ERGODICITY FOR A STOCHASTIC GEODESIC EQUATION IN THE TANGENT BUNDLE OF THE 2D SPHERE

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Abstract. We study ergodic properties of stochastic geometric wave equations on a particular model with the target being the $2D$ sphere while considering the space variable-independent solutions only. This simplification leads to a degenerate stochastic equation in the tangent bundle of the $2D$ sphere. Studying this equation, we prove existence and non-uniqueness of invariant probability measures for the original problem and we obtain also results on attractivity towards an invariant measure. We also present a structure-preserving numerical scheme to approximate solutions and provide computational experiments to motivate and illustrate the theoretical results.

1. Introduction

Wave equations subject to random excitations have been largely studied in the last fourty years for its applications in physics, relativistic quantum mechanics or oceanography, see e.g. [10], [11], [12], [13], [17], [27], [28], [32], [35], [34], [37], [16], [26], [25], [33], [36], [38]. The mathematical research has paid attention predominantly to stochastic wave equations whose solutions took values in Euclidean spaces, however many physical theories and models in modern physics such as harmonic gauges in general relativity, non-linear σ-models in particle systems, electro-vacuum Einstein equations or Yang-Mills field theory require the target space of the solutions to be a Riemannian manifold, see e.g. [19] and [39]. Stochastic wave equations with values in Riemannian manifolds were first studied in [8] (see also [7]) where existence and uniqueness of global strong solutions were proved for equations defined on the one-dimensional Minkowski space $\mathbb{R}^{1+1}$ and arbitrary Riemannian manifold. Later, in [9], global existence was proved for equations on a general Minkowski space $\mathbb{R}^{1+d}$ with the target space being restricted to homogeneous spaces (for instance, a sphere) and, in [7], global existence of weak solutions was proved for equations on $\mathbb{R}^{1+1}$ with an arbitrary target. The last two works admitted rougher noises than in [8], but for the price of not dealing with the question of uniqueness and of worse spatial regularity of the solutions.

In the present paper, we intend to open the door and enter into the study of ergodic properties of solutions of these equations. In particular, we are interested in existence and uniqueness (or multitude) of invariant measures of the Markov semigroup associated to solutions of a stochastic geometric equation and we also want to address the questions of ergodic properties and of the rates of convergence to an attracting law, if there is any.

This goal however seems to be fairly complicated and too ambitious to achieve at once, hence we will proceed *a minori ad majus* and we will study just space independent solutions of a damped stochastic geometric wave equation in the 2D sphere. This particular exemplary equation is, in our opinion, quite illustrative to understand what one can expect in the general case. In this way, the stochastic equation will reduce to a degenerate second order stochastic differential equation with values in the tangent bundle $T\mathbb{S}^2$. We will prove that there exist plenty of invariant measures and that the system always converges in total variation to a limit law. If we however restrict the state space to a suitable submanifold in $T\mathbb{S}^2$ then there exists just one unique invariant measure

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(the normalized surface measure on this submanifold) which attracts every initial distribution in total variation with an exponential rate.

A further goal of this paper is to construct a numerical scheme for solving a class of SDEs on manifolds – the geodesic equation on the sphere $\mathbb{S}^2$ with stochastic forcing. A convergent discretisation in space and time for a first order stochastic Landau-Lifshitz-Gilbert equation where solutions take values in $\mathbb{S}^2$ is proposed in [?, ?]; the present case is however very different, and the structure preserving discretization given in Section 6.1 is inspired by the 'discrete Lagrange multiplier' strategy developed in [4].

Computational examples for the stochastic geodesic equation on the sphere are provided in Section 6.2 to motivate long-time asymptotics, which is then studied analytically in the later sections.

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2. Notation and conventions

If Y is a topological space, we will denote by $B_b(Y)$ the space of real bounded Borel functions on Y, by $C_b(Y)$ the space of real bounded continuous functions on Y, by $\mathscr{B}(Y)$ the Borel σ-algebra over Y. We will work on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration $(\mathscr{F}_t)$ such that $\mathscr{F}_0$ contains all $\mathbb{P}$-negligible sets in $\mathscr{F}$ and W will be a standard $(\mathscr{F}_t)$-Wiener process. Throughout this paper, all initial conditions are assumed to be $\mathscr{F}_0$-measurable.

3. The problem

Let M be a compact m-dimensional Riemannian manifold embedded in a Euclidean space $\mathbb{R}^n$. Denote by T_pM the tangent space at $p \in M$, by $N_pM = (T_pM)^\perp$ the normal space at $p \in M$, by $TM = \bigcup_{p\in M} T_pM$ and $T^kM = \bigcup_{p\in M}(T_pM)^k$ the tangent bundle and the k-tangent bundle of M resp., by $S_p : T_pM \times T_pM \to N_pM$, $p \in M$ the second fundamental form of M in $\mathbb{R}^n$ and let W be, for simplicity, a one-dimensional Wiener process. According to [8], the general Cauchy problem for a stochastic geometric wave equation has the form

$$du_t = [\Delta u - \sum_{i=1}^{m} S_u(u_{x_i}, u_{x_i}) + S_u(u_t, u_t) + F_u(Du)]\, dt + G_u(Du)\, dW \tag{3.1}$$

$$u \in M \tag{3.2}$$

$$(u(0), u_t(0)) \in TM \tag{3.3}$$

where F is a drift, G a diffusion and Du denotes the $(m+1)$-tuple $(u_t, u_{x_1}, \dots, u_{x_m})$ in the equation (3.1). For the equation to make sense, it is required that $F : T^{m+1}M \to TM$ and $G : T^{m+1}M \to TM$ are Borel measurable and that $F_p(X_0, \dots, X_m)$ and $G_p(X_0, \dots, X_m)$ belong to the tangent space T_pM for every $p \in M$ and every $X_0, \dots, X_m \in T_pM$.

In case M is the unit sphere in $\mathbb{R}^3$ then the second fundamental form satisfies $S_p(X, Y) = -\langle X, Y\rangle p$. If we set $F_p(X_0, X_1, X_2) = -\frac{1}{2}X_0$, $G_p(X_0, X_1, X_2) = p \times X_0$ then the equation (3.1) with the constraints (3.2), (3.3) has the form

$$du_t = [\Delta u + (|\nabla u|^2 - |u_t|^2)u - \frac{1}{2}u_t]\, dt + u \times u_t\, dW, \qquad |u| = 1, \qquad u(0) \perp u_t(0). \tag{3.4}$$

If we consider just space independent solutions, i.e. solutions independent of the spatial variables, then (3.4) reduces to an Itô SDE

$$du' = [-|u'|^2 u - \frac{1}{2}u']\, dt + (u \times u')\, dW, \qquad |u| = 1, \qquad u(0) \perp u'(0) \tag{3.5}$$

or, equivalently, to a Stratonovich SDE

$$du' = -|u'|^2 u\, dt + (u \times u') \circ dW, \qquad |u| = 1, \qquad u(0) \perp u'(0) \tag{3.6}$$

which is the stochastic geodesic equation for the unit sphere[1]. Let us rewrite (3.6) to two equations of first order equations

$$dz = f(z)\,dt + g(z)\circ dW, \qquad z\in T\mathbb{S}^2, \qquad z(0)\in T\mathbb{S}^2 \tag{3.7}$$

where $T\mathbb{S}^2 \subseteq \mathbb{R}^6$ is the tangent bundle of $\mathbb{S}^2$, i.e. $T\mathbb{S}^2 = \{(u,v) : |u| = 1, u\perp v\}$ and

$$z = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad f(z) = \begin{pmatrix} v \\ -|v|^2 u \end{pmatrix}, \qquad g(z) = \begin{pmatrix} 0 \\ u\times v \end{pmatrix}. \tag{3.8}$$

Remark 3.1. Observe that restrictions of f and g to $T\mathbb{S}^2$ are vector fields on the manifold $T\mathbb{S}^2$. Hence (3.7) is a correctly defined stochastic differential equation on the manifold $T\mathbb{S}^2$, cf. [24, Chapter V].

The equation (3.5) and its equivalent formulations (3.6), (3.7) will be *the object of study* of the present paper. It is also important to realize while reading the paper that (3.5) is a particular case of the stochastic geometric wave equation (3.1)-(3.3).

4. Basic properties of solutions of the SDE

We will study existence of global solutions, dependence on initial conditions, some further qualitative properties of solutions of the equation (3.7) and the Feller property of the associated Markov semigroup.

4.1. **Global existence.** The nonlinearities of the equation (3.7) are locally Lipschitz on $\mathbb{R}^6$ hence, by the standard existence result (see e.g. [24, Lemma 2.1]), the equation (3.7) considered without the constraint,

$$dz = f(z)\,dt + g(z)\circ dW, \qquad z(0)\in T\mathbb{S}^2, \tag{4.1}$$

has a unique local solution z in $\mathbb{R}^6$ defined up to an explosion time $\tau>0$, i.e.

$$\limsup_{t\uparrow\tau} |z(t)| = \infty \quad \text{almost surely on} \quad [\tau<\infty]. \tag{4.2}$$

Proposition 4.1. *The solution to* (4.1) *is unique, global and satisfies* $z = (u,v)\in T\mathbb{S}^2$*, i.e. it is a solution to the equation* (3.7)*. Moreover,* $|v(t)| = |v(0)|$ *for every* $t\geq 0$ *almost surely.*

Proof. Applying the Itô formula to $|u|^2$, we obtain that $\phi = |u|^2 - 1$ satisfies almost surely on $[0,\tau)$ the ODE

$$\phi'' = -2|v|^2\phi - \frac{1}{2}\phi', \qquad \phi(0) = 0, \qquad \phi'(0) = 0. \tag{4.3}$$

Hence, by the uniqueness of the solutions to the equation (4.3), we obtain that $\phi = 0$ on $[0,\tau)$, consequently, $|u| = 1$ on $[0,\tau)$ almost surely. In particular, differentiating $|u|^2 = 1$, we obtain that $u\perp v = 0$ on $[0,\tau)$ almost surely. Now, applying the Itô formula to $|v|^2$, we obtain that $\varphi = |v|^2$ satisfies on $[0,\tau)$ almost surely the equation

$$\varphi' = -(1 + 2\langle u,v\rangle)|v|^2 + |u\times v|^2.$$

The right hand side equals to

$$-(1 + 2\langle u,v\rangle)|v|^2 + |u|^2|v|^2 - \langle u,v\rangle^2 = 0$$

as $u\perp v$ and $|u| = 1$ almost surely. Hence $|v|$ is pathwise constant. In particular, $\tau = \infty$ almost surely by (4.2). □

4.2. **The Markov and the Feller property.** Define $Y = \mathbb{R}^n$. It is well known that if $\tilde{f}$, $\tilde{g}$ are C^∞ vector fields on $\mathbb{R}^n$ with a compact support and u^ξ denotes the solution of the equation

$$dX = \tilde{f}(X)\,dt + \tilde{g}(X)\circ dW, \qquad X(0) = \xi \tag{4.4}$$

for an $\mathscr{F}_0$-measurable Y-valued random variable ξ then the solutions of the equation (4.4) satisfy the Markov property and define a Feller semigroup[2] on Y by which we mean that

[1] The geodesic equation for the unit sphere has the form $u'' = -|u'|^2 u$, $|u| = 1$, $u'(0)\perp u(0)$.

[2] We allow here a little inaccuracy. More precisely, the semigroup is defined on the space of bounded Borel functions on Y.

(a) the transition function

$$q_{t,x}(A) = \mathbb{P}\,[u^x(t) \in A], \qquad t \geq 0,\ x \in Y,\ A \in \mathscr{B}(Y)$$

is jointly measurable in $(t,x) \in [0,\infty) \times Y$ for every $A \in \mathscr{B}(Y)$,

(b) the endomorphisms on $B_b(Y)$

$$Q_t\varphi(x) = \mathbb{E}\,\varphi(u^x(t)), \qquad t \geq 0,\ x \in Y,\ \varphi \in B_b(Y)$$

satisfy the semigroup property, i.e. $Q_t \circ Q_s = Q_{t+s}$ for every $t, s \geq 0$,

(c) $Q_t\varphi$ is continuous on Y whenever $t \geq 0$ and $\varphi \in C_b(Y)$,

(d) $\mathbb{E}\,[\varphi(u^\xi(t))|\mathscr{F}_s] = (Q_{t-s}\varphi)(u^\xi(s))$ holds a.s. for every $\varphi \in B_b(Y)$, $0 \leq s \leq t$ and an $\mathscr{F}_0$-measurable Y-valued random variable ξ,

see e.g. [15, Section 9.2.1]. In fact, (a) and (c) follow simply from the fact that

$$Q_t\varphi(x) \text{ is jointly continuous in } (t,x) \text{ on } [0,\infty) \times Y \text{ if } \varphi \in C_b(Y), \tag{4.5}$$

see again [15, Section 9.2.1] for the proof of (4.5), and the semigroup property (b) follows from the Markov property (d).

Moreover, if $\varphi \in C^2(Y)$ with derivatives of order $0, 1, 2$ bounded then

$$\rho(t,x) = Q_t\varphi(x) \text{ belongs to } C^{1,2}([0,\infty) \times Y) \tag{4.6}$$

with ρ, $\frac{\partial \rho}{\partial t}$, $\frac{\partial \rho}{\partial x_i}$, $\frac{\partial^2 \rho}{\partial x_i \partial x_j}$ bounded for every $i, j \in \{1, \dots, n\}$ and it is a solution to the backward Kolmogorov equation

$$\frac{\partial U}{\partial t} = \sum_{i=1}^n \tilde{f}_i \frac{\partial U}{\partial x_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \tilde{g}_i \frac{\partial}{\partial x_i}\left(\tilde{g}_j \frac{\partial U}{\partial x_j}\right), \qquad U(0,x) = \varphi(x) \text{ for every } x \in Y \tag{4.7}$$

unique in the class $C^{1,2}([0,\infty) \times Y)$, see e.g. [15, Section 9.3].

Unfortunately, the coefficients of the equation (3.7) are not compactly supported so we cannot simply conclude that the solutions of (3.7) satisfy the Markov property and define a Feller semigroup in the sense (a)-(d) above. Yet, it is true, as will be shown below.

Definition 4.2. *From now on, z^ξ denotes the solution of* (3.7) *with the initial condition ξ, $p_{t,x}(A) = \mathbb{P}\,[z^x(t) \in A]$ and $P_t\varphi(x) = \mathbb{E}\,\varphi(z^x(t))$ are defined for $\varphi \in B_b(T\mathbb{S}^2)$, $t \geq 0$, $x \in T\mathbb{S}^2$ and $A \in \mathscr{B}(T\mathbb{S}^2)$.*

Proposition 4.3. *The solutions of* (3.7) *satisfy the Markov property and define a Feller semigroup on $T\mathbb{S}^2$. In fact, $P_t\varphi(x)$ is jointly continuous in (t,x) on $[0,\infty) \times T\mathbb{S}^2$ for every $\varphi \in C_b(T\mathbb{S}^2)$ and*

$$\mathbb{E}\,[\varphi(z^\xi(t))|\mathscr{F}_s] = (P_{t-s}\varphi)(z^\xi(s)) \text{ almost surely}$$

holds for every $\varphi \in B_b(T\mathbb{S}^2)$, $0 \leq s \leq t$ and every initial $T\mathbb{S}^2$-valued initial condition ξ.

Proof. Let us prove the joint continuity assertion first. Assume that $(t_n, x_n) \to (t,x)$ in $[0,\infty) \times T\mathbb{S}^2$ and let $\sup_n |x_n| \leq l$. Let $\tilde{f}$, $\tilde{g}$ be compactly supported C^∞ vector fields on $\mathbb{R}^6$ so that $f = \tilde{f}$ and $g = \tilde{g}$ on the ball of radius l in $\mathbb{R}^6$. Now $|z^{x_n}(t)| = |x_n| \leq l$ and $|z^x(t)| = |x| \leq l$ holds for every $t \geq 0$ a.s. by Proposition 4.1 and hence z^{x_n}, z^x are also solutions to the equation

$$dX = \tilde{f}(X)\,dt + \tilde{g}(X) \circ dW.$$

So, if $\varphi \in C_b(T\mathbb{S}^2)$ and $\tilde{\varphi} \in C_b(\mathbb{R}^6)$ is any extension of φ (which always exists by the Tietze theorem) then

$$\lim_{n\to\infty} P_{t_n}\varphi(x_n) = \lim_{n\to\infty} \mathbb{E}\,\tilde{\varphi}(z_n(t_n)) = \mathbb{E}\,\tilde{\varphi}(z(t)) = P_t\varphi(x)$$

by (4.5).

To prove the Markov property, let $\xi = (\xi^1, \xi^2)$ be a $T\mathbb{S}^2$-valued initial condition and define $\xi_k = (\xi^1, \xi^2 \mathbf{1}_{[|\xi^2| \leq k]})$. Then ξ_k take values in $T\mathbb{S}^2$ and by Proposition 4.1, $|z^{\xi_k}(t)| = |\xi_k| \leq \sqrt{1+k^2}$. Let $\tilde{f}$, $\tilde{g}$ be compactly supported C^∞ vector fields on $\mathbb{R}^6$ so that $f = \tilde{f}$ and $g = \tilde{g}$ on the ball of radius $\sqrt{1+k^2}$ in $\mathbb{R}^6$ and define $Q_t\phi(y) = \mathbb{E}\,\phi(u^y(t))$ for $\phi \in B_b(\mathbb{R}^6)$, $y \in \mathbb{R}^6$, $t \geq 0$ and u^y the solutions to $dX = \tilde{f}(X)\,dt + \tilde{g}(X) \circ dW$, $X(0) = y$. By the first part of the proof, we know that

$P_t\varphi(x) = Q_t\tilde{\varphi}(x)$ holds for every $x \in T\mathbb{S}^2$ such that $|x| \le \sqrt{1+k^2}$, $\varphi \in B_b(T\mathbb{S}^2)$, $\tilde{\varphi} \in B_b(\mathbb{R}^6)$, $\varphi = \tilde{\varphi}$ on $T\mathbb{S}^2$ and $t \ge 0$.

Now $z^{\xi_k} = u^{\xi_k}$ and if we define $A_k = [|\xi^2| \le k]$ and $\tilde{\varphi} \in B_b(\mathbb{R}^6)$ extends $\varphi \in B_b(T\mathbb{S}^2)$ then

$$\mathbf{1}_{A_k}\mathbb{E}\left[\varphi(z^\xi(t))|\mathscr{F}_s\right] = \mathbb{E}\left[\mathbf{1}_{A_k}\varphi(z^\xi(t))|\mathscr{F}_s\right] = \mathbb{E}\left[\mathbf{1}_{A_k}\varphi(z^{\xi_k}(t))|\mathscr{F}_s\right] = \mathbf{1}_{A_k}\mathbb{E}\left[\varphi(z^{\xi_k}(t))|\mathscr{F}_s\right] =$$

$$\mathbf{1}_{A_k}\mathbb{E}\left[\tilde{\varphi}(u^{\xi_k}(t))|\mathscr{F}_s\right] = \mathbf{1}_{A_k}(Q_{t-s}\tilde{\varphi})(u^{\xi_k}(s)) = \mathbf{1}_{A_k}(P_{t-s}\varphi)(z^{\xi_k}(s)) = \mathbf{1}_{A_k}(P_{t-s}\varphi)(z^{\xi}(s)) \text{ a.s.}$$

by the Markov property of solutions of the equation (4.4). To obtain the result, let $k \to \infty$. □

5. Multitude of invariant measures

Now we are ready to prove that the equation (3.7) and, consequently, also the equation (3.4) have many invariant measures due to the geometric nature of the equation.

Definition 5.1. *Let Y be a Polish space, $r_{t,x}(\cdot)$ probability measures on $\mathscr{B}(Y)$ indexed by $(t,x) \in [0,\infty) \times Y$ such that $r_{t,x}(A)$ is jointly measurable in (t,x) on $[0,\infty) \times Y$ for every $A \in \mathscr{B}(Y)$ and the operators*

$$R_t\varphi(x) = \int_Y \varphi\, dr_{t,x}, \qquad \varphi \in B_b(Y),\ t \ge 0$$

satisfy the semigroup property on $B_b(Y)$. We introduce the adjoint endomorphisms R_t^* acting on the space of probability measures on $\mathscr{B}(Y)$

$$R_t^*\nu(A) = \int_Y r_{t,x}(A)\, d\nu(x), \qquad t \ge 0,\ A \in \mathscr{B}(Y).$$

A probability measure ν on $\mathscr{B}(Y)$ is called invariant provided that

$$R_t^*\nu = \nu \text{ for all } t \ge 0 \text{ and } A \in \mathscr{B}(Y).$$

A probability measure on $\mathscr{B}(Y)$ is called ergodic provided that it is an extreme point in the convex set of invariant probability measures.

Remark 5.2. To make the meaning of the above definition clear, apply the Markov property in Proposition 4.3 with $s = 0$. If ξ is an $\mathscr{F}_0$-measurable $T\mathbb{S}^2$-valued random variable with a distribution ν then $P_t^*\nu$ is the law of $z^\xi(t)$.

At this moment, we introduce subsets of the tangent bundle $T\mathbb{S}^2$

$$M_r = \{(u,v) \in T\mathbb{S}^2 : |v| = r\}, \qquad r \ge 0. \tag{5.1}$$

Remark 5.3 (Invariance). If $r > 0$ and $x \in M_r$ then $z^x(t) \in M_r$ for every $t \ge 0$ almost surely. If $|u| = 1$ then $z^{(u,0)}(t) = (u,0)$ for every $t \ge 0$ almost surely. These conclusions follow directly from Proposition 4.1.

Corollary 5.4. *Let $r > 0$. For every $t \ge 0$, P_t is an endomorphism on $B_b(M_r)$.*

Corollary 5.5. *Let $x \in M_0$. Then δ_x is an invariant measure.*

We are going to prove that there is more to see, than what was disclosed by Corollary 5.5, on the sets M_r as far as invariant measures are concerned.

Remark 5.6. Observe that, for every $r > 0$, the mappings f and g in (3.8) are vector fields on the manifold M_r. In particular, Proposition 4.1 is now a direct consequence of the general result [24, Theorem 1.1, Chapter V].

In view of Remark 5.6, we can introduce the following second order differential operator on M_r.

Definition 5.7. *Define the second order differential operator*

$$\mathcal{A}\varphi = f(\varphi) + \frac{1}{2}g(g(\varphi)) \tag{5.2}$$

for $\varphi \in C^2(M_r)$ for $r > 0$.

The next result follows from [24, Chapter V, Theorem 3.1] but, rather than checking the assumptions in [24, Chapter V, Section 3], we will give, for our purposes and for the reader's comfort, the short proof here.

Proposition 5.8. *Let $r > 0$ and let $\varphi \in C^2(M_r)$. Then $\rho(t,x) = P_t\varphi(x)$ belongs to $C^{1,2}([0,\infty) \times M_r)$ and satisfies the backward Kolmogorov equation*

$$\frac{\partial \rho}{\partial t} = \mathcal{A}\rho \quad on \quad [0,\infty) \times M_r, \qquad \rho(0,\cdot) = \varphi. \tag{5.3}$$

On the other hand, if $\rho \in C^{1,2}([0,\infty) \times M_r)$ satisfies (5.3) then $\rho(t,x) = P_t\varphi(x)$ on $[0,\infty) \times M_r$.

Proof. Let $k \in \mathbb{N}$ and let $\tilde f$ and $\tilde g$ be C^∞ vector fields on $\mathbb{R}^6$ such that $\tilde f = f$ and $\tilde g = g$ on the centered ball in $\mathbb{R}^6$ of radius $R = \sqrt{1+r^2}$. Denote by u^x the solution of $dX = \tilde f(X)\,dt + \tilde g(X) \circ dW$, $X(0) = x$ and let Q_t be the associated Markov operators. Let $\tilde\varphi \in C^2(\mathbb{R}^6)$ be a compactly supported extension of φ. Then $z^x = u^x$ for every $x \in M_r$ by Proposition 4.1, $J(t,x) = Q_t\tilde\varphi(x) \in C^{1,2}([0,\infty) \times \mathbb{R}^6)$ by (4.6), hence $J(t,x) = \rho(t,x)$ for $(t,x) \in [0,\infty) \times M_r$. In particular, $\rho \in C^{1,2}([0,\infty) \times M_r)$ and (5.3) holds by (4.7).

To prove the converse assertion, extend ρ to a function in $C^{1,2}([0,\infty)\times\mathbb{R}^6)$, let $t > 0$ and apply the Itô formula to $\rho(t-r, z^x(r))$ for $r \in [0,t]$, obtaining

$$\varphi(z^x(t)) = \rho(0, z^x(t)) = \rho(t,x) + \int_0^t g(\rho)(t-r, z^x(r))\,dW.$$

Taking expectations on both sides yields the claim. □

The next assertion is obvious if $Q \in \mathbb{R}^3 \otimes \mathbb{R}^3$ is a unitary matrix with $\det Q = 1$ due to the invariance of the equation (3.7) for positively oriented unitary matrices. But it also holds if $\det Q = -1$. To prove this, we are going to use the uniqueness of the solutions of the backward Kolmogorov equation.

Corollary 5.9. *Let Q be a 3×3-unitary matrix. Denote by $\widetilde Q = \operatorname{diag}[Q,Q] \in \mathbb{R}^6 \otimes \mathbb{R}^6$. Then*

$$p(t, \widetilde Q x, A) = p(t, x, [\widetilde Q \in A])$$

holds for every $(t,x) \in [0,\infty) \times M_r$, every $A \in \mathscr{B}(M_r)$ and every $r > 0$.

Proof. Let $\varphi \in C^2(M_r)$ and define $\rho(t,x) = P_t\varphi(x)$ for $(t,x) \in [0,\infty) \times M_r$. Then ρ verifies (5.3). Now define $\varrho(t,x) = \rho(t, \widetilde Q x)$ for $(t,x) \in [0,\infty) \times M_r$ which we can do since $\widetilde Q$ is a diffeomorphism on M_r. Then $\varrho \in C^{1,2}([0,\infty) \times M_r)$ and

$$\frac{\partial \varrho}{\partial t}(t,x) - \mathcal{A}\varrho(t,x) = \frac{\partial \rho}{\partial t}(t, \widetilde Q x) - \mathcal{A}\rho(t, \widetilde Q x) = 0 \quad \text{on} \quad [0,\infty) \times M_r, \qquad \varrho(0,\cdot) = \varphi(\widetilde Q \cdot).$$

So, form the uniqueness part of Proposition 5.8, we obtain that

$$P_t\varphi(\widetilde Q x) = P_t(\varphi \circ \widetilde Q)(x) \quad \text{on} \quad [0,\infty) \times M_r. \tag{5.4}$$

By density of $C^2(M_r)$ in $C(M_r)$ we get that (5.4) holds for every $\varphi \in C(M_r)$ and consequently for every $\varphi \in B_b(M_r)$. □

Now we are ready to describe some analytic properties of the Markov semigroup (P_t) on M_r.

Theorem 5.10. *Let $r > 0$. Then (P_t) is a C_0-semigroup on $C(M_r)$, $P_t[C^2(M_r)] \subseteq C^2(M_r)$, $C^2(M_r)$ is contained in the domain of the infinitesimal generator A of (P_t) and $A = \mathcal{A}$ on $C^2(M_r)$.*

Proof. The C_0 property follows from the joint continuity in Proposition 4.3 and the invariance of $C^2(M_r)$ under P_t from Proposition 5.8. By the Itô formula,

$$P_t\varphi(x) = \varphi(x) + \int_0^t P_s(\mathcal{A}\varphi)(x)\,ds, \qquad t \ge 0,\ x \in M_r,$$

so φ belongs to the domain of the infinitesimal generator A of (P_t) and $A\varphi = \mathcal{A}\varphi$. □

Corollary 5.11. *Let $r > 0$. Then there exists an invariant measure with support in M_r.*

Proof. Let θ be a Borel probability measure with support in M_r. The semigroup (P_t) is Feller on $B_b(T\mathbb{S}^2)$, the average probability measures $\frac{1}{T}\int_0^T P_s^*\theta\,ds$ are supported in M_r, hence they are tight and therefore any of its weak cluster points is an invariant probability measure according to the Krylov-Bogolyubov theorem, see e.g. Corollary 3.1.2 in [14]. □

We have proved so far that the tangent bundle $T\mathbb{S}^2$ decomposes to invariant sets

$$T\mathbb{S}^2 = \bigcup_{x\in M_0} \{x\} \cup \bigcup_{r>0} M_r$$

where on each if these sets there exists an invariant measure.

6. Numerical simulations

We present a numerical scheme to approximate problem (3.7). It is the consequent simulations that lead us to conjecture that (P_t^*) restricted to M_r attracts every initial distribution on M_r to the normalized surface measure on M_r. In particular, this would mean that the normalized surface measure on M_r is the unique invariant measure on M_r, cf. Corollary 5.11.

6.1. **Numerical approximation.** Let $I_k := \{t_n\}_{n=0}^N$ denote an equi-distant mesh of size $k > 0$ covering $[0, T]$. The following Algorithm A gives a non-dissipative, symmetric discretization of (3.5) with solutions $\{(U^n, V^n);\, n \geq 0\}$. We denote $d_t\varphi^{n+1} := \frac{1}{k}(\varphi^{n+1} - \varphi^n)$. Throughout this section, $C > 0$ denotes a constant which does not depend on k and T.

Algorithm A. Let (U^0, V^0) be such that $(U^0, V^0) = 0$, $|U^0| = 1$, $|V^0| = r$, and define $U^{-1} := U^0 - kV^0$. For every $n \geq 0$, find the $\mathbb{R}^{3+3+1}$-valued random variable $(U^{n+1}, V^{n+1}, \lambda^{n+1})$, such that

$$\begin{aligned} V^{n+1} - V^n &= k\frac{\lambda^{n+1}}{2}(U^{n+1} + U^{n-1}) + \frac{1}{4}(U^{n+1} + U^{n-1}) \times (V^{n+1} + V^n)\Delta_{n+1}W\,, \\ d_tU^{n+1} &= V^{n+1}\,, \end{aligned} \tag{6.1}$$

where $\Delta_{n+1}W := W(t_{n+1}) - W(t_n) \sim \mathscr{N}(0, k)$, and

$$\lambda^{n+1} = \begin{cases} 0 & \text{for } \frac{1}{2}(U^{n+1} + U^{n-1}) = 0\,, \\ -\frac{(V^n, V^{n+1})}{|\frac{1}{2}(U^{n+1}+U^{n-1})|^2} & \text{for } \frac{1}{2}(U^{n+1} + U^{n-1}) \neq 0 \text{ and } n \geq 1\,, \\ -\frac{(V^0, V^1) - \frac{1}{2}|V^0|^2}{|\frac{1}{2}(U^1+U^{-1})|^2} & \text{for } \frac{1}{2}(U^1 + U^{-1}) \neq 0 \text{ and } n = 0\,. \end{cases} \tag{6.2}$$

For our simulations, we use $(U^0, V^0) := \big(u(0), v(0)\big)$. Next, we show the existence of a sequence of triples $\big\{\big(U^{n+1}, V^{n+1}, \lambda^{n+1}\big);\, 0 \leq n \leq N-1\big\}$ which solves (6.1)–(6.2). This definition (6.2) of the discrete Lagrange multiplier λ^{n+1} ensures that $|U^{n+1}| = 1$ for $n \geq 0$; here, the definition of λ^1 according to $(6.2)_3$ accounts for the fact that $1 \neq |U^{-1}| \leq 1 + rk$ in general. Finally, $|V^{n+1}| = |V^1| = |V^0| + Ck$ for all $n \geq 0$ is valid, such that solutions $\{(U^n, V^n);\, 0 \leq n \leq N\}$ of Algorithm A inherit the properties of the solutions $\big\{\big(u(t), v(t)\big);\, t \in [0, T]\big\}$ of (3.6) stated in Proposition 4.1.

Proposition 6.1. *Let $k \leq k_0(r)$ be sufficiently small. For all $n \geq 0$ there exists an $\mathbb{R}^{3+3+1}$-valued random variable $\big(U^{n+1}, V^{n+1}, \lambda^{n+1}\big)$ which solves (6.1)–(6.2). Moreover, iterates satisfy $|U^{n+1}| = |U^0|$, and $|V^{n+1}| = |V^1|$ for $1 \leq n \leq N-1$, where $||V^1| - |V^0|| \leq Ck$.*

The proof is by induction, and uses Brouwer's fixed point argument to show existence, and a proper 'testing' of (6.1), in combination with the definition (6.2) to verify the given properties.

Proof. Induction assumption. Fix $n \geq 1$; for the sake of better presentation, we consider $n = 0$ at the end of the proof. Let $\{(U^\ell, V^\ell);\, 0 \leq \ell \leq n\}$ be a solution of (6.1)–(6.2) which satisfies $|U^\ell| = 1$ for $0 \leq \ell \leq n$ and $|V^\ell| = \tilde{r} := |V^1| \leq 2r$ for $1 \leq \ell \leq n$. Further, let $k \leq k_0(\tilde{r})$ be such that $k|V^\ell| \leq \frac{1}{4}$.

1. Step: Construction of $(U^{n+1}, V^{n+1}, \lambda^{n+1})$. In a preparatory step, we define $A^{n+1} := \frac{1}{2}(U^{n+1} + U^{n-1})$ and rewrite the leading term in (6.1) as

$$V^{n+1} - V^n = \frac{1}{k}\big(U^{n+1} - 2U^n + U^{n-1}\big) = \frac{2}{k}(A^{n+1} - U^n)\,. \tag{6.3}$$

Hence, (6.1) may be rewritten as

$$\begin{aligned}\frac{2}{k}\big(A^{n+1}-U^n\big) &= k\lambda^{n+1}A^{n+1}+\frac{1}{2}A^{n+1}\times\big(V^{n+1}+V^n\big)\Delta_{n+1}W\\ &= k\lambda^{n+1}A^{n+1}+A^{n+1}\times\big(V^n-\frac{1}{k}U^n\big)\Delta_{n+1}W \qquad (6.4)\end{aligned}$$

Obviously, we have found (U^{n+1},V^{n+1}) once A^{n+1} is constructed, which is a zero of the mapping $\mathcal{F}^\omega_{0,n}:\mathbb{R}^3\to\mathbb{R}^3$,

$$\mathcal{F}^\omega_{0,n}(\widetilde{A}) := \frac{2}{k}\big(\widetilde{A}-U^n\big)-k\widetilde{\lambda}_{0,n}\widetilde{A}-\widetilde{A}\times\big(V^n-\frac{1}{k}U^n\big)\Delta_{n+1}W\,, \qquad (6.5)$$

where according to (6.3),

$$\widetilde{\lambda}_{0,n}\equiv\widetilde{\lambda}_{0,n}(\widetilde{A}) := -\frac{|V^n|^2+\frac{2}{k}(V^n,\widetilde{A}-U^n)}{|\widetilde{A}|^2}\,. \qquad (6.6)$$

Since $\mathcal{F}^\omega_{0,n}:\mathbb{R}^3\to\mathbb{R}^3$ is not a continuous mapping, we consider a modification $\mathcal{F}^\omega_{\varepsilon,n}$ with some $\frac{1}{8}\le\varepsilon\le\frac{1}{4}$, where $\widetilde{\lambda}_{0,n}$ in $\mathcal{F}^\omega_{0,n}$ is replaced by

$$\widetilde{\lambda}_{\varepsilon,n}\equiv\widetilde{\lambda}_{\varepsilon,n}(\widetilde{A}) := -\frac{|V^n|^2+\frac{2}{k}(V^n,\widetilde{A}-U^n)}{\max\{\varepsilon,|\widetilde{A}|^2\}}\,. \qquad (6.7)$$

a) Solvability of $\mathcal{F}^\omega_{\varepsilon,n}(\widetilde{A})=0$ for every $\frac{1}{8}\le\varepsilon\le\frac{1}{4}$. The map $\mathcal{F}^\omega_{\varepsilon,n}:\mathbb{R}^3\to\mathbb{R}^3$ is continuous. Moreover, by computing

$$\begin{aligned}\frac{2}{k}\big(\widetilde{A}-U^n,\widetilde{A}\big) &\ge \frac{2}{k}\big(|\widetilde{A}|-|U^n|\big)|\widetilde{A}|\,,\\ -k\big(\widetilde{\lambda}_{\varepsilon,n}\widetilde{A},\widetilde{A}\big) &\ge -k|V^n|^2-2|V^n|\big(|\widetilde{A}|+|U^n|\big)\,,\end{aligned}$$

we may conclude by the induction assumption that there exists a deterministic number $R_n:=R_n(\tilde r)>0$ such that for $k\le\widetilde{k}_0(\tilde r)$ holds

$$\big(\mathcal{F}^\omega_{\varepsilon,n}(\widetilde{A}),\widetilde{A}\big)\ge 0 \qquad \forall\,\widetilde{A}\in\big\{\mathcal{A}\in\mathbb{R}^3:\ |\mathcal{A}|\ge R_n\big\}\,.$$

By Brouwer's fixed point theorem, there exists $\widetilde{A}^*$ such that $\mathcal{F}^\omega_{\varepsilon,n}(\widetilde{A}^*)=0$ where $\frac{1}{8}\le\varepsilon\le\frac{1}{4}$. — We now show that $\widetilde{A}^*$ also solves $\mathcal{F}^\omega_{0,n}(\widetilde{A}^*)=0$ provided $k\le k_0(r)\le\widetilde{k}_0(\tilde r)$. For this purpose, we use the definitions

$$U^{n+1}_\varepsilon := 2\widetilde{A}^*-U^{n-1} \quad\text{and}\quad V^{n+1}_\varepsilon := \frac{1}{k}\big(U^{n+1}_\varepsilon-U^n\big) \qquad (6.8)$$

to write (see (6.1))

$$V^{n+1}_\varepsilon-V^n = k\frac{\lambda^{n+1}_\varepsilon}{2}(U^{n+1}_\varepsilon+U^{n-1})+\frac{1}{4}(U^{n+1}_\varepsilon+U^{n-1})\times(V^{n+1}_\varepsilon+V^n)\Delta_{n+1}W\,, \qquad (6.9)$$

where

$$\lambda^{n+1}_\varepsilon := -\frac{(V^n,V^{n+1}_\varepsilon)}{\max\{\varepsilon,|\frac{1}{2}(U^{n+1}_\varepsilon+U^{n-1})|^2\}} = -\frac{1}{k}\frac{(V^n,[U^{n+1}_\varepsilon+U^{n-1}]-[U^n+U^{n-1}])}{\max\{\varepsilon,|\frac{1}{2}(U^{n+1}_\varepsilon+U^{n-1})|^2\}}\,. \qquad (6.10)$$

It now suffices to show that $\frac{1}{2}\le|\frac{1}{2}(U^{n+1}_\varepsilon+U^{n-1})|^2=\max\{\varepsilon,|\frac{1}{2}(U^{n+1}_\varepsilon+U^{n-1})|^2\}$, since in this case $\widetilde{\lambda}_{\varepsilon,n}(\widetilde{A}^*)=\widetilde{\lambda}_{0,n}(\widetilde{A}^*)$.

b) $\widetilde{A}^*$ also satisfies $\mathcal{F}^\omega_{0,n}(\widetilde{A}^*)=0$ provided $k\le k_0(\tilde r)$. By (6.9), the inverse triangle inequality, the induction assumption, and for $k\le k_0(\tilde r)$,

$$\begin{aligned}\big|\frac{1}{2}(U^{n+1}_\varepsilon+U^{n-1})\big| &= \big|\frac{k}{2}V^{n+1}_\varepsilon+\frac{1}{2}(U^n+U^{n-1})\big| = \big|\frac{k}{2}(V^{n+1}_\varepsilon+V^n)+U^{n-1}\big|\\ &\ge |U^{n-1}|-\Big(\frac{k}{2}|V^n|+\frac{k}{2}|V^{n+1}_\varepsilon|\Big)\\ &\ge 1-\frac{1}{4}-\frac{k}{2}|V^{n+1}_\varepsilon|\,. \qquad (6.11)\end{aligned}$$

There remains to show that $\frac{k}{2}|V_\varepsilon^{n+1}| \leq \frac{1}{4}$. For this purpose, multiply (6.9) with $V_\varepsilon^{n+1} + V^n$ and use binomial formula to get

$$
\begin{aligned}
|V_\varepsilon^{n+1}|^2 - |V^n|^2 &= k\frac{\lambda_\varepsilon^{n+1}}{2}\Big(U_\varepsilon^{n+1} + U^{n-1}, V_\varepsilon^{n+1} + V^n\Big) \\
&= k\frac{\lambda_\varepsilon^{n+1}}{2}\Big(kV_\varepsilon^{n+1} + U^n + U^{n-1}, V_\varepsilon^{n+1} + V^n\Big)\,. \qquad (6.12)
\end{aligned}
$$

Note that since $\frac{|a|}{\max\{\varepsilon,|a|^2\}} \leq \frac{1}{\sqrt{\varepsilon}}$, we get by $(6.10)_2$

$$
\frac{k}{2}|\lambda_\varepsilon^{n+1}| \leq \Big(\frac{1}{\sqrt{\varepsilon}} + \frac{1}{2\varepsilon}\big(|U^n| + |U^{n-1}|\big)\Big)|V^n| \leq \frac{1}{\sqrt{\varepsilon}}\Big(1 + \frac{1}{\sqrt{\varepsilon}}\Big)\tilde{r} := C_\varepsilon \tilde{r}\,,
$$

such that the following bound follows from (6.12):

$$
\begin{aligned}
|V_\varepsilon^{n+1}|^2 &\leq \tilde{r}^2 + C_\varepsilon\tilde{r}\Big(k[1+\frac{1}{2}]|V_\varepsilon^{n+1}|^2 + \frac{k}{2}|V^n|^2\Big) \\
&\quad + \frac{1}{2}|V_\varepsilon^{n+1}|^2 + \frac{C_\varepsilon^2}{2}\tilde{r}^2\big|U^n + U^{n-1}\big|^2 \\
&\quad + \frac{C_\varepsilon}{2}\tilde{r}\Big(\big|U^n + U^{n-1}\big|^2 + |V^n|^2\Big)\,. \qquad (6.13)
\end{aligned}
$$

Consequently, by induction assumption, for $k \leq k_0(\tilde{r})$, and since $\frac{1}{8} \leq \varepsilon \leq \frac{1}{4}$,

$$
\frac{1}{4}|V_\varepsilon^{n+1}|^2 \leq C\big(1+\tilde{r}\big)^2\,. \qquad (6.14)
$$

Therefore, we may choose $k \leq k_0(\tilde{r})$ sufficiently small to validate $\frac{k}{2}|V_\varepsilon^{n+1}| \leq \frac{1}{4}$. By the arguments given before, this settles the existence of a triple $(U^{n+1}, V^{n+1}, \lambda^{n+1})$ which solves (6.1)–(6.2) for the index $n+1$.

2. Step: Properties of (U^{n+1}, V^{n+1}). We start with showing $|U^{n+1}| = 1$. Taking the scalar product of $(6.1)_1$ with $\frac{1}{2k}(U^{n+1} + U^{n-1})$, using $(6.1)_2$, the binomial formula, and elementary calculations lead to

$$
\begin{aligned}
&\lambda^{n+1}\Big|\frac{1}{2}(U^{n+1} + U^{n-1})\Big|^2 = \frac{1}{2}\big(d_t V^{n+1}, U^{n+1} + U^{n-1}\big) \\
&= \frac{1}{2k^2}\Big[|U^{n+1}|^2 + 2(U^{n+1}, U^{n-1}) - 2(U^{n+1}, U^n) - 2(U^n, U^{n-1}) + |U^{n-1}|^2\Big] \qquad (6.15)\\
&= \frac{1}{2k^2}\Big[|U^{n+1}|^2 - 2k(U^{n+1}, V^n) - 2(U^n, U^{n-1}) + |U^{n-1}|^2\Big]\,.
\end{aligned}
$$

By induction assumption, the last term may be replaced by the identity $|U^{n-1}|^2 = |U^n|^2 = 1$. Hence, (6.15) equals to

$$
\begin{aligned}
&= \frac{1}{2k^2}\Big[|U^{n+1}|^2 - |U^n|^2 - 2k(U^{n+1}, V^n) - 2(U^n, U^{n-1}) + 2|U^n|^2\Big] \\
&= \frac{1}{2k^2}\Big[|U^{n+1}|^2 - |U^n|^2 - 2k(U^{n+1}, V^n) + 2k(U^n, V^n)\Big] \\
&= \frac{1}{2k^2}\Big[|U^{n+1}|^2 - 1 - 2k^2(V^{n+1}, V^n)\Big]\,.
\end{aligned}
$$

The definition of λ^{n+1} in (6.2) then implies $|U^{n+1}| = 1$.

In order to verify $|V^{n+1}| = \tilde{r}$, we take the scalar product of $(6.1)_1$ with $V^{n+1} + V^n = \frac{1}{k}(U^{n+1} - U^{n-1})$ and use binomial formula,

$$
|V^{n+1}|^2 - |V^n|^2 = \frac{\lambda^{n+1}}{2}\big[|U^{n+1}|^2 - |U^{n-1}|^2\big] = 0\,. \qquad (6.16)
$$

This settles the inductive argument for $n \geq 1$.

Modifications for $n = 0$.

Step 1'. In order to construct a triple (U^1, V^1, λ^1), we proceed as in Step 1, with the following exceptions in (6.6), (6.7), (6.10):

$$\widetilde{\lambda}_{\varepsilon,0} := -\frac{\frac{1}{2}|V^0|^2 + \frac{2}{k}(V^0, \widetilde{A} - U^0)}{\max\{\varepsilon, |\widetilde{A}|^2\}}, \qquad \lambda^1_\varepsilon := -\frac{(V^0, V^1_\varepsilon) - \frac{1}{2}|V^0|^2}{\max\{\varepsilon, |\frac{1}{2}(U^1_\varepsilon + U^{-1})|^2\}}.$$

The estimate of $|V^1_\varepsilon|^2 \leq C(1+r)^2 \leq C(1+\tilde{r})^2$ in (6.14) follows accordingly since the additional term $-\frac{1}{2}|V^0|^2$ in the nominator of λ^1_ε has modulus $\frac{1}{2}r^2$. The remaining arguments from *Step 1* now apply to establish the existence of the triple (U^1, V^1, λ^1). Note, in particular, that according to (6.11) we have

$$|\frac{1}{2}(U^1 + U^{-1})| \geq \frac{1}{2}. \tag{6.17}$$

Step 2'. A slightly modified version of (6.15) leads to the calculation

$$\begin{aligned}
\lambda^1\big|\frac{1}{2}(U^1 + U^{-1})\big|^2 &= \frac{1}{2}\big(d_t V^1, U^1 + U^{-1}\big) \\
&= \frac{1}{2k^2}\Big[|U^1|^2 - 2k(U^1, V^0) - 2(U^0, U^{-1}) + |U^{-1}|^2\Big] \\
&= \frac{1}{2k^2}\Big[|U^1|^2 - 2k(U^1, V^0) - 2|U^0|^2 + 2k(U^0, V^0) + |U^{-1}|^2\Big] \\
&= \frac{1}{2k^2}\Big[\big[|U^1|^2 - |U^0|^2\big] - 2k^2(V^1, V^0) - |U^0|^2 + |U^{-1}|^2\Big].
\end{aligned}$$

Note that $|U^{-1}|^2 = |kV^0 - U^0|^2$ need not be 1. By binomial formula, and since $(U^0, V^0) = 0$,

$$\begin{aligned}
&= \frac{1}{2k^2}\Big[\big[|U^1|^2 - 1\big] - 2k^2(V^1, V^0) - |U^0|^2 + |U^0|^2 - 2k(V^0, U^0) + k^2|V^0|^2\Big] \\
&= \frac{1}{2k^2}\Big[\big[|U^1|^2 - 1\big] - 2k^2(V^1, V^0) + k^2|V^0|^2\Big],
\end{aligned}$$

such that $|U^1| = 1$ now follows from $(6.2)_3$.

Next, we proceed as in (6.16) to bound $|V^1|$. By the definition of U^{-1}, and $(U^0, V^0) = 0$,

$$|U^{-1}|^2 = (U^0 - kV^0, U^0 - kV^0) = |U^0|^2 - 2k(V^0, U^0) + k^2|V^0|^2 = |U^0|^2 + k^2|V^0|^2. \tag{6.18}$$

We take the scalar product of $(6.1)_1$ with $V^1 + V^0 = k^{-1}(U^1 - U^{-1})$ and employ (6.18), and $|U^1| = |U^0| = 1$,

$$|V^1|^2 - |V^0|^2 = \frac{\lambda^1}{2}\big[|U^1|^2 - |U^{-1}|^2\big] = k^2\frac{\lambda^1}{2}|V^0|^2. \tag{6.19}$$

In order to bound $|\lambda^1|$ we use (6.17) and the triangle and Young's inequalities

$$|\lambda^1| \leq 4\left(|V^0||V^1| + \frac{1}{2}|V^0|^2\right) \leq \left(2|V^1|^2 + 4|V^0|^2\right). \tag{6.20}$$

Using (6.20) we get from (6.19) that

$$|V^1|^2 \leq |V^0|^2 + k^2|V^0|^2\left(|V^1|^2 + 2|V^0|^2\right). \tag{6.21}$$

Since $|V^0| = r$, it follows from (6.21) that for $k \leq k_0(r)$

$$|V^1|^2 \leq |V^0|^2 + k^2\frac{3r^4}{1 - r^2k_0^2} \leq \left(|V^0| + k\sqrt{\frac{3r^4}{1 - r^2k_0^2}}\right)^2. \tag{6.22}$$

Hence, it follows from (6.22) that $||V^1| - |V^0|| \leq C(r)k$. The modulus of $|V^1|$ is then exactly preserved for $n > 0$, see (6.16).

$\square$

6.2. **Numerical experiments.** We use Algorithm A to provide simulations for (3.5) in the form

$$d\dot{u} = -|\dot{u}|^2 u\, dt + \sqrt{D}(u \times \dot{u}) \circ dW,$$

where D is a fixed constant that controls the intensity of the noise term. Instead of (6.2), we use an equivalent form

$$\lambda^{n+1} = \frac{-\frac{1}{k}(V^n, U^{n+1} + U^{n-1}) + \frac{1}{2k^2}(1 - |U^{n-1}|^2)}{\left|\frac{1}{2}(U^{n+1} + U^{n-1})\right|^2}, \qquad (n \geq 0). \tag{6.23}$$

The above formula is equivalent to the formulation (6.2); since $|U^\ell|^2 = 1$, $\ell \geq 0$ we obtain for $n > 0$ that $-\frac{1}{k}(V^n, U^{n+1} + U^{n-1}) + \frac{1}{2k^2}(1 - |U^{n-1}|^2) = -\frac{1}{k}(V^n, U^{n+1} + U^{n-1}) = -(V^n, V^{n+1}) + \frac{1}{k}(V^n, U^n + U^{n-1}) = -(V^n, V^{n+1})$. The equivalence for $n = 0$ follows similarly on recalling that $(U^0, V^0) = 0$. The formulation (6.23) is more convenient for numerical computations, since in this reformulation the round off errors and errors due to inexact solution of the nonlinear system (6.1) do not accumulate over time in the constraint $|U^n| = 1$. The solution of the nonlinear scheme (6.1)-(6.23) is obtained up to machine accuracy by a simple fixed-point algorithm, cf. [3].

The stochastic process $\{(U^n, V^n),\, n \geq 0\}$ is computed by the classical Monte-Carlo sampling algorithm; we denote by N_{mc} the number of simulated sample paths of the corresponding stochastic process. In order to obtain an approximation of the marginal probability density function of the stochastic process $\{U^n,\, n >\geq 0\}$, the unit sphere $\mathbb{S}^2$ is divided into segments $\omega_{ij} \subset \mathbb{S}^2$ associated with points

$$x_{ij} = \Big(\sin(i\pi/16)\cos(j\pi/16), \sin(i\pi/16)\sin(j\pi/16), \cos(i\pi/16)\Big),$$

$i = 0, \ldots, 16$, $j = 0, \ldots, 31$ such that $\omega_{ij} = \{x \in \mathbb{S}^2 | \quad x_{ij} = \arg\min_{x_{lm}} |x - x_{lm}|\}$. For the above partition of the sphere, at a fixed time level $t_n = nk$, we construct a piecewise constant empirical probability density function $\hat{f}^n(x) : \mathbb{S}^2 \to \mathbb{R}$ of $U^n \in \mathbb{S}^2$ as

$$\hat{f}^n(x)|_{\omega_{ij}} = \hat{f}^n(x_{ij}) = \frac{\#\{l |\, U^{n,l} \in \omega_{ij}\}}{|\omega_{ij}| N},$$

for $i = 0, \ldots, 16$, $j = 0, \ldots, 31$, where $\#\Omega$ denotes the cardinality of the set Ω and $\{U^{n,l},\, n \geq 0\}$ is the l-th realization (sample path) of the the stochastic process $\{U^n,\, n \geq 0\}$.

The marginal probability density function $\hat{f}^n$ of $\{U^n,\, n \geq 0\}$ was constructed as an average of $N_{mc} = 20000$ sample paths. For all computations in this section we take the time step size $k = 0.001$ and the initial conditions $U^0 = (0, 1, 0)$, $V^0 = (1, 0, 0)$. The marginal probability density function $\hat{f}^0$ associated with the above initial conditions is a Dirac delta function concentrated at U^0.

In Figure 2 we display the computed probability density $\hat{f}^n$ for $D = 1$, $T = 60$ at different time levels. Initially the probability density function is advected in the direction of the initial velocity V^0 and is simultaneously being diffused. For early times, the diffusion seems to act predominantly in the direction perpendicular to the initial velocity. In Figure 1 we display the time averaged marginal probability density function $\overline{f}$ over the last 100 time levels (i.e., we compute $\overline{f}(x) = \frac{1}{100}\sum_{T/k-100}^{T/k} \hat{f}^n(x)$), the function $t_n \to \mathbb{E}[U^n]$ and a zoom at $t_n \to \mathbb{E}[U^n]$ near the center of the sphere.

The evolution of the probability density $\{\hat{f}^n,\, n \geq 0\}$ for $D = 10$, $T = 60$ is shown in Figure 3. Similarly to the previous experiment the probability density function diffuses and becomes uniform for large times. Some advection in the direction of the initial velocity can still be observed, however, the overall process has a predominantly diffuse character. We observe a damping effect which is due to the effects of the random forcing term, see Figure 7. In Figure 3 we display the time averaged probability density function $\overline{f}$, the function $t_n \to \mathbb{E}[U^n]$ and a zoom at $t_n \to \mathbb{E}[U^n]$ for $n \geq 0$ near the center.

Figure 6 contains the computed functions of $t_n \to \mathbb{E}[U^n]$ for $D = 0.1$ and $D = 100$. The respective probability densities asymptotically converge towards the uniform distribution for large times.

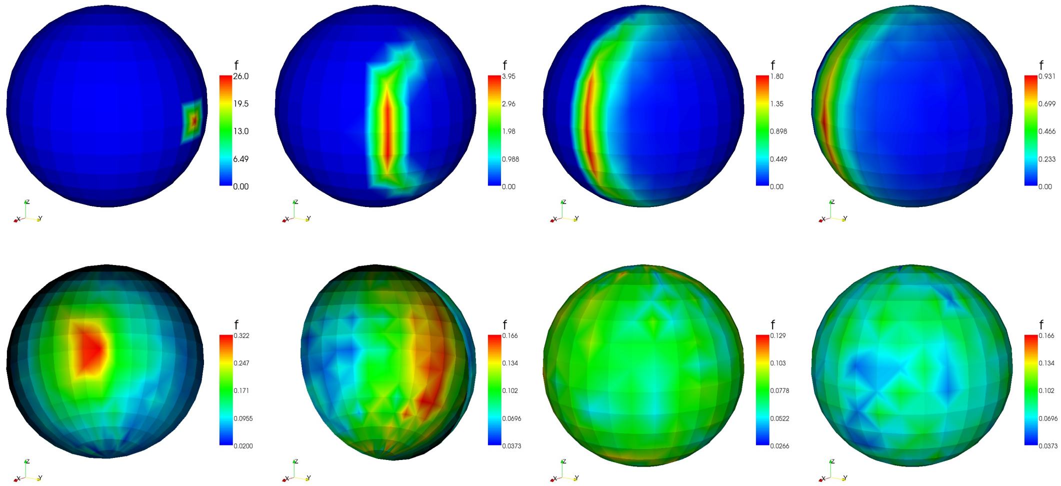

FIGURE 1. Approximate marginal probability density $\hat{f}^n$ of $\{U^n, n \geq 0\}$ for $D = 1$ at times $t_n = 0, 1, 1.5, 2.1, 4.3, 5.5, 10, 60$.

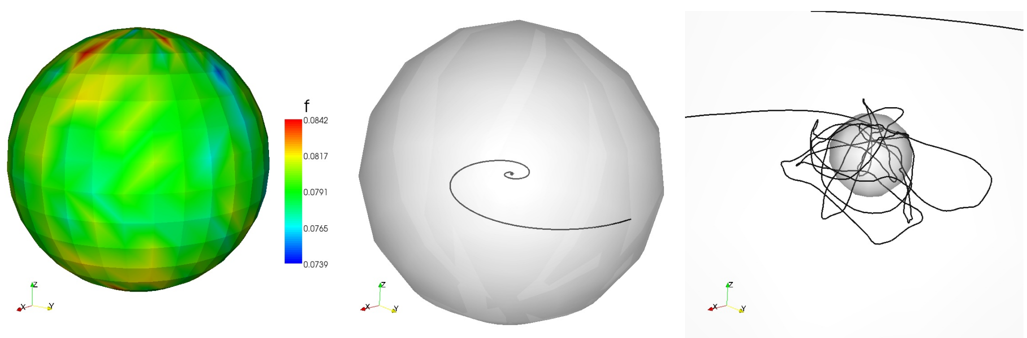

FIGURE 2. Time averaged marginal marginal probability density $\overline{f}$ of $\{U^n, n \geq 0\}$ (left), $t_n \to \mathbb{E}[U^n]$ (middle), and a zoom at $\mathbb{E}[U^n]$ with a sphere with radius 0.01 (right), $D = 1$.

In Figure 7 we show the graphs of the time evolution of the approximate error $\mathcal{E}^n_{\max} : t_n \to \max_{x \in \mathbb{S}^2} |\hat{f}^n(x) - f^{\mathbb{S}^2}|$ for $D = 0.01, 0.1, 1, 10, 100$ with $f^{\mathbb{S}^2}$ being the uniform distribution on the unit sphere. The quantity $\mathcal{E}^n_{\max}$ serves as a measure of the speed of convergence towards the uniform probability distribution $f^{\mathbb{S}^2}$. Note that the oscillations in the error graphs are due to the approximation of the probability density. The numerical experiments provide evidence that the probability densities for all D converge towards the uniform probability density $f^{\mathbb{S}}$ for $t \to \infty$. The probability density evolutions for decreasing values of D have an increasingly "advective" character, and the evolutions for increasing values have an increasingly "diffusive" character. It is also interesting to note, that the convergence in time towards the uniform distribution becomes slower for increasing and decreasing values of D when compared with the fastest converging evolutions for $D = 1$ or $D = 10$.

In the final experiment we study the long time behavior of the pair $\{(U^n, V^n), n \geq 0\}$for $D = 1$, $N_{mc} = 20000$. Towards this end, we introduce a partition of the manifold M_1 defined in (5.1). First, we consider a partition of the unit sphere into segments $\{\omega_i, i = 1, \ldots, 6\}$ associated with the points $\tilde{x}_i = (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$ in such a way that $x \in \mathbb{S}^2$ belongs to ω_i if and only if $|x - \tilde{x}_i| = \min_{1 \leq j \leq 6} |x - \tilde{x}_j|$. Next, we denote by T_i the tangent planes to points $\tilde{x}_i$. Fixing an $i \in \{1, \ldots, 6\}$, the orthogonal projections of vectors $\{\tilde{x}_1, \ldots, \tilde{x}_6\}$ onto the tangent

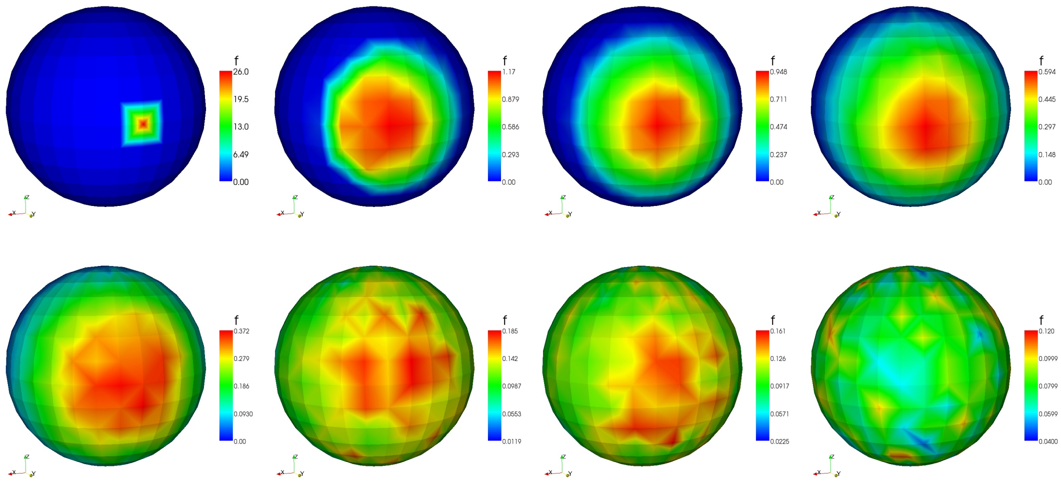

FIGURE 3. Approximate probability density $\hat{f}^n$ of $\{U^n,\, n \geq 0\}$ for $D = 10$ at $t_n = 0, 0.9, 1.2, 2, 3.1, 8, 10, 60$.

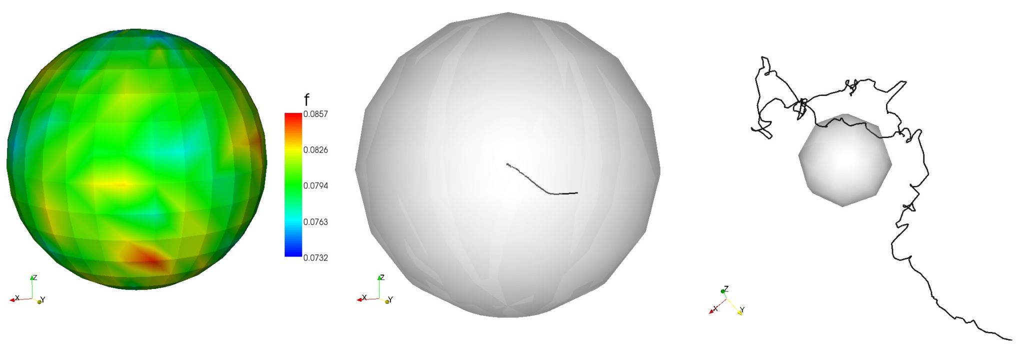

FIGURE 4. Time averaged marginal probability density $\overline{f}$ of $\{U^n,\, n \geq 0\}$ (left), $t_n \to \mathbb{E}[U^n]$ (middle), and a zoom at $t_n \to \mathbb{E}[U^n]$ for $n \geq 0$ with a sphere with radius 0.01 (right), $D = 10$.

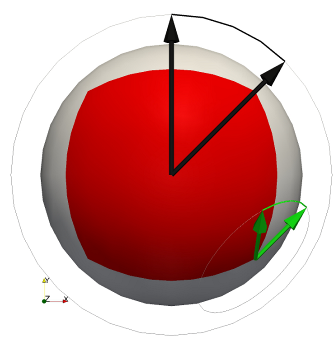

FIGURE 5. The partition of the submanifold M_1 of $T\mathbb{S}^2$: ω_i in red, a segment γ_i^j in black, the green arc indicates the elements of M_i^j starting from a point in the down-right corner of ω_i.

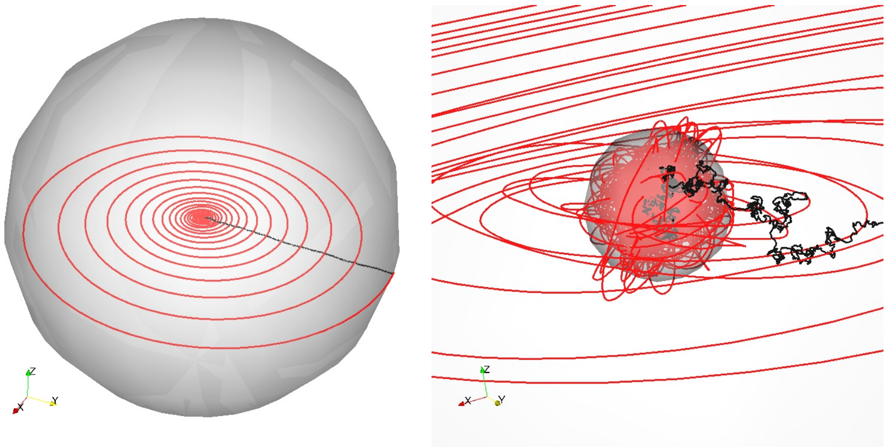

FIGURE 6. The function $t_n \to \mathbb{E}[U^n]$ (left), and a zoom near the center with a sphere with radius 0.01 (right) for $D = 100$ (black line), $D = 0.1$ (red line).

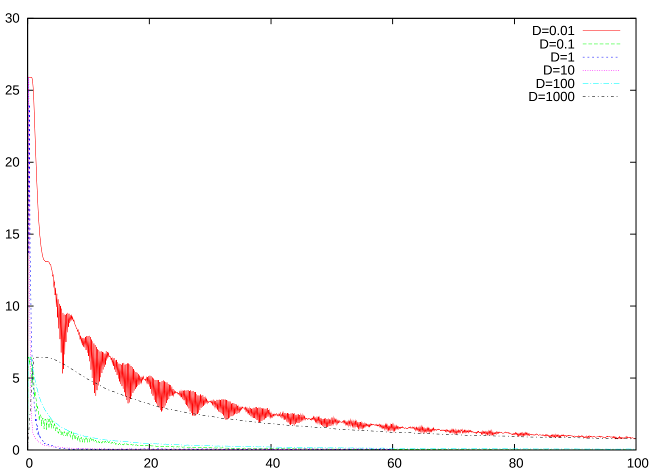

FIGURE 7. Evolution of $\mathcal{E}^n_{\max}$, $n \geq 0$ different values of the coefficient D.

plane T_i delimit 4 sectors on T_i. We subsequently halve each sector obtaining thus 8 equiangular sectors $\gamma_i^1, \ldots, \gamma_i^8$ in T_i. Now we introduce the following partition of M_1 into 6×8 segments (see Figure 5): a point $(p, \xi) \in T\mathbb{S}^2$ belongs to M_i^j if $p \in \omega_i$ and the orthogonal projection of ξ onto the tangent plane T_i belongs to the sector γ_i^j. The approximate probability density function $\hat{f}^n_{M_1}$ of $\{(U^n, V^n),\, n \geq 0\}$ is computed analogically to the marginal probability density function $\hat{f}^n$ of $\{U^n,\, n \geq 0\}$. It can be verified by symmetries of this partition that the normalized surface volume of each M_i^j is equal to $1/48$. For $n = 60000$ (i.e., at time $t_n = 60$) we have for $i = 1, \ldots, 6$, $j = 1, \ldots, 8$ that $\#\{l | U^{n,l} \in \omega_i\} \in (3380, 3260) \approx N_{mc}/6 = 3333$ and that $\#\{l | (U^{n,l}, V^{n,l}) \in M_i^j\} \in (386, 455) \approx N_{mc}/6/8 = 417$, see Figure 8 left and Figure 8 right, respectively. This result indicates that the point-wise probability measure for (U^n, V^n), $n \geq 0$ converges to the invariant measure $\overline{\nu}$ which is the uniform measure on the set M_1. Figure 9 reveals that the (suitably rescaled) approximate error $\mathcal{E}^n_{M_1,\max} = |\hat{f}^n_{M_1} - \overline{\nu}|$ for $\{(U^n, V^n),\, n \geq 0\}$ has similar evolution as the corresponding error $\mathcal{E}^n_{\max}$ for U^n. Moreover, it seems that the convergence of the error in time is exponential, see Figure 9.

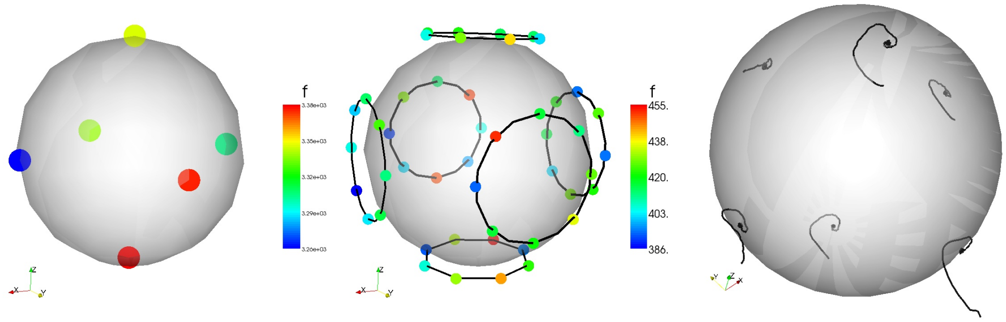

FIGURE 8. Probability density function $\hat{f}^n_{M_1}$ of $\{(U^n, V^n),\, n \geq 0\}$ at time $T = 60$ (left and middle), and the evolution of $t_n \to \int_{\omega_i} \mathbb{E}[V^n]$, $i = 1, \ldots, 6$, $n \geq 0$ (right).

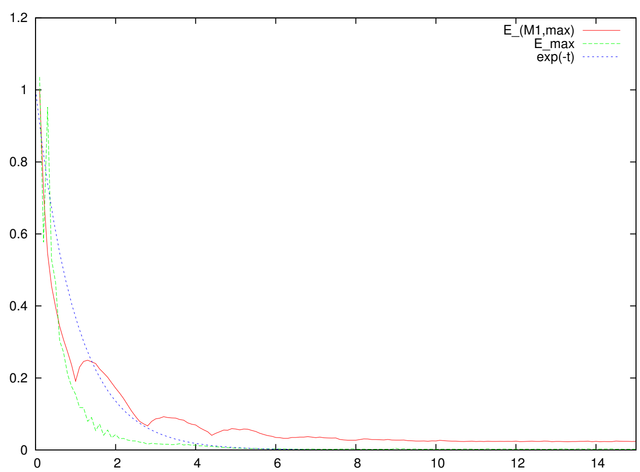

FIGURE 9. Time evolutions of $\mathcal{E}^n_{M_1,\max}$ (rescaled) and $\mathcal{E}^n_{\max}$, $n \geq 0$.

7. INVARIANT MEASURES ON M_r, $r > 0$

It is known that equations on manifolds with non-degenerate diffusions have a unique invariant probability law, that this invariant measure is absolutely continuous with respect to the surface measure and the density is C^∞-smooth and strictly positive, see e.g. [2] or [24, Proposition 4.5]. Unfortunately, the equation (3.7) on M_r has a degenerate diffusion - there is just one vector field g in the diffusion but M_r is a 3-dimensional manifold. In other words, there is not enough noise in the equation in order the above cited results on the nice ergodic behaviour could be applied in our case. We must therefore proceed in another way to confirm the conjectures of Section 6.

Convention 7.1. *In the present section, we restrict the operators (P_t) and (P^*_t) to the invariant space M_r where $r > 0$ is fixed. More precisely, P_t is understood as an endomorphism on $B_b(M_r)$ and P^*_t is an endomorphism on the space of probability measures on $\mathscr{B}(M_r)$, cf. Theorem 5.10. Also M_r is understood as a submanifold in $\mathbb{R}^6$.*

Definition 7.2. *We denote by λ_r the normalized surface (Riemannian) measure on M_r.*

7.1. Uniqueness.

We are going to prove, using the geometric version of the Hörmander theorem A.3 that λ_r is the unique invariant measure on M_r. But let us first, before we proceed with the study of the qualitative properties of the adjoint Markov semigroup (P^*_t), establish some further geometric properties of the drift and the diffusion vector fields f and g defined in (3.8).

Lemma 7.3. *M_r is a connected 3-dimensional submanifold in $\mathbb{R}^6$ and the vector fields f and g on M_r satisfy*

$$[g,f] = \begin{pmatrix} u \times v \\ 0 \end{pmatrix}, \qquad [f,[g,f]] = r^2 g, \qquad [g,[g,f]] = -f, \qquad \operatorname{div} f = \operatorname{div} g = \operatorname{div}[g,f] = 0$$

where $[\cdot,\cdot]$ is the Jacobi bracket.

Proof. Obviously, any (p,ξ_1) and (p,ξ_2) in M_r can be connected by a rotation curve in the circle $\{(p,\xi) : \xi \perp p,\, |\xi| = r\}$ and if $|p| = |q| = 1$ and $\gamma : [a,b] \to \mathbb{S}^2$ is a curve connecting p and q with $|\dot\gamma| = r$ then $\Gamma = (\gamma, \dot\gamma)$ is a curve connecting $(p, \dot\gamma(a))$ and $(q, \dot\gamma(b))$ in M_r. Altogether, any two points in M_r can be connected by an at most two times broken curve.

Observe that f, g and $[g,f]$ are orthogonal tangent vector fields on M_r. If we define $E_1 = f/(r^2+r^4)^{\frac12}$, $E_2 = g/r$, $E_3 = [G,F]/r$ then $\{E_1, E_2, E_3\}$ is an othonormal frame on M_r and

$$\operatorname{div} Y = \sum_{j=1}^{3} \langle d_{E_j} Y, E_j \rangle_{\mathbb{R}^6} = 0, \qquad Y \in \{f, g, [f,g]\}$$

where $d_X Y(p) = \lim_{t\to 0} t^{-1}[Y(p+tX) - Y(p)]$. □

Definition 7.4. *Let $S^1, \dots, S^m$ be vector fields on a manifold M. Denote by $(S^1,\dots,S^m)$ the least algebra for the Jacobi bracket $[X,Y] = XY - YX$ that contains $\{S^1,\dots,S^m\}$ and denote*

$$\mathscr{L}(S^1,\dots,S^m)(p) = \{S_p : S \in \mathscr{L}(S^1,\dots,S^m)\} \subseteq T_pM, \qquad p \in M.$$

Corollary 7.5. *$\mathscr{L}(f,g)(z) = T_z M_r$ holds for every $z \in M_r$.*

The following result is known[3] but we can give its straight analytic proof in few lines now.

Proposition 7.6. *A probability measure ν on $\mathscr{B}(M_r)$ is invariant if and only if*

$$\int_{M_r} \mathcal{A} h \, d\nu = 0 \quad \text{for every} \quad h \in C^2(M_r) \tag{7.1}$$

where the operator $\mathcal{A}$ was defined in (5.2).

Proof. This is an immediate consequence of the C_0-semigroup property of (P_t) on $C(M_r)$, the invariance of $C^2(M_r)$ under (P_t), the fact that $P_t \circ \mathcal{A} = \mathcal{A} \circ P_t$ on $C^2(M_r)$ for every $t \ge 0$ and density of $C^2(M_r)$ in $B_b(M_r)$ as all proved in Theorem 5.10. □

Proposition 7.7. *Let $R \in C^2(M_r)$. Then the measure $d\nu = R\, d\lambda_r$ satisfies (7.1) iff R is constant on M_r.*

Proof. Using the standard formulae

$$\int_{M_r} Yh \, d\lambda_r = -\int_{M_r} h \operatorname{div} Y \, d\lambda_r, \qquad \operatorname{div}(hY) = Y(h) + h \operatorname{div} Y$$

that hold for any smooth vector field Y on M_r and any smooth function h on M_r, applying Lemma 7.3 and Proposition 7.6 and using the fact that $C^2(M_r)$ is dense in $L^1(M_r, \lambda_r)$, we get that ν satisfies (7.1) iff the identity

$$fR = \frac{1}{2} g(gR) \tag{7.2}$$

holds on M_r. But

$$\int_{M_r} R(fR - \frac{1}{2} g(gR)) \, d\lambda_r = \frac{1}{2} \int_{M_r} |gR|^2 \, d\lambda_r$$

as f and g are divergence-free, so we conclude that (7.2) holds iff $gR = fR = 0$. If R is constant, this equality surely holds. For the converse implication, by definition of the Lie bracket, $[g,f]R = 0$ holds. Since f_z, g_z and $[g,f]_z$ span $T_z M_r$ for every $z \in M_r$ by Lemma 7.3, we obtain that R is locally constant. But M_r is connected by Lemma 7.3, hence R is constant. □

Theorem 7.8. *λ_r is the unique invariant probability measure on M_r.*

[3] See e.g. (4.58) on p. 292 in [24].

Proof. Let ν be an invariant measure. Since (7.1) holds and the geometric version of the Hörmander theorem A.3 is applicable due to Corollary 7.5, we conclude that ν has a smooth density R with respect to λ_r. But then $R = 1$ on M_r by Proposition 7.7. □

8. The transition probabilities on M_r, $r > 0$

In this section, we continue the study of the Markov semigroup (P_t) and its adjoint semigroup (P_t^*) restricted to M_r as set forth in Convention 7.1, with $r > 0$ fixed. We are going to show that the transition probabilities $p_{t,x}$ restricted to $\mathscr{B}(M_r)$ for $x \in M_r$ are absolutely continuous with respect to the normalized surface measure λ_r on M_r for every $(t,x) \in (0,\infty) \times M_r$ and that the density $p(t,x,\cdot)$ satisfies $p \in C^\infty((0,\infty) \times M_r \times M_r)$. The density $p(t,x,\cdot)$ should be denoted by $p_r(t,x,\cdot)$ to indicate the dependence on $r > 0$ but we will not use this notation since r is fixed in this section and we will not use the densities elsewhere in this paper.

An expert could be simply advised to apply the abstract results based on the geometric Hörmander theorem in [22, Theorem 3] but we prefer to guide the reader through, to explain the actual structure of the problem better.

For, let us define the adjoint operator

$$\mathcal{A}^* h = -f(h) + \frac{1}{2} g(g(h)), \qquad h \in C^2(M_r) \tag{8.1}$$

to the operator $\mathcal{A}$ defined in (5.2). Indeed, by Lemma 7.3,

$$\int_{M_r} (\mathcal{A}h_1)h_2 \, d\lambda_r = \int_{M_r} h_1 \mathcal{A}^* h_2 \, d\lambda_r, \qquad \forall h_1, h_2 \in C^2(M_r) \tag{8.2}$$

as f and g are divergence-free on M_r.

Theorem 8.1. *The transition probabilities $p_{t,x}$ are absolutely continuous with respect to the normalized surface measure λ_r on M_r for every $(t,x) \in (0,\infty) \times M_r$ and the density $p(t,x,\cdot)$ satisfies $p \in C^\infty((0,\infty) \times M_r \times M_r)$.*

Proof. Consider the Riemannian manifold $N = (0,\infty) \times M_r \times M_r$ and define the Radon measure

$$\Gamma(A) = \int_0^\infty \int_{M_r} \int_{M_r} \mathbf{1}_A(t,x,z) \, dp_{t,x}(z) \, d\lambda_r(x) \, dt = \mathbb{E} \int_0^\infty \int_{M_r} 1_A(t,x,z^x(t)) \, dt \, d\lambda_r(x), \ A \in \mathscr{B}(N).$$

Every function $h \in C^\infty(N)$ has variables (t,x,z) and we are going to indicate by $\mathcal{A}_z$ that the operator $\mathcal{A}$ is applied on the variable z and by $\mathcal{A}_x^*$ that the operator $\mathcal{A}^*$ is applied on the variable x of the function $h(t,x,z)$.

By the Itô formula,

$$\int_0^\infty \int_{M_r} \left(\frac{\partial H}{\partial t} + \mathcal{A}H \right) dp_{t,x} \, dt = 0 \qquad \forall H \in C^\infty_{comp}((0,\infty) \times M_r) \tag{8.3}$$

holds for every $x \in M_r$ hence

$$\int_N \left(\frac{\partial h}{\partial t} + \mathcal{A}_z h \right) d\Gamma = 0 \qquad \forall h \in C^\infty_{comp}(N). \tag{8.4}$$

Let $h_1 \in C^\infty_{comp}(0,\infty)$, $h_2, h_3 \in C^\infty(M_r)$ and define $H(t,x) = h_1(t)h_2(x)$, $h(t,x,z) = h_1(t)h_2(x)h_3(z)$ and $v(t,x) = P_t h_3(x)$. Then

$$\int_N \left(\frac{\partial h}{\partial t} + \mathcal{A}_x^* h \right) d\Gamma = \int_0^\infty \int_{M_r} \left(\frac{\partial H}{\partial t} + \mathcal{A}^* H \right) v \, d\lambda_r \, dt = \int_0^\infty \int_{M_r} H \left(-\frac{\partial v}{\partial t} + \mathcal{A}v \right) d\lambda_r \, dt = 0$$

by (5.3) and the duality (8.2). In fact,

$$\int_N \left(\frac{\partial h}{\partial t} + \mathcal{A}_x^* h \right) d\Gamma = 0, \qquad \forall h \in C^\infty_{comp}(N) \tag{8.5}$$

by a density argument as shown in Proposition B.1.

Altogether we have obtained that

$$\int_N \left(2\frac{\partial h}{\partial t} + \mathcal{A}_x^* h + \mathcal{A}_z h \right) d\Gamma = 0, \qquad \forall h \in C^\infty_{comp}(N).$$

In order to apply the geometric Hörmander theorem A.3, we define the vector fields

$$Y(t,x,z)=\begin{pmatrix}2\\-f(x)\\f(z)\end{pmatrix},\qquad X^1(t,x,z)=\begin{pmatrix}0\\g(x)\\0\end{pmatrix},\qquad X^2(t,x,z)=\begin{pmatrix}0\\0\\g(z)\end{pmatrix}$$

where the vector field Y corresponds to the operator $h\mapsto 2\frac{\partial h}{\partial t}-f_x(h)+f_z(h)$, the vector field X^1 to the operator $h\mapsto g_x(h)$ and the vector field X^2 to the operator $h\mapsto g_z(h)$. Defining also $h=[g,f]$ on M_r, we get by Lemma 7.3 that

$$[Y,X^1]=\begin{pmatrix}0\\h(x)\\0\end{pmatrix},\quad [Y,X^2]=-\begin{pmatrix}0\\0\\h(z)\end{pmatrix},\quad [X^1,X^2]=0,$$

$$[Y,[Y,X^1]=-r^2X^1,\quad [Y,[Y,X^2]]=-r^2X^2,\quad [X^1,[Y,X^1]=-\begin{pmatrix}0\\f(x)\\0\end{pmatrix},\quad [X^1,[Y,X^2]]=0,$$

$$[X^2,[Y,X^1]=0,\quad [X^2,[Y,X^2]]=\begin{pmatrix}0\\0\\f(z)\end{pmatrix},\quad [[Y,X^1],[Y,X^2]]=0.$$

At this stage we see that

$$\mathscr{L}(Y,X^1,X^2)(t,x,z)=\mathbb{R}\times T_xM_r\times T_zM_r=T_{(t,x,z)}N,\qquad \forall (t,x,z)\in N$$

so the geometric Hörmander theorem A.3 is applicable and Γ has a smooth density $p\in C^\infty(N)$ with respect to $dt\otimes\lambda_r\otimes\lambda_r$.

Let $\varphi\in C(M_r)$. Then, by the standard measure theoretical properties of integrals,

$$P_t\varphi(x)=\int_{M_r}\varphi(z)p(t,x,z)\,d\lambda_r(z)\tag{8.6}$$

holds for $dt\otimes\lambda_r$-almost every (t,x). But since both sides are continuous in (t,x) (the right hand side by Theorem 5.10), the identity (8.6) holds for every $(t,x)\in(0,\infty)\times M_r$. By standard procedure, we extend (8.6) to hold for every $\varphi\in B_b(M_r)$ and every $(t,x)\in(0,\infty)\times M_r$. □

The following result recasts Corollary 5.9 in terms of the transition densities.

Corollary 8.2. *Let Q be a 3×3-unitary matrix. Denote by $\widetilde{Q}=\operatorname{diag}[Q,Q]$. Then*

$$p(t,x,y)=p(t,\widetilde{Q}x,\widetilde{Q}y)$$

holds for every $(t,x,y)\in(0,\infty)\times M_r\times M_r$.

Proof. We just realize that $\widetilde{Q}$ is a measure preserving diffeomorphism on M_r (as a restriction of an isometry on $\mathbb{R}^6$) and then we apply Corollary 5.9. □

9. Controlability in M_r, $r>0$

In this section, we are going to examine the supports of the probability measures $p_{t,x}$ on $\mathscr{B}(M_r)$ for $x\in M_r$. Again, in this section, the Markov semigroup (P_t) and its adjoint semigroup (P_t^*) are restricted to M_r as in Convention 7.1, with $r>0$ fixed.

Theorem 9.1. *Let $t\geq 2\pi/r$. Then* $\operatorname{supp} p_{t,x}=M_r$ *holds for every $x\in M_r$.*

9.1. **General support result.** Let $x \in M_r$ and denote by $V^{x,a}$ the solutions of the ordinary differential equation

$$X' = f(X) + a(t)g(X), \qquad X(0) = x \tag{9.1}$$

on M_r where $a \in L^1_{loc}([0,\infty))$ and f and g are defined in (3.8).

Remark 9.2. It can be checked analogously as in the proof of Proposition 4.1 that the solutions $V^{x,a}$ take values in M_r and are therefore global.

The next lemma tells us that, to describe the support of the probabilities $p_{t,x}$ for $x \in M_r$, it is sufficient and necessary to study solutions of the ordinary differential equation (9.1).

Lemma 9.3. *Let $t > 0$ and $x \in M_r$. Then*

$$\operatorname{supp} p_{t,x} = \overline{\{V^{x,a}(t) : a \in L^1(0,t)\}}^{M_r}. \tag{9.2}$$

Proof. Let $\tilde{f}$, $\tilde{g}$ be smooth compactly supported vector fields on $\mathbb{R}^6$ and denote by μ the law of the solution of the equation

$$dX = \tilde{f}(X) + \tilde{g}(X) \circ dW, \qquad X(0) = x \tag{9.3}$$

on $\mathscr{B}(C([0,t];\mathbb{R}^6))$. Let also $a \in L^1(0,t)$ and denote by v^a the solution of

$$X' = \tilde{f}(X) + a(t)\tilde{g}(X), \qquad X(0) = x. \tag{9.4}$$

Then, according to the Support theorem of Stroock and Varadhan [40] (see also [1], [5], [6], [20], [29] for generalizations or shorter proofs),

$$\operatorname{supp} \mu = \overline{\{v^a : a \in L^1(0,t)\}}$$

where the closure and the support are taken in $C([0,t];\mathbb{R}^6)$. Since $v^{a_n} \to v^a$ uniformly on $[0,t]$ if $a_n \to a$ in $L^1(0,t)$ and A is a dense subset in $L^1(0,t)$, it also holds

$$\operatorname{supp} \mu = \overline{\{v^a : a \in A\}}.$$

To get back to our problem (3.7), let $\tilde{f}$, $\tilde{g}$ be smooth compactly supported vector fields on $\mathbb{R}^6$ such that $\tilde{f} = f$ and $\tilde{g} = g$ on the centered ball in $\mathbb{R}^6$ of the radius $R = |x|$. Then the solution X coincides with z^x being the solution of (3.7) with $z^x(0) = x$. Also, by uniqueness, $V^{x,a} = v^a$. Thus we conclude that

$$\operatorname{supp} (\text{Law } z^x) = \overline{\{V^{x,a} : a \in L^1(0,t)\}} \tag{9.5}$$

where both the support and the closure are taken in $C([0,t];M_r)$ being a closed subset of $C([0,t];\mathbb{R}^6)$.

Now consider the projection $\pi_t : C([0,t];M_r) \to M_r : \xi \mapsto \xi(t)$. Since π is continuous,

$$\overline{\pi_t[\operatorname{supp} (\text{Law } z^x)]} = \operatorname{supp} (\text{Law } \pi_t(z^x)) = \operatorname{supp} p_{t,x},$$

and by continuity of π_t and (9.5), we also have

$$\overline{\pi_t[\operatorname{supp} (\text{Law } z^x)]} = \overline{\pi_t[\overline{\{V^{x,a} : a \in L^1(0,t)\}}]} = \overline{\pi_t[\{V^{x,a} : a \in L^1(0,t)\}]} = \overline{\{V^{x,a}(t) : a \in L^1(0,t)\}}.$$

□

9.2. **The control problem.** In view of Lemma 9.3, it remains to prove that the ordinary differential equation (9.1) can be controlled to hit every point in M_r after time $2\pi/r$. It turns out that it is necessary to enter deeper to the geometry of the $2D$ sphere.

For consider the equation (9.1) with a constant control $a \in \mathbb{R}$

$$w'' = -|w'|^2 w + a w \times w' \tag{9.6}$$

and with the initial condition $w(0) = p$, $w'(0) = \xi$ for $x = (p,\xi) \in M_r$. It can be guessed (and consequently checked) from rotational symmetries of (9.6) that the unique solution has the form

$$w^{x,a}(t) = \frac{a}{b} E_1^{x,a} + \frac{r}{b} E_2^{x,a} \cos(bt) + \frac{r}{b} E_3^{x,a} \sin(bt) \tag{9.7}$$

$$E_1^{x,a} = \frac{a}{b} p + \frac{1}{b} p \times \xi, \quad E_2^{x,a} = \frac{r}{b} p - \frac{a}{rb} p \times \xi, \quad E_3^{x,a} = \frac{1}{r} \xi$$

where $b = \sqrt{r^2 + a^2}$. Since $\{E_1^{x,a}, E_2^{x,a}, E_3^{x,a}\}$ is orthonormal with $\det(E_1^{x,a}, E_2^{x,a}, E_3^{x,a}) = 1$, we deduce that $w^{x,a}$ is a parametrization of a circle on $\mathbb{S}^2$ with the derivative of constant length r.

Lemma 9.4. *A C^2-smooth curve such that $|w|_{\mathbb{R}^3} = 1$ and $|w'|_{\mathbb{R}^3} = r$ satisfies the equation* (9.6) *for some control $a \in \mathbb{R}$ iff it parametrizes a non-degenerate circle*[4] *on $\mathbb{S}^2$.*

Hence, solutions of (9.6) can be regarded as oriented circles in $\mathbb{S}^2$.

Definition 9.5. *In the sequel, we are going to consider pairs (K, Y) where K is a non-degenerate circle in $\mathbb{S}^2$, i.e. K is understood as a one-dimensional submanifold in $\mathbb{S}^2$, and Y is a vector field on the manifold K with $|Y_p| = r$ for every $p \in K$, i.e. Y determines an orientation of the manifold K. Such pairs are going to be called "oriented circles" in $\mathbb{S}^2$ for simplicity.*

Remark 9.6. Any non-degenerate circle K in $\mathbb{S}^2$ can be described in a unique way as $K = (v + P) \cap \mathbb{S}^2$ where P is a two-dimensional subspace in $\mathbb{R}^3$, $v \in \mathbb{R}^3$ is perpendicular to P and $|v| < 1$. Here the vector v is the center of the circle K and P is the plane of the circle. Obviously, if $\bar{v} \in \mathbb{R}^3$ then $K = (\bar{v} + P) \cap \mathbb{S}^2$ iff $\bar{v} - v \in P$. Also

$$T_z K = \{p \in P : p \perp z\} = \{p \in P : p \perp (z - v)\}, \qquad \forall z \in K.$$

If we define $\theta = \sqrt{1 - |v|^2}$, $\{p_1, p_2\}$ is an orthonormal basis in P and

$$Y_z = \frac{r}{\theta} \left[-\langle z, p_1 \rangle p_2 + \langle z, p_2 \rangle p_1 \right], \qquad z \in K$$

then $\{Y, -Y\}$ are the only two vector fields on K of length r.

Lemma 9.7. *Let $x = (p, \xi) \in M_r$ and define the circle on $\mathbb{S}^2$*

$$K = (p + \operatorname{span}\{E_2^{x,a}, E_3^{x,a}\}) \cap \mathbb{S}^2$$

in the notation of (9.7) *and the vector field on K of length r*

$$Y(z) = -b\langle z, E_3^{x,a} \rangle E_2^{x,a} + b\langle z, E_2^{x,a} \rangle E_3^{x,a}, \qquad z \in K^{x,a}$$

where $b = \sqrt{r^2 + a^2}$. Then K is the orbit of $w^{x,a}$ and $Y(w^{x,a}) = (w^{x,a})'$ holds on $\mathbb{R}$.

The following technical result tells us that we can move continuously from one element in M_r to another, along two oriented circles in $\mathbb{S}^2$ with just one "switch" from one circle to the other.

Proposition 9.8. *In the terminology of Definition 9.5, let (K, Y) be an oriented circle in $\mathbb{S}^2$ and let $(p, \xi) \in M_r$ satisfy $p \notin K$. Then there exists $z \in K$ and an oriented circle (T, B) in $\mathbb{S}^2$ such that $z, p \in T$, $B_z = Y_z$ and $B_p = \xi$.*

Proof. Denote by Q_z the vector space generated by $\{p - z, Y_z\}$ for $z \in K$. Since $p - z$ and Y_z are linearly independent, Q_z is two-dimensional. Now $T_z = (p + Q_z) \cap \mathbb{S}^2$ is a non-degenerate circle in $\mathbb{S}^2$ as it contains two distinct points $p, z \in \mathbb{S}^2$. Fixing $z \in K$, we are going to show that there exists a vector field B of length r on T_z such that $B_z = Y_z$. For, if we define

$$R_z = r^2(p - z) - \langle p - z, Y_z \rangle Y_z, \qquad z \in K$$

then $R_z \neq 0$ by linear independence of $\{p - z, Y_z\}$ and we can set $V_z = \frac{R_z}{|R_z|}$. So $\{V_z, \frac{1}{r} Y_z\}$ is an orthonormal basis in Q_z. Let q_z be the orthogonal projection of p onto Q_z and define $p_z = p - q_z$, $\theta_z = \sqrt{1 - |p_z|^2}$. So $T = (p_z + Q_z) \cap \mathbb{S}^2$. According to Remark 9.6,

$$B_z(\tau) = \frac{1}{\theta_z} \left[\langle \tau, Y_z \rangle V_z - \langle \tau, V_z \rangle Y_z \right], \qquad \tau \in T_z$$

is a vector field of length r on T_z. Since $z - p$ and $z - p_z$ belong to Q_z and $p_z \perp Q_z$, we have $z = p_z + \langle z, V_z \rangle V_z + \frac{1}{r^2} \langle z, Y_z \rangle Y_z = p_z + \langle z, V_z \rangle V_z$ as $z \perp Y_z$, hence

$$1 = |z|^2 = |p_z|^2 + \langle z, V_z \rangle^2, \qquad \theta_z = |\langle z, V_z \rangle|.$$

But

$$|R_z| \langle z, V_z \rangle = \langle z, R_z \rangle = r^2 \langle z, p - z \rangle = r^2(\langle z, p \rangle - 1) \leq 0$$

[4]Here "non-degenerate" means that the radius of the circle is strictly positive.

so we conclude that $\theta_z = -\langle z, V_z\rangle$. From this we obtain that $B_z(z) = -\frac{1}{\theta_z}\langle z, V_z\rangle Y_z = Y_z$. Eventually, $B_z(p) = \frac{1}{\theta_z}\left[\langle p, Y_z\rangle V_z - \langle p, V_z\rangle Y_z\right]$. It remains to prove that the mapping

$$L : K \to \{\zeta \in T_p\mathbb{S}^2 : |\zeta| = r\} : z \mapsto B_z(p)$$

is a surjection. Since K and $\{\zeta \in T_p\mathbb{S}^2 : |\zeta| = r\}$ are homeomorphic with $\mathbb{S}^1$ and L is continuous, it is sufficient to prove that L is locally injective by Proposition C.1. Here we can easily see that L_z spans the one-dimensional vector space $Q_z \cap \{p\}^\perp$.

So let us study injectivity of L. Let $K = (v + U) \cap \mathbb{S}^2$ where U is a two-dimensional subspace in $\mathbb{R}^3$, $v \perp U$ and $|v| < 1$. Let $z_1 \in K$. Then there exists an orthonormal basis u_1, u_2 in U such that $z_1 = v + \xi u_1$ where $1 = |v|^2 + \xi^2$ and $Y(z_1) = ru_2$. If $z_2 \in K$ satisfies $z_1 \neq z_2$ then there exists a unique $\Delta \in (-\pi, \pi] \setminus \{0\}$ such that

$$z_2 = v + \xi u_1 \cos\Delta + \xi u_2 \sin\Delta$$

and, from this,

$$Y(z_2) = r[-u_1 \sin\Delta + u_2 \cos\Delta].$$

Then $Q_{z_1} \cap Q_{z_2}$ is a one-dimensional space spanned by

$$A = (z_1 - p)\sin\Delta + \frac{\xi}{r}(1 - \cos\Delta)Y(z_1) = (z_2 - p)\sin\Delta - \frac{\xi}{r}(1 - \cos\Delta)Y(z_2).$$

Obviously, the vector A belongs also to $\{p\}^\perp$ iff

$$\psi(\Delta) := \frac{\sin\Delta}{1 - \cos\Delta} = \frac{\xi\langle p, u_2\rangle}{1 - \langle p, z_1\rangle}. \tag{9.8}$$

Now $\psi : (-\pi, \pi] \setminus \{0\} \to \mathbb{R}$ is a bijection and the right hand side of (9.8) is bounded by a constant $C_{p,K}$ irrespective of z_1, z_2, u_1 or u_2, as $p \notin K$. So Δ satisfying the identity (9.8) must verify to $|\Delta| \geq \varepsilon_{p,K} > 0$ and, consequently, $|z_1 - z_2| \geq \varepsilon'_{p,K} > 0$. In particular, L is locally injective and, consequently, L is surjective. The identity (9.8) then also implies that

$$\{z \in K \setminus \{z_1\} : L(z) \in \{-L(z_1), L(z_1)\}\} = \{z \in K \setminus \{z_1\} : \dim Q_{z_1} \cap Q_z \cap \{p\}^\perp = 1\}$$

contains exactly one element z_2, which, by surjectivity of L, must satisfy $L(z_1) = -L(z_2)$. In particular, L is injective. □

9.3. **Proof of Theorem 9.1.** Let $(p_1, \xi_1), (p_3, \xi_3) \in M_r$. We are going to show that, choosing a suitable piece-wise constant control a in the equation (9.1), we can reach (p_3, ξ_3) from (p_1, ξ_1) by the solution (9.1) with this control a in any time $T \geq 2\pi/r$. We are going to proceed in steps.

First let $a_1 = 0$ and move (p_1, ξ_1) along the solution of (9.6) with the constant control a_1 to some (p_2, ξ_2) in a very short time just to arrange $p_2 \neq p_3$.

Next let a_2 be an extremely large constant control so that the orbit K_2 of the solution $w^{(p_2,\xi_2),a_2}$ does not contain p_3. This can be done by choosing a large control a as the diameter of the orbit is $2r/\sqrt{r^2 + a^2}$ by (9.7). This solution defines an oriented circle (K_2, Y_2) in $\mathbb{S}^2$ and $p_3 \notin K_2$. Hence, by Proposition 9.8, there exists an oriented circle (K_3, Y_3) in $\mathbb{S}^2$ such that $z \in K_2 \cap K_3$, $p_3 \in K_3$, $Y_2(z) = Y_3(z)$ and $Y_3(p_3) = \xi_3$. This circle K_3 is associated to a control $a_3 \in \mathbb{R}$.

Let a be the piece-wise constant control with steps a_1, a_2 and a_3 at times τ_1, τ_2 and τ_3 so that the solution X to (9.1) with this control satisfies $X(0) = (p_1, \xi_1)$, $X(\tau_1) = (p_2, \xi_2)$, $X(\tau_2) = (z, Y_2(z)) = (z, Y_3(z))$ and $X(\tau_3) = (p_3, \xi_3)$. Now τ_1 was as small as we wanted, $\tau_2 - \tau_1$ too because a_2 was large and the periodicity of the solutions to (9.1) with a constant control a is $2\pi/\sqrt{r^2 + a^2}$ by (9.7). Hence $\tau_3 - \tau_2$ is not larger that $2\pi/r$ since we do not let the solution run the full period. Altogether, $\tau_3 < T$.

Let $a_4 \in \mathbb{R}$ be a control such that $T - \tau_3 \in \{2\pi k/\sqrt{r^2 + a_4^2} : k \geq 0\}$ and let $a = a_4$ on $(\tau_3, T]$. Then $X(T) = X(\tau_3) = (p_3, \xi_3)$. In other words, we let the solution revolve to wait for the time T, to wind up in the point of the departure (p_3, ξ_3).

10. Exponential ergodicity in M_r, $r > 0$

In this section, again, we consider the Markov semigroup (P_t) and its adjoint semigroup (P_t^*) restricted to M_r as in Convention 7.1, with $r > 0$ fixed. We are going to prove the exponential convergence to the invariant measure λ_r in total variation via the Doeblin theorem and a minorization condition due to [31] and [30].

Lemma 10.1. *The transition densities satisfy $p > 0$ on $(2\pi/r, \infty) \times M_r \times M_r$.*

Proof. We develop the idea of [31, Section 5.2] and the proof of [30, Lemma 2.3]. According to Theorem 8.1, the transition densities $p(t, x, \cdot)$ are smooth in all three variables. Let $t_1 > 2\pi/r$ and $t_2 > 0$ satisfy $t = t_1 + t_2$. Let also $x_0, y_0 \in M_r$ be such that $p(t_2, \cdot, \cdot) \geq \varepsilon$ on a neighbourhood $O_{x_0} \times O_{y_0}$ for some $\varepsilon > 0$. Then, from the Chapman-Kolmogorov identity

$$p(t, x, y) = \int_{M_r} p(t_1, x, z) p(t_2, z, y) \, d\lambda_r(z) \geq \varepsilon p(t_1, x, O_{x_0}) > 0, \qquad \forall x \in M_r, \ \forall y \in O_{y_0}$$

since the support of $p_{t_1,x}$ is M_r by Theorem 9.1. Now if $p(t, x_1, y_1) = 0$ for some $x_1, y_1 \in M_r$, let $Q \in \mathbb{R}^3 \otimes \mathbb{R}^3$ be one of the two unitary matrices for which $\widetilde{Q} = Q \otimes Q = \operatorname{diag}[Q, Q]$ satisfies $\widetilde{Q} y_1 = y_0$. Then $0 = p(t, x_1, y_1) = p(t, \widetilde{Q} x_1, y_0)$ by Corollary 8.2, which is a contradiction. □

Theorem 10.2. *There exist positive constants c_r, α_r such that*

$$\|P_t^* \nu - \lambda_r\| \leq c_r e^{-\alpha_r t} \|\nu - \lambda_r\|, \qquad \forall t \geq 0 \tag{10.1}$$

holds for every probability measure ν on $\mathscr{B}(M_r)$, where $\|\cdot\|$ is the norm in total variation on M_r.

Proof. Set $\tau = 4\pi/r$. According to Lemma 10.1, there exists $\varepsilon > 0$ such that $p_{\tau,x}(A) \geq \varepsilon \lambda_r(A)$ holds for every $x \in M_r$ and every $A \in \mathscr{B}(M_r)$. Hence, by the Doeblin theorem[5], (P_t^*) has a unique invariant probability measure μ on M_r and there exist positive constants c_r and α_r such that

$$\|P_t^* \nu - \mu\| \leq c_r e^{-\alpha_r t} \|\nu - \mu\|, \qquad \forall t \geq 0$$

holds for every probability measure ν on $\mathscr{B}(M_r)$. But λ_r is the unique invariant probability measure on M_r by Theorem 7.8. □

11. Invariant measures and attractivity on $T\mathbb{S}^2$

In this last section, we are going to study the global dynamics on the full target space $T\mathbb{S}^2$. We will identify the set of invariant probability measures on $\mathscr{B}(T\mathbb{S}^2)$, the set of ergodic probability measures on $\mathscr{B}(T\mathbb{S}^2)$ and it will be shown that the dual Markov semigroup is always attractive.

Definition 11.1. *Extend λ_r from $\mathscr{B}(M_r)$ to $\mathscr{B}(T\mathbb{S}^2)$, in the unique way to obtain a probability measure on $\mathscr{B}(T\mathbb{S}^2)$, i.e. $A \mapsto \lambda_r(A \cap M_r)$. Let us denote this extension still by λ_r.*

Definition 11.2. *If ν is a probability measure on $T\mathbb{S}^2$, we define the probability measures*

$$\nu_*(U) = \nu\{(p, \xi) \in T\mathbb{S}^2 : |\xi| \in U\}, \qquad U \in \mathscr{B}([0, \infty))$$

$$\bar{\nu}(A) = \nu(A \cap M_0) + \int_{(0,\infty)} \lambda_r(A \cap M_r) \, d\nu_*, \qquad A \in \mathscr{B}(T\mathbb{S}^2)$$

in the notation of (5.1).

Remark 11.3. One can check by the definition of λ_r that the mapping $r \mapsto \lambda_r(A \cap M_r)$ is Borel measurable on $(0, \infty)$ for every $A \in \mathscr{B}(T\mathbb{S}^2)$ by the Fubini theorem.

Theorem 11.4. *Let z be a solution of* (3.7) *on $T\mathbb{S}^2$ with an initial distribution ν on $\mathscr{B}(T\mathbb{S}^2)$. Then the laws of $z(t)$ converge in total variation on $T\mathbb{S}^2$ to $\bar{\nu}$ as $t \to \infty$. Moreover, ν is invariant for* (3.7) *iff $\nu = \bar{\nu}$ and $\{\delta_x, \lambda_r : x \in M_0, \, r > 0\}$ is the set of ergodic probability measures for* (3.7).

[5] See e.g. [18, Theorem 4] for a particularly simple proof of the Doeblin theorem.

Proof. Let $F : [0,\infty) \times \mathscr{B}(T\mathbb{S}^2) \to [0,1]$ be a regular version of a conditional probability measure $\nu(\cdot||\xi| = r)$ on $\mathscr{B}(T\mathbb{S}^2)$ for $r \geq 0$, i.e. $F(r,\cdot)$ is a probability measure on $\mathscr{B}(T\mathbb{S}^2)$ for every $r \geq 0$, $F(\cdot, A)$ is Borel measurable on $[0,\infty)$ for every $A \in \mathscr{B}(T\mathbb{S}^2)$ and

$$\nu(A \cap \{(p,\xi) : |\xi| \in U\}) = \int_U F(r,A)\, d\nu_*(r) \tag{11.1}$$

holds for every $A \in \mathscr{B}(T\mathbb{S}^2)$ and $U \in \mathscr{B}[0,\infty)$. The equality (11.1) implies that

$$\int_{T\mathbb{S}^2} h(|\xi|,p,\xi)\, d\nu(p,\xi) = \int_{[0,\infty)} \left(\int_{T\mathbb{S}^2} h(r,y)\, dF_r(y) \right) d\nu_*(r) \tag{11.2}$$

holds for every bounded measurable $h : [0,\infty)\times T\mathbb{S}^2 \to \mathbb{R}$. In particular, setting $h(r,p,\xi) = \mathbf{1}_{[r=|\xi|]}$, we obtain that $\nu_*(O) = 1$ where $O = \{r \geq 0 : F(r, M_r) = 1\}$. So (11.2) implies that

$$\begin{aligned} (P_t^*\nu)(A) &= \int_{T\mathbb{S}^2} p(t,x,A)\, d\nu = \int_O \left(\int_{M_r} p(t,x,A)\, dF_r(x) \right) d\nu_*(r) = \int_O (P_t^* F_r)(A \cap M_r)\, d\nu_*(r) \\ &= \nu(A \cap M_0) + \int_{O\cap(0,\infty)} (P_t^* F_r)(A \cap M_r)\, d\nu_*(r) \end{aligned}$$

holds for every $t \geq 0$ and $A \in \mathscr{B}(T\mathbb{S}^2)$. By a contradiction argument, we get that $P_t^*\nu$ converge in total variation on $T\mathbb{S}^2$ to $\bar{\nu}$, by Theorem 10.2.

To prove the invariance part of the claim, realize that

$$\int_{T\mathbb{S}^2} h\, d\bar{\nu} = \int_{M_0} h\, d\nu + \int_{(0,\infty)} \left(\int_{M_r} h\, d\lambda_r \right) d\nu_*$$

holds for every bounded measurable $h : T\mathbb{S}^2 \to \mathbb{R}$ by the definition of the measure $\bar{\nu}$. Hence, setting $h(x) = p(t,x,A)$, we get that

$$(P_t^*\bar{\nu})(A) = \nu(A \cap M_0) + \int_{(0,\infty)} \lambda_r(A \cap M_r)\, d\nu_* = \bar{\nu}(A)$$

holds for every $A \in \mathscr{B}(T\mathbb{S}^2)$ by Theorem 7.8. In particular, $\bar{\nu}$ is invariant. If ν is invariant then $\nu = \lim_{t\to\infty} P_t^*\nu = \bar{\nu}$ by the first part of the proof.

Concerning the ergodic measures (according to Definition 5.1), the probability measures $\{\delta_x, \lambda_r : x \in M_0,\, r > 0\}$ are invariant by the second part of the proof and ergodicity follows from Remark 11.5 as ergodic probability measures are the extremal points of the set of all invariant probability measures (see e.g. Proposition 3.2.7 in [14]). Indeed, the probability measure ν_a is ergodic for (3.7) iff a is an extremal point in the convex set of probability measures on $\mathscr{B}(M_0\dot{\cup}(0,\infty))$. This occurs iff a is a Dirac measure, i.e. either $a = \delta_x$ for some $x \in M_0$ (hence $\nu_a = \delta_x$) or $a = \delta_r$ for some $r > 0$ (hence $\nu_a = \lambda_r$). □

Remark 11.5. Invariant measures for (3.7) can be uniquely described as measures

$$\nu_a(A) = a\,(A \cap M_0) + \int_{(0,\infty)} \lambda_r(A \cap M_r)\, da, \qquad A \in \mathscr{B}(T\mathbb{S}^2)$$

where a is a Borel probability measure on the Polish space[6] $X = M_0\dot{\cup}(0,\infty)$, i.e. $G \subseteq X$ is open iff $G \cap M_0$ is open in M_0 and $G \cap (0,\infty)$ is open in $(0,\infty)$. X is Polish as so are M_0 and $(0,\infty)$. The assignment $a \mapsto \nu_a$ is a bijection onto the set of invariant probability measures.

Appendix A. Lie algebra

Let U be an open set on a C^∞-manifold.

- The set $\mathscr{L}$ of all smooth tangent vector fields on U is a vector space with the Jacobi bracket. Any vector subspace of $\mathscr{L}$ closed under the Jacobi bracket is called *a Lie algebra*.
- If $\mathcal{X}$ is a set of smooth tangent vector fields on U, then we denote by $\mathscr{L}(\mathcal{X})$ the smallest Lie algebra containing $\mathcal{X}$.

[6] Topological spaces that can be metrized by a complete separable metric are called Polish spaces.

- If $\mathcal{A}\subseteq\mathscr{L}$ and $p\in U$, then we define $\mathcal{A}(p)=\{A_p : A\in\mathcal{A}\}$.

Proposition A.1. *Define* $L_0=\operatorname{span}\{\mathcal{X}\}$ *and* $L_n=\operatorname{span}\{L_{n-1}\cup\{[A,B]:A,B\in L_{n-1}\}\}$. *Then* $\bigcup L_n=\mathscr{L}(\mathcal{X})$.

Proposition A.2. *Let* $X_1,\dots,X_m,Y\in\mathscr{L}$ *and let* $f_i\in C^\infty(U)$. *Then*

$$\mathscr{L}(X_1,\dots,X_m,Y)(p)=\mathscr{L}(X_1,\dots,X_m,Y+\sum_{j=1}^m f_jX_j)(p),\qquad p\in U.$$

Proof. Let us write $\mathcal{A}^1=\{X_1,\dots,X_m,Y\}$, $\mathcal{A}^2=\{X_1,\dots,X_m,Y+\sum_{j=1}^m f_jX_j\}$,

$$\mathscr{C}^i=\left\{\sum_{k=1}^K h_kL_k : h_k\in C^\infty(U), L_k\in\mathscr{L}(\mathcal{A}^i), K\in\mathbb{N}\right\}.$$

Apparently, $\mathscr{C}^i$ is a Lie algebra for $i\in\{1,2\}$, $\mathcal{A}^i\subseteq\mathscr{C}^j$ whenever $\{i,j\}=\{1,2\}$ hence $\mathscr{L}(\mathcal{A}^i)\subseteq\mathscr{C}^j$ whenever $\{i,j\}=\{1,2\}$. But then

$$\mathscr{L}(\mathcal{A}^i)(p)\subseteq\mathscr{C}^j(p)\subseteq\mathscr{L}(\mathcal{A}^j)(p).$$

□

Theorem A.3 (Hörmander)**.** *Let M be a Riemannian manifold with a countable topological basis, let $X^1,\dots,X^m,Y$ be smooth vector fields on M, let Z be a smooth function on M and let μ be a Radon measure on $\mathscr{B}(M)$ such that*

$$\int_M\left\{Zh+Y(h)+\sum_{i=1}^m X^i(X^i(h))\right\}d\mu=0,\qquad\forall h\in C^\infty_{comp}(M)\tag{A.1}$$

and

$$\operatorname{span}\{L_p : L\in\mathscr{L}(X^1,\dots,X^m,Y)\}=T_pM,\qquad\forall p\in M.$$

Then μ has a C^∞-smooth density with respect to the Riemannian measure on M.

Proof. Let $\varphi:O\to U$ be a diffeomorphism from an open set $O\subseteq\mathbb{R}^d$ onto an open set $U\subseteq M$, denote by ϕ the inverse of φ, define $\theta(A)=\mu[\varphi[A]]$ for $A\in\mathscr{B}(O)$, decompose $X^i_\varphi=\sum_{j=1}^d x^i_j\partial^j_\varphi$, $Y_\varphi=\sum_{j=1}^d y_j\partial^j_\varphi$ on O and define $z=Z(\varphi)$ and

$$Q=-y+2\sum_{i=1}^m(\operatorname{div}x^i)x^i,\qquad S=z-\operatorname{div}y+\sum_{i=1}^m\operatorname{div}[(\operatorname{div}x^i)x^i].$$

Then (A.1) implies for smooth functions h with compact support in U (which always satisfy the identity $h=\Phi\circ\phi$ on U for some $\Phi\in C^\infty_{comp}(O)$) that

$$\int_O\left\{z\Phi+\sum_{j=1}^d y_j\frac{\partial\Phi}{\partial z_j}+\sum_{i=1}^m\sum_{j=1}^d\sum_{k=1}^d x^i_j\frac{\partial}{\partial z_j}\left(x^i_k\frac{\partial\Phi}{\partial z_k}\right)\right\}d\theta=0,\qquad\forall\Phi\in C^\infty_{comp}(O),\tag{A.2}$$

i.e.

$$S\theta+Q(\theta)+\sum_{i=1}^m x^i(x^i(\theta))=0$$

holds in the sense of distributions on O. According to Proposition A.2,

$$\operatorname{span}\{L_z : L\in\mathscr{L}(x^1,\dots,x^m,y)\}=\operatorname{span}\{L_z : L\in\mathscr{L}(x^1,\dots,x^m,Q)\}=\mathbb{R}^d,\qquad\forall z\in O.$$

Hence, by the Hörmander theorem [21], θ is absolutely continuous with respect to the Lebesgue measure and the density ρ belongs to $C^\infty(O)$. If we define $L=\sqrt{\det g_{ij}}$ on U then

$$\nu(B)=\int_O\mathbf{1}_B(\varphi)\rho\,dx=\int_B\frac{\rho(\phi)}{L}\,dx,\qquad B\in\mathscr{B}(U).$$

By a localization argument, we obtain that μ has a density $R\in C^\infty(M)$ with respect to dx. □

Appendix B. Density of product functions

Proposition B.1. *Let M be a compact submanifold in $\mathbb{R}^m$. Then*

$$\mathcal{P} = \operatorname{span}\{h_1(t)h_2(x)h_3(z) : h_1 \in C^\infty_{comp}(0,\infty),\ h_2, h_3 \in C^\infty(M)\}$$

is dense in the space $C^\infty_{comp}((0,\infty)\times M\times M)$ in the following sense. Let $h \in C^\infty_{comp}((0,\infty)\times M\times M)$. Then there exist $\chi_n \in \mathcal{P}$ such that

$$\chi_n \to h \quad \textit{and} \quad X_m \dots X_1\chi_n \to X_m \dots X_1 h$$

uniformly on $(0,\infty)\times M\times M$ for every vector fields $X_1,\dots,X_m$ on $(0,\infty)\times M\times M$.

Proof. Let $0 < a < b$ be such that the support of h is contained in $(a,b)\times M\times M$ and extend h to a smooth compactly supported function in $\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m$. This can be done by standard methods of local extensions and a partition of unity as M is assumed to be compact. Denote by h_1 such an extension. The support of h_1 fits in a some large cube $Q = (-N,N)^{1+m+m}$ and we can replicate h_1 to each cube $2Nk+Q$ for $k \in \mathbb{Z}^{1+m+m}$ to obtain a smooth $2N$-periodic function h_2 such that $h_1 = h_2$ in Q. Now we can apply the Fejér's theorem on Fourier series to find a sequence of functions

$$\xi_n \in \operatorname{span}\{v_1(t)v_2(x)v_3(z) : v_1 \in C^\infty_{2N\text{-per}}(0,\infty),\ h_2, h_3 \in C^\infty_{2N\text{-per}}(\mathbb{R}^m)\}$$

such that $\xi_n \to h_2$ in $C^\infty(\mathbb{R}^{1+m+m})$. If $\rho \in C^\infty(\mathbb{R})$ has support in $(0,\infty)$ and $\rho = 1$ on $[a,b]$ then we can define $\chi_n(t,x,z) = \rho(t)\xi_n(t,x,z)$. The restrictions of χ_n to $(0,\infty)\times M\times M$ belong to $\mathcal{P}$ and approximate h in the asserted sense. □

Appendix C. Continuous surjections between circles

Proposition C.1. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be continuous and locally injective. Then f is a surjection.*

Proof. Since $\mathbb{S}^1$ is compact and f is continuous, $f[\mathbb{S}^1]$ is also a compact. But local injectivity of f implies that $f[\mathbb{S}^1]$ is open. Hence f is a surjection as $\mathbb{S}^1$ is connected. □

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