Extended abstract
Constraint Satisfaction Problems, Twisted Subgroups and Transversals

Barbara Baumeister

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We cite from [6]: “Near subgroups of finite groups were introduced by Feder and Vardi [7] as a tool to study the computational complexity of constraint satisfaction problems. Aschbacher [2] addressed some questions raised in [7] and showed that near subgroups possess much structure. More recently, Feder [5] showed that near subgroups do indeed characterize the polynomial time solvable cases of group theoretic constraint satisfaction problems, using new structural results for near subgroups obtained by Aschbacher [3].”

In our paper we explore further the structure of twisted subgroups which are strongly related to near subgroups. There is a correspondence between twisted subgroups, Bol loops and groups $G$ satisfying the following: $G$ has a subgroup $U$ which has a transversal $T$ such that

(1) $1 \in T$

(2) $T$ is closed under conjugation by $G$,

see [4].

If we replace (2) by

(2') $T$ is closed under conjugation by $U$

and if $T$ consists beside of the identity only of involutions (elements of order 2), then we obtain a gyrodecomposition of $G$, see [6].
We study the groups satisfying (1) and (2) or (1) and (2') which are also of independent interest in group theory. The reader is directed to [1] for notation and terminology.

Gil Kaplan could show the following:

**Theorem 1** [8] Let $G$ be a group and let $p$ be a prime. Let $P$ be a Sylow $p$-subgroup of $G$. Assume that $P$ has a transversal $T$ in $G$ which is normalized by $P$. Then $P$ has a normal $p$-complement.

In this paper we generalize this result to:

**Theorem 2** Let $U$ be a nilpotent Hall subgroup of $G$ which has a by $U$ normalized transversal in $G$. Then $U$ has a normal complement in $G$.

What is happening if $U$ is not nilpotent, but soluble? The following example shows that a soluble Hall subgroup $U$ which has a by $U$ normalized transversal does not have a normal complement in general:

**Example 1** Let $G = S_5$ and $U$ the stabilizer of 5 in $G$. Set $T = \{1\} \cup (45)^U$. Then $U$ is a soluble Hall subgroup of $G$ and $T$ an $U$-invariant transversal to $U$ in $G$. Clearly, $U$ has no normal complement in $G$.

Assume that $U$ is a $\pi$-Hall subgroup of $G$ which admits a normal complement $N$. Then $N$ is a $\pi'$-subgroup and therefore, it is a $U$-invariant transversal to $U$ in $G$ which is contained in $O^{\pi}(G)$.

We prove that this necessary condition is already sufficient.

**Theorem 3** Let $U$ be a Hall subgroup of $G$ which is not perfect. Then $G$ has a normal complement to $U$ if and only if $U$ admits a transversal $T \subseteq O^{\pi}(G)$ with $T^U = T$.

Example 1 is not a counterexample this Theorem, as in the example $U$ is a $\{2, 3\}$-Hall subgroup of $G$, $O^{\{2,3\}}(G) = A_5$ and there is no $U$-invariant transversal to $U$ in $G$ which is contained in $A_5$.

Clearly, the immediat question arises: what can be said if $U$ is a non-perfect Hall subgroup with has a $U$-invariant transversal, but none of the normalized transversals is contained in $O^\pi(G)$? In the paper we further investigate these groups. Clearly, we can generalize example 1: Let $G = S_p$ for a prime $p$, let $U$ be the stabilizer of $p$ in $G$ and let $T = \{1\} \cup (p - 1, p)^U$. 


Then $U$ is a non-perfect Hall subgroup of $G$, $\mathcal{T}$ an $U$-invariant transversal to $U$ in $G$ which is not contained in $O^p'(G) = A_p$ and $U$ has no complement in $G$.

Moreover, we study the case that $U$ is a perfect Hall subgroup.

References


