutation group Perm(U) (resp. fPerm (U)) on U. Define

$$
\begin{aligned}
gl(S) &= (\Phi_S, G_S, X_S) \\
fgl(S) &= (\Phi_S, fG_S, X_S).
\end{aligned}
$$

Then $gl(S)$ and $fgl(S)$ are global actions called **simplicial actions**. They are not in general functorial.

Well order now the vertices $|X_S|$ of S and let $c\Phi_S$ denote the subset of Φ_S of all simple subcomplexes α such that $\mathrm{Part}(X_S)_\alpha$ contains only finite sets. The smallest element of Φ_S, say $*$, clearly lies in $c\Phi_S$. Give $c\Phi_S$ a new partial ordering such that $\alpha \leqq \beta \Leftrightarrow \alpha = *$. Thus if $\alpha \neq * \neq \beta$ then either $\alpha = \beta$ or there is no relation between α and β. For $\alpha \in c\Phi_S$, define $(cX_S)_\alpha = (X_S)_\alpha$ and

$$
(cG_S)_\alpha = \prod_{U \in Part(X_S)_\alpha} cPerm(U).
$$

Define

$$
cgl(S) = (c\Phi_S, cG_S, cX_S).
$$

Then $cgl(S)$ is a global action called a **cyclic simplicial action**. It is not in general functorial.

There are three kinds of morphisms for global actions. There is first of all a general concept. Then there is a structural concept which gives rise to so-called regular morphisms. Global actions which are regularly isomorphic are essentially the same. Finally there is a concept lying between the two above, which leads to the definition of normal morphism. The set $Mor(A, B)$ of all morphisms from a global action A to a global action B will be given the structure of a global action such that the assignment $(A, B) \mapsto Mor(A, B)$ is functorial in A over all morphisms and functorial in B over precisely normal morphisms.

We define first the general concept of morphism. This concept is geometric in flavor and depends on the notion of local frame. A local frame at a point $\mathrm{p} \in |X_A|$ signifies a range of possible moves available at p. It is defined as follows.

DEFINITION 2.5 Let A be a global action.Let $\mathrm{x} \in (X_A)_\alpha$. A **local frame** at x in α or simply an α-**frame** at x is a sequence $\mathrm{x} = \mathrm{x}_o, \cdots, \mathrm{x}_p$ of points in $(X_A)_\alpha$, with the property that for each i $(1 \leqq i \leqq p)$ there is a $g_i \in (G_A)_\alpha$ such that $g_i\mathrm{x} = \mathrm{x}_i$. (Clearly $\mathrm{x}_0, \mathrm{x}_1, \cdots, \mathrm{x}_p$ is an α-frame at $\mathrm{x}_0 \Leftrightarrow \mathrm{x}_0, \cdots, \mathrm{x}_i, \cdots, \mathrm{x}_p$ is an α-frame at x_i.) A **morphism** f $: A \to B$ of global actions is a function f $: |A| \to |B|$ which preserves local frames. Specifically if $\mathrm{x}_0, \cdots, \mathrm{x}_p$ is an α-frame at x_0 then $f(\mathrm{x}_0), \cdots, f(\mathrm{x}_p)$ is an β-frame at $f(\mathrm{x}_0)$ for some $\beta \in \Phi_B$.

A path in a global action A is intuitively a sequence $\mathrm{x}_0, \cdots, \mathrm{x}_p$ of points in $|A|$ such that there are coordinates $\alpha_0, \cdots, \alpha_p \in \Phi_A$ and group elements $\sigma_i \in (G_A)_{\mathrm{x}_i} (i = 0, \cdots, p-1)$ such that $\sigma_i\mathrm{x}_i = \mathrm{x}_{i+1}$. The figure below illustrates a path when $p = 3$.

$|A|$

$(X_A)_{\alpha_0}$ $(X_A)_{\alpha_1}$ $(X_A)_{\alpha_2}$

σ_0 σ_1 σ_2

x_0 x_1 x_2 x_3

The notion is formalized as follows.

DEFINITION 2.6 A **path** in a global action A is a morphism $\omega : L \to A$, where L is the line action in (2.3), such that there are integers $n \leq N \in \mathbb{Z} = |L|$ with the property that $\omega(l) = \omega(n)$ for all $l \leq n$ and $\omega(l) = \omega(N)$ for all $N \leqq l$. Thus a path is stably constant on the left and on the right.

DEFINITION 2.7 A **regular** morphism $\eta : A \to B$ of global actions is a triple $(\eta_\Phi, \eta_G, \eta_X)$ satisfying the following conditions.

(2.7.1) $\eta_\Phi : \Phi_A \to \Phi_B$ is a relation preserving function.

(2.7.2) $\eta_G : G_A \to (G_B)_{\eta_\Phi(\ )}$ is a natural transformation of group valued functions on Φ_A where $(G_B)_{\eta_\Phi(\ )}$ denotes the composition of η_Φ with G_B. This means by definition that if $\alpha \leqq \alpha' \in \Phi_A$ then the diagram

$$\begin{tikzcd}
(G_A)_\alpha \arrow[r, "(\eta_G)(\alpha)"] \arrow[d] & (G_B)_{\eta_\Phi(\alpha)} \arrow[d] \\
(G_A)_{\alpha'} \arrow[r, "\eta_G(\alpha')"'] & (G_B)_{\eta_\Phi(\alpha')}
\end{tikzcd}$$

commutes

(2.7.3) $\eta_X : |A| \to |B|$ is a function such that $\eta_X((X_A)_\alpha) \subsetneqq (X_B)_{\eta_\Phi(\alpha)}$ for all $\alpha \in \Phi_A$.

(2.7.4) For each $\alpha \in \Phi_A$, the pair $(\eta_G, \eta_X) : (G_A)_\alpha \curvearrowright (X_A)_\alpha \to (G_B)_{\eta_\Phi(\alpha)} \curvearrowright (X_B)_{\eta_\Phi(\alpha)}$ is a morphism of group actions. Clearly a regular morphism is one in the general sense of (2.7).

A **regular isomorphism** $\eta : A \to B$ is a regular morphism such that there is a regular morphism $\eta' : B \to A$ called the **regular inverse** of η with the property that η'_Φ is inverse to η_Φ, η'_X is inverse to η_X, and for each $\alpha \in \Phi_A$, $\eta'_G(\eta_\Phi(\alpha))$ is inverse to $\eta_G(\alpha)$.

It is of course not true in general that a regular morphism which is an isomorphism in the general sense is a regular isomorphism.

Example 2.8 Let $G \curvearrowright X$ and $G' \curvearrowright X'$ be group actions. Let $\{G_\alpha \mid \alpha \in \Phi\}$ be a set of subgroups of G such that $G_\alpha = G_\beta \Leftrightarrow \alpha = \beta$. Let $\{G'_{\alpha'} \mid \alpha' \in \Phi'\}$ be a set of subgroups of G' sucht that $G'_{\alpha'} = G'_{\beta'} \Leftrightarrow \alpha' = \beta'$. Let (Φ, G, X) and $(\Phi', G'.X')$ denote the global actions constructed in (2.2). Let $\mathrm{f} : G \to G'$ be a group homomorphism and $g : X \to X'$ be a function such that (f, g) defines a morphism $(f, g) : G \curvearrowright X \to G' \curvearrowright X'$ of ordinary group actions. Let $\varphi :$ subgroups $(G) \to$ subgroups (G') be a function such that if $K \subseteq L \subseteq G$ are subgroups of G then $K \subseteq \varphi(K) \subseteq \varphi(L)$. Suppose that $\{\varphi(\mathrm{f}(G_\alpha)) \mid \alpha \in \Phi\} = \{G'_{\alpha'} \mid \alpha' \in \Phi'\}$. Then (f, g) defines in an obvious way a regular morphism $(\Phi, G, X) \to (\Phi', G', X')$.

Example 2.9 Let $\mathrm{f} : S \to T$ be a morphism of abstract simplicial complexes. Then f defines morphisms $gl(\mathrm{f}) : gl(S) \to gl(T), \mathrm{f}gl(\mathrm{f}) : \mathrm{f}gl(S) \to \mathrm{f}gl(T)$, and $clg(\mathrm{f}) : cgl(S) \to cgl(T)$ of global actions which do not in general have a regular structure. Moreover each of the assignments $S \mapsto gl(S), S \mapsto \mathrm{f}gl(S)$, and $S \mapsto cgl(S)$' defines a functor ((abstract simplicial complexes)) $\to$ ((global actions)).

Next we define the concepts of chart and frame and then use them to define the notion of normal morphism.

Definition 2.10 Let A and B be global actions. Let $\mathrm{f} : A \to B$ be a morphism of global actions. A **framing** of f is a function $\beta : |A| \to \Phi_B$ such that the following conditions hold.

(2.10.1) $\mathrm{f}(\mathrm{x}) \in (X_B)_{\beta(\mathrm{x})}$ for all $\mathrm{x} \in |A|$.

(2.10.2) If $\mathrm{x}, \mathrm{x}_1, \cdots, \mathrm{x}_p$ is a local frame in A then $\mathrm{f}(\mathrm{x}), \mathrm{f}(\mathrm{x}_1), \cdots, \mathrm{f}(\mathrm{x}_p)$ is a local frame in $\mathrm{b} \in \Phi_B$ for some $b \geqq \beta(\mathrm{x}), \beta(\mathrm{x}_p), \cdots, \beta(\mathrm{x}_p)$.

An A-**chart** in B is a pair (f, β) consisting of a morphism $\mathrm{f} : A \to B$ of global actions and a framing $\beta : |A| \to \Phi_B$ of f.

DEFINITION-LEMMA 2.11 Let (f, β) be an A-chart in B.

If

$$\sigma = (\sigma_{\mathrm{x}}) \in \prod_{\mathrm{x} \in |A|} (G_B)_{\beta(\mathrm{x})}$$

define

$$\begin{aligned} \sigma \mathrm{f} : |A| &\to |B|. \\ \mathrm{x} &\mapsto \sigma_{\mathrm{x}} \mathrm{f}(\mathrm{x}) \end{aligned}$$

Then $\sigma\mathrm{f}$ is a morphism $A \to B$ of global actions and $(\sigma\mathrm{f}, \beta)$ is an A-chart in B.

PROOF Since $\sigma_{\mathrm{x}} \in (G_B)_{\beta(\mathrm{x})}$, it follows that $\sigma\mathrm{f}(\mathrm{x}) \in (X_B)_{\beta(\mathrm{x})}$. Thus the pair $(\sigma\mathrm{f}, \beta)$ satisfies (2.10.1). To show that $\sigma\mathrm{f}$ is a morphism of global actions and that $(\sigma\mathrm{f}, \beta)$ is an A-frame in B, it suffices to show that (2.10.2) is satisfied. Let $\mathrm{x}_o, \cdots, \mathrm{x}_p$ be a local frame at $\mathrm{x}_o \in |A|$. By definition $\mathrm{f}(\mathrm{x}_o), \cdots, \mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\mathrm{f}(\mathrm{x}_o)$ for some $b \geq \beta(\mathrm{x}_o), \cdots, \beta(\mathrm{x}_p)$. Let $\rho_{\mathrm{x}_o}, \cdots, \rho_{\mathrm{x}_p}$ denote respectively the images of $\sigma_{\mathrm{x}_o}, \cdots, \sigma_{\mathrm{x}_p}$ in $(G_B)_{\mathrm{b}}$ under the canonical homomorphisms $(G_B)_{\beta(\mathrm{x}_i)} \to (G_B)_{\mathrm{b}}$ $(0 \leqq \mathrm{i} \leqq \mathrm{p})$. Clearly $\rho_{\mathrm{x}_o}\mathrm{f}(\mathrm{x}_o), \cdots, \rho_{\mathrm{x}_p}\mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\rho_{\mathrm{x}_o}\mathrm{f}(\mathrm{x}_o)$. But $\rho_{\mathrm{x}_i}\mathrm{f}(\mathrm{x}_i) = \sigma_{\mathrm{x}_i}\mathrm{f}(\mathrm{x}_i)$. Thus $\sigma\mathrm{f}(\mathrm{x}_o), \cdots, \sigma\mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\sigma\mathrm{f}(\mathrm{x}_o)$ and $b \geqq \beta(\mathrm{x}_o), \cdots, \beta(\mathrm{x}_p)$. $\square$

DEFINITION 2.12 Let (f, β) be an A-chart in B. An A-**frame at** f **on** (f, β) is a set $\mathrm{f} = \mathrm{f}_o, \mathrm{f}_1, \cdots, \mathrm{f}_p : A \to B$ of morphisms for which there are elements $\sigma_1, \cdots, \sigma_p \in \prod_{\mathrm{x} \in |A|} (G_B)_{\beta(\mathrm{x})}$ such that $\sigma_i \mathrm{f} = \mathrm{f}_i$ $(1 \leqq i \leqq \mathrm{p})$. (In view of Lemma (2.11), $\mathrm{f} = \mathrm{f}_o, f_1, \cdots, \mathrm{f}_\mathrm{p}$ is also an A-frame at f_i on (f_i, β) for any i $(0 \leqq \mathrm{i} \leqq \mathrm{p})$.)

The next lemma is very useful.

LOCAL-GLOBAL LEMMA 2.13 Let (f, β) be an A-chart in B. Then $\mathrm{f} = \mathrm{f}_o, \mathrm{f}_1, \cdots, \mathrm{f}_p$ is an A-frame at f on $(\mathrm{f}, \beta) \Leftrightarrow$ for each $\mathrm{x} \in |A|, \mathrm{f}(\mathrm{x}), \mathrm{f}_1(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})$ is a local frame at $\mathrm{f}(\mathrm{x})$ in $\beta(\mathrm{x})$.

PROOF The assertions are trivial consequences of Lemma (2.11).

DEFINITION 2.14 Let A, B and C be global actions. An A-**normal** morphism $\mathrm{g} : B \to C$ of global actions is one which preserves A-frames, i.e. if $\mathrm{f}, \mathrm{f}_1, \cdots, \mathrm{f}_{\mathrm{p}}$ is an A-frame at f on (f, β) then there is a framing $\gamma : |A| \to \Phi_C$ of gf such that $\mathrm{gf}, \mathrm{gf}_1, \cdots, \mathrm{gf}_{\mathrm{p}}$ is an A-frame on the A-chart (gf, γ). A **normal** morphism $\mathrm{g} : B \to C$ is one which preserves A-frames for any global action A. An A-**normal** (resp. **normal**) **isomorphism** is an A-normal (resp. normal) morphism which has an A-normal (resp. normal) inverse.

It is not true in general that an A-normal (resp. normal) morphism which is an isomorphism in the usual sense is an A-normal (resp. normal) isomorphism.

LEMMA 2.15 A regular morphism is normal.

PROOF Let $\eta : B \to C$ be a regular morphism. If (f, β) is an A-chart in B then it follows straightforward that $(\eta_X \mathrm{f}, \eta_\Phi \beta)$ is an A-chart in C. Let $\mathrm{f}, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be an A-frame at f on (f, β) and let $\sigma_1, \cdots, \sigma_p \in \prod_{\mathrm{x} \in |A|} (G_B)_{\beta(\mathrm{x})}$ such that $\sigma_{\mathrm{i}} \mathrm{f} = \mathrm{f}_{\mathrm{i}}$ $(1 \leqq \mathrm{i} \leqq \mathrm{p})$. If $\sigma = (\sigma_{\mathrm{x}}) \in \prod_{\mathrm{x} \in |A|} (G_B)_{\beta(\mathrm{x})}$, define $\eta_G(\sigma) = (\eta_G(\beta(\mathrm{x}))(\sigma_{\mathrm{x}})) \in \prod_{\mathrm{x} \in |A|} (G_C)_{\eta_\Phi(\beta(\mathrm{x}))}$. Then $\eta_G(\sigma_i)(\eta_X \mathrm{f}) = \eta_X \mathrm{f}_{\mathrm{i}}$ $(1 \leqq i \leqq \mathrm{p})$, by (2.7.4). Thus $\eta_X \mathrm{f}, \eta_X \mathrm{f}_1, \cdots, \eta_X \mathrm{f}_p$ is an A-frame at $\eta_X \mathrm{f}$ on $(\eta_X \mathrm{f}, \eta_\Phi \beta)$. $\square$

Next we use the concept of framing to give the set of all morphisms from a global action A to a global action B the structure of a global action.

DEFINITION 2.16 Let A and B be global actions. Let $|Mor(A, B)|$ denote the set of all morphisms from A to B. Define a global action

$$Mor(A, B) = (\Phi_{(A,B)}, G_{(A,B)}, X_{(A,B)})$$

as follows. Its enveloping set is $|Mor(A, B)|$. Define

$$\Phi_{(A,B)} = \{\beta : |A| \to \Phi_B\}.$$

Give $\Phi_{(A,B)}$ the reflexive relation defined by $\beta \leqq \beta' \Leftrightarrow \beta(\mathrm{x}) \leqq \beta'(\mathrm{x})\ \forall \mathrm{x} \in |A|$. For $\beta \in \Phi_{(A,B)}$, define

$$(G_{(A,B)})_\beta = \prod_{\mathrm{x} \in |A|} (G_B)_{\beta(\mathrm{x})}.$$

If $\beta \leqq \beta'$, there is for each $\mathrm{x} \in |A|$ a canonically defined homomorphism $(G_B)_{\beta(\mathrm{x})} \to (G_B)_{\beta'(\mathrm{x})}$ and therefore a homomorphism $(G_{(A,B)})_\beta \longrightarrow (G_{(A,B)})_{\beta'}$. For $\beta \in \Phi_{(A,B)}$, define

$$(X_{(A,B)})_\beta = \{\mathrm{f} : A \to B \mid \mathrm{f} \in |Mor(A,B)|, \beta \text{ is a framing of f}\}.$$

By (2.11), if $\sigma \in (G_{(A,B)})_\beta$ and $\mathrm{f} \in (X_{(A,B)})_\beta$ then $\sigma\mathrm{f} \in (X_{(A,B)})_\beta$ and so there is an action of $(G_{(A,B)})_\beta$ on $(X_{(A,B)})_\beta$. All the conditions for a global action are obviously satisfied. Moreover the global action $Mor(A,B)$ is covariant, contravariant, or functorial whenever the same holds for B.

Proposition 2.17 As a functor in two variables with values in global actions, $Mor(,)$ is contravariant and regular over all morphisms in the first variable and covariant over all normal morphisms in the second variable. More precisely the following holds.

(2.17.1) Let C be a global action and let $\mathrm{f} : A \to B$ be a morphism of global actions. Then f defines a regular morphism

$$\eta = Mor(\mathrm{f}, 1_C) : Mor(B,C) \to Mor(A,C)$$

as follows. Define the relation preserving morphism

$$\begin{aligned} \eta_\Phi : \Phi_{(B,C)} &\to \Phi_{(A,C)}. \\ \beta &\mapsto \beta\mathrm{f} \end{aligned}$$

Define the natural transformation

$$\eta_G : G_{(B,C)} \to G_{(A,C)}$$

by

$$\begin{array}{ccc} \eta_G(\beta) : (G_{(B,C)})_\beta & \longrightarrow & (G_{(A,C)})_{\eta_\Phi(\beta)} \\ \| & & \| \\ \prod_{y \in |B|} (G_C)_{\beta(y)} & & \prod_{\mathrm{x} \in |A|} (G_C)_{\beta\mathrm{f}(\mathrm{x})} \end{array}$$

where

$\eta_G(\beta)|_{(G_C)_{\beta(y)}}$ is the diagonal homomorphism

$$(G_C)_{\beta(y)} \to \prod_{\mathrm{x}\in|A|,\ f(\mathrm{x})=y} (G_C)_{\beta\mathrm{f}(\mathrm{x})},$$

under the convention that the empty product of groups, which can occur on the right hand side of the arrow above, is the trivial group. Define

$$\begin{aligned} \eta_X : |Mor(B,C)| &\to |Mor(A,C)|. \\ \mathrm{g} &\mapsto \mathrm{gf} \end{aligned}$$

Then $\eta = (\eta_\Phi, \eta_G, \eta_X)$ is a regular morphism of global actions.

(2.17.2) Let A be a global action and let $\mathrm{g} : B \to C$ be a morphism of global actions. Then the function

$$Mor(1_A, \mathrm{g}) : |Mor(A,B)| \to |Mor(A,C)|$$

is a morphism $Mor(A,B) \to Mor(A,C)$ of global actions $\Leftrightarrow$ g is A-normal.

PROOF (2.17.1) Straightforward and routine. Details are left to the reader.

(2.17.2) Let (f,β) be an A-chart in B and let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_\mathrm{p}$ be an A-frame on (f,β). By definition of the term local frame, $\mathrm{f}_0, \cdots, \mathrm{f}_\mathrm{p}$ is also a local β-frame in the global action $Mor(A,B)$ and conversely, any local frame in $Mor(A,B)$ is an A-frame on some A-chart in B. Thus the function $Mor(1_A, \mathrm{g}) : |Mor(A,B)| \to |Mor(A,C)|$ is a morphism of global actions $\Leftrightarrow$ it preserves A-frames $\Leftrightarrow$ g is A-normal.□

The next lemma is needed to show that the exponential map on global actions is regular.

Lemma 2.18 If $\mathrm{g} : B \to C$ is a regular morphism then for any global action A, the morphism $Mor(1_A, \mathrm{g}) : Mor(A,B) \to Mor(A,C)$ is regular.

PROOF By (2.15) and (2.17.2), the morphism $Mor(1_A, \mathrm{g}) : Mor(A,B) \to Mor(A,C)$ exists. Let $(\eta_\Phi, \eta_G, \eta_X = \mathrm{g})$ be the regular structure of g. We define a regular structure $(\mu_\Phi, \mu_g, \mu_X = Mor(1_A, \mathrm{g}))$ for $Mor(1_A, \mathrm{g})$ as follows.

Define the coordinate morphism

$$\begin{aligned} \mu_\Phi : \Phi_{(A,B)} &\to \Phi_{(A,C)}. \\ \beta &\mapsto \eta_\Phi \beta \end{aligned}$$

Define the natural transformation

$$\mu_G : G_{(A,B)} \to G_{(A,C)}$$

by the commutative diagram

$$\begin{tikzcd}
(G_{(A,B)})_\beta \arrow[r, "\mu_G(\beta)"] \arrow[d, equal] & (G_{(A,C)})_{\mu_\Phi(\beta)} \arrow[d, equal] \\
\prod\limits_{\mathrm{x}\in|A|} (G_{(A,B)})_{\beta(\mathrm{x})} \arrow[r, "\prod\limits_{\mathrm{x}\in|A|} \eta_G(\beta(\mathrm{x}))"'] & \prod\limits_{\mathrm{x}\in|A|} (G_{(A,C)})_{\eta_\Phi(\beta(\mathrm{x}))}
\end{tikzcd}$$

One checks straightforward that $(\mu_\Phi, \mu_G, Mor(1_A, \mathrm{g}))$ is a regular morphism.$\square$

For the results below on the exponential law, the notion of product is needed. We construct this next.

Definition-Lemma 2.19 Let A and B be global actions. Their **product** $A \times B$ is constructed as follows.

$$\Phi_{A\times B} = \Phi_A \times \Phi_B$$

and $(\alpha, \beta) \leqq (\alpha', \beta') \Leftrightarrow \alpha \leqq \alpha'and\beta \leqq \beta'$.

$$\begin{aligned} G_{A\times B} &= G_A \times G_B \\ |A \times B| &= |A| \times |B| \\ X_{A\times B} &= X_A \times X_B. \end{aligned}$$

For any coordinate $(\alpha, \beta) \in \Phi_{A\times B}$, there is an obvious action of $(G_{A\times B})_{(\alpha,\beta)}$ on $(X_{A\times B})_{(\alpha,\beta)}$, namely the one defined coordinatewise. One checks easily that $A\times B$ satisfies the universal property of a product.

The following notation will be used below. If S and T are sets, let

$$(S, T) = Mor_{((sets))}(S, T).$$

If U is also a set then there is a canonical isomorphism

$$E : (U,(S,T)) \xrightarrow{\cong} (U \times S, T) \qquad (2.20)$$
$$f \longmapsto Ef$$

of sets such that $Ef(u,s) = f(u)(s)$. Its inverse is obviously the function

$$E' : (U \times S, T) \longrightarrow (U,(S,T))$$
$$f \longmapsto E'f$$

where $(E'f(u))(s) = f(u,s)$.

Definition 2.21 Let A, B and C be global actions. Define a regular morphism

$$E : Mor(A, Mor(B,C)) \to Mor(A \times B, C)$$

as follows. Denote the structural components of the global action $Mor(A, Mor(B,C))$ by $(\Phi_{(A,(B,C))}, G_{(A,(B,C))}, X_{(A,(B,C))})$. Define

$$\begin{array}{ccc} E_\Phi : \Phi_{(A,(B,C))} & \longrightarrow & \Phi_{(A\times B,C)} \\ \| & & \| \\ (|A|,(|B|,\Phi_C)) & & (|A|\times|B|,\Phi_C) \end{array}$$

to be the set theoretic exponential isomorphism (2.20). Clearly E_Φ preserves the reflexive relation. Define the natural transformation

$$E_G : G_{(A,(B,C))} \to (G_{(A\times B,C)})_{E_\Phi(\ )}$$

such that

$$\begin{array}{ccc} E_G(\alpha) : (G_{(A,(B,C))})_\alpha & \longrightarrow & (G_{(A\times B,C)})_{E_\Phi(\alpha)} \\ \| & & \| \\ \prod_{x\in|A|}(\prod_{y\in|B|}(G_C)_{\alpha(x)(y)}) & & \prod_{(x,y)\in|A|\times|B|}(G_C)_{(E_\Phi\alpha)(x,y)} \end{array}$$

maps the factor $(G_C)_{\alpha(\mathrm{x})(y)}$ via the identity map onto the factor $(G_C)_{(E_\Phi\alpha)(\mathrm{x},y)} = (G_C)_{\alpha(\mathrm{x})(y)}$. One verifies easily that the composite mapping $|Mor(A, Mor(B,C))| \to (|A|, (|B|, |C|))$ @ > (2.20) >> $(|A| \times |B|, |C|)$ takes its image in $|Mor(A \times B, C)|$ and we define

$$E_X : |Mor(A, Mor(B,C))| \to |Mor(A \times B, C)|$$

to be the resulting mapping. One checks straightforward that

$$E = (E_\Phi, E_G, E_X)$$

is a regular morphism. (It fails in general to be an isomorphism (resp. regular isomorphism) because E_X is not necessarily surjective (resp. $E_X((X_{(A,(B,C))})_\alpha)$ is not necessarily all of $(X_{(A\times B,C)})_{E_\Phi(\alpha)}$).

Let $A_n, \cdots, A_1$ be an arbitrary sequence of global actions. Iterating the procedure above, one defines for any $n \geq 2$ a regular morphism

$$E_n : Mor(A_n, Mor(A_{n-1}, \cdots, Mor(A_1, C)) \cdots) \to Mor(A_n \times \cdots \times A_1, C)$$

as follows. For $n = 2$, the morphism is defined above. Suppose $n > 2$ and that the morphism has been defined for every natural number N where $2 \leqq N \leqq n-1$. Let E_{n-1} denote the morphism for the sequence $A_{n-1}, \cdots, A_1$. Define E_n for the sequence $A_n, A_{n-1}, \cdots, A_1$ as the composite of the regular morphism $Mor(1_{A_n}, E_{n-1})$ (see (2.18)) and the regular morphism $E_2 : Mor(A_n, Mor(A_{n-1} \times \cdots \times A_1, B)) \to Mor(A_n \times \cdots \times A_1, B)$.

Many global actions arising in nature satisfy the following condition.

Definition 2.22 Let A be a global action. If $\Delta \subseteqq \Phi_A$, let $\Phi_A{}^{\geqq\Delta} = \{\alpha \in \Phi_A | \alpha \geqq \beta \ \forall \ \beta \in \Delta\}$. A is called an **strong infimum action** if for any finite subset $\Delta \subseteqq \Phi_A$ and any finite nonempty set $U \subseteqq |A|$ such that $(X_A)_\beta \cap U \neq \varnothing$ for each $\beta \in \Delta$, the set $\{\alpha \in \Phi_A{}^{\geqq\Delta} | U \text{ an } \alpha-\text{frame}\}$ is either empty or contains an initial element. A is called an **infimum action** if it satisfies the condition above at least for $\Delta = \varnothing$ (empty set).

Theorem 2.23 The exponential map

$$E_n : Mor(A_n, Mor(A_{n-1}, \cdots, Mor(A_1, C)) \cdots) \to Mor(A_n \times \cdots \times A_1, C)$$

in (2.21) has a normal (resp. regular) inverse if C satisfies the infimum (resp. strong infimum) condition.

PROOF See [1] Theorem 3.23.

We close by defining when two morphisms are homotopic.

DEFINITION 2.24 Let f, $g : A \to B$ be morphisms of global actions. Then f is **homotopic** to g if there is a morphism $F : A \times L \to B$ of global actions and integers $n \leqq N \in \mathbb{Z} = |L|$ such that $F|_{A\times\{l\}} =$ f for all $l \leqq n$ and $F|_{A\times\{l\}} = g$ for all $N \leqq l$.

References

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