POLYTOPES ASSOCIATED TO DIHEDRAL GROUPS

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Abstract. We investigate the convex hull of $n \times n$-permutation matrices corresponding to the symmetry group of a regular $n$-gon. We give their complete facet description. As an application, we show that they are Gorenstein polytopes, and determine their $h^*$-vectors.

1. Introduction

To any finite group $G$ of real $(n \times n)$-permutation matrices we can associate the permutation polytope $P(G)$ given by the convex hull of these matrices in the vector space $\mathbb{R}^n$. A well-known example of such a polytope is the Birkhoff polytope $B_n$, which is defined as the convex hull of all $(n \times n)$-permutation matrices [BG77, BS96]. This polytope appears in various contexts in mathematics, see e.g. [Tin86, Onn93, Pak00, BS03, Ath05], from optimization over statistics to enumerative combinatorics. It is also a famous example of a Gorenstein polytope (see Section 5) which turn up in connection to Mirror symmetry.

Guralnick and Perkinson [GP06] studied polytopes associated to general subgroups $G$ and proved results about their dimension, their graph and the diameter. Based on this, the authors of this paper gave in [BHNP09] a systematic exposition of general permutation polytopes. In particular, they studied which groups lead to affinely equivalent polytopes, considered products of groups and polytopes, classified low-dimensional cases, and formulated several open conjectures.

In order to get an intuition about what one can expect in general, it is necessary to focus on special classes of $G$. A seemingly very difficult case is when $G$ equals the group of even permutation matrices. Just to provide an exponential lower bound on the number of facets is already a daunting task, for this see [HP04]. Another basic situation occurs when $G$ is cyclic. In [BHNP11] the authors showed that also these polytopes have a surprisingly complex and not yet fully understood facet structure.

In [CP04] Collins and Perkinson studied polytopes given by Frobenius groups. A special case is the dihedral group $D_n$ for $n$ odd, which was considered in more detail by Steinkamp [Ste99]. Since $D_n$ is the automorphism group of a regular $n$-gon, the cases where $n$ is even and odd are quite different.

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In this note, we completely determine the face structure of $P(D_n)$ for arbitrary $n$ (Theorem 3.3, Theorem 4.1). As an application we show that these polytopes are also Gorenstein polytopes, and we completely determine their Ehrhart polynomials in terms of their $h^*$-vectors (Theorem 5.3, Corollary 5.4).

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2. Notation and preliminary results

Let $S_n$ be the permutation group on $n \geq 3$ elements. Any permutation $\sigma \in S_n$ can be represented by an $(n \times n)$-matrix $M_\sigma$ with entries in $\{0, 1\}$ that has exactly one 1 in each row and column. We can view such a matrix as a vector in $\mathbb{R}^{n^2}$. For a subgroup $G$ of $S_n$ we define the polytope

$$P_G := \text{conv}(M_\sigma \mid \sigma \in G).$$

This is a 0/1-polytope, so all matrices are in fact vertices of the polytope.

We denote by the dihedral group $D_n$ the subgroup of $S_n$ corresponding to the symmetry group of the regular $n$-gon. This group is generated by two elements. If $n$ is odd, then these may taken to be the rotation $\rho$ of the $n$-gon by $360/n$ degree, and the reflection $\tau$ along a line through one vertex and the midpoint of the opposite edge. If $n$ is even, then the second generator $\tau$ is instead the reflection along a line through two opposite vertices.

The associated permutation polytope is the convex hull of the corresponding matrices,

$$DP_n := \text{conv}(M_\sigma \mid \sigma \in D_n).$$

The dihedral group $D_n$ has $2n$ elements

$$\rho^0, \rho^1, \rho^2, \ldots, \rho^{n-1}, \tau \rho^{n-1}, \tau \rho^{n-2}, \tau \rho^{n-3}, \ldots, \tau \rho^0.$$

We label the vertices by $v_0, \ldots, v_{n-1}, w_0, \ldots, w_{n-1}$ in this order. Let us give a more convenient way to write these matrices.

Let $I$ be the $n$-dimensional identity matrix and $R$ be the $n \times n$-matrix that has 0’s everywhere except at the $n$ entries $(i, j)$, where $0 \leq i, j \leq n - 1$ and
Reading the matrices $M_{\sigma}$ row by row, we can identify $M_{\sigma}$ with a (row) vector in $\mathbb{R}^{n^2}$. For instance, the $2 \times 2$-identity matrix would be identified with $(1 \ 0 \ 0 \ 1)$. As the reader checks easily, under this identification the vertices of $\text{DP}_n$ are (in the order given above) the rows of the $2n \times n^2$-matrix

$$
R = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
$$

Permuting the coordinates (corresponding to a linear automorphism of $\mathbb{R}^{n^2}$) we may write the vertices in the form

$$
V = \begin{bmatrix}
I & I & I & \ldots & I \\
I & R^{-2} & R^{-4} & \ldots & R^{-(n-1)} \\
\end{bmatrix}
$$

Clearly, all vertices are already determined by their first $2n$ coordinates, so we could project onto $\mathbb{R}^{2n}$ without changing the combinatorics of the polytope. Hence, we observe that the dimension of $\text{DP}_n$ is at most $2n$.

### 3. The situation for odd $n$

In this section we completely describe $\text{DP}_n$ for $n$ odd. As it will turn out, it is useful to introduce a new polytope that will serve as a basic building block for both situations of even and odd $n$.

**Definition 3.1.** Let $Q_n$ be the polytope defined as the convex hull of the rows of the $2n \times n^2$-matrix

$$
W := \begin{bmatrix}
I & I & I & \ldots & I \\
I & R^1 & R^2 & \ldots & R^{n-1} \\
\end{bmatrix}.
$$

While $Q_n$ differs from $\text{DP}_n$ for even $n$, we deduce from (1) that, for $n$ odd, $Q_n$ is up to a permutation of coordinates just the polytope $\text{DP}_n$.

**Proposition 3.2.** For odd $n$, the polytopes $\text{DP}_n$ and $Q_n$ are affinely isomorphic. \[\square\]

The following theorem examines the structure of $Q_n$ for arbitrary $n$. For $n$ odd, this result is a special case of Thm. 4.4 in [CP04].

Let us fix some convenient notation. We denote by $\Delta_r$ the $r$-dimensional simplex. We also use for any two integers $s, k$, the term $[s]_k \in \{0, \ldots, k-1\}$
to denote the remainder of $s$ modulo $k$. The combinatorial type of a free sum of two polytopes is the combinatorial dual to their product.

**Theorem 3.3** (Collins & Perkinson [CP04]). Let $n$ be arbitrary. The polytopes $Q_n$ have dimension $2n - 2$ and are free sums of two copies of $\Delta_{n-1}$. Their affine hull is given by the equations

\[(\text{aff}) \quad 1 = \sum_{i=ln}^{(l+1)n-1} x_i \]

\[(A_{j,k}) \quad 0 = x_{kn+[j]} - x_{(k+1)n+[j]} - x_{(k+1)n+[j+2]} + x_{(k+2)n+[j+2]} \]

for $0 \leq l \leq n - 1$, $0 \leq j \leq n - 2$, $0 \leq k \leq n - 3$.

An irredundant system of inequalities defining the polytope inside its affine hull is given by the inequalities

\[x_i \geq 0 \]

for $0 \leq i \leq n^2 - 1$.

**Proof.** All the given equations are satisfied by the vertices of $Q_n$. There are $n$ equations of type (aff) and $n^2 - 2n + 2$ equations of type $(A_{j,k})$. They are easily seen to be linearly independent, so the dimension of $Q_n$ is at most $2n - 2$. On the other hand, $Q_n$ does not contain the origin, and deleting any row of $W$ leaves us with an affinely independent set of row vectors (observe that deleting a row leaves us with a column that contains exactly one 1). Hence, $\dim(Q_n) = 2n - 2$ and the given equations define the affine hull of $Q_n$ in $\mathbb{R}^{n^2}$.

Further, any $2n - 1$ of the $2n$ rows of $W$ span the affine hull of $Q_n$, so any facet of $Q_n$ has at most $2n - 2$ vertices. Since the inequalities $x_j \geq 0$ are 0 on exactly $2n - 2$ of the rows, so they all define facets.

In order to prove that $Q_n$ is a free sum of simplices we observe that the first $n$ and the last $n$ vertices define $(n - 1)$-dimensional simplices sitting in transversal subspaces (intersecting in the matrix corresponding to the row vector $(1/n, \ldots, 1/n)$). Therefore, the combinatorial dual of $Q_n$ corresponds to the product of $\Delta_{n-1}$ by itself. In particular, $Q_n$ has precisely $n^2$ facets, so the facet description given above is complete. \(\square\)

4. The situation for even $n$

Recall that the join of two polytopes $P$ and $Q$ is the convex hull of $P \cup Q$ after embedding $P$ and $Q$ in a space such that the dimension of the convex hull is as maximal as possible, i.e., it equals $\dim(P) + \dim(Q) + 1$. For instance, the join of two intervals is a tetrahedron.

**Theorem 4.1.** Let $n$ be even. The polytope $DP_n$ is a join of two copies of $Q_{n/2}$. In particular, its dimension is $2n - 3$. 

Combined with Theorem 3.3, this result gives a complete description of the facet inequalities and the affine hull equations of $\text{DP}_n$ for $n$ even.

**Proof.** Permuting the coordinates, we can transform $V$ (see (1)) into

$$\begin{bmatrix} I & I & I & \cdots & I & I & I & \cdots & I \\ R^0 & R^2 & R^4 & \cdots & R^{n-2} & R^0 & R^2 & \cdots & R^{n-2} \end{bmatrix}$$

Clearly, projecting onto the first $\frac{n}{2}$ coordinates yields an affine isomorphism of $\text{DP}_n$ onto the convex hull of the rows of the $2n \times \frac{n^2}{2}$-matrix

$$\begin{bmatrix} I & I & I & \cdots & I \\ R^0 & R^2 & R^4 & \cdots & R^{n-2} \end{bmatrix}.$$  

In the representation given by this matrix let us partition the set of $2n$ vertices (labelled from 0 to $2n-1$) into two sets: consisting of $n$ even vertices and $n$ odd vertices. Then we permute the $\frac{n^2}{2}$ coordinates in such a way that in the first set of rows (corresponding to even vertices) all nonzero entries are in the first half (i.e., in the first $\frac{n^2}{4}$ columns). Then all nonzero entries in the second set of rows (corresponding to the odd vertices) will be in the second half (i.e., in the last $\frac{n^2}{4}$ columns). By a permutation of the coordinates within the first half we get that the rows of even vertices yields precisely the vertex set of $Q_{n/2} \times \{0\}$ (for $0 \in \mathbb{R}^{\frac{n^2}{2}}$). In the same way, the coordinates in the second half can be permuted so that the rows of odd vertices equal the vertices of $\{0\} \times Q_{n/2}$ (for $0 \in \mathbb{R}^{\frac{n^2}{2}}$). Since 0 is not in the affine hull of $Q_{n/2}$, we deduce that $\text{DP}_n$ is a join of two copies of $Q_{n/2}$. Hence, its dimension equals $2 \dim(Q_{n/2}) + 1 = 2(n - 2) + 1 = 2n - 3$ by Theorem 3.3.  

**5. Lattice Properties**

$\text{DP}_n$ and $Q_n$ are *lattice polytopes*, i.e. their vertices lie in the lattice $\mathbb{Z}^{n^2}$ of integral vectors. It is readily checked that all above affine isomorphisms respect lattice points. In this section, we will show that these lattice polytopes have especially nice properties which allow to completely describe their Ehrhart $h^*$-vectors.

A $d$-dimensional lattice polytope $P$ containing 0 in its interior is *reflexive*, if its polar (or dual) polytope

$$P^* := \{ x \in \mathbb{R}^d \mid \langle x, v \rangle \geq -1 \ \forall \ v \in P \}$$

is again a lattice polytope (in the dual lattice). This notion was introduced by Batyrev in [Bat94]. A generalization of this is the class of Gorenstein polytopes. A lattice polytope is a *Gorenstein polytope of codegree $k$*, if there is a positive integer $k$ and an interior lattice point $m$ in $kP$ such that $kP - m$ is a reflexive polytope. Such polytopes play an important role in the classification of Calabi-Yau manifolds for string theory. See [BN08] for basic properties. The next proposition tells us that the polytopes $Q_n$ are in this class.
Proposition 5.1. Let \( n \) be arbitrary. The polytope \( Q_n \) is Gorenstein of codegree \( n \) and normalized volume \( n \).

Proof. By Theorem 3.3, the point \( \frac{1}{n}(1,1,\ldots,1) \) is an interior point of \( Q_n \) with equal integral distance 1 to all facets, and \( m := (1,1,\ldots,1) \) is the unique interior lattice point in \( nQ_n \). Hence \( nQ_n - m \) is a reflexive polytope.

By Theorem 3.3, all facets of \( Q_n \) are simplices of facet width 1, hence they are all unimodular. As we have seen, multiplying with \( n \) gives (up to translation) a reflexive polytope with the unique interior lattice point \( m = (1,1,\ldots,1) \). The normalized volume of \( nQ_n \) is the sum of the volumes of \( n^2 \) pyramids over facets with apex \( m \). But in \( nQ_n \) each facet has normalized volume \( n^{n-1} \), and the apex has lattice distance one from the facet, so each pyramid has normalized volume \( n^{n-1} \). There are \( n^2 \) of these pyramids, and dividing by \( n^n \) to get from \( nQ_n \) back to \( Q_n \) gives the normalized volume \( n \) of \( Q_n \). \( \square \)

A polytope \( P \) is compressed if every so-called pulling triangulation is regular and unimodular. Equivalently, \( P \) is compressed if for any supporting inequality \( a^T x \leq b \) with a primitive integral normal \( a \) the polytope is contained in the set \( \{ x \mid b - 1 \leq a^T x \leq b \} \). For a more detailed explanation of these terms we refer to [DRS10]. This property has strong implications on the associated toric ideal, see e.g. [Stu96]. The next proposition follows immediately from Theorem 1.1 of [OH01] and Theorem 3.3.

Proposition 5.2. Let \( n \) be arbitrary. The polytope \( Q_n \) is compressed. \( \square \)

The Ehrhart polynomial \( L_P(k) := |kP \cap \mathbb{Z}^d| \) of a \( d \)-dimensional polytope counts the number of integral points in integral dilates of \( P \). It is well known that the generating function of \( L_P \) is given by

\[
\sum_{m \geq 0} L_P(m)t^m = \frac{h^*(t)}{(1-t)^{d+1}}
\]

for some polynomial \( h^* \) of degree at most \( d \) with integral non-negative coefficients, see [BR07a]. Hence, determining the Ehrhart polynomial is equivalent to finding the \( h^* \)-vector (also called \( \delta \)-vector) of coefficients of \( h^*(t) \). The following theorem shows that in our case this vector has a particularly nice form.

Theorem 5.3. Let \( n \) be arbitrary. The \( h^* \)-vector of \( Q_n \) satisfies \( h^*_i = 1 \) for \( 0 \leq i \leq n - 1 \) and \( h^*_i = 0 \) otherwise.

Proof. Since the codegree of \( Q_n \) is \( n \) and its dimension is \( 2n - 2 \) by Theorem 3.3, the maximal non-zero entry of the \( h^* \)-vector has to be \( h^*_{n-1} \), see [BR07a]. By a theorem of Bruns and Römer [BR07b] we know that the \( h^* \)-vector of a Gorenstein polytope that has a regular unimodular triangulation is symmetric and unimodal. In particular, \( h^*_i \geq 1 \) for \( i = 0,\ldots,n - 1 \). Since by Proposition 5.1 the sum of the entries of the \( h^* \)-vector equals \( n \), the statement follows. \( \square \)
The $h$*-vector of the join of two lattice polytopes $P, Q$ (in the sense of $h$*-polynomials) equals the product of the two $h$*-vectors of $P$ and $Q$, if the splitting of $\text{lin}((P \cup Q) \times \{1\})$ into $\text{lin}(P \times \{1\}) \oplus \text{lin}(Q \times \{1\})$ respects the lattice structure [BR07a]. This property holds for the join in Theorem 4.1. Therefore, we get the following corollary:

**Corollary 5.4.** Let $n$ be even. The $h$*-vector of $DP_n$ equals

$$\left(1, 2, 3, \ldots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} - 1, \ldots, 2, 1\right).$$

In particular, the polytope $DP_n$ is Gorenstein of codegree $n$ and normalized volume $n^2/4$. □

6. Substructures

In [BHNP09] the authors discussed the question which subgroups of a permutation group yield faces. An obvious class of such subgroups are stabilizers:

Take a partition $[n] := \{1, \ldots, n\} = \bigsqcup I_i$. Then the polytope of the stabilizer of the subsets $I_i$

$$\text{stab}(G; (I_i)_i) := \{\sigma \in G : \sigma(I_i) = I_i \text{ for all } i\} \leq G$$

is a face of $P(G)$. The authors have the feeling that the following should hold:

**Conjecture 5.8** [BHNP09] Let $G \leq S_n$. Suppose $H \leq G$ is a subgroup such that $P(H) \preceq P(G)$ is a face. Then $H = \text{stab}(G; (I_i)_i)$ for a partition $[n] = \bigsqcup I_i$.

They verified the conjecture for $G = S_n$ as well as for $G \leq S_n$ a cyclic group, see Proposition 5.9 [BHNP09]. Meanwhile Jessica Nowack and Daniel Heinrich studied this question for the dihedral groups in their Diploma theses.

**Proposition 6.1.** [Now11, Hei11] The Conjecture 5.8 holds for $G = D_n \leq S_n$ for every $n$.

**Proof.** For $n$ odd Heinrich first shows if $H$ is the the subgroup of all rotations of $G$, then $P_H$ is not a face of $P_G$. The remaining subgroups are precisely the stabilizers of their orbits, see [Hei11] Theorem 7.1.1.

For $n$ even the main work is to show that the subgroup of all rotations, the subgroup of the squares of the rotations and finally the subgroup generated by the squares of the rotations and by the reflections through two edges are precisely those subgroups $H$ of $G$ for which $P_H$ is not a face of $P_G$. Nowack shows for the remaining subgroups that they are precisely the stabilizers of their orbits, see [Now11] Section 4.2. □
References


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