HOMOGENIZATION OF EVOLUTIONARY STOKES-CAHN-HILLIARD EQUATIONS FOR TWO-PHASE POROUS MEDIA FLOW

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Abstract. We consider homogenization of a phase-field model for two-phase immiscible, incompressible porous media flow with surface tension effects. The pore-scale model consists of a strongly coupled system of time-dependent Stokes-Cahn-Hilliard equations. In the considered model the fluids are separated by an evolving diffuse interface of a finite width, which is assumed to be independent of the scale parameter ε. We obtain upscaled equations for the considered model by a rigorous two-scale convergence approach.

1. Introduction

Flow of mixtures of fluids, solids and gases in porous media occurs in a large variety of physical, biological and industrial processes. Understanding and accurate prediction of multiphase multi-component flows in porous media are therefore of considerable interest for many scientific and engineering applications.

Phase-field approach is a popular tool for the modeling and simulation of multiphase flow problems, see for instance [6], [21], [28], [9] for an overview. A phase field model for the evolution of a mixture of two-incompressible immiscible fluids occupying a domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$ on the time interval $(0,T)$ consists of a system of Stokes-Cahn-Hilliard equations

$$\partial_t \mathbf{q} - \mu\Delta\mathbf{q} + \nabla p = -u\nabla w \qquad \text{in } (0,T)\times\Omega, \tag{1.1a}$$

$$\nabla\cdot\mathbf{q} = 0 \qquad \text{in } (0,T)\times\Omega, \tag{1.1b}$$

$$\partial_t u + \mathbf{q}\cdot\nabla u = \Delta w \qquad \text{in } (0,T)\times\Omega, \tag{1.1c}$$

$$w = -\lambda^2\Delta u + f(u) \qquad \text{in } (0,T)\times\Omega, \tag{1.1d}$$

where μ is the viscosity, λ is the interfacial width parameter and, $\mathbf{q}$ and w are the unknown velocity and chemical potential, respectively. The order parameter u plays the role of a microscopic concentration (or volume fraction). The order parameter is assumed to attain physically meaningful values -1 and 1 in the parts of the domain occupied by the pure fluids and $|u| < 1$ within a thin interfacial layer (so-called diffuse interface) of a uniform width that is proportional to the parameter λ. The nonlinearity $f(u) = F'(u)$, where F is a homogeneous free energy functional that penalizes the deviation from the physical constraint $|u| \le 1$. A common choice for F is a quadratic double-well free energy functional

$$F(s) = \frac{1}{4}(s^2-1)^2\,. \tag{1.2}$$

Other choices such as a logarithmic or a non-smooth (obstacle) free energy functional are also possible, see [11, 16]. Equations (1.1a)-(1.1b) are the incompressible Stokes equations, where the nonlinear term $u\nabla w$ models the surface tension effects, cf. [26] and [9, Section 2.4]. Equations (1.1c)-(1.1d) are a Cahn-Hilliard type equations with advection effect modeled by the term $\mathbf{q}\cdot\nabla u$. Suitable choices of boundary and initial conditions for the model (1.1) will be discussed below.

A prototypical macroscopic model for a single phase porous media flow is the Darcy's equation. This model is well understood and is derived by rigorous homogenization results, see for instance [25]. For numerical homogenization approaches for single phase Stokes flow we mention

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[8], [10], [24], [1]. Traditionally two-phase flow is modeled by the relative permeability Darcy's law [25, Chapter 5], which is a heuristic approach with well-known limitations, see for instance [37]. Homogenization theory for two-phase flow is less developed and so far effective models have only been justified by heuristic asymptotic expansion methods. For the homogenization result of sharp interface models for two-phase flows we refer to [7, 25]. Homogenization of phase-field models for two-phase flow using the formal asymptotic expansion method has been considered in [42], [17]. We also mention the homogenization result for the evolutionary single-phase Stokes equations (i.e., (1.1) with $u \equiv 0$) in [4]; for more recent developments see for instance [44], [27]. For homogenization of evolutionary Navier-Stokes equations in porous media we refer to recent works [45], [20] and the references therein. Rigorous homogenization of two-phase emulsion with fixed geometry of (microscale) interfaces and surface tension effects has been considered in [31], [32]. Upscaled models for Cahn-Hilliard type equations have been derived in [41], [40] via the asymptotic expansion method and in [30] via the two-scale convergence approach.

The aim of this paper is to obtain an upscaled model for two-phase porous media flow with surface tension described by (1.1). Our approach is based on rigorous concepts of two-scale convergence. The two-scale convergence approach has been used extensively for the homogenization of various models, cf. [36, 3, 5, 4, 38, 39, 18, 35, 34, 19, 29, 14, 15, 17, 46] and references therein.

The paper is organized as follows. We introduce the geometry of the porous medium and the considered phase-field model in the Section 2. In Section 3, we collect notation and mathematical preliminaries required for the subsequent analysis. Section 4 contains the analytical results related to the pore-scale model. Finally, Section 5 is dedicated to the derivation of the upscaled model.

2. The pore-scale model

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded, connected set with a smooth boundary. We consider the unit reference cell $Y := (0,1)^d \subset \mathbb{R}^d$, such that $Y = Y_s \cup Y_p$ where Y_s is a solid part and Y_p a pore part, s.t. $Y_p \cap Y_s = \emptyset$, and denote the solid boundary as $\Gamma_s = \partial Y_s$, see Figure 2.1. Given a scale parameter $\varepsilon > 0$ we define the pore space by $\Omega^\varepsilon := \cup_{\mathbf{k}\in\mathbb{Z}^d} Y_{p_k} \cap \Omega$, where $Y_{p_k} := \varepsilon\{Y_p + \mathbf{k}\}$, and the solid part as $\Omega^\varepsilon_s := \cup_{\mathbf{k}\in\mathbb{Z}^d} Y_{s_k} \cap \Omega = \Omega \setminus \Omega^\varepsilon$, where $Y_{s_k} := \varepsilon\{Y_s + \mathbf{k}\}$. We assume that Ω^ε is connected and has a smooth boundary. We consider the situation where the pore space Ω^ε is occupied by two immiscible fluids separated by an evolving macroscopic interface $\Gamma : [0,T] \to \Omega$ represented by the blue part in Figure 2.1. We denote the characteristic function of Y_p by χ and hence $\chi^\varepsilon(x) := \chi(\frac{x}{\varepsilon})$, $x \in \Omega$ is the characteristic function of Ω^ε. Throughout the paper we denote the time-interval as $I := [0,T)$.

We consider a situation where the porous medium is filled with a mixture of two immiscible, incompressible fluids separated by an evolving macroscopic interface and include the effects of surface tension on the motion of the interface. We model the flow of the fluid mixture on the pore-scale using a phase-field approach motivated by the Stokes-Cahn-Hilliard system (1.1).

The velocity of the fluid mixture $\mathbf{q}^\varepsilon = \mathbf{q}^\varepsilon(t,x)$, $(t,x) \in I \times \Omega^\varepsilon$ satisfies the Stokes law

$$\partial_t \mathbf{q}^\varepsilon - \varepsilon^2 \mu \Delta \mathbf{q}^\varepsilon + \nabla p^\varepsilon = -u^\varepsilon \nabla w^\varepsilon \qquad \text{in } I \times \Omega^\varepsilon, \tag{2.1}$$

$$\nabla \cdot \mathbf{q}^\varepsilon = 0 \qquad \text{in } I \times \Omega^\varepsilon, \tag{2.2}$$

where p^ε is the fluid pressure and the term $u^\varepsilon \nabla w^\varepsilon$ models the surface tension forces which act on the interface between the different fluids, cf. [26], [42], [17].

The order parameter u^ε, which plays the role of microscopic concentration, and the chemical potential w^ε satisfy the Cahn-Hilliard equation

$$\partial_t u^\varepsilon - \nabla \cdot (\nabla w^\varepsilon - \mathbf{q}^\varepsilon u^\varepsilon) = 0 \qquad \text{in } I \times \Omega^\varepsilon, \tag{2.3}$$

$$w^\varepsilon = -\lambda^2 \Delta u^\varepsilon + f(u^\varepsilon) \qquad \text{in } I \times \Omega^\varepsilon, \tag{2.4}$$

with $f(s) = s^3 - s = F'(s)$, where F is the double-well free energy (1.2).

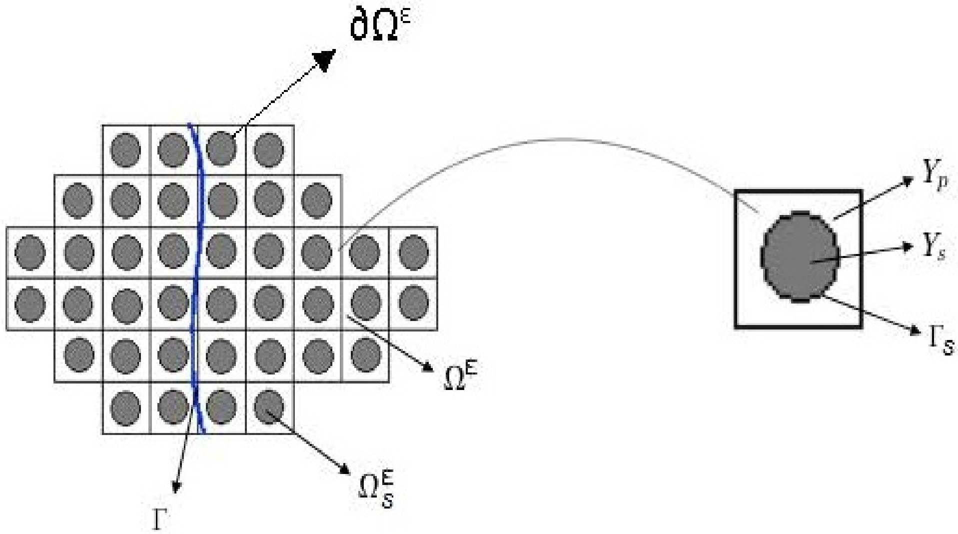

FIGURE 2.1. (left) Porous medium $\Omega^\varepsilon = \Omega \setminus \Omega_s^\varepsilon$ (left) as a periodic covering of the reference cell $Y = Y_p \cap Y_s$ (right). The blue interface γ is the interface between two fluid phases occupying the the pore space Ω^ε.

The complete system of Stokes-Cahn-Hilliard equations reads as

$$\partial_t \mathbf{q}^\varepsilon - \varepsilon^2 \mu \Delta \mathbf{q}^\varepsilon + \nabla p^\varepsilon = -u^\varepsilon \nabla w^\varepsilon \qquad \text{in } I \times \Omega^\varepsilon, \tag{2.5a}$$

$$\nabla \cdot \mathbf{q}^\varepsilon = 0 \qquad \text{in } I \times \Omega^\varepsilon, \tag{2.5b}$$

$$\mathbf{q}^\varepsilon = 0 \qquad \text{on } I \times \partial\Omega^\varepsilon, \tag{2.5c}$$

$$\mathbf{q}^\varepsilon(0,x) = \mathbf{q}_0^\varepsilon(x) \qquad \text{in } \Omega^\varepsilon, \tag{2.5d}$$

$$\partial_t u^\varepsilon - \Delta w^\varepsilon + \mathbf{q}^\varepsilon \cdot \nabla u^\varepsilon = 0 \qquad \text{in } I \times \Omega^\varepsilon, \tag{2.5e}$$

$$w^\varepsilon = -\lambda^2 \Delta u^\varepsilon + f(u^\varepsilon) \qquad \text{in } I \times \Omega^\varepsilon, \tag{2.5f}$$

$$\mathbf{n} \cdot \nabla u^\varepsilon = 0 \qquad \text{on } I \times \partial\Omega^\varepsilon, \tag{2.5g}$$

$$\mathbf{n} \cdot \nabla w^\varepsilon = 0 \qquad \text{on } I \times \partial\Omega^\varepsilon, \tag{2.5h}$$

$$u^\varepsilon(0,x) = u_0^\varepsilon(x) \qquad \text{in } \Omega^\varepsilon. \tag{2.5i}$$

Remark 2.1. *The model (2.5) describes the situation where the two fluids are separated by an evolving interface, the considered situation is displayed in Figure 2.1. As already noted in the introduction, in the phase-field model (2.5) the fluids are separated by a "diffuse" interface of uniform width proportional to the parameter λ, which is ε-independent. The "sharp" interface between the fluids is then defined implicitly as the zero level set of the order parameter $\Gamma(t) = \{x \in \Omega; u^\varepsilon(t,x) = 0\}$. The advantage of the phase-field approximation are, for instance, the mass conservation and the ability to deal with topological changes of the interface, cf.* [9].

Remark 2.2. *The following reformulations of the nonlinear terms in (2.5) are useful for the analysis of the model.*

The advection term in (2.5a) is written in an equivalent form by using the incompressibility condition $\nabla \cdot \mathbf{q}^\varepsilon = 0$, since

$$\nabla \cdot [\mathbf{q}^\varepsilon u^\varepsilon] = [\nabla \cdot \mathbf{q}^\varepsilon] u^\varepsilon + \mathbf{q}^\varepsilon \cdot \nabla u^\varepsilon = \mathbf{q}^\varepsilon \cdot \nabla u^\varepsilon .$$

Using the identity $\nabla(u^\varepsilon w^\varepsilon) = u^\varepsilon \nabla w^\varepsilon + w^\varepsilon \nabla u^\varepsilon$ the surface tension term $-u^\varepsilon \nabla w^\varepsilon$ in (2.5a) can be replaced by $w^\varepsilon \nabla u^\varepsilon$, where the additional gradient term is absorbed into the pressure. For more details about the modelling of surface tension effects in phase-field models see, e.g., [6], [26] *and* [9, Section 2.4].

Remark 2.3. *The model (2.5) describes a situation where the viscosities of different fluids are equal, i.e., the viscosity coefficient μ is a constant. To generalize (2.5a) to the case of two fluids with distinct viscosities $0<\mu_1,\mu_2<\infty$ we define $\mu(u^\varepsilon)=\mu_1(1-u^\varepsilon)+\mu_2(1+u^\varepsilon)$ and replace the second order term in (2.5a) by $\nabla\cdot\big(\mu(u^\varepsilon)[\nabla\mathbf{q}^\varepsilon+(\nabla\mathbf{q}^\varepsilon)^T]\big)$. We note that it is straightforward to extend all results presented in this paper for the model with variable viscosity. The only modification for the upscaled model with variable viscosity would be to replace the second order term $\mu\Delta_y\mathbf{q}$ in the equation (5.8a) by $\nabla_y\cdot\big(\mu(u(x))[\nabla_y\mathbf{q}(x,y)+(\nabla_y\mathbf{q}(x,y))^T]\big)$.*

3. Notation and Mathematical Preliminaries

3.1. **Function Spaces.** For a Banach space X, let X^* denote its dual and the duality pairing is denoted by $\langle .\,,\,.\rangle_{X^*\times X}$; to simplify the notation we use $\langle .\,,\,.\rangle$ where the notation is clear from the context. We denote by $L^r(\Omega)$ and $H^{l,r}(\Omega)$ the usual Lebesgue and Sobolev spaces; for $r=2$ we denote $H^l:=H^{l,2}$ and by L^2_0 we denote the space of L^2 functions with zero mean. As usual we denote $H^{-1}(\Omega):=H^1_0(\Omega)^*$. The space of divergence free vector fields is denoted by $\mathbf{H}^1_{div}(\Omega)=\{\boldsymbol{\zeta}\in H^1_0(\Omega)^d;\nabla\cdot\boldsymbol{\zeta}=0\}$ with its dual space denoted by $\mathbf{H}^{-1}_{div}$. Further, let $C^\alpha_\#(Y)$ denote the set of all Y-periodic α-times continuously differentiable functions in y. We denote by $C^\infty_0(I\times\Omega;C^\infty_\#(Y))$ the space of all Y-periodic continuously differentiable functions in t, x, y with compact support inside Ω. The symbols $\hookrightarrow$, $\hookrightarrow\hookrightarrow$ and $\stackrel{d}{\hookrightarrow}$ denote the continuous, compact and dense embeddings respectively.

Below we summarize some known results that will be used in the paper.

Lemma 3.1 (cf. p. 106f in [43])**.** *Let B be a Banach space and B_0 and B_1 be reflexive spaces with $B_0\subset B\subset B_1$. Suppose further that $B_0\hookrightarrow\hookrightarrow B\hookrightarrow B_1$. For $1<r,s<\infty$ and $0<T<\infty$ define $X:=\{u\in L^r(I;B_0):\partial_t u\in L^s(I;B_1)\}$. Then $X\hookrightarrow\hookrightarrow L^r((0,T);B)$.*

Lemma 3.2 (Extension theorem, cf. [34])**.** *Any function $u^\varepsilon\in H^{1,r}(\Omega^\varepsilon)$, $1\le r\le\infty$ can be extended to a function $\tilde u^\varepsilon\in H^{1,r}(\Omega)$ defined on all of Ω such that $\tilde u^\varepsilon|_{\Omega^\varepsilon}=u^\varepsilon$ and there exists a constant C independent of ε and u*

$$\|\tilde u^\varepsilon\|_{H^{1,r}(\Omega)}\le C\|u^\varepsilon\|_{H^{1,r}(\Omega^\varepsilon)}. \tag{3.1}$$

In particular, for $u^\varepsilon\in L^2(\Omega^\varepsilon)$, then the extension $\tilde u^\varepsilon$ satisfies

$$\|\tilde u^\varepsilon\|_{L^2(\Omega)}\le C\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)}, \tag{3.2}$$

where the constant is independent of ε and u^ε.

The generalization of the extension theorem for time dependent functions is stated below.

Lemma 3.3 (Extension theorem, cf. [34])**.** *There exists a bounded and linear extension operator $E^\varepsilon_t:L^2(I;H^1(\Omega^\varepsilon))\to L^2(I;H^1(\Omega))$ such that for all $u^\varepsilon\in L^2(I;H^1(\Omega^\varepsilon))$, we have*

$$E^\varepsilon_t u^\varepsilon|_{I\times\Omega^\varepsilon}=u^\varepsilon, \tag{3.3a}$$

$$\|E^\varepsilon_t u^\varepsilon\|_{L^2(I;H^1(\Omega))}\le C\|u^\varepsilon\|_{L^2(I;H^1(\Omega^\varepsilon))}, \tag{3.3b}$$

where C is independent of ε and u^ε.

The restriction theorem below can be found, e.g., in [5, Lemma 5.2].

Lemma 3.4 (Restriction theorem)**.** *There exists a linear restriction operator $R^\varepsilon:H^1_0(\Omega)^d\to H^1_0(\Omega^\varepsilon)^d$ such that $R^\varepsilon u(x)=u(x)|_{\Omega^\varepsilon}$ for $u\in H^1_0(\Omega)^d$ and $\nabla\cdot R^\varepsilon u=0$ if $\nabla\cdot u=0$. Furthermore, the restriction satisfies the following bound*

$$\|R^\varepsilon u\|_{L^2(\Omega^\varepsilon)}+\varepsilon\|\nabla R^\varepsilon u\|_{L^2(\Omega^\varepsilon)}\le C\Big(\|u\|_{L^2(\Omega)}+\varepsilon\|\nabla u\|_{L^2(\Omega)}\Big), \tag{3.4}$$

with an ε-independent constant C.

Lemma 3.5 (cf. Theorem 2.10 in [34]). *Assume that* $1 \le r < n$ *and* $u \in H^{1,r}(\Omega^\varepsilon)$. *Then* $u^\varepsilon \in L^{r^*}(\Omega^\varepsilon)$ *and there is a positive constant* C *independent of* ε *and* u

$$\|u^\varepsilon\|_{L^{r^*}(\Omega^\varepsilon)} \le C\|u^\varepsilon\|_{H^{1,r}(\Omega^\varepsilon)}, \text{ where } r^* = \frac{dr}{n-r}. \tag{3.5}$$

In other words, $H^{1,r}(\Omega^\varepsilon) \hookrightarrow L^{r^*}(\Omega^\varepsilon)$ *with embedding constant* C *independent of* ε *and* u.

We will often use the following inequality which follows from Lemma 3.5 for $d = 3$ and $r = 2$:

$$\|u^\varepsilon\|_{L^4(\Omega^\varepsilon)} \le C\|u^\varepsilon\|_{H^1(\Omega^\varepsilon)}. \tag{3.6}$$

3.2. **Two-scale convergence.** Below we recall some well-known results about the two-scale convergence.

Definition 3.1. *A sequence of functions* $(u^\varepsilon)_{\varepsilon>0}$ *in* $L^r(I\times\Omega)$ *is said to be two-scale convergent to a limit* $u \in L^r(I\times\Omega\times Y)$ *if*

$$\lim_{\varepsilon\to 0}\int_{I\times\Omega} u^\varepsilon(t,x)\phi(t,x,\frac{x}{\varepsilon})\,dx\,dt = \int_{I\times\Omega\times Y} u(t,x,y)\,\phi(t,x,y)\,dx\,dt\,dy \tag{3.7}$$

for all $\phi \in L^s(I\times\Omega; C_\#(Y))$.

By $\stackrel{2}{\rightharpoonup}$, $\stackrel{w}{\rightharpoonup}$ and $\to$ we denote the two-scale, weak and strong convergence of a sequence respectively.

Lemma 3.6 (cf. [33]). *For every bounded sequence* $(u^\varepsilon)_{\varepsilon>0}$ *in* $L^r(I\times\Omega)$ *there exists a subsequence* $(u^\varepsilon)_{\varepsilon>0}$ *(still denoted by same symbol) and a* $u \in L^r(I\times\Omega\times Y)$ *such that* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$.

Lemma 3.7 (cf. [33]). *Let* $(u^\varepsilon)_{\varepsilon>0}$ *be strongly convergent to* $u \in L^r(I\times\Omega)$, *then* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$.

Lemma 3.8 (cf. [33]). *Let* $(u^\varepsilon)_{\varepsilon>0}$ *be a sequence in* $L^r(I; H^{1,r}(\Omega))$ *such that* $u^\varepsilon \stackrel{w}{\rightharpoonup} u$ *in* $L^r(I; H^{1,r}(\Omega))$. *Then* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$ *and there exists a subsequence* $(u^\varepsilon)_{\varepsilon>0}$, *still denoted by same symbol, and a* $u_1 \in L^r(I\times\Omega; H^{1,r}_\#(Y))$ *such that* $\nabla_x u^\varepsilon \stackrel{2}{\rightharpoonup} \nabla_x u + \nabla_y u_1$.

Lemma 3.9 (cf. [3]). *Let* $(u^\varepsilon)_{\varepsilon>0}$ *be a bounded sequence of functions in* $L^r(I\times\Omega)$ *such that* $\varepsilon\nabla u^\varepsilon$ *is bounded in* $L^r(I\times\Omega)^d$. *Then there exist a function* $u \in L^r(I\times\Omega; H^{1,r}_\#(Y))$ *such that* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$, $\varepsilon\nabla_x u^\varepsilon \stackrel{2}{\rightharpoonup} \nabla_y u$.

Definition 3.2 (cf. [22, 23]). *Let* $u^\varepsilon \in L^r(\Omega)$, $1 \le r \le \infty$. *We define the unfolding operator* $\mathcal{T}^\varepsilon : L^r(\Omega) \to L^r(\Omega\times Y)$ *as*

$$\mathcal{T}^\varepsilon u^\varepsilon(x,y) = u^\varepsilon(t^\varepsilon(x,y)) \qquad \text{for} \quad x \in Y_{s_k} \subset \Omega \tag{3.8a}$$

$$\mathcal{T}^\varepsilon u^\varepsilon(x,y) = u^\varepsilon(x) \qquad \text{for} \quad Y_{s_k} \cap \partial\Omega \ne \emptyset. \tag{3.8b}$$

where $t^\varepsilon(x,y) = \varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y$, $[s]$ *being the lower integer part of* s.

We note that the unfolding operator $\mathcal{T}^\varepsilon$ transforms a single variable function u on Ω into a two-variable function $\mathcal{T}^\varepsilon u$ on $\Omega\times Y$, s.t. $u^\varepsilon(x) = \mathcal{T}^\varepsilon u^\varepsilon(x, x-\varepsilon\left[\frac{x}{\varepsilon}\right])$. Some basic properties of periodic unfolding and the relation to the two-scale convergence are summarized, e.g., in [23, Lemma 5.1]. Further information about unfolding operators and applications to homogenization can be found in [22, 23], [15], [13].

4. Properties of the pore-scale model

The weak solution of the pore-scale model (2.5) is defined below.

Definition 4.1 (Weak Solution). *The triplet* $(\mathbf{q}^\varepsilon, u^\varepsilon, w^\varepsilon) \in L^2(I;\mathbf{H}^1_{div}(\Omega^\varepsilon)) \cap H^1(I;\mathbf{H}^{-1}_{div}(\Omega^\varepsilon)) \times L^\infty(I;H^1(\Omega^\varepsilon)) \cap H^1(I;H^1(\Omega^\varepsilon)^*) \times L^2(I;H^1(\Omega^\varepsilon))$ *is a weak solution of* (2.5) *if it satisfies* $\mathbf{q}^\varepsilon(0,x) = \mathbf{q}^\varepsilon_0(x)$, $u^\varepsilon(0,x) = u^\varepsilon_0(x)$ *and*

$$\int_I \langle \partial_t \mathbf{q}^\varepsilon, \boldsymbol{\psi}\rangle\, dt + \varepsilon^2 \mu \int_{I\times\Omega^\varepsilon} \nabla \mathbf{q}^\varepsilon : \nabla \boldsymbol{\psi}\, dx\, dt = -\int_{I\times\Omega^\varepsilon} u^\varepsilon \nabla w^\varepsilon \cdot \boldsymbol{\psi}\, dx\, dt, \tag{4.1a}$$

$$\int_I \langle \partial_t u^\varepsilon, \varphi\rangle\, dt + \int_{I\times\Omega^\varepsilon} \nabla w^\varepsilon \cdot \nabla\varphi\, dx\, dt - \int_{I\times\Omega^\varepsilon} u^\varepsilon \mathbf{q}^\varepsilon \cdot \nabla \varphi\, dx\, dt = 0, \tag{4.1b}$$

$$\int_{I\times\Omega^\varepsilon} w^\varepsilon \phi\, dx\, dt = \lambda^2 \int_{I\times\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla \phi\, dx\, dt + \int_{I\times\Omega^\varepsilon} f(u^\varepsilon)\phi\, dx\, dt, \tag{4.1c}$$

for all $\boldsymbol{\psi} \in L^2(I;\mathbf{H}^1_{div}(\Omega^\varepsilon))$ *and* $\varphi, \phi \in L^2(I;H^1(\Omega^\varepsilon))$.

Furthermore, with each weak solution $(\mathbf{q}^\varepsilon, u^\varepsilon, w^\varepsilon)$ *we associate a pressure* $p^\varepsilon := \partial_t P^\varepsilon$, $P^\varepsilon \in L^\infty(I;L^2_0(\Omega^\varepsilon))$ *which satisfies (2.5a) in the distributional sense (4.16).*

The theorem below summarizes basic existence and regularity properties of the weak solution of the pore-scale model (2.5) which are necessary for the derivation of the upscaled model. The proof of the theorem can be found, e.g., in [12], [21], [2]; we present the main steps of the proof for the convenience of the reader.

Theorem 4.1. *Let* $\mathbf{q}^\varepsilon_0 \in L^2(\Omega^\varepsilon)$, $u^\varepsilon_0 \in H^1(\Omega^\varepsilon)$, *s.t.* $|u^\varepsilon_0| \le 1$ *a.e. in* Ω. *Then there exists a weak solution* $(\mathbf{q}^\varepsilon, u^\varepsilon, w^\varepsilon)$ *of the problem (2.5) in the sense of Definition 4.1. The weak solution satisfies the following estimate*

$$\begin{aligned}\|\mathbf{q}^\varepsilon\|_{L^\infty(I;L^2(\Omega^\varepsilon)^d)} + \sqrt{\mu}\varepsilon \|\nabla \mathbf{q}^\varepsilon\|_{L^2(I\times\Omega^\varepsilon)^{d\times d}} + \|\partial_t \mathbf{q}^\varepsilon\|_{L^2(I;\mathbf{H}^{-1}_{div}(\Omega^\varepsilon))} \\ + \|w^\varepsilon\|_{L^2(I;H^1(\Omega^\varepsilon))} + \lambda \|\nabla u^\varepsilon\|_{L^\infty(I;L^2(\Omega^\varepsilon)^d)} + \|u^\varepsilon\|_{L^\infty(I;L^4(\Omega^\varepsilon))} + \|\partial_t u^\varepsilon\|_{L^2(I;H^1(\Omega^\varepsilon)^*)} \le C\end{aligned} \tag{4.2}$$

for all $\varepsilon > 0$, *where the constant* C *is independent of* ε.

Furthermore, there exists a pressure $p^\varepsilon = \partial_t P^\varepsilon$, *where* $P^\varepsilon \in L^\infty(I;L^2_0(\Omega))$ *such that (2.5a) is satisfied in the distributional sense. The pressure satisfies the estimate*

$$\sup_{t\in[0,T]} \|\nabla P^\varepsilon(t)\|_{H^{-1}(\Omega^\varepsilon)^d} \le C \qquad \forall\, \varepsilon > 0, \tag{4.3}$$

with an ε*-independent constant* C.

Proof. (i) We set $\boldsymbol{\psi} = \mathbf{q}^\varepsilon$, $\phi = w^\varepsilon$ and $\varphi = \partial_t u^\varepsilon$ in (4.1a), (4.1b) and (4.1c) and get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^\varepsilon} |\mathbf{q}^\varepsilon|^2\, dx + \mu\varepsilon^2 \int_{\Omega^\varepsilon} |\nabla \mathbf{q}^\varepsilon|^2\, dx - \varepsilon \int_{\Omega^\varepsilon} w^\varepsilon \nabla u^\varepsilon \cdot \mathbf{q}^\varepsilon\, dx = 0, \tag{4.4}$$

$$\int_{\Omega^\varepsilon} \partial_t u^\varepsilon w^\varepsilon\, dx + \int_{\Omega^\varepsilon} |\nabla w^\varepsilon|^2\, dx + \varepsilon \int_{\Omega^\varepsilon} \mathbf{q}^\varepsilon \cdot \nabla u^\varepsilon w^\varepsilon\, dx = 0, \tag{4.5}$$

$$-\int_{\Omega^\varepsilon} \partial_t u^\varepsilon w^\varepsilon\, dx\, dt + \frac{1}{2}\lambda^2 \frac{d}{dt}\int_{\Omega^\varepsilon} |\nabla u^\varepsilon|^2\, dx\, dt + \frac{d}{dt}\int_{\Omega^\varepsilon} F(u^\varepsilon)\, dx, = 0. \tag{4.6}$$

We add (4.4), (4.5) and (4.6) and integrate over $(0,t)$ and get

$$\begin{aligned}&\frac{1}{2}\lambda^2 \int_{\Omega^\varepsilon} |\nabla u^\varepsilon(t)|^2\, dx + \int_{\Omega^\varepsilon} F(u^\varepsilon(t))\, dx + \int_0^t \int_{\Omega^\varepsilon} |\nabla w^\varepsilon|^2\, dx\, dt + \frac{1}{2}\int_{\Omega^\varepsilon} |\mathbf{q}^\varepsilon(t)|^2 \\ &+ \mu\varepsilon^2 \int_0^t \int_{\Omega^\varepsilon} |\nabla \mathbf{q}^\varepsilon|^2\, dx\, dt = \frac{1}{2}\lambda^2 \int_{\Omega^\varepsilon} |\nabla u^\varepsilon(0)|^2\, dx + \int_{\Omega^\varepsilon} F(u^\varepsilon(0))\, dx + \frac{1}{2}\int_{\Omega^\varepsilon} |\mathbf{q}^\varepsilon(0)|^2\, dx.\end{aligned} \tag{4.7}$$

Since $u_0 \in H^1(\Omega^\varepsilon)$, $|u_0| \le 1$ and $\mathbf{q}_0 \in L^2(\Omega^\varepsilon)^d$ and are bounded in the respective spaces independently of ε the above equation (4.7) implies

$$\begin{aligned}&\frac{1}{2}\lambda^2 \int_{\Omega^\varepsilon} |\nabla u^\varepsilon(t)|^2\, dx + \int_{\Omega^\varepsilon} F(u^\varepsilon(t))\, dx + \int_0^t \int_{\Omega^\varepsilon} |\nabla w^\varepsilon|^2\, dx\, dt + \frac{1}{2}\int_{\Omega^\varepsilon} |\mathbf{q}^\varepsilon(t)|^2 \\ &+ \mu\varepsilon^2 \int_0^t \int_{\Omega^\varepsilon} |\nabla \mathbf{q}^\varepsilon|^2\, dx\, dt \le C.\end{aligned} \tag{4.8}$$

On noting that $F(u^\varepsilon)\geq 0$, inequality (4.8) implies

$$\lambda\|\nabla u^\varepsilon\|_{L^\infty(I;L^2(\Omega^\varepsilon))}+\|\mathbf{q}^\varepsilon\|_{L^\infty(I;L^2(\Omega^\varepsilon))}+\|\nabla w^\varepsilon\|_{L^2(I\times\Omega^\varepsilon)}+\sqrt{\mu}\varepsilon\|\nabla\mathbf{q}^\varepsilon\|_{L^2(I\times\Omega^\varepsilon)}\leq C. \tag{4.9}$$

(ii) The bound $\int_{\Omega^\varepsilon}F(u^\varepsilon(t))\,dx\leq C$ from (4.8) and Young's inequality imply

$$\int_{\Omega^\varepsilon}|u^\varepsilon|^4\,dx\leq C+2\delta\int_{\Omega^\varepsilon}|u^\varepsilon|^4\,dx+(C_\delta+1)|\Omega^\varepsilon|.$$

Since $|\Omega^\varepsilon|\leq|\Omega|$, we get after choosing, e.g., $\delta=\frac{1}{4}$ in the above inequality that

$$\|u^\varepsilon\|_{L^\infty(I;L^4(\Omega^\varepsilon))}\leq C\qquad\forall\,\varepsilon>0. \tag{4.10}$$

(iii) From (4.1a) get for all $\boldsymbol{\psi}\in\mathbf{H}^1_{div}(\Omega^\varepsilon)$

$$\begin{aligned}|\langle\partial_t\mathbf{q}^\varepsilon,\boldsymbol{\psi}\rangle|&\leq\mu\varepsilon^2\|\nabla\mathbf{q}^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|\nabla\boldsymbol{\psi}\|_{L^2(\Omega^\varepsilon)}\\&\quad+\|u^\varepsilon\|_{L^4(\Omega^\varepsilon)}\|\nabla w^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|\boldsymbol{\psi}\|_{L^4(\Omega^\varepsilon)^d}\\&\leq\mu\varepsilon^2\|\nabla\mathbf{q}^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|\boldsymbol{\psi}\|_{H^1_0(\Omega^\varepsilon)^d}\\&\quad+\|u^\varepsilon\|_{L^4(\Omega^\varepsilon)}\|\nabla w^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|\boldsymbol{\psi}\|_{H^1_0(\Omega^\varepsilon))^d}.\end{aligned}$$

Consequently

$$\sup_{\|\boldsymbol{\psi}\|_{\mathbf{H}^1_{div}(\Omega^\varepsilon)}\leq 1}|\langle\partial_t\mathbf{q}^\varepsilon,\boldsymbol{\psi}\rangle|\leq\mu\varepsilon^2\|\nabla\mathbf{q}^\varepsilon\|_{L^2(\Omega^\varepsilon)}+\|u^\varepsilon\|_{L^4(\Omega^\varepsilon)}\|\nabla w^\varepsilon\|_{L^2(\Omega^\varepsilon)}. \tag{4.11}$$

We integrate the square of (4.11) on $(0,T)$ and get by (i) and (3.6) the bound

$$\|\partial_t\mathbf{q}^\varepsilon\|_{L^2(I;\mathbf{H}^{-1}_{div}(\Omega^\varepsilon))}\leq C\qquad\forall\varepsilon>0, \tag{4.12}$$

where the constant C is independent of ε.

(iv) Using the Cauchy-Schwarz inequality we get from (4.1b) that

$$|\langle\partial_t u^\varepsilon,\phi\rangle|\leq\|\nabla w^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|\nabla\phi\|_{L^2(\Omega^\varepsilon)}+\|u^\varepsilon\|_{L^4(\Omega^\varepsilon)}\|\mathbf{q}^\varepsilon\|_{L^4(\Omega^\varepsilon)}\|\nabla\phi\|_{L^2(\Omega^\varepsilon)}.$$

Similarly as in (iii), using (3.6) and the estimates from part (i) we obtain the bound

$$\|\partial_t u^\varepsilon\|_{L^2(I;H^1(\Omega^\varepsilon)^*)}\leq C\qquad\forall\,\varepsilon>0.$$

(v) Since $f(u^\varepsilon)=(u^\varepsilon)^3-u^\varepsilon$ we get from (4.1c) with $\phi=1$

$$\int_{\Omega^\varepsilon}w^\varepsilon\,dx=\int_{\Omega^\varepsilon}f(u^\varepsilon)\,dx\leq C\big(\|u^\varepsilon|^4_{L^4(\Omega^\varepsilon)}+|\Omega|\big)\leq C, \tag{4.13}$$

where C is independent of ε by part (ii). Hence, by the Poincaré and triangle inequalities using (4.13) and part (ii) we obtain

$$\|w^\varepsilon\|_{L^2(I\times\Omega^\varepsilon)}\leq C\qquad\forall\,\varepsilon>0. \tag{4.14}$$

(vi) By a classical argument, cf. [47, Proposition III.1.1], the identity (4.1a) implies the existence of a pressure $p^\varepsilon:=\partial_t P^\varepsilon\in W^{-1,\infty}((0,T),L^2_0(\Omega^\varepsilon))$ such that

$$\begin{aligned}-\int_{\Omega^\varepsilon}P^\varepsilon(t)\nabla\cdot\boldsymbol{\psi}\,dx\quad&=\quad-\int_{\Omega^\varepsilon}(\mathbf{q}^\varepsilon(t)-\mathbf{q}^\varepsilon_0)\cdot\boldsymbol{\psi}\,dx\\&\quad-\varepsilon^2\mu\int_0^t\int_{\Omega^\varepsilon}\nabla\mathbf{q}^\varepsilon(s):\nabla\boldsymbol{\psi}\,dx\,ds-\int_0^t\int_{\Omega^\varepsilon}u^\varepsilon\nabla(s)w^\varepsilon(s)\cdot\boldsymbol{\psi}\,dx\,ds\qquad\forall\boldsymbol{\psi}\in H^1_0(\Omega^\varepsilon)^d.\end{aligned} \tag{4.15}$$

Hence, we get

$$\begin{aligned}\langle\nabla P^\varepsilon(t),\boldsymbol{\psi}\rangle:=-\int_{\Omega^\varepsilon}P^\varepsilon(t)\,ds\nabla\cdot\boldsymbol{\psi}\,dx&\leq(\|\mathbf{q}^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)}+\|\mathbf{q}^\varepsilon_0\|_{L^2(\Omega^\varepsilon)})\|\boldsymbol{\psi}\|_{L^2(\Omega^\varepsilon)}\\&\quad+\varepsilon^2\mu\int_0^t\|\nabla\mathbf{q}^\varepsilon\|_{L^2(\Omega^\varepsilon)}\,ds\|\nabla\boldsymbol{\psi}\|_{L^2(\Omega^\varepsilon)}+\int_0^t\|u^\varepsilon\|_{L^4(\Omega^\varepsilon)}\|\nabla w^\varepsilon\|_{L^2(\Omega^\varepsilon)}\,ds\|\boldsymbol{\psi}\|_{L^4(\Omega^\varepsilon)}.\end{aligned}$$

By (3.6), (4.9), (4.10) it follows that

$$\langle\nabla P^\varepsilon(t),\boldsymbol{\psi}\rangle\leq C\|\boldsymbol{\psi}\|_{H^1_0(\Omega^\varepsilon)^d},$$

which implies (4.3).

Collecting the estimates from (i)-(vi) concludes the proof. $\square$

Remark 4.1. *We multiply (4.15) by $\partial_t\phi$ for a $\phi \in C_0^\infty(I)$ and integrate in time and obtain*

$$
\begin{aligned}
(4.16) \qquad -\int_I\int_{\Omega^\varepsilon} P^\varepsilon(t)\nabla\cdot\boldsymbol{\psi}\,dx\,\partial_t\,\phi(t)dt &= \int_I \langle \partial_t\mathbf{q}^\varepsilon(t),\boldsymbol{\psi}\rangle\,\phi(t)\,dt \\
&+\varepsilon^2\mu\int_I\int_{\Omega^\varepsilon}\nabla\mathbf{q}^\varepsilon(t):\nabla\boldsymbol{\psi}\,dx\,\phi(t)\,dt + \int_I\int_{\Omega^\varepsilon} u^\varepsilon(t)\nabla w^\varepsilon(t)\cdot\boldsymbol{\psi}\,dx\,\phi(t)\,dt\,.
\end{aligned}
$$

The above formulation implies that (2.5a) is satisfied in the distributional sense with $P^\varepsilon(t) = \int_0^t p^\varepsilon(s)\,ds$. Furthermore, the formulation (4.16) is equivalent to (4.1a) for $\boldsymbol{\psi}\in\mathbf{H}^1_{div}(\Omega^\varepsilon)$. Due to the limited time regularity of the pressure $p^\varepsilon \in W^{-1,\infty}((0,T),L^2_0(\Omega^\varepsilon))$ we will use the formulation (4.16) to derive the two-scale limit of (2.5a).

Remark 4.2. *Note that if $u^\varepsilon \in L^\infty(I\times\Omega^\varepsilon)$ then $u^\varepsilon\nabla w^\varepsilon \in L^2(I\times\Omega^\varepsilon)$ and we obtain improved regularity $\partial_t\mathbf{q}^\varepsilon \in L^2(I\times\Omega^\varepsilon)$, $p^\varepsilon\in L^2(I;L^2_0(\Omega^\varepsilon))$ as in* [4, Proposition 1.3]. *The property $|u^\varepsilon|\le 1$ is physically reasonable assumption, however, it is not obvious whether an ε-independent L^∞ bound for u^ε holds for the system (2.5) with the double-well potential, cf.* [21]*, therefore we do not assume it and work with the weakest regularity assumptions. We note that a uniform bound $|u^\varepsilon|\le 1$ in $I\times\Omega$ holds for the double-obstacle type energy, cf.* [9], [2]*, but we do not consider this situation here.*

5. Derivation of the upscaled model

In this section we rigorously derive the upscaled model for $\varepsilon\to 0$ in Theorems 5.1. We start with the construction of an extension of solution from Ω^ε to Ω in the lemma below.

Lemma 5.1. *There exists a positive constant C depending on u_0, $\mathbf{q}_0$, λ and μ but independent of ε and extensions $\widetilde{u}^\varepsilon$, $\widetilde{w}^\varepsilon$, $\widetilde{\mathbf{q}}^\varepsilon$, $\widetilde{P}^\varepsilon$ of the solution u^ε, w^ε, $\mathbf{q}^\varepsilon$, P^ε to $I\times\Omega$ such that*

$$
\begin{aligned}
(5.1)\quad &\|\widetilde{\mathbf{q}}^\varepsilon\|_{L^\infty(I;L^2(\Omega)^d)} + \sqrt{\mu}\varepsilon\|\nabla\widetilde{\mathbf{q}}^\varepsilon\|_{L^2(I\times\Omega)^{d\times d}} + \|\partial_t\widetilde{\mathbf{q}}^\varepsilon\|_{L^2(I;\mathbf{H}^{-1}_{div}(\Omega))} + \|\widetilde{w}^\varepsilon\|_{L^2(I;H^1(\Omega))} \\
&+\lambda\|\nabla\widetilde{u}^\varepsilon\|_{L^\infty(I;L^2(\Omega)^d)} + \|\widetilde{u}^\varepsilon\|_{L^\infty(I;L^4(\Omega))} + \|\partial_t\widetilde{u}^\varepsilon\|_{L^2(I;H^1(\Omega)^*)} + \sup_{t\in[0,T]}\|\widetilde{P}^\varepsilon(t)\|_{L^2_0(\Omega)} \le C\,.
\end{aligned}
$$

Proof. (i) The existence of extensions for $\widetilde{u}^\varepsilon$, $\widetilde{w}^\varepsilon$, $\widetilde{\mathbf{q}}^\varepsilon$ is guaranteed by Lemma 3.3 which together with the a-priori estimate (4.2) implies the bound

$$
\begin{aligned}
&\|\widetilde{\mathbf{q}}^\varepsilon\|_{L^\infty(I;L^2(\Omega)^d)} + \sqrt{\mu}\varepsilon\|\nabla\widetilde{\mathbf{q}}^\varepsilon\|_{L^2(I\times\Omega)^{d\times d}} + \|\widetilde{w}^\varepsilon\|_{L^2(I;H^1(\Omega))} \\
&\quad+\lambda\|\nabla\widetilde{u}^\varepsilon\|_{L^\infty(I;L^2(\Omega)^d)} + \|\widetilde{u}^\varepsilon\|_{L^\infty(I;L^4(\Omega))} \le C\,.
\end{aligned}
$$

(ii) Next, we consider the extension of $\partial_t u^\varepsilon$ from $L^2(I;H^1(\Omega^\varepsilon)^*)$ to $L^2(I;H^1(\Omega)^*)$. For $\Theta\in H^1(\Omega^\varepsilon)^*$ define the extension operator $F^\varepsilon : H^1(\Omega^\varepsilon)^*\to H^1(\Omega)^*$ as

$$
(5.2)\qquad \langle F^\varepsilon\Theta,\phi\rangle_{H^1(\Omega)^*\times H^1(\Omega)} = \langle\Theta,\mathcal{R}^\varepsilon\phi\rangle_{H^1(\Omega^\varepsilon)^*\times H^1(\Omega^\varepsilon)},
$$

where $\mathcal{R}^\varepsilon : H^1(\Omega)\to H^1(\Omega^\varepsilon)$ is the trivial restriction operator $\mathcal{R}^\varepsilon\phi = \phi|_{\Omega^\varepsilon}$ for $\phi\in H^1(\Omega)$. Since $\|\mathcal{R}^\varepsilon\phi\|_{H^1(\Omega^\varepsilon)}\le\|\phi\|_{H^1(\Omega)}$ it follows that

$$
(5.3)\qquad \|F^\varepsilon\Theta\|_{H^1(\Omega)^*}\le\|\Theta\|_{H^1(\Omega^\varepsilon)^*}.
$$

Using (5.2) we define the extension $\widetilde{\partial_t u^\varepsilon}$ of $\partial_t u^\varepsilon$ in $L^2(I;H^1(\Omega)^*)$ as

$$
\int_I\langle\widetilde{\partial_t u^\varepsilon},\phi\rangle_{H^1(\Omega)^*\times H^1(\Omega)} := \int_I\langle F^\varepsilon\partial_t u^\varepsilon,\phi\rangle_{H^1(\Omega)^*\times H^1(\Omega)}\,,
$$

and by the linearity of the restriction operator $\mathcal{R}^\varepsilon$ it follows that $\widetilde{\partial_t u^\varepsilon} = \partial_t\widetilde{u}^\varepsilon$. Hence, the estimate for $\partial_t\widetilde{u}^\varepsilon$ in (5.1) follows from (5.3) and the estimate (4.2).

Analogically, using the properties of the restriction operator from Lemma 3.4 we can define the extension of $\partial_t\mathbf{q}^\varepsilon$ from $L^2(I;\mathbf{H}^{-1}_{div}(\Omega^\varepsilon))$ to $L^2(I;\mathbf{H}^{-1}_{div}(\Omega))$ and obtain the corresponding bound for $\partial_t\widetilde{\mathbf{q}}^\varepsilon$.

(iii) To construct the extension of the pressure P^ε we employ the restriction operator R^ε from Lemma 3.4 and define $G^\varepsilon \in H^{-1}(\Omega)^d$ as

$$\langle G^\varepsilon, \boldsymbol{\psi}\rangle_{H^{-1}(\Omega)^d\times H^1_0(\Omega)^d} = \langle \nabla_x P^\varepsilon, R^\varepsilon \boldsymbol{\psi}\rangle_{H^{-1}(\Omega^\varepsilon)^d\times H^1_0(\Omega^\varepsilon)^d} \quad \text{for any } \boldsymbol{\psi} \in H^1_0(\Omega)^d. \tag{5.4}$$

Estimate (4.3) implies the ε-independent bound

$$\sup_{t\in[0,T]} \|G^\varepsilon(t)\|_{H^{-1}(\Omega^\varepsilon)^d} \le C\,. \tag{5.5}$$

Since $\langle G^\varepsilon, \psi\rangle_{H^{-1}(\Omega)^d\times H^1_0(\Omega)^d} = 0$ for $\boldsymbol{\psi} \in \mathbf{H}^1_{div}(\Omega)$ (cf. Remark 4.1) it follows that $G^\varepsilon \equiv \nabla \widetilde{P}^\varepsilon$, i.e., G^ε is the gradient of a function $\widetilde{P}^\varepsilon$ in $L^2_0(\Omega)$, cf. [47, Proposition I.1.1]. In particular, it can be shown (cf. [5, proof of Proposition 4.1]) that the extension $\widetilde{P}^\varepsilon$ is given by

$$\widetilde{P}^\varepsilon(t,x) := \begin{cases} P^\varepsilon(t,x) & \text{in } \Omega^\varepsilon, \\ \dfrac{1}{|Y_{p_k}|}\displaystyle\int_{Y_{p_k}} P^\varepsilon(t,y)\,dy & \text{in each } Y_{s_k}. \end{cases} \tag{5.6}$$

Finally, the bound (5.5) implies that $\sup_{t\in[0,T]} \|\tilde{P}^\varepsilon(t)\|_{L^2_0(\Omega)} \le C$ (with C independent of ε), cf. [47, Proposition III.1.1]. □

Lemma 5.2. *Let $(\mathbf{q}^\varepsilon, P^\varepsilon, u^\varepsilon, w^\varepsilon)_{\varepsilon>0}$ be the extension of the weak solution from Lemma 5.1 (denoted by the same symbol). Then there exists some functions $\mathbf{q} \in L^2(I\times\Omega; H^1_\#(Y))^d$, $u, w \in L^2(I; H^1(\Omega))$, $P \in L^2(I\times\Omega\times Y)$, $u_1, w_1 \in L^2(I\times\Omega; H^1_\#(Y))$ and a subsequence of $(\mathbf{q}^\varepsilon, P^\varepsilon, u^\varepsilon, w^\varepsilon)_{\varepsilon>0}$ (not relabeled) such that the following convergence results hold:*

(5.7a) *(i) $(u^\varepsilon)_{\varepsilon>0}$ is two-scale convergent to u,*

(5.7b) *(ii) $(\mathbf{q}^\varepsilon)_{\varepsilon>0}$ is two-scale convergent to $\mathbf{q}$,*

(5.7c) *(iii) $(w^\varepsilon)_{\varepsilon>0}$ is two-scale convergent to w,*

(5.7d) *(iv) $(P^\varepsilon)_{\varepsilon>0}$ is two-scale convergent to P,*

(5.7e) *(v) $(\nabla_x u^\varepsilon)_{\varepsilon>0}$ is two-scale convergent to $\nabla_x u + \nabla_y u_1$,*

(5.7f) *(vi) $(\nabla_x w^\varepsilon)_{\varepsilon>0}$ is two-scale convergent to $\nabla_x w + \nabla_y w_1$ and*

(5.7g) *(vii) $(\varepsilon\nabla_x \mathbf{q}^\varepsilon)_{\varepsilon>0}$ is two-scale convergent to $\nabla_y \mathbf{q}$*

in the sense of (3.7) respectively.

Proof. The convergence follows from the estimate (5.1) and Lemmas 3.6, 3.8 and 3.9. □

In the next lemma we discuss the convergence of nonlinear terms for $\varepsilon \to 0$.

Lemma 5.3. *The following convergence results hold:*

(i) *$(u^\varepsilon)_{\varepsilon>0}$ is strongly convergent to u in $L^2(I\times\Omega)$.*
(ii) *$\mathcal{T}^\varepsilon(u^\varepsilon)_{\varepsilon>0}$ converges to u strongly in $L^2(I\times\Omega\times Y)$, whereas $\mathcal{T}^\varepsilon[\nabla_x w^\varepsilon]$ and $\mathcal{T}^\varepsilon\mathbf{q}^\varepsilon$ converge respectively to $\nabla_x w + \nabla_y w_1$ and $\mathbf{q}$ weakly in $L^2(I\times\Omega\times Y)$.*
(iii) *The nonlinear terms $f(u^\varepsilon)$, $u^\varepsilon\nabla_x w^\varepsilon$ and $\mathbf{q}^\varepsilon u^\varepsilon$ two-scale converge to $f(u)$, $u(\nabla_x w + \nabla_y w_1)$ and $\mathbf{q}u$, respectively.*

Proof. (i) From Lemma (5.2) and Lemma (5.1) it follows that, up to a subsequence, still denoted by the same subscript, $(u_\varepsilon)_{\varepsilon>0}$ is weakly convergent to u and is bounded in $L^2(I; H^1(\Omega))$. By the estimate (5.1) we have that $(\partial_t u^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(I; H^1(\Omega)^*)$. Therefore, by Lemma 3.1 there exists a subsequence $(u^\varepsilon)_{\varepsilon>0}$ that is strongly convergent to u in $L^2(I\times\Omega)$.

(ii) By (i) u^ε converges strongly to the limit u. Hence, by [15, Proposition 2.9 (ii)], [23, Lemma 5.1 (f)] it follows that $\mathcal{T}^\varepsilon u^\varepsilon$ is strongly convergent to u. Similarly, [23, Lemma 5.1] and the estimates of Lemma 5.2 imply that $\mathcal{T}^\varepsilon\nabla w^\varepsilon$ and $\mathcal{T}^\varepsilon q^\varepsilon$ are weakly convergent and their weak limits coincide with the corresponding two-scale limits from Lemma 5.2. Similarly, we obtain that $\mathcal{T}^\varepsilon[\nabla_x w^\varepsilon]$ converges weakly to $\nabla_x w + \nabla_y w_1$ in $L^2(I\times\Omega\times Y)$, cf. [23, Lemma 6.1].

(iii) We note that by the equivalence of the two-scale convergence and the weak convergence of the unfolded operator, cf. [15, Proposition 2.14], the convergence $f(u^\varepsilon) \stackrel{2}{\rightharpoonup} f(u)$ is equivalent to the weak convergence $\mathcal{T}^\varepsilon f(u^\varepsilon) \equiv f(\mathcal{T}^\varepsilon u^\varepsilon) \rightharpoonup f(u)$. The strong convergence $\mathcal{T}^\varepsilon u^\varepsilon \to u$ in $L^2(I\times\Omega\times Y)$ from part (ii) implies a.e. pointwise (sub-)convergence of $\mathcal{T}^\varepsilon u^\varepsilon$ in $I\times\Omega\times Y$. Due to the integral preserving property of the unfolding operator, cf. [23, Lemma 4.2 b)], we have that $\|\mathcal{T}^\varepsilon u^\varepsilon\|_{L^4(I\times\Omega\times Y)} = \|u^\varepsilon\|_{L^4(I\times\Omega)} \leq C$ by the estimate (5.1). Since $f(s)=s^3-s$ the convergence

$$\int_{I\times\Omega\times Y} f(\mathcal{T}^\varepsilon u^\varepsilon)\varphi \, dx\, dy\, dt \to \int_{I\times\Omega\times Y} f(u)\varphi\, dx\, dy\, dt \qquad \text{for} \qquad \varphi\in C^\infty(I\times\Omega; C^\infty_\#(Y)),$$

follows from the pointwise convergence and the bound $\|\mathcal{T}^\varepsilon u^\varepsilon\|_{L^4(I\times\Omega\times Y)}$ by the generalized dominated convergence theorem.

The two-scale convergence $u^\varepsilon\nabla_x w^\varepsilon \stackrel{2}{\rightharpoonup} u(\nabla_x w+\nabla_y w_1)$ is equivalent to the weak convergence $\mathcal{T}^\varepsilon u^\varepsilon \mathcal{T}^\varepsilon[\nabla_x w^\varepsilon] \rightharpoonup u(\nabla_x w+\nabla_y w_1)$, cf. [15, Proposition 2.14]. Hence, for $\boldsymbol{\varphi}\in C^\infty(I\times\Omega; C^\infty_\#(Y))^d$ we estimate

$$\begin{aligned}
&\int_{I\times\Omega\times Y} \mathcal{T}^\varepsilon u^\varepsilon \mathcal{T}^\varepsilon[\nabla_x w^\varepsilon]\cdot\boldsymbol{\varphi}\, dx\, dy\, dt - \int_{I\times\Omega\times Y} u(\nabla_x w+\nabla_y w_1)\boldsymbol{\varphi}\, dx\, dy\, dt\\
&\leq \Big|\int_{I\times\Omega\times Y} (\mathcal{T}^\varepsilon u^\varepsilon - u)\mathcal{T}^\varepsilon[\nabla_x w^\varepsilon]\cdot\boldsymbol{\varphi}\, dx\, dy\, dt\Big| + \Big|\int_{I\times\Omega\times Y} u(\mathcal{T}^\varepsilon[\nabla_x w^\varepsilon] - \nabla_x w + \nabla_y w_1)\cdot\boldsymbol{\varphi}\, dx\, dy\, dt\Big|\\
&= I_1 + I_2\,.
\end{aligned}$$

Since $\|\mathcal{T}^\varepsilon[\nabla_x w^\varepsilon]\|_{L^2(I\times\Omega\times Y)} = \|\nabla_x w^\varepsilon\|_{L^2(I\times\Omega)} \leq C$, we get for $\varepsilon\to 0$

$$I_1 \leq \|\boldsymbol{\varphi}\|_{L^\infty(I\times\Omega\times Y)}\|\mathcal{T}^\varepsilon u^\varepsilon - u\|_{L^2(I\times\Omega\times Y)}\|\mathcal{T}^\varepsilon[\nabla_x w^\varepsilon]\|_{L^2(I\times\Omega\times Y)} \to 0\,,$$

because of the strong convergence $\mathcal{T}^\varepsilon u^\varepsilon \to u$ in $L^2(I\times\Omega\times Y)$. Furthermore

$$I_2 = \Big|\int_{I\times\Omega\times Y} (\mathcal{T}^\varepsilon[\nabla_x w^\varepsilon] - \nabla_x w + \nabla_y w_1)\cdot(u\boldsymbol{\varphi})\, dx\, dy\, dt\Big| \to 0\,,$$

since by part (ii) $\mathcal{T}^\varepsilon[\nabla_x w^\varepsilon] \rightharpoonup \nabla_x w + \nabla_y w_1$ in $L^2(I\times\Omega\times Y)$.

Analogically we obtain the convergence $\mathbf{q}^\varepsilon u^\varepsilon \stackrel{2}{\rightharpoonup} \mathbf{q}u$. $\square$

The main result of this paper is stated in the following theorem.

Theorem 5.1. *Let the extended initial condition converge as $\mathbf{q}_0^\varepsilon \to \mathbf{q}_0$, $u_0^\varepsilon \to u_0$ for $\varepsilon\to 0$ in $L^2(\Omega)$. Then there exists a $p_1 := \partial_t P_1$, $P_1 \in L^\infty(I; L^2_0(\Omega; L^2_\#(Y_p)))$ such that the limiting functions $\mathbf{q}$, u, w, $p := \partial_t P$, u_1, w_1 from Lemma 5.2 satisfy the following system of equations in the distributional sense (to simplify the notation we omit the dependence of the solution on the time variable t)*

(5.8a)
$$\partial_t\mathbf{q}(x,y) - \mu\Delta_y\mathbf{q}(x,y) + \nabla_y p_1(x,y) + \nabla_x p(x) = -u(x)\big(\nabla_x w(x) + \nabla_y w_1(x,y)\big) \qquad \textit{in } I\times\Omega\times Y_p,$$

$$\nabla_y\cdot\mathbf{q}(x,y) = 0 \qquad \textit{in } I\times\Omega\times Y_p, \tag{5.8b}$$

$$\nabla_x\cdot\overline{\mathbf{q}}(x) = 0 \qquad \textit{in } I\times\Omega, \tag{5.8c}$$

$$\mathbf{q}(x,y) = 0 \qquad \textit{on } I\times\Omega\times\Gamma_s, \tag{5.8d}$$

$$\overline{\mathbf{q}}(x)\cdot\mathbf{n}_{\partial\Omega} = 0 \qquad \textit{on } I\times\partial\Omega, \tag{5.8e}$$

$$\mathbf{q}(0,x) = \mathbf{q}_0(x) \qquad \textit{in } \Omega, \tag{5.8f}$$

$$\partial_t u(x) - \Delta_x w(x) - \nabla_x\cdot\overline{\nabla_y w_1}(x) = -\nabla_x\cdot(\overline{\mathbf{q}}(x)u(x)) \qquad \textit{in } I\times\Omega, \tag{5.8g}$$

$$w(x) + \lambda^2\left(\Delta_x u(x) + \nabla_x\cdot\overline{\nabla_y u_1}(x)\right) = f(u(x)) \qquad \textit{in } I\times\Omega, \tag{5.8h}$$

$$\left[\nabla_x u(x) + \overline{\nabla_y u_1}(x)\right]\cdot\mathbf{n}_{\partial\Omega} = 0 \qquad \textit{on } I\times\partial\Omega, \tag{5.8i}$$

$$[\nabla_x w(x) + \overline{\nabla_y w_1}(x) - \overline{\mathbf{q}}(x)u(x)]\cdot\mathbf{n}_{\partial\Omega} = 0 \qquad \textit{on } I\times\partial\Omega, \tag{5.8j}$$

$$u(0,x) = u_0(x) \qquad \textit{in } \Omega, \tag{5.8k}$$

where $\overline{\xi}(x) = \dfrac{1}{|Y_p|}\displaystyle\int_{Y_p} \xi(x,y)\; dy,\; x\in\Omega$ *denotes the mean of the quantity* ξ *over the pore space* Y_p *and*

$$-\Delta_y w_1(x,y) + \nabla_y\cdot(\mathbf{q}(x,y)u(x)) = \nabla_y\cdot\nabla_x w(x) \qquad in\ I\times\Omega\times Y_p\,, \tag{5.9}$$

$$-\Delta_y u_1(x,y) = \nabla_y\cdot\nabla_x u(x) \qquad in\ I\times\Omega\times Y_p\,. \tag{5.10}$$

Remark 5.1. *The homogenized system (5.8) is a two-scale model where equations (5.8c), (5.8g), (5.8h) can be viewed as a macroscale Darcy-Cahn-Hilliard system and the equations (5.8a), (5.8b), (5.9)-(5.10) are microscale problems defined on* Y_p *where the macroscale variable* x *enters as a parameter. We also note that the equations (5.8g), (5.8h) depend implicitly on the microscale variable* y *via the coupling with the microscale problems (5.8a),(5.9)-(5.10).*

Proof. We show that the limiting functions from Lemma 5.2 and Lemma 5.3 satisfy (5.8), (5.9), (5.10).

(i) We first consider the homogenization of the Cahn-Hilliard part of the system. Let us choose the functions $\phi_0\in C_0^\infty(I\times\Omega)$ and $\phi_1\in C_0^\infty(I\times\Omega;C_\#^\infty(Y))$. We take $\varphi(t,x,\frac{x}{\varepsilon}) = \phi_0(t,x) + \varepsilon\phi_1(t,x,\frac{x}{\varepsilon})$ in (4.1b) and obtain

$$\begin{aligned}
&-\int_{I\times\Omega^\varepsilon} u^\varepsilon(t,x)(\partial_t\phi_0(t,x,\frac{x}{\varepsilon})+\varepsilon\partial_t\phi_1(t,x,\frac{x}{\varepsilon}))\,dx\,dt\\
&\quad+\int_{I\times\Omega^\varepsilon}\nabla_x w^\varepsilon(t,x)\cdot\Big(\nabla_x\phi_0(t,x,\frac{x}{\varepsilon})+\varepsilon\nabla_x\phi_1(t,x,\frac{x}{\varepsilon})+\nabla_y\phi_1(t,x,\frac{x}{\varepsilon})\Big)\,dx\,dt\\
&\quad\quad-\int_{I\times\Omega^\varepsilon}\mathbf{q}^\varepsilon(t,x)u^\varepsilon(t,x)\cdot\Big(\nabla_x\phi_0(t,x,\frac{x}{\varepsilon})+\varepsilon\nabla_x\phi_1(t,x,\frac{x}{\varepsilon})+\nabla_y\phi_1(t,x,\frac{x}{\varepsilon})\Big)\,dx\,dt=0\,,
\end{aligned}$$

or equivalently (using the extensions of solution to Ω with the same notation)

$$\begin{aligned}
&-\int_{I\times\Omega}\chi(\frac{x}{\varepsilon})u^\varepsilon(t,x)\partial_t\phi_0(t,x,\frac{x}{\varepsilon})\,dx\,dt\\
&\quad+\int_{I\times\Omega}\chi(\frac{x}{\varepsilon})\nabla_x w^\varepsilon(t,x)\cdot\Big(\nabla_x\phi_0(t,x,\frac{x}{\varepsilon})+\varepsilon\nabla_x\phi_1(t,x,\frac{x}{\varepsilon})+\nabla_y\phi(t,x,\frac{x}{\varepsilon})\Big)\,dx\,dt\\
&\quad\quad-\int_{I\times\Omega}\chi(\frac{x}{\varepsilon})\mathbf{q}^\varepsilon(t,x)u^\varepsilon(t,x)\cdot\Big(\nabla_x\phi_0(t,x,\frac{x}{\varepsilon})+\varepsilon\nabla_x\phi_1(t,x,\frac{x}{\varepsilon})+\nabla_y\phi(t,x,\frac{x}{\varepsilon})\Big)\,dx\,dt=0.
\end{aligned}$$

We pass $\varepsilon\to 0$ in the two-scale sense. The terms containing ε are bounded and the limits converge to 0. Hence, we get

$$\begin{aligned}
&-\int_{I\times\Omega\times Y_p} u(t,x)\partial_t\phi_0(t,x)\,dx\,dy\,dt\\
&\quad+\int_{I\times\Omega\times Y_p}(\nabla_x w(t,x)+\nabla_y w_1(t,x,y))\cdot(\nabla_x\phi_0(t,x)+\nabla_y\phi_1(t,x,y))\,dx\,dy\,dt\\
&\quad\quad-\int_{I\times\Omega\times Y_p}\mathbf{q}(t,x,y)\;u(t,x)\cdot(\nabla_x\phi_0(t,x)+\nabla_y\phi_1(t,x,y))\,dx\,dy\,dt=0.
\end{aligned}\tag{5.11}$$

We choose $\phi_0=0$ in (5.11) and get

$$\int_{I\times\Omega\times Y_p}(\nabla_x w(t,x)+\nabla_y w_1(t,x,y)-\mathbf{q}(t,x,y)\;u(t,x))\cdot\nabla_y\phi_1(t,x,y)\,dx\,dy\,dt=0\,, \tag{5.12}$$

which implies (5.9). Similarly, setting $\phi_1=0$ in (5.11) yields (5.8g).

To show (5.8h) and (5.10), we set $\phi = \phi_0 + \varepsilon\phi_1$ in (4.1c) and obtain for $\varepsilon \to 0$

$$\begin{aligned}\int_{I\times\Omega\times Y_p} & w(t,x,y)\phi_0(t,x)\,dx\,dy\,dt \\ &= \lambda^2 \int_{I\times\Omega\times Y_p} (\nabla_x u(t,x) + \nabla_y u_1(t,x,y)) \cdot (\nabla_x \phi_0(t,x) + \nabla_y \phi_1(t,x,y))\,dx\,dy\,dt \\ &\qquad + \int_{I\times\Omega\times Y_p} f(u(t,x))\phi_0(t,x)\,dx\,dy\,dt. \end{aligned} \tag{5.13}$$

Setting $\phi_0 = 0$ in (5.13) implies

$$\int_{I\times\Omega\times Y_p} (\nabla_x u(t,x) + \nabla_y u_1(t,x,y)) \cdot \nabla_y \phi_1(t,x,y)\,dx\,dy\,dt = 0, \tag{5.14}$$

which is the weak formulation of (5.10). Taking $\phi_1 = 0$ in (5.13) yields the weak formulation of (5.8h).

The boundary conditions (5.8i), (5.8j) follow after an application of the integration by parts formula in (5.11), (5.13), respectively.

(ii) We perform the two-scale limit in the Stokes equations. We choose a $\boldsymbol{\psi} \in C_0^\infty(\Omega; C_\#^\infty(Y))^d$, $\phi \in C_0^\infty(I)$ in (4.16). Then, due to Lemma 5.1 and Lemma 5.2 we obtain for $\varepsilon \to 0$

$$\begin{aligned} 0 &= \int_I \int_{\Omega\times Y_p} P(t,x,y)\nabla_y \cdot \boldsymbol{\psi}(x,y)\partial_t\phi(t)\,dx\,dy\,dt \\ &= \lim_{\varepsilon\to 0} \int_I \int_{\Omega^\varepsilon} P^\varepsilon(t,x)\left(\nabla_x \cdot \boldsymbol{\psi}(x,\frac{x}{\varepsilon}) + \frac{1}{\varepsilon}\nabla_y \cdot \boldsymbol{\psi}(x,\frac{x}{\varepsilon})\right)\partial_t\phi(t)\,dx\,dt. \end{aligned} \tag{5.15}$$

It follows from (5.15) that the two-scale limit of the pressure P is independent of y, i.e., $P(t) \in L_0^2(\Omega)$ for a.a. $t \in I$.

Next, we proceed similarly as in part (i). We take $\boldsymbol{\psi} \in C_0^\infty(\Omega; C_\#^\infty(Y))^d$ such that $\nabla_y \cdot \boldsymbol{\psi}(x,y) = 0$ in (4.16), and obtain using Lemma 5.2 and Lemma 5.3 for $\varepsilon \to 0$ that

$$\begin{aligned} -\int_{I\times\Omega\times Y_p} & \mathbf{q}(t,x,y)\boldsymbol{\psi}(x,y)\partial_t\phi(t)\,dx\,dy\,dt + \int_{I\times\Omega\times Y_p} P(t,x)\nabla_x \cdot \boldsymbol{\psi}(x,y)\partial_t\phi(t)\,dx\,dt \\ &+ \mu \int_{I\times\Omega\times Y_p} \nabla_y \mathbf{q}(t,x,y) : \nabla_y \boldsymbol{\psi}(x,y)\phi(t)\,dx\,dy\,dt \\ &= -\int_{I\times\Omega\times Y_p} u(t,x)(\nabla_x w(t,x) + \nabla_y w_1(t,x,y)) \cdot \boldsymbol{\psi}(x,y)\phi(t)\,dx\,dy\,dt. \end{aligned}$$

Furthermore, using [4, Lemma 3.8] we obtain similarly as in part (vi) of the proof of Theorem 4.1 the existence of a pressure $P_1 \in L^\infty(I; L_0^2(\Omega; L_\#^2(Y_p)))$ such that

$$\begin{aligned} -\int_{I\times\Omega\times Y_p} & \mathbf{q}(t,x,y)\boldsymbol{\psi}(x,y)\partial_t\phi(t)\,dx\,dy\,dt + \mu \int_{I\times\Omega\times Y_p} \nabla_y \mathbf{q}(t,x,y) : \nabla_y \boldsymbol{\psi}(x,y)\phi(t)\,dx\,dy\,dt \\ + \int_{I\times\Omega\times Y_p} & P(t,x)\nabla_x \cdot \boldsymbol{\psi}(x,y)\partial_t\phi(t)\,dx\,dt + \int_{I\times\Omega\times Y_p} P_1(t,x,y)\nabla_y \cdot \boldsymbol{\psi}(x,y)\partial_t\phi(t)\,dx\,dt \\ &= -\int_{I\times\Omega\times Y_p} u(t,x)(\nabla_x w(t,x) + \nabla_y w_1(t,x,y)) \cdot \boldsymbol{\psi}(x,y)\phi(t)\,dx\,dy\,dt. \end{aligned} \tag{5.16}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega; C_\#^\infty(Y))^d$, $\phi \in C_0^\infty(I)$. Hence, (5.16) implies that (5.8a) holds in the distributional sense with $p = \partial_t P$, $p_1 = \partial_t P_1$.

The remaining identities (5.8b)-(5.8e) follow as in [4, Theorem 3.1]. $\square$

Remark 5.2. *The pore-scale model considered in* [42] *is similar to the model (2.5) except for the fact that they considered stationary Stokes equations and more general boundary conditions along with a different scaling of the advection term. Using the splitting*

$$w^\varepsilon = -\Delta u^\varepsilon \tag{5.17}$$

and the asymptotic expansion method they obtain a macroscale model (cf. equations (29)-(31) in [42])

$$\begin{aligned} \textstyle\int_{Y_p} \left\{\partial_t[\Delta_x^{-1} w] - \Delta_x[w + f(u)] + \nabla_x \cdot \nabla_y[w_1 + f'(u)u_1]\right\} &= 0\,, \\ w &= \textstyle\int_{Y_p} \{-\Delta_x u - \nabla_x \cdot \nabla_y u_1\}\,. \end{aligned} \tag{5.18}$$

and a micro-scale model

$$\begin{aligned} -\Delta_y u_1 &= \nabla_y \cdot \nabla_x u\,, \\ -\Delta_y[w_1 + f'(u)u_1] - \nabla_y \cdot \nabla_x[w + f(u)] &= -\mathbf{q} \cdot \nabla_y[\Delta_x^{-1} w_1] - (\mathbf{q} - \overline{\mathbf{q}}) \cdot \nabla_x[\Delta_x^{-1} w]. \end{aligned} \tag{5.19}$$

where $\overline{\mathbf{q}}$ is the mean velocity. Due to the different splitting (note we use (2.5f) instead of (5.17)) a direct comparison with (5.8) is not obvious. Nevertheless, it is possible to identify some common features between the two models.

By denoting $w^* := w + f(u)$, $w_1^* := w_1 + f'(u)u_1$, $u^* := \Delta_x^{-1} w$, one can rewrite $(5.18)_1$ as ($\overline{\nabla_y w^*}$ denotes the mean over the pore space Y_p)

$$\partial_t u^* - \Delta_x w^* + \nabla_x \cdot \overline{\nabla_y w_1^*} = 0\,,$$

which is an equation that shares common features with equation (5.8g) (note that the advection term is different due to different scaling in the respective models). Similarly, for instance, for the micro problem $(5.19)_2$, with the additional notation $u_1^* = \Delta_x^{-1} w_1$, we get

$$-\Delta_y w_1^* - \nabla_y \cdot \nabla_x w^* = -\mathbf{q} \cdot \nabla_y u_1^* - (\mathbf{q} - \overline{\mathbf{q}}) \cdot \nabla_x u^*\,,$$

which is an analogue of (5.9) (again with a different advection term).

For $\mathbf{q} = 0$ the above equations (5.18), (5.19) reduce to the Cahn-Hilliard equation derived [41]. In addition, it appears that our upscaled model (5.8g)-(5.8h) with $\mathbf{q} = 0$, agrees with the upscaled Cahn-Hilliard model derived in [30], however the paper doesn't consider homogenization of the Cahn-Hilliard equation in porous medium explicitly.

We also note that in the case of single-phase flow the upscaled equations (5.8a) reduce to the time-dependent Darcy model obtained in [4].

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