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On the Modular Representations of the General Linear and Symmetric Groups

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1. Introduction

This paper is the result of our attempt to extend the classical theory of polynomial representations of the general linear group $GL_n(\mathbb{C})$ and of complex representations of the symmetric group S_r to the more general case in which the field of complex numbers is replaced by an infinite field of arbitrary characteristic. The characteristic zero situation is a well-established theory created at the beginning of this century by Young, Schur and Frobenius (see Weyl [15]). The characteristic p case has been investigated by several authors: Brauer, Nesbitt, Thrall, Littlewood, Robinson, Kerber, and others, but the question is far from being completely understood.

Let A be an infinite field, and let $\bar{V}$ be the n-dimensional A-vector space of which $GL_n(A)$ is the group of automorphisms. Let $\bar{V}^{\otimes r}$ be the r-fold tensor product of $\bar{V}$ over A. $\bar{V}^{\otimes r}$ is both a $GL_n(A)$ and an $A[S_r]$-module[1] (S_r acts on $\bar{V}^{\otimes r}$ by permuting the factors, and this action clearly commutes with that of $GL_n(A)$). Let λ be a partition of r into at most n parts.

In 3.2 and 3.6 we define subspaces $\bar{V}_\lambda$ (resp. $\bar{\mathscr{V}}_\lambda$) of $\bar{V}^{\otimes r}$ which are $GL_n(A)$ (resp. $A[S_r]$) submodules of $\bar{V}^{\otimes r}$. The definitions involve only relations with integral coefficients and we prove that $\bar{V}_\lambda$, $\bar{\mathscr{V}}_\lambda$ have dimensions independent of the field A; in fact we prove that the $\bar{V}_\lambda$, $\bar{\mathscr{V}}_\lambda$ corresponding to a field A of characteristic $p>0$ can be regarded as the reductions modulo p of the corresponding modules over $\mathbb{C}$.

If $A=\mathbb{C}$, $\bar{V}_\lambda$ can be regarded as the image of the appropriate Young symmetrizer (primitive idempotent in $\mathbb{C}[S_r]$) but this does not hold if char $A>0$.

We also note that $\bar{\mathscr{V}}_\lambda$ is isomorphic to the classical Specht-module involving Vandermonde determinants. It is however more convenient for us to regard $\bar{\mathscr{V}}_\lambda$ as a subspace of $\bar{V}^{\otimes r}$. Actually we can define $\bar{\mathscr{V}}_\lambda$ as the set of tensors in $\bar{V}^{\otimes r}$ which are U-invariant of weight λ. (Here U denotes the group of upper unitriangular matrices in $GL_n(A)$ and we

[1] Given a ring A and a group G we denote by $A[G]$ the group algebra of G over A.

identify the partition $\lambda=(\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_u > 0)$, $u \leqq n$, with the dominant weight $(\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_u \geqq \cdots \geqq \lambda_n \geqq 0)$ of $GL_n(A)$ defined by $\lambda_i = 0, u < i \leqq n$.)

If $A=\mathbb{C}$, the modules $\bar{V}_\lambda$, $\bar{\mathscr{V}}_\lambda$ are irreducible, and we get in this way all irreducible polynomial $GL_n(\mathbb{C})$-modules (resp. irreducible $\mathbb{C}[S_r]$-modules). If char $A=p>0$ the modules $\bar{V}_\lambda$, $\bar{\mathscr{V}}_\lambda$ are not necessarily irreducible.

In fact, in 3.12 we give sufficient conditions for the existence of a non-zero $GL_n(A)$-homomorphism $\bar{V}_{\lambda'} \to \bar{V}_\lambda$ which, by the duality Theorem 3.7, is equivalent, at least for $p \neq 2$, to the existence of a non-zero $A[S_r]$-homomorphism $\bar{\mathscr{V}}_\lambda \to \bar{\mathscr{V}}_{\lambda'}$. We regard this as our main theorem. It has been known previously only for $n \leqq 3$ (Braden [3]).

In 3.8 we give a necessary condition for the existence of such homomorphisms; this has been conjectured by Verma in the context of semisimple algebraic groups and proved by Humphreys [6] for p sufficiently large. Our result 3.8 is new only for $p \leqq n$; the case $p > n$ is contained in Humphreys' result.

Both 3.8 and 3.12 can be formulated in terms of an action of the affine Weyl group W_a of type A_{n-1} on the lattice of weights of $GL_n(A)$ (see 4.1). This action of the affine Weyl group is closely related to the process of raising squares in a partition diagram (see 4.2).

It seems certain that the affine Weyl group plays a central role in the modular representation theory of the general linear and symmetric groups. This has been pointed out by Verma [14] in the context of semisimple algebraic groups, but it does not seem to have been observed before in the case of the symmetric groups. It is striking how far the analogy between GL_n and S_r goes. There is practically no result for GL_n in this paper which has no analogue for S_r.

The main technical tool in this paper is the use of the $\mathbb{Z}$-form $\mathscr{U}_\mathbb{Z}$ of the universal enveloping algebra of the Lie algebra $\mathfrak{gl}(n)$. This has been defined by Kostant in the context of semisimple Lie algebras.

In 2.2 we produce n explicit polynomial generators for the centre of $\mathscr{U}_\mathbb{Z} \otimes Q$ which are closely related to the classical Capelli element. The explicit knowledge of these generators is precisely what is needed to prove 3.8.

In 2.3 we define some elements $T_j^i(t) \in \mathscr{U}_\mathbb{Z}$ which can be combined to provide the non-zero maps of 3.12. Our elements $T_j^i(t)$ have a strange similarity with the Capelli-type elements defined in 2.2.

In 2.9 we define some elements $S_j^i \in \mathscr{U}_\mathbb{Z}$ which are slightly less symmetric than $T_j^i(t)$ but are actually equivalent to them. The advantage of the elements S_j^i is that they are more appropriate for the purpose of generalization. Actually, in Chapter 5 we discuss the possibility of defining elements analogous to S_j^i in the case of arbitrary simple Lie algebras.

Finally, we would like to thank Dr. M. Beetham for useful discussions, and for pointing out a slight error in an earlier definition of $\overline{V}_\lambda$.

2. Computations in the Enveloping Algebra

2.1. Let $\mathscr{U}[\mathfrak{gl}(n,Q)]$, or simply $\mathscr{U}_Q$, be the universal enveloping algebra of the Lie algebra of all $n\times n$ matrices over Q.

$\mathscr{U}_Q$ can be described as the associative Q-algebra with generators θ^i_j, $1\leqq i,\ j\leqq n$ (corresponding to elementary matrices) and relations

$$\theta^i_h\,\theta^k_j-\theta^k_j\,\theta^i_h=\delta^i_j\,\theta^k_h-\delta^k_h\,\theta^i_j,\qquad 1\leqq i,\ j,\ k,\ h\leqq n. \tag{1}$$

A Q basis for $\mathscr{U}_Q$ is given by

$$\theta^{(N)}=\prod_{1\leqq i<j\leqq n}\frac{(\theta^i_j)^{N^i_j}}{(N^i_j)!}\prod_{1\leqq i\leqq n}\binom{\theta^i_i}{N^i_i}\prod_{n\geqq i>j\geqq 1}\frac{(\theta^i_j)^{N^i_j}}{(N^i_j)!} \tag{2}$$

where $N=(N^i_j)_{1\leqq i,\,j\leqq n}$ runs through all $n\times n$ matrices with non-negative integers as entries and the symbol $\binom{x}{n}$ means

$$\frac{1}{n!}\,x(x-1)\cdot\ldots\cdot\big(x-(n-1)\big).$$

The factors in (2) must be arranged in lexicographic order in the first and third product. (The factors in the second product commute with each other.) More precisely, in the first product (resp. third product) the factor involving θ^i_j comes before the factor involving $\theta^{i'}_{j'}$ if and only if $j<j'$ or $j=j'$ and $i<i'$.

The elements $\theta^{(N)}$ span over $\mathbb{Z}$ a subgroup $\mathscr{U}_{\mathbb{Z}}\subset\mathscr{U}_Q$ which is a subring. This is the Kostant $\mathbb{Z}$-form of the enveloping algebra $\mathscr{U}_Q$. For a definition of the $\mathbb{Z}$-form of the enveloping algebra of an arbitrary semisimple complex Lie algebra see Kostant's paper [9].

Let $\mathscr{U}^-_{\mathbb{Z}}$ (resp. $\mathscr{U}^+_{\mathbb{Z}}$) be the subgroup of $\mathscr{U}_{\mathbb{Z}}$ spanned by $\theta^{(N)}$ with $N^i_j=0$ for $i\geqq j$ (resp. $i\leqq j$). Similarly, let $\mathscr{U}^0_{\mathbb{Z}}$ be the subspace of $\mathscr{U}_{\mathbb{Z}}$ spanned by $\theta^{(N)}$ with $N^i_j=0$ for $i\neq j$. Then $\mathscr{U}^\pm_{\mathbb{Z}}$, $\mathscr{U}^0_{\mathbb{Z}}$ are subrings of $\mathscr{U}_{\mathbb{Z}}$ and $\mathscr{U}_{\mathbb{Z}}=\mathscr{U}^-_{\mathbb{Z}}\otimes\mathscr{U}^0_{\mathbb{Z}}\otimes\mathscr{U}^+_{\mathbb{Z}}$ as a $\mathbb{Z}$-module.

Note that for any integers α_i $(0\leqq i\leqq n)$ and $N\geqq 1$ we have

$$\binom{\sum_{1\leqq i\leqq n}\alpha_i\,\theta^i_i+\alpha_0}{N}\in\mathscr{U}^0_{\mathbb{Z}}. \tag{3}$$

This follows immediately from the formal identities:

$$\binom{x+y}{N}=\sum_{0\leqq N_1\leqq N}\binom{x}{N_1}\binom{y}{N-N_1}\quad\text{and}\quad\binom{-x}{N}=(-1)^N\binom{x+N-1}{N}.$$

2.2. The Centre of $\mathcal{U}_Q$. It is well known that the centre $\mathcal{Z}(\mathcal{U}_Q)$ of $\mathcal{U}_Q$ is a polynomial algebra over Q in n generators but the explicit generators do not seem to appear in the literature. In a recent letter to Atiyah, S. Ramanan noted that the classical Capelli element, given by formula (4) below for $t=-n$, can be regarded as an element of the centre of $\mathcal{U}_Q$. This is an immediate consequence of the Capelli identity proved in Weyl's book [15], or can be checked by direct computation. We propose the following generalization:

Theorem. *The element*

$$C(t)=\begin{vmatrix} \theta_1^1-1-t & \theta_1^2 & \cdots & \theta_1^n \\ \theta_2^1 & \theta_2^2-2-t & \cdots & \theta_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \theta_n^1 & \cdots & \cdots & \theta_n^n-n-t \end{vmatrix} \tag{4}$$
$$=\sum_{\sigma\in S_n}\varepsilon(\sigma)\left(\theta_{\sigma(1)}^1-\delta_{\sigma(1)}^1(1+t)\right)\left(\theta_{\sigma(2)}^2-\delta_{\sigma(2)}^2(2+t)\right)\cdot\ldots\cdot\left(\theta_{\sigma(n)}^n-\delta_{\sigma(n)}^n(n+t)\right)$$

belongs to the centre of $\mathcal{U}_Q$ for any $t\in Q$.

(Here $\varepsilon(\sigma)$ is the sign of σ.)

Proof. The general case can be reduced to the case $t=-n$ as follows. Define an algebra automorphism $\varphi_t\colon \mathcal{U}_Q\to\mathcal{U}_Q$ by $\varphi_t(\theta_j^i)=\theta_j^i$ for $i\neq j$ and $\varphi_t(\theta_i^i)=\theta_i^i-t-n$. It is clear that φ_t preserves the relations (1) so it extends uniquely to the whole of $\mathcal{U}_Q$. Clearly

$$\varphi_t(C(-n))=C(t) \quad\text{and}\quad \varphi_t(\mathcal{Z}(\mathcal{U}_Q))=\mathcal{Z}(\mathcal{U}_Q).$$

Hence from $C(-n)\in\mathcal{Z}(\mathcal{U}_Q)$ it follows that $C(t)\in\mathcal{Z}(\mathcal{U}_Q)$, for all $t\in Q$.

Expanding with respect to the powers of t we get:

Corollary. *Let*

$$C_k=\sum_{1\leqq i_1<\cdots<i_k\leqq n}\begin{vmatrix} \theta_{i_1}^{i_1}-i_1 & \cdots & \cdots & \theta_{i_1}^{i_k} \\ \theta_{i_2}^{i_1} & \theta_{i_2}^{i_2}-i_2 & \cdots & \theta_{i_2}^{i_k} \\ \vdots & & \ddots & \vdots \\ \theta_{i_k}^{i_1} & \cdots & \cdots & \theta_{i_k}^{i_k}-i_k \end{vmatrix} \qquad (1\leqq k\leqq n).$$

Then C_k belongs to $\mathcal{U}_{\mathbb{Z}}\cap\mathcal{Z}(\mathcal{U}_Q)$.

Remark. It is not difficult to prove that $\mathcal{Z}(\mathcal{U}_Q)$ is the polynomial algebra over Q with generators $C_1, C_2, \ldots, C_n$.

2.3. We define now some elements of $\mathcal{U}_{\mathbb{Z}}$ which will play a fundamental role in this paper.

Definition. For $1 \leqq i < j \leqq n$, $t \in \mathbb{Z}$ *define*

$$(5)\quad T_j^i(t) = \begin{vmatrix} \theta_{i+1}^i & \theta_{i+1}^{i+1} - (i+1+t) & 0 \dots 0 & \\ & & \ddots \vdots & \\ \theta_{i+2}^i & \theta_{i+2}^{i+1} & \theta_{i+2}^{i+2} - (i+2+t) \;\; 0 & \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & \theta_{j-1}^{j-1} - (j-1+t) \\ \theta_j^i & \theta_j^{i+1} & \dots\dots & \theta_j^{j-1} \end{vmatrix}$$

$$= \sum \theta_{i_1}^i \theta_{i_2}^{i_1} \cdot \dots \cdot \theta_{i_k}^{i_{k-1}} \theta_j^{i_k} (t + j_1 - \theta_{j_1}^{j_1})(t + j_2 - \theta_{j_2}^{j_2}) \cdot \dots \cdot (t + j_l - \theta_{j_l}^{j_l})$$

where the sum is over all sequences $i < i_1 < i_2 < \dots < i_k < j$ with i, j fixed and the set $\{j_1, j_2, \dots, j_l\}$ $(l = j - i - 1 - k)$ is the complement of $\{i_1, i_2, \dots, i_k\}$ in $\{i+1, i+2, \dots, j-1\}$.

Note that the sum has 2^{j-i-1} terms.

More generally, for any $1 \leqq i < j \leqq m \leqq n$ and $\tau = (\tau_1, \tau_2, \dots, \tau_n) \in \mathbb{Z}^n$ define

$$(6)\quad {}_mT_j^i(\tau) = \sum_{i < i_1 < \dots < i_k < j} \theta_{i_1}^i \theta_{i_2}^{i_1} \cdot \dots \cdot \theta_{i_k}^{i_{k-1}} \theta_m^{i_k} (\tau_{j_1} - \theta_{j_1}^{j_1})(\tau_{j_2} - \theta_{j_2}^{j_2}) \cdot \dots \cdot (\tau_{j_l} - \theta_{j_l}^{j_l})$$

where the sets $\{i_1, \dots, i_k\}$, $\{j_1, \dots, j_l\}$ are as in (5). Although our main interest lies in the elements (5) we are forced to introduce the more general elements (6) for purely technical reasons.

We also put ${}_jT_j^i(\tau) = T_j^i(\tau)$. Note that

$$(7)\qquad T_j^i(\tau) = T_j^i(t) \quad \text{if } \tau = (t+1, t+2, \dots, t+n).$$

We make the conventions: ${}_mT_j^i(\tau) = 1$, if $i = j = m$; ${}_mT_j^i(\tau) = 0$ if $i = j < m$ or $i > j$.

The following relations are easy consequences of the definitions (assume $i < j$):

$$(8)\qquad {}_mT_j^i(\tau) = T_{j-1}^i(\tau)\,\theta_m^{j-1} + {}_mT_{j-1}^i(\tau)(\tau_{j-1} - \theta_{j-1}^{j-1}) \qquad (m \geqq j).$$

$$(9)\qquad {}_mT_j^i(\tau) = \theta_m^j\, T_j^i(\tau) - T_j^i(\tau)\,\theta_m^j \qquad (m > j).$$

From (8) and (9) one deduces:

$$(10)\quad {}_mT_j^i(\tau) = \theta_m^{j-1}\, T_{j-1}^i(\tau) + {}_mT_{j-1}^i(\tau)(\tau_{j-1} - \theta_{j-1}^{j-1} - 1) \qquad (m \geqq j),$$

$$(11)\quad {}_mT_j^i(\tau) = T_{j-1}^i(\tau)\,\theta_m^{j-1}(1 - \tau_{j-1} - \theta_{j-1}^{j-1}) + \theta_m^{j-1}\, T_{j-1}^i(\tau)(\tau_{j-1} - \theta_{j-1}^{j-1}) \qquad (m \geqq j).$$

2.4. Our aim in this section is to prove commutation formulae involving θ_{a-1}^a and products of elements of the form $T_j^i(\tau)$:

Let $1 \leqq i < j \leqq n$, $2 \leqq a \leqq n$,

$$\tau = (\tau_1, \tau_2, \dots, \tau_n) \in \mathbb{Z}^n$$

$$\tau^{(h)} = (\tau_1, \tau_2, \dots, \tau_{a-2}, \tau_{a-1}+h, \tau_a - h, \tau_{a+1}, \dots, \tau_n) \quad h \geqq 1,$$

$$(12) \quad c\begin{pmatrix} i & a \\ j & a-1 \end{pmatrix}, \tau\Big) = (\tau_{a-1} - \tau_a + 1)\,\nu + \delta^i_{a-1}(\theta^i_i - \tau_i) + \delta^a_j(\tau_j - \theta^j_j - 1) \in \mathcal{U}^0_{\mathbb{Z}}$$

where
$$\nu = \begin{cases} 1 & \text{if } i \leqq a-1 < a \leqq j \\ 0 & \text{otherwise.} \end{cases}$$

Then the following commutation formula holds:

Lemma. $\theta^a_{a-1} T^i_j(\tau) = T^i_j(\tau^{(1)})\,\theta^a_{a-1} + T^i_{a-1}(\tau)\,T^a_j(\tau)\,c\left(\begin{matrix} i & a \\ j & a-1 \end{matrix}, \tau\right)$.

Proof. We can assume $i \leqq a-1 < a \leqq j$, otherwise the result is obvious. We use induction on j. The result is obvious for $j = i+1$. Assume $j \geqq i+2$ and that the result is true for $j-1$. Applying (11) for $m = j$, we have:

$$\begin{aligned}
X &\overset{\text{def}}{=} \theta^a_{a-1} T^i_j(\tau) - T^i_j(\tau^{(1)})\,\theta^a_{a-1} \\
&= \theta^a_{a-1}\big(T^i_{j-1}(\tau)\,\theta^{j-1}_j(1 - \tau_{j-1} - \theta^{j-1}_{j-1}) + \theta^{j-1}_j T^i_{j-1}(\tau)(\tau_{j-1} - \theta^{j-1}_{j-1})\big) \\
&\quad - T^i_{j-1}(\tau^{(1)})\,\theta^{j-1}_j\,\theta^a_{a-1}(1 - \tau_{j-1} + \theta^{j-1}_{j-1}) \\
&\quad - \theta^{j-1}_j T^i_{j-1}(\tau^{(1)})\,\theta^a_{a-1}(\tau_{j-1} - \theta^{j-1}_{j-1}) \\
&= \text{(by induction hypothesis)} \\
&= T^i_{j-1}(\tau^{(1)})\big(\theta^{j-1}_j \theta^a_{a-1} + \theta^a_j(\theta^{j-1}_{j-1} - \theta^j_j)\big)(1 - \tau_{j-1} - \theta^{j-1}_{j-1}) \\
&\quad + T^i_{a-1}(\tau)\,T^a_{j-1}(\tau)\,c\left(\begin{matrix} i & a \\ j-1 & a-1 \end{matrix}, \tau\right)\theta^{j-1}_j(1 - \tau_{j-1} + \theta^{j-1}_{j-1}) \\
&\quad + \theta^{j-1}_j T^i_{j-1}(\tau^{(1)})\,\theta^a_{a-1}(\tau_{j-1} - \theta^{j-1}_{j-1}) \\
&\quad + \theta^{j-1}_j T^i_{a-1}(\tau)\,T^a_{j-1}(\tau)\,c\left(\begin{matrix} i & a \\ j-1 & a-1 \end{matrix}, \tau\right)(\tau_{j-1} - \theta^{j-1}_{j-1}) \\
&\quad + \theta^a_j(\theta^{j-1}_{j-1} - \theta^j_j)\,T^i_{j-1}(\tau)(\tau_{j-1} - \theta^{j-1}_{j-1}) \\
&\quad - T^i_{j-1}(\tau^{(1)})\,\theta^{j-1}_j\,\theta^a_{a-1}(1 - \tau_{j-1} - \theta^{j-1}_{j-1}) \\
&\quad - \theta^{j-1}_j T^i_{j-1}(\tau^{(1)})\,\theta^a_{a-1}(\tau_{j-1} - \theta^{j-1}_{j-1}).
\end{aligned}$$

Case 1. $i < a-1 < a < j-1$

$$\begin{aligned}
X &= T^i_{a-1}(\tau)\big[T^a_{j-1}(\tau)\,\theta^{j-1}_j(\tau_{a-1} - \tau_a + 1)(1 - \tau_{j-1} + \theta^{j-1}_{j-1}) \\
&\qquad + \theta^{j-1}_j T^a_{j-1}(\tau)(\tau_{a-1} - \tau_a + 1)(\tau_{j-1} - \theta^{j-1}_{j-1})\big] \\
&= T^i_{a-1}(\tau)\,T^a_j(\tau)(\tau_{a-1} - \tau_a + 1).
\end{aligned}$$

Case 2. $i<a-1<a=j-1$

$$\begin{aligned}X &= T^i_{j-1}(\tau)(\tau_{j-2}-\theta^{j-2}_{j-2})\,\theta^{j-1}_j(1-\tau_{j-1}+\theta^{j-1}_{j-1})\\ &\quad+\theta^{j-1}_j\,T^i_{j-2}(\tau)(\tau_{j-2}-\theta^{j-1}_{j-1})(\tau_{j-1}-\theta^{j-1}_{j-1})\\ &= T^i_{j-2}(\tau)\,\theta^{j-1}_j(\tau_{j-2}-\theta^{j-1}_{j-1}+1)(1-\tau_{j-1}+\theta^{j-1}_{j-1})\\ &\quad+T^i_{j-2}(\tau)\,\theta^{j-1}_j(\tau_{j-2}-\theta^{j-1}_{j-1})(\tau_{j-1}-\theta^{j-1}_{j-1})\\ &= T^i_{j-2}(\tau)\,T^{j-1}_j(\tau)(\tau_{j-2}-\tau_{j-1}+1).\end{aligned}$$

Case 3. $i<a-1<a=j$

$$\begin{aligned}X &= T^i_{j-1}(\tau)(\theta^{j-1}_{j-1}-\theta^j_j)(1-\tau_{j-1}+\theta^{j-1}_{j-1})\\ &\quad+(\theta^{j-1}_{j-1}-\theta^j_j)\,T^i_{j-1}(\tau)(\tau_{j-1}-\theta^{j-1}_{j-1})\\ &= T^i_{j-1}(\tau)[(\theta^{j-1}_{j-1}-\theta^j_j)(1-\tau_{j-1}+\theta^{j-1}_{j-1})\\ &\quad+(\theta^{j-1}_{j-1}-\theta^j_j+1)(\tau_{j-1}-\theta^{j-1}_{j-1})]\\ &= T^i_{j-1}(\tau)(\tau_{j-1}-\theta^j_j).\end{aligned}$$

Case 4. $i=a-1<a<j-1$

$$\begin{aligned}X &= T^a_{j-1}(\tau)(\theta^i_i-\tau_{i+1}+1)\,\theta^{j-1}_j(1-\tau_{j-1}+\theta^{j-1}_{j-1})\\ &\quad+\theta^{j-1}_j\,T^a_{j-1}(\tau)(\theta^i_i-\tau_{i+1}+1)(\tau_{j-1}-\theta^{j-1}_{j-1})\\ &= [T^a_{j-1}(\tau)\,\theta^{j-1}_j(1-\tau_{j-1}+\theta^{j-1}_{j-1})+\theta^{j-1}_j\,T^a_{j-1}(\tau)(\tau_{j-1}-\theta^{j-1}_{j-1})](\theta^i_i-\tau_{i+1}+1)\\ &= T^a_{j-1}(\tau)(\theta^i_i-\tau_{i+1}+1).\end{aligned}$$

Case 5. $i=a-1<a=j-1$

$$\begin{aligned}X &= \theta^{i+1}_i\,T^i_{i+2}(\tau)-T^i_{i+2}(\tau^{(1)})\,\theta^{i+1}_i\\ &= \theta^{i+1}_i\big(\theta^i_{i+1}\,\theta^{i+1}_{i+2}+\theta^i_{i+2}(\tau_{i+1}-\theta^{i+1}_{i+1})\big)\\ &\quad-\big(\theta^i_{i+1}\,\theta^{i+1}_{i+2}+\theta^i_{i+2}(\tau_{i+1}-\theta^{i+1}_{i+1}-1)\big)\,\theta^{i+1}_i\\ &= \theta^{i+1}_{i+2}(\theta^i_i-\tau_{i+1}+1)\end{aligned}$$

and the lemma is proved.

We may now generalize this result to obtain a commutation formula involving θ^a_{a-1} and certain products of elements of the form $T^i(\tau)$.

Define

$$T_j^i(\tau-(d-1),\dots,\tau)=T_j^i(\tau-(d-1))\cdot\ldots\cdot T_j^i(\tau-1)\,T_j^i(\tau)$$

where $d\geq 1$ is an integer and $\tau-m=(\tau_1-m,\tau_2-m,\dots,\tau_n-m)$ for $m\in\mathbb{Z}$.

Corollary.

$$\begin{aligned}\theta_{a-1}^a\,T_j^i(\tau-(d-1),\dots,\tau)&=T_j^i(\tau^{(1)}-(d-1),\dots,\tau^{(1)})\,\theta_{a-1}^a\\ &+\sum_{h=0}^{d-1}T_j^i(\tau^{(1)}-(d-1),\dots,\tau^{(1)}-(h+1))\,T_{a-1}^i(\tau-h)\,T_j^a(\tau-h)\\ &\cdot T_j^i(\tau-(h-1),\dots,\tau)\big(c(\tau)-2h\,\delta_j^a\big)\end{aligned}\tag{13}$$

where $c(\tau)=c\begin{pmatrix}i & a\\ j & a-1\end{pmatrix},\tau\Big)$ *is given by* (12).

Proof. First we note the relation

$$c(\sigma)\,T_j^i(\tau-(e-1),\dots,\tau)=T_j^i(\tau-(e-1),\dots,\tau)\big(c(\sigma+e)-2e\,\delta_j^a\big)\tag{14}$$

which follows easily from (1). The corollary follows by applying the Lemma repeatedly and using (14).

2.5. We wish to obtain the additive expression for $T_j^i(\tau-(d-1),\dots,\tau)$ as an integral combination of the basis elements of $\mathscr{U}_{\mathbb{Z}}$. In order to do this we need the following

Lemma. *Assume* $m_1,m_2,\dots,m_d\geq j>i$. *Then*

$${}_{m_d}T_j^i(\tau-(d-1))\cdot\ldots\cdot{}_{m_2}T_j^i(\tau-1)\,{}_{m_1}T_j^i(\tau)$$

is symmetric in $m_1,m_2,\dots,m_d$ *and equals*

$$\begin{aligned}\sum_{1\leq i_1<i_2<\dots<i_k\leq d}&T_{j-1}^i(\tau-(d-1))\cdot\ldots\cdot T_{j-1}^i(\tau-k)\,{}_{m_{i_k}}T_{j-1}^i(\tau-(k-1))\\ &\cdot\ldots\cdot{}_{m_{i_1}}T_{j-1}^i(\tau)\,\theta_{m_{j_1}}^{j-1}\,\theta_{m_{j_2}}^{j-1}\cdot\ldots\cdot\theta_{m_{j_l}}^{j-1}(\tau_{j-1}-\theta_{j-1}^{j-1})(\tau_{j-1}-\theta_{j-1}^{j-1}-1)\\ &\cdot\ldots\cdot(\tau_{j-1}-\theta_{j-1}^{j-1}-k+1)\end{aligned}$$

where the sum is over all subsets $\{i_1,i_2,\dots,i_k\}$ *of* $\{1,2,\dots,d\}$ *and* $\{j_1,j_2,\dots,j_l\}$ *is the complementary subset* $(l=d-k)$.

Proof. In the course of this proof we shall use the following abbreviation:

$$\begin{aligned}{}_{m_d}T_j^i(\tau-(d-1))\cdot\ldots\cdot{}_{m_1}T_j^i(\tau)&=T(m_d,\dots,m_1)\qquad(m_a\geq j)\\ {}_{m'_d}T_{j-1}^i(\tau-(d-1))\cdot\ldots\cdot{}_{m'_1}T_{j-1}^i(\tau)&=\tilde T(m'_d,\dots,m'_1)\qquad(m'_a\geq j-1).\end{aligned}$$

First note that

$$\theta_m^{j-1}\, T_{j-1}^i(\sigma) = T_{j-1}^i(\sigma)\,\theta_m^{j-1} + m\, T_{j-1}^i(\sigma), \qquad m > j-1$$

(cf. (9))

$$\theta_m^{j-1}\,{}_{m'}T_{j-1}^i(\sigma) = {}_{m'}T_{j-1}^i(\sigma)\,\theta_m^{j-1}, \qquad\qquad m, m' > j-1.$$

These formulae imply by induction

$$\begin{aligned} &\theta_m^{j-1}\,\tilde{T}(\underbrace{j-1,\ldots,j-1}_{d-k}, m_{i_k},\ldots,m_{i_1}) \\ &\quad = \sum_{h=1}^{d-k} \tilde{T}(j-1,\ldots,j-1,m,j-1,\ldots,j-1,m_{i_k},\ldots,m_{i_1}) \qquad (15)\\ &\qquad + \tilde{T}(j-1,\ldots,j-1,m_{i_k},\ldots,m_{i_1})\,\theta_m^{j-1} \end{aligned}$$

where in the sum over h, m occurs in the h-th place.

We prove the lemma by induction. The cases $d=1$ or $j=i+1$ are obvious. Assume the result known for (j,d) and for $(j-1,d+1)$. We prove it for $(j,d+1)$.

Using (8) and the induction hypothesis for (j,d) we have

$$\begin{aligned} &T(m_{d+1}, m_d, \ldots, m_1) = {}_{m_{d+1}}T_j^i(\tau-d)\, T(m_d,\ldots,m_1) \\ &= [T_{j-1}^i(\tau-d)\,\theta_{m_{d+1}}^{j-1} + {}_{m_{d+1}}T_{j-1}^i(\tau-d)(\tau_{j-1}-\theta_{j-1}^{j-1}-d)] \\ &\quad\cdot \sum_{1\le i_1<\cdots<i_k\le d} \tilde{T}(j-1,\ldots,j-1,m_{i_k},\ldots,m_{i_1})\,\theta_{m_{j_1}}^{j-1}\cdot\ldots\cdot\theta_{m_{j_l}}^{j-1}\binom{\tau_{j-1}-\theta_{j-1}^{j-1}}{k}k! \end{aligned}$$

$=$(using (15) and the symmetry statement for $j-1$, $d+1$)

$$\begin{aligned} &= \sum_{1\le i_1<\cdots<i_k\le d} (d-k)\,\tilde{T}(\underbrace{j-1,\ldots,j-1}_{d-k}, m_{d+1}, m_{i_k},\ldots,m_{i_1}) \\ &\quad\cdot \theta_{m_{j_1}}^{j-1}\cdot\ldots\cdot\theta_{m_{j_l}}^{j-1}\binom{\tau_{j-1}-\theta_{j-1}^{j-1}}{k}k! \\ &\quad + \sum_{1\le i_1<\cdots<i_k\le d} \tilde{T}(\underbrace{j-1,\ldots,j-1}_{d-k+1}, m_{i_k},\ldots,m_{i_1})\,\theta_{m_{d+1}}^{j-1}\,\theta_{m_{j_1}}^{j-1}\cdot\ldots\cdot\theta_{m_{j_l}}^{j-1} \\ &\quad\cdot\binom{\tau_{j-1}-\theta_{j-1}^{j-1}}{k}k! \\ &\quad + \sum_{1\le i_1<\cdots<i_k\le d} \tilde{T}(\underbrace{j-1,\ldots,j-1}_{d-k}, m_{d+1}, m_{i_k},\ldots,m_{i_1})\,\theta_{m_{j_1}}^{j-1}\cdot\ldots\cdot\theta_{m_{j_l}}^{j-1} \\ &\quad\cdot\binom{\tau_{j-1}-\theta_{j-1}^{j-1}}{k}k!\cdot(\tau_{j-1}-\theta_{j-1}^{j-1}-k). \end{aligned}$$

Collecting the first and third sum together we find

$$\begin{aligned}
&T(m_{d+1}, m_d, \dots, m_1) \\
&= \sum_{1 \leqq i_1 < \dots < i_k \leqq d} \tilde{T}(\underbrace{j-1, \dots, j-1}_{d-k}, m_{d+1}, m_{i_k}, \dots, m_{i_1})\, \theta_{m_{j_1}}^{j-1} \cdot \dots \cdot \theta_{m_{j_l}}^{j-1} \\
&\quad \cdot \binom{\tau_{j-1} - \theta_{j-1}^{j-1}}{k+1} (k+1)! \\
&\quad + \sum_{1 \leqq i_1 < \dots < i_k \leqq d} \tilde{T}(\underbrace{j-1, \dots, j-1}_{d-k+1}, m_{i_k}, \dots, m_{i_1})\, \theta_{m_{d+1}}^{j-1}\, \theta_{m_{j_1}}^{j-1} \cdot \dots \cdot \theta_{m_{j_l}}^{j-1} \\
&\quad \cdot \binom{\tau_{j-1} - \theta_{j-1}^{j-1}}{k} k!
\end{aligned}$$

which proves the desired formula. The symmetry assertion follows from the formula.

The above lemma gives immediately an inductive proof of the following formula valid for $1 \leqq i < j \leqq n$ and $j \leqq m_1, \dots, m_d \leqq n$:

$$\begin{aligned}
(16) \qquad & {}_{m_d}T_j^i(\tau - (d-1)) \cdot \dots \cdot {}_{m_2}T_j^i(\tau - 1)\, {}_{m_1}T_j^i(\tau) \\
& = \sum \theta_{b_1}^{a_1}\, \theta_{b_2}^{a_2} \cdot \dots \cdot \theta_{b_h}^{a_h}\, C_{b_1, b_2, \dots, b_h}^{a_1, a_2, \dots, a_h}
\end{aligned}$$

where the sum is over all arrays $\begin{matrix} a_1, a_2, \dots, a_h \\ b_1, b_2, \dots, b_h \end{matrix}$ of integers with following properties:

(A) $i < b_1 \leqq b_2 \leqq \dots \leqq b_{h-d} < j$
$b_{h-d+1} = m_d, \dots, \; b_{h-1} = m_2, \; b_h = m_1$
$a_1 < b_1, \; a_2 < b_2, \dots, \; a_{h-d} < b_{h-d}$
$a_{h-d+1} < j, \dots, \; a_{h-1} < j, \; a_h < j.$

(B) If u_k (resp. v_k) denotes the number of a's (resp. b's) which are equal to k, then

$$u_k = v_k \leqq d \quad \text{for } i < k < j$$
$$u_i = d.$$

The C-factor in the formula (16) has the value:

$$C_{b_1, b_2, \dots, b_h}^{a_1, a_2, \dots, a_h} = \prod_{i<k<j} \; \prod_{0 \leqq s \leqq d - u_k} (\tau_k - \theta_k^k - s).$$

Note that the number of terms in the sum (16) is $2^{(j-i-1)d}$. We now substitute $m_d = \dots = m_2 = m_1 = j$. In order to simplify the sum (16) observe that $\theta_b^a\, \theta_b^{a'} = \theta_b^{a'}\, \theta_b^a$ so that the factors in each product $\theta_{b_1}^{a_1}\, \theta_{b_2}^{a_2} \dots$ can be rearranged in lexicographic order. Collecting together terms involving the same product in the θ's (up to order) we find:

Theorem.

$$\begin{aligned}
& T_j^i(\tau - (d-1)) \cdot \dots \cdot T_j^i(\tau - 1)\, T_j^i(\tau) \\
(17) \qquad & = \sum_{(N)} d! \prod_{i<c<j} \Big[\big(\sum_a N_c^a\big)!\, \big(d - \sum_a N_c^a\big)!\Big] \prod_{i \leqq a < b \leqq j} \frac{(\theta_b^a)^{N_b^a}}{(N_b^a)!} \prod_{i<c<j} \binom{\tau_c - \theta_c^c}{d - \sum_a N_c^a}
\end{aligned}$$

where the sum is over all matrices (N_b^a) *with non negative integer entries such that* $N_b^a=0$ *unless* $i\leqq a<b\leqq j$, $\sum_a N_c^a=\sum_b N_b^c\leqq d$ *for all* $i<c<j$ *and* $\sum_a N_j^a=\sum_b N_b^i=d$.

Another application of the lemma is the following

Proposition.

$$T_{j-1}^i(\sigma^{(1)}-1)\,T_j^i(\sigma)=T_j^i(\sigma^{(1)}-1)\,T_{j-1}^i(\sigma) \tag{18}$$

where

$$1\leqq i<j-1<j\leqq m\leqq n$$

and

$$\sigma^{(1)}=(\sigma_1,\sigma_2,\ldots,\sigma_{j-2},\,\sigma_{j-1}+1,\,\sigma_j-1,\,\sigma_{j+1},\ldots,\sigma_n).$$

Proof. First note

$$T_{j-1}^i(\sigma-1)\,{}_jT_{j-1}^i(\sigma)={}_jT_{j-1}^i(\sigma-1)\,T_{j-1}^i(\sigma) \tag{19}$$

which is a special case of the Lemma ($d=2$). Using (19), (8) and (9) we have:

$$\begin{aligned}
&T_{j-1}^i(\sigma^{(1)}-1)\,T_j^i(\sigma)-T_j^i(\sigma^{(1)}-1)\,T_{j-1}^i(\sigma)\\
&\quad=T_{j-1}^i(\sigma^{(1)}-1)\big(T_{j-1}^i(\sigma)\,\theta_j^{j-1}+{}_jT_{j-1}^i(\sigma)(\sigma_{j-1}-\theta_{j-1}^{j-1})\big)\\
&\qquad-\big(T_{j-1}^i(\sigma^{(1)}-1)\,\theta_j^{j-1}+{}_jT_{j-1}^i(\sigma^{(1)}-1)(\sigma_{j-1}-\theta_{j-1}^{j-1})\big)\,T_{j-1}^i(\sigma)\\
&\quad=T_{j-1}^i(\sigma-1)\big(T_{j-1}^i(\sigma)\,\theta_j^{j-1}-\theta_j^{j-1}\,T_{j-1}^i(\sigma)\big)\\
&\qquad+T_{j-1}^i(\sigma-1)\,{}_jT_{j-1}^i(\sigma)(\sigma_{j-1}-\theta_{j-1}^{j-1})-(\sigma_{j-1}-(\theta_{j-1}^{j-1}+1))\\
&\quad=T_{j-1}^i(\sigma-1)\,{}_jT_{j-1}^i(\sigma)(1-1)=0.
\end{aligned}$$

2.6. We now consider the commutation of the powers of θ_{a-1}^a with $T_j^i(\tau-(d-1),\ldots,\tau)$. We first consider the case $a=j$.

Take $a=j$ in the formula (13). Using (18) we can bring $T_{j-1}^i(\tau-h)$ to the front, commuting it successively with the factors of

$$T_j^i\big(\tau^{(1)}-(d-1),\ldots,\tau^{(1)}-(h+1)\big).$$

We find

$$\begin{aligned}
&\theta_{j-1}^j\,T_j^i\big(\tau-(d-1),\ldots,\tau\big)-T_j^i\big(\tau^{(1)}-(d-1),\ldots,\tau^{(1)}\big)\,\theta_{j-1}^j\\
&\quad=T_{j-1}^i\big(\tau-(d-1)\big)\,T_j^i\big(\tau-(d-2),\ldots,\tau\big)\sum_{h=0}^{d-1}\big(c(\tau)-2h\big)\\
&\quad=T_{j-1}^i\big(\tau-(d-1)\big)\,T_j^i\big(\tau-(d-2),\ldots,\tau\big)\,d\big(c(\tau)-(d-1)\big).
\end{aligned} \tag{20}$$

This can be generalized as follows:

Lemma.

$$(21)\quad \begin{aligned}(\theta^j_{j-1})^l\, T^i_j(\tau-(d-1),\dots,\tau) = \sum_{0\leqq s\leqq \min(l,d)} & T^i_{j-1}(\tau-(d-1),\dots,\tau-(d-s)) \\ & \cdot T^i_j(\tau^{(l-s)}-(d-s-1),\dots,\tau^{(l-s)})\cdot(\theta^j_{j-1})^{l-s}\binom{d}{s} c^{(l)}_s(\tau)\end{aligned}$$

where

$$c^{(l)}_s(\tau)=\frac{l!}{s!}\big(c(\tau)-d+l\big)\cdot\ldots\cdot\big(c(\tau)-d+l-s+1\big).$$

Proof. The case $l=1$ has been just proved. Assume (21) to be known for l. We prove it for $l+1$. Apply θ^j_{j-1} to both sides of (21) and use (20):

$$\begin{aligned}&(\theta^j_{j-1})^{l+1}\, T^i_j(\tau-(d-1),\dots,\tau)\\ &\quad=\sum_s T^i_{j-1}(\tau-(d-1),\dots,\tau-(d-s))\, T^i_j(\tau^{(l-s)}-(d-s-1),\dots,\tau^{(l-s)})\\ &\qquad\cdot(\theta^j_{j-1})^{l+1-s}\binom{d}{s} c^{(l)}_s(\tau)\\ &\quad+\sum_s T^i_{j-1}(\tau-(d-1),\dots,\tau-(d-s))\, T^i_{j-1}(\tau-(d-s-1))\\ &\qquad\cdot\, T^i_j(\tau^{(l-s)}-(d-s-2),\dots,\tau^{(l-s)})\cdot(d-s)\big(c(\tau^{(l-s)})-(d-s-1)\big).\\ &\qquad\cdot(\theta^j_{j-1})^{l-s}\binom{d}{s} c^{(l)}_s.\end{aligned}$$

Use now

$$\big(c(\tau^{(l-s)})-(d-s-1)\big)(\theta^j_{j-1})^{l-s}=(\theta^j_{j-1})^{l-s}\big(c(\tau)-d-s+2l+1\big).$$

We get the following recurrence formula

$$c^{(l+1)}_s(\tau)=s\big(c(\tau)-d-s+2l+2\big)\,c^{(l)}_{s-1}(\tau)+c^{(l)}_s(\tau)$$

from which the desired formula follows immediately.

2.7. It follows from (17) that

$$\frac{1}{d!}\,T^i_j(\tau-(d-1),\dots,\tau)\in\mathscr{U}_{\mathbb{Z}}\cdot\mathscr{U}^0_{\mathbb{Z}}.$$

The following result is fundamental for the applications in Chapter 3.

Theorem. *Assume* $n\geqq a>b\geqq 1$, $1\leqq i<j\leqq n$, $l\geqq 1$. *Then*

$$\frac{(\theta^a_b)^l}{l!}\,\frac{1}{d!}\,T^i_j(t-(d-1))\cdot\ldots\cdot T^i_j(t-1)\,T^i_j(t)$$

belongs to the left $\mathscr{U}_{\mathbb{Z}}$-ideal I generated by $\dfrac{(\theta_{b'}^{a'})^s}{s!}$, $a \geqq a' > b' \geqq b$, $s \geqq 1$

$$\binom{\theta_i^i - t - i}{h},\quad h \geqq 1 \quad \textit{and} \quad (\theta_i^i - i) - (\theta_j^j - j) - d.$$

Proof. First note that the general case follows by induction from the case $b = a - 1$ using the formula

$$\frac{(\theta_b^a)^l}{l!} = \sum_{1 \leqq h \leqq l} (-1)^{h+1} \frac{(\theta_b^a)^{l-h}}{(l-h)!} \cdot \frac{(\theta_b^{a-1})^h}{h!} \frac{(\theta_{a-1}^a)^h}{h!} + (-1)^l \frac{(\theta_{a-1}^a)^l}{l!} \frac{(\theta_b^{a-1})^l}{l!} \qquad (a \geqq b+2) \tag{22}$$

which is proved by a straightforward induction on l. Note that (22) shows also that $\mathscr{U}_{\mathbb{Z}}^{+}$ is generated as a ring by

$$\frac{(\theta_{a-1}^a)^l}{l!}, \qquad 2 \leqq a \leqq n,\ l \geqq 1.$$

Assume now that $b = a - 1$.

Case 1. $a = j$. We can rewrite (21) in the form:

$$\begin{aligned} &\frac{(\theta_{j-1}^j)^l}{l!} \frac{1}{d!} T_j^i(\tau - (d-1), \ldots, \tau) \\ &= \sum_{0 \leqq s \leqq \min(l,d)} \frac{1}{s!} T_{j-1}^i(\tau - (d-1), \ldots, \tau - (d-s)) \frac{1}{(d-s)!} \\ &\quad \cdot T_j^i(\tau^{(l-s)} - (d-s-1), \ldots, \tau^{(l-s)}) \\ &\quad \cdot \frac{(\theta_{j-1}^j)^{l-s}}{(l-s)!} (c(\tau) - d + l) \cdot \ldots \cdot (c(\tau) - d + l - s + 1). \end{aligned}$$

We take now $\tau = (t+1, t+2, \ldots, t+n)$ (cf. (7)). The terms corresponding to $s < l$ are left $\mathscr{U}_{\mathbb{Z}}$-multiples of $\dfrac{(\theta_{j-1}^j)^{l-s}}{(l-s)!}$; the term corresponding to $s = l$ is a left $\mathscr{U}_{\mathbb{Z}}$-multiple of

$$\begin{aligned} c(\tau) - d + 1 &= \delta_{j-1}^i(\theta_i^i - t - i) - (\theta_j^j - t - j + d) \\ &= (\theta_i^i - i) - (\theta_j^j - j) - d + (\delta_{j-1}^i - 1)(\theta_i^i - t - i) \end{aligned}$$

and the theorem is proved in this case.

Case 2. $a \neq j$. Applying formula (13) with $\tau=(t+1, t+2, \ldots, t+n)$ and $a \neq j$ we find

$$\theta_{a-1}^{a} \frac{1}{d!} T_j^i(t-(d-1)) \cdot \ldots \cdot T_j^i(t-1)\, T_j^i(t) = x\theta_{a-1}^{a} + u(\theta_i^i - t - i)$$

where $x, u \in \mathscr{U}_{\mathbb{Z}}^- \cdot \mathscr{U}_{\mathbb{Z}}^0$.

Note that the inclusion $u \in \mathscr{U}_{\mathbb{Z}}^- \cdot \mathscr{U}_{\mathbb{Z}}^0$ follows from the fact that

$$u(\theta_i^i - t - i) \in \mathscr{U}_{\mathbb{Z}}^- \cdot \mathscr{U}_{\mathbb{Z}}^0$$

combined with the fact that u does not involve θ_i^i.

The result follows now clearly from the following

Lemma. *Let $y \in \mathscr{U}_{\mathbb{Z}}^- \cdot \mathscr{U}_{\mathbb{Z}}^0$ be such that $\theta_{a-1}^a y = x\theta_{a-1}^a + u(\theta_i^i - c)$ where $x, u \in \mathscr{U}_{\mathbb{Z}}^- \cdot \mathscr{U}_{\mathbb{Z}}^0$, $c \in \mathbb{Z}$. Then for any $l \geqq 1$ we must have:*

$$\frac{(\theta_{a-1}^a)^l}{l!} y = \sum_{1 \leqq s \leqq l} x_s \frac{(\theta_{a-1}^a)^s}{s!} + \sum_{h \geqq 1} u_h \binom{\theta_i^i - c}{h} \tag{23}$$

where $x_s, u_h \in \mathscr{U}_{\mathbb{Z}}^- \cdot \mathscr{U}_{\mathbb{Z}}^0$, for all s, h and $u_h = 0$ for h large

Proof. It is easy to see that we can find unique elements x_s, $u_h \in (\mathscr{U}_{\mathbb{Z}}^- \cdot \mathscr{U}_{\mathbb{Z}}^0) \otimes Q$ such that (23) is satisfied and u_h does not involve θ_i^i for any h. Expressing x_s, u_h in terms of the basis (2) it follows easily that x_s, u_h must lie in $\mathscr{U}_{\mathbb{Z}}^- \cdot \mathscr{U}_{\mathbb{Z}}^0$.

2.8. It is natural to try to find the greatest integer $N(i, j, d)$ such that $\frac{1}{N(i, j, d)} T_j^i(\tau-(d-1), \ldots, \tau) \in \mathscr{U}_{\mathbb{Z}}$. From (17) and from the fact that (2) is a $\mathbb{Z}$-basis of $\mathscr{U}_{\mathbb{Z}}$ it follows that

$$N(i, j, d) = d!\,(\xi(d))^{j-i-1}$$

where $\xi(d)$ is the greatest common divisor of the numbers

$$N!(d-N)!, \quad 0 \leqq N \leqq d.$$

Some elementary number theory shows that

$$\xi(d) = \frac{(d+1)!}{\prod_p p^{[\log_p(d+1)]}}$$

(product over all primes). For example: $\xi(1)=\xi(2)=1$, $\xi(3)=\xi(4)=2$, $\xi(5)=\xi(6)=12$, $\xi(7)=48$, $\xi(8)=144$, etc. We note that the conclusion of Theorem 2.7 becomes false if in the statement one replaces $\frac{1}{d!}$ by $\frac{1}{N(i, j, d)}$. (Take for example $l=2$, $j=i+2$, $d=3$.)

2.9. Let $\mathcal{U}'_{\mathbb{Z}}$ be the subring of $\mathcal{U}_{\mathbb{Z}}$ generated by

$$\binom{\theta^i_j}{N^i_j}, \qquad 1 \leqq i, j \leqq n, \ i \neq j, \ N^i_j \geqq 1$$

and

$$\binom{\theta^i_i - \theta^{i+1}_{i+1}}{N^i_i}, \quad 1 \leqq i \leqq n-1, \ N^i_i \geqq 1.$$

Then $\mathcal{U}'_{\mathbb{Z}}$ is the Kostant $\mathbb{Z}$-form of the enveloping algebra of $\mathfrak{sl}_n$. It is easy to see that $\mathcal{U}'_{\mathbb{Z}}$ is precisely the commutator subring of $\mathcal{U}_{\mathbb{Z}}$. The elements $T^i_j(t) \in \mathcal{U}_{\mathbb{Z}}$ defined by (5) do not in general lie in $\mathcal{U}'_{\mathbb{Z}}$.

Let

$$S^i_j = \sum \theta^i_{i_1} \theta^{i_1}_{i_2} \cdot \ldots \cdot \theta^{i_{k-1}}_{i_k} \theta^{i_k}_j \big((\theta^i_i - i) - (\theta^{j_1}_{j_1} - j_1)\big) \cdot \big((\theta^i_i - i) - (\theta^{j_2}_{j_2} - j_2)\big) \cdot \ldots \cdot \big((\theta^i_i - i) - (\theta^{j_l}_{j_l} - j_l)\big)$$

where the sum is over the same set of indices $i_1 < i_2 < \cdots < i_k$ as in (5); $j_1, j_2, \ldots, j_l$ are also defined in (5). Then $S^i_j \in \mathcal{U}'_{\mathbb{Z}}$ and $\theta^a_{a-1} S^i_j$ belongs to the left $\mathcal{U}'_{\mathbb{Z}}$-ideal generated by θ^a_{a-1} and $(\theta^i_i - i) - (\theta^j_j - j) - 1$ for any a, $2 \leqq a \leqq n$. More generally the element $\frac{(S^i_j)^d}{d!}$ belongs to $\mathcal{U}'_{\mathbb{Z}}$ and $\frac{(\theta^a_b)^l}{l!} \frac{(S^i_j)^d}{d!}$ $(a > b; l \geqq 1)$ belongs to the left $\mathcal{U}'_{\mathbb{Z}}$-ideal generated by $\frac{(\theta^{a'}_{b'})^s}{s!}$ $(a \geqq a' > b' \geqq b, \ s \geqq 1)$ and $(\theta^i_i - i) - (\theta^j_j - j) - d$. We have the equality

$$\frac{(S^i_j)^d}{d!} = \sum_{(N)} \prod_{i<c<j} \Big[\big(\sum_a N^a_c\big)!\big(d - \sum_a N^a_c\big)!\Big] \prod_{i \leqq a < b \leqq j} \frac{(\theta^a_b)^{N^a_b}}{(N^a_b)!} \cdot \prod_{i<c<j} \binom{(\theta^i_i - i) - (\theta^c_c - c)}{d - \sum_a N^a_c}.$$

These statements can be easily deduced from (5), (17) and 2.7 using the following obvious

Lemma. *Fix i, $1 \leqq i \leqq n$ and $t \in \mathbb{Z}$. Then any $u \in \mathcal{U}_{\mathbb{Z}}$ can be written uniquely in the form:*

$$u = \sum_{l \geqq 0} u_l \binom{\theta^i_i - t - i}{l}, \qquad u_l \in \mathcal{U}'_{\mathbb{Z}}.$$

Notice also that

$$\frac{(S^i_j)^d}{d!} = \frac{1}{d!} T^i_j\big(t - (d-1)\big) \cdot \ldots \cdot T^i_j(t-1)\, T^i_j(t) + \sum_{l \geqq 1} v_l \binom{\theta^i_i - t - i}{l}$$

where $v_l \in \mathcal{U}'_{\mathbb{Z}}$.

For the main application in 3.12 the elements

$$\frac{1}{d!} T_j^i\big(t-(d-1)\big)\cdot\ldots\cdot T_j^i(t-1)\, T_j^i(t), \frac{(S_j^i)^d}{d!}$$

are indistinguishable since there we work modulo the left $\mathscr{U}_{\mathbb{Z}}$-ideal generated by $\binom{\theta_i^i-t-i}{l}$, $l\geqq 1$.

3. The Tensor Space

3.1. Let V be a free $\mathbb{Z}$-module of rank n and let $TV=\sum_{r\geqq 0} V^{\otimes r}$ be the tensor algebra of V; we use the notation $V^{\otimes r}=V\underset{\mathbb{Z}}{\otimes} V\underset{\mathbb{Z}}{\otimes}\cdots\underset{\mathbb{Z}}{\otimes} V$ (r factors). The symmetric group S_r acts in $V^{\otimes r}$ by place permutation: if $\big(\sigma(1),\sigma(2),\ldots,\sigma(r)\big)$ is a permutation of $(1,2,\ldots,r)$ we have

$$\sigma(v_1\otimes v_2\otimes\cdots\otimes v_r)=v_{\sigma(1)}\otimes v_{\sigma(2)}\otimes\cdots\otimes v_{\sigma(r)}$$

for any $v_1,v_2,\ldots,v_r\in V$. In this way $V^{\otimes r}$ becomes a left $\mathbb{Z}[S_r]$ module. Let $GL(V)$ be the group of automorphisms of V. Then $GL(V)$ acts naturally on $V^{\otimes r}$. Let $X_1,X_2,\ldots,X_n$ be a basis for V; the non-commutative monomials $X_{i_1}X_{i_2}\cdot\ldots\cdot X_{i_r}$ in 1-1 correspondence with sequences $(i_1,i_2,\ldots,i_r)$ of integers between 1 and n form a basis for $V^{\otimes r}$. (Here we use the abbreviation $X_{i_1}\otimes X_{i_2}\otimes\cdots\otimes X_{i_r}=X_{i_1}X_{i_2}\cdot\ldots\cdot X_{i_r}$.) In terms of the basis (X_i), $GL(V)\cong GL_n(\mathbb{Z})$ and if $g\in GL_n(\mathbb{Z})$ we have

$$g(X_{j_1}X_{j_2}\cdot\ldots\cdot X_{j_r})=\sum_{(i_1,i_2,\ldots,i_r)} g_{j_1}^{i_1} g_{j_2}^{i_2}\cdot\ldots\cdot g_{j_r}^{i_r} X_{i_1}X_{i_2}\cdot\ldots\cdot X_{i_r} \tag{24}$$

where $g(X_j)=\sum_i g_j^i X_i$.

Let θ_j^i: $TV\to TV$ be the unique derivation of the tensor algebra such that $\theta_j^i(X_i)=X_j$, $\theta_j^i(X_h)=0$ $(h\neq i)$; it is clear that $\theta_j^i(V^{\otimes r})\subset V^{\otimes r}$.

θ_j^i is related to the $GL_n(\mathbb{Z})$ action on $V^{\otimes r}$ as follows:

Let $g_j^i(u)\in GL_n(\mathbb{Z})$, $u\in\mathbb{Z}$ be defined by $g_j^i(u)(X_i)=X_i+uX_j$, $g_j^i(u)(X_h)=X_h$ $(h\neq i)$ in case $i\neq j$ and by

$$g_j^i(u)(X_i)=(1+u)X_i,\quad g_j^i(u)(X_h)=X_h\quad (h\neq i),\ u\neq 0,$$

in case $i=j$.

Then on $V^{\otimes r}$

$$\begin{aligned} g_j^i(u)&=\sum_{s\geqq 0}\frac{(\theta_j^i)^s}{s!}u^s && \textit{in case } i\neq j\\ g_j^i(u)&=\sum_{s\geqq 0}\binom{\theta_i^i}{s}u^s && \textit{in case } i=j. \end{aligned} \tag{25}$$

(These are finite sums since $\frac{(\theta^i_j)^s}{s!}$ and $\binom{\theta^i_i}{s}$ are zero on $V^{\otimes r}$ for large s.) This shows that $\frac{(\theta^i_j)^s}{s!}$ $(i \neq j)$ and $\binom{\theta^i_i}{s}$ map $V^{\otimes r}$ into itself rather than into $V^{\otimes r} \otimes Q$. It is clear that the operators $\theta^i_j \in \mathrm{End}(V^{\otimes r})$ satisfy the relations (1) so that we get a left $\mathscr{U}_{\mathbb{Z}}$-module structure on $V^{\otimes r}$ commuting with the action of $\mathbb{Z}[S_r]$.

Theorem. (i) *The natural ring homomorphism*

$$\mathbb{Z}[S_r] \to \mathrm{End}_{\mathscr{U}_{\mathbb{Z}}}(V^{\otimes r})$$

is an isomorphism provided $n \geqq r$.

(ii) *The natural ring homomorphism* $\mathscr{U}_{\mathbb{Z}} \to \mathrm{End}_{\mathbb{Z}[S_r]}(V^{\otimes r})$ *is surjective.*

Proof. It is easy to see in terms of coordinates that for any commutative ring A,

$$\mathrm{End}_{\mathscr{U}_A}(V^{\otimes r} \otimes A) \cong \mathrm{End}_{\mathscr{U}_{\mathbb{Z}}}(V^{\otimes r}) \otimes A$$

and

$$\mathrm{End}_{A[S_r]}(V^{\otimes r} \otimes A) \cong \mathrm{End}_{\mathbb{Z}[S_r]}(V^{\otimes r}) \otimes A$$

(where by definition $\mathscr{U}_A = \mathscr{U}_{\mathbb{Z}} \otimes A$; similarly we put

$$\mathscr{U}_A^{\pm} = \mathscr{U}_{\mathbb{Z}}^{\pm} \otimes A, \ \mathscr{U}_A^0 = \mathscr{U}_{\mathbb{Z}}^0 \otimes A).$$

It is then enough to prove that for A an arbitrary infinite field, the natural homomorphisms

$$\text{(26)} \qquad \begin{array}{ll} \text{(i')} & A[S_r] \to \mathrm{End}_{\mathscr{U}_A}(V^{\otimes r} \otimes A) \quad (n \geqq r) \\ \text{(ii')} & \mathscr{U}_{\mathbb{Z}} \otimes A \to \mathrm{End}_{A[S_r]}(V^{\otimes r} \otimes A) \end{array}$$

are surjective. (The first one is clearly injective.)

Here we have used the following general fact: let u: $L_1 \to L_2$ be a homomorphism between two free $\mathbb{Z}$-modules of finite rank. Then u is surjective if and only if $u \otimes 1$: $L_1 \otimes A \to L_2 \otimes A$ is surjective for any algebraically closed field A.

We have the following

Lemma. *If A is an infinite field, the natural homomorphisms*

$$\text{(i'')} \qquad A[S_r] \to \mathrm{End}_{GL_n(A)}(V^{\otimes r} \otimes A) \quad (n \geqq r)$$

and

$$\text{(ii'')} \quad A[GL_n(A)] \to \mathrm{End}_{A[S_r]}(V^{\otimes r} \otimes A)$$

are surjective.

To conclude the proof of the theorem note that (i″) and (ii″) imply (i′) and (ii′) using (25) and the fact that $g^i_j(u)$, $u\in A$, $i\neq j$ and $g^i_i(u)$, $u\neq -1$ generate $GL_n(A)$.

Proof of the Lemma. (i″) Let $t\in \operatorname{End}_{GL_n(A)}(V^{\otimes r}\otimes A)$. Then

$$t(X_{j_1}X_{j_2}\cdot\ldots\cdot X_{j_r})=\sum_{(j_1,\ldots,j_r)} t^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} X_{i_1}\cdot\ldots\cdot X_{i_r}$$

where $t^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}\in A$. Let $g\colon V\xrightarrow{\approx} V$ be given by $gX_j=\sum g^i_j X_i$.

Then g acts on $V^{\otimes r}$ by the formula (24). Since t and g commute on $V^{\otimes r}$ we must have:

$$\sum_{\substack{(i_1,\ldots,i_r)\\(k_1,\ldots,k_r)}} t^{k_1,\ldots,k_r}_{i_1,\ldots,i_r} g^{i_1}_{j_1} g^{i_2}_{j_2}\cdot\ldots\cdot g^{i_r}_{j_r} X_{k_1}\cdot\ldots\cdot X_{k_r}$$

$$=\sum_{\substack{(i_1,\ldots,i_r)\\(k_1,\ldots,k_r)}} g^{k_1}_{i_1} g^{k_2}_{i_2}\cdot\ldots\cdot g^{k_r}_{i_r} t^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} X_{k_1}\cdot\ldots\cdot X_{k_r}.$$

It follows that

$$\sum_{(i_1,\ldots,i_r)} t^{k_1,\ldots,k_r}_{i_1,\ldots,i_r} g^{i_1}_{j_1}\cdot\ldots\cdot g^{i_r}_{j_r}=\sum_{(j_1,\ldots,j_r)} t^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} g^{k_1}_{j_1}\cdot\ldots\cdot g^{k_r}_{j_r} \tag{27}$$

for any $(k_1,\ldots,k_r)$, $(j_1,\ldots,j_r)$, *and any* g^i_j *with* $\det(g^i_j)\neq 0$. Since A is infinite (27) must be true even without the restriction $\det(g^i_j)\neq 0$. So (27) can be regarded as an identity in the indeterminates g^i_j.

Comparing coefficients in (27), it follows that $t^{k_1,\ldots,k_r}_{i_1,\ldots,i_r}=0$ if $(k_1,\ldots,k_r)$ is not a permutation of $(i_1,\ldots,i_r)$.

Assume now that $k_1,\ldots,k_r$ are distinct and $(i_1,\ldots,i_r)$ are distinct. It follows from (27) that

$$t^{k_1,\ldots,k_r}_{\sigma(k_1),\ldots,\sigma(k_r)}=t^{\sigma^{-1}(i_1),\ldots,\sigma^{-1}(i_r)}_{i_1,\ldots,i_r},\qquad \sigma\in S_r.$$

Hence there exists a unique function $\varphi\colon S_r\to A$ such that $t^{k_1,\ldots,k_r}_{\sigma(k_1),\ldots,\sigma(k_r)}=\varphi(\sigma)$ whenever $k_1,\ldots,k_r$ are distinct.

Assume now that $k_1,\ldots,k_r$ are arbitrary but $i_1,\ldots,i_r$ are distinct (this is only possible if $r\leqq n$). It follows from (27) that

$$t^{k_1,\ldots,k_r}_{\sigma(k_1),\ldots,\sigma(k_r)} \quad \text{is a sum of expressions} \quad t^{\tau(i_1),\ldots,\tau(i_r)}_{i_1,\ldots,i_r}$$

where τ runs over a certain subset of S_r.

It follows that all coefficients of t are linear combinations of $r!$ parameters $\varphi(\sigma)$, $\sigma\in S_r$.

This shows that $\dim \operatorname{End}_{GL_n(A)}(V^{\otimes r}\otimes A)\leqq r!$ and (i″) is proved. For a straightforward proof of (ii″) we refer to Thrall [13].

3.2. Weyl Modules. Let A be a commutative ring with $1 \neq 0$ i.e. with non empty spectrum and let $\overline{V}$ be a free A-module of finite rank, n say.

Let $\mu = (\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_s > 0)$ be a partition of $r = \mu_1 + \mu_2 + \cdots + \mu_s$ into s integers. (s is the number of parts of the partition.) It will be useful to make the convention that μ_h is defined for all $h \geqq 1$, $\mu_h = 0$ for $h > s$. μ defines a decomposition of the set $\{1, 2, \ldots, r\}$ into subsets $I_1, I_2, \ldots, I_s$ where

$$I_h = \{u \in \mathbb{Z}, 1 \leqq u - (\mu_1 + \mu_2 + \cdots + \mu_{h-1}) \leqq \mu_h\}, \qquad 1 \leqq h \leqq s.$$

Let $\overline{V}^* = \mathrm{Hom}(\overline{V}, A)$. There is a canonical pairing $\overline{V}^{\otimes r} \underset{A}{\otimes} \overline{V}^{*\otimes r} \to A$ which will be denoted by $\langle\ ,\ \rangle$.

Here $\overline{V}^{\otimes r} = \overline{V} \underset{A}{\otimes} \overline{V} \underset{A}{\otimes} \cdots \underset{A}{\otimes} \overline{V}$ (r factors). The symmetric group S_r acts on $\overline{V}^{\otimes r}$ by the formula

$$\sigma(\overline{v}_1 \otimes \overline{v}_2 \otimes \cdots \otimes \overline{v}_r) = \overline{v}_{\sigma(1)} \otimes \overline{v}_{\sigma(2)} \otimes \cdots \otimes \overline{v}_{\sigma(r)}$$

and on $\overline{V}^{*\otimes r}$ by a similar formula, so that we have

$$\begin{aligned}&\langle \sigma(\overline{v}_1 \otimes \overline{v}_2 \otimes \cdots \otimes \overline{v}_r), \overline{v}'_1 \otimes \overline{v}'_2 \otimes \cdots \otimes \overline{v}'_r \rangle \\ &\quad = \langle \overline{v}_1 \otimes \overline{v}_2 \otimes \cdots \otimes \overline{v}_r, \overline{v}'_{\rho(1)} \otimes \overline{v}'_{\rho(2)} \otimes \cdots \otimes \overline{v}'_{\rho(r)} \rangle\end{aligned}$$

for $\overline{v}_1, \ldots, \overline{v}_r \in \overline{V}$, $\overline{v}'_1, \ldots, \overline{v}'_r \in \overline{V}^*$ ($\rho = \sigma^{-1}$).

Define $\overline{V}^\mu$ to be the set of all tensors $\overline{X} \in \overline{V}^{\otimes r}$ satisfying conditions (28) and (29) below:

$$\text{(28)} \qquad \langle \overline{X}, \overline{v}'_1 \otimes \overline{v}'_2 \otimes \cdots \otimes \overline{v}'_r \rangle = 0$$

whenever $\overline{v}'_1, \ldots, \overline{v}'_r \in \overline{V}^*$ *are such that there exist* $i \neq j$ *in the same subset* I_h $(1 \leqq h \leqq s)$ *such that* $\overline{v}'_i = \overline{v}'_j$.

(29) *For any* $1 \leqq h \leqq s-1$ *and any* J, $J \subset I_{h+1}$, $J \neq \varnothing$ *we have*

$$\sum_{\sigma \in \mathcal{G}(J)} \varepsilon(\sigma)\, \sigma\, \overline{X} = 0$$

where σ *runs over the set* $\mathcal{G}(J)$ *of all permutations of* $\{1, 2, \ldots, r\}$ *which are the identity outside* $I_h \cup J$ *and such that* $\sigma(i) < \sigma(j)$ *for* $i < j$ *in* I_h *and for* $i < j$ *in* J.

Note that if $\mu = (1)$ the conditions (28) and (29) are empty so that $\overline{V}^{(1)} = \overline{V}$. On the other hand if $\mu_h > n$ for some h then $\overline{V}^\mu = 0$ (cf. (28)). We also remark that in case 2 is invertible in A, (28) is equivalent to the equation $\sigma \overline{X} = \varepsilon(\sigma) \overline{X}$ where σ is any transposition interchanging $i \neq j$ in the same subset I_h. The symmetric and exterior powers are special cases of this construction, in fact $S^r \overline{V} = \overline{V}^\mu$ for $\mu = (1 \geqq 1 \geqq \cdots \geqq 1)$, r components, and $\Lambda^r \overline{V} = \overline{V}^\mu$ for $\mu = (r)$.

Note that in the presence of the alternacy condition (28) and assuming $n!$ invertible in A, condition (29) is equivalent to

(30) $(\sum_\sigma \varepsilon(\sigma)\sigma)\bar{X}=0$ *where* σ *runs through all permutations of* $\{1, 2, \ldots, r\}$ *which are the identity outside* $I_h \cup J$.

In fact this is just Eq. (29) multiplied by the numerical factor $\mu_h!\,|J|!$ which is invertible in A since we can assume $\mu_h \leqq n$, $|J| \leqq \mu_{h+1} \leqq n$.

It is clear that condition (30) for J arbitrary is equivalent to condition (30) for J such that $|J|=1$.

It follows that $\bar{V}^\mu$ can be defined by conditions (28) and (29) with $|J|=1$, provided $n!$ is invertible in A.

Returning now to the general case we prove that in the presence of (28), condition (29) is equivalent to

$$(31)\qquad \Big(\sum_{\sigma\in\mathscr{G}'(J)} \varepsilon(\sigma)\,\sigma\Big)\,\bar{X}=(-1)^{|J|}\,\bar{X}$$

where σ *runs through the set* $\mathscr{G}'(J)$ *of all permutations in* $\mathscr{G}(J)$ (*see* (29)) *such that* $\sigma(J)\subset I_h$.

To prove the equivalence of (29) and (31) let u_J be the left hand side of equality (31). Note that u_J makes sense also when J is empty so that (31) states that

$$(32)\qquad u_J=(-1)^{|J|}\,u_\varnothing \qquad \text{for any } J\subset I_{h+2}.$$

It is clear that in the presence of (28), condition (29) is equivalent to:

$$(33)\qquad \sum_{J'\subset J} u_{J'}=0 \quad \text{for } J \text{ non-empty fixed,} \qquad J\subset I_{h+1}.$$

Assume first that (32) is true. We have

$$\sum_{J'\subset J} u_{J'}=\Big(\sum_{J'\subset J}(-1)^{|J'|}\Big)\,u_\varnothing=0$$

since $J\neq\varnothing$ hence (33) follows. Assume now that (33) is true. If $|J|=1$, (33) is the same as (32), assume now (32) true whenever $|J|\leqq a$, $a\geqq 1$ and let J be such that $|J|=a+1$. Then

$$u_J=-\sum_{\substack{J'\subset J\\ |J'|\leqq a}} u_{J'}=-\Big(\sum_{\substack{J'\subset J\\ |J'|\leqq a}}(-1)^{|J'|}\Big)\,u_\varnothing=(-1)^{|J|}\,u_\varnothing$$

and the equivalence of (32) and (33) and hence that of (29) and (31) is proved.

Note that in case $\mu_h=\mu_{h+1}$ and $J=I_{h+1}$ the equality (31) takes the form of a symmetry condition

$$(34)\qquad \sigma\bar{X}=\bar{X}, \qquad \bar{X}\in\bar{V}^\mu$$

where σ is the unique permutation in $\mathscr{G}'(J)$.

We shall call $\overline{V}^\mu$ the Weyl module associated to $\overline{V}$ and μ. It is clear that $\overline{V} \to \overline{V}^\mu$ is a covariant functor from the category of all free A-modules of finite rank to the category of all A-modules. (We shall actually prove that $\overline{V}^\mu$ is also a free A-module of finite rank.) Indeed if $\overline{V}, \overline{V}'$ are two free A-modules of finite rank and $t: \overline{V} \to \overline{V}'$ is an A-homomorphism it is clear that $t^{\otimes r}: \overline{V}^{\otimes r} \to \overline{V}'^{\otimes r}$ takes $\overline{V}^\mu$ into $\overline{V}'^\mu$. In particular $\overline{V}^\mu$ becomes a $GL(\overline{V})$-module.

Example. Take $\mu = (2 \geqq 2)$. $\overline{V}^\mu$ can be described in this case as the set of all tensors

$$\overline{X} = \sum_{i,j,k,l} x(i\,j\,k\,l)\,\overline{X}_i \overline{X}_j \overline{X}_k \overline{X}_l \in \overline{V}^{\otimes 4},$$

$1 \leqq i, j, k, l \leqq n$, $x(i\,j\,k\,l) \in A$ such that

$$x(i\,j\,k\,l) = -x(j\,i\,k\,l) = -x(i\,j\,l\,k), \qquad x(i\,i\,k\,l) = x(i\,j\,k\,k) = 0,$$

$$x(i\,j\,k\,l) + x(j\,k\,i\,l) + x(k\,i\,j\,l) = 0, \qquad x(i\,j\,k\,l) = x(k\,l\,i\,j).$$

These four conditions are precisely the identities satisfied by the Riemannian curvature tensor in Riemannian geometry. The third condition is known as the Bianchi identity.

Note that the symmetry condition $x(ijkl) = x(klij)$ follows from the other conditions provided 2 is invertible in A.

3.3. We wish to describe some A-basis for $\overline{V}^\mu$ and to do so we recall some classical notions concerned with partitions.

Let λ be the partition $(\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_u > 0)$. *The partition diagram* $[\lambda]$ associated to λ consists of $\lambda_1 + \lambda_2 + \cdots + \lambda_u = r$ squares arranged in consecutive rows so that the first row has λ_1 squares, the second row has λ_2 squares and so on. The rows are counted from top to bottom and arranged so that they all start from the same left extremity. The columns are counted then from left to right. It is clear that the i-th column must have $\tilde{\lambda}_i$ squares where $\tilde{\lambda}$ is the partition dual to λ. We assume that $\tilde{\lambda} = \mu$, $\mu = (\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_s > 0)$.

A *λ-tableau* is by definition a way to distribute r natural numbers (not necessarily distinct) in the r squares of $[\lambda]$, one number in each square.

A λ-tableau is said to be *standard* if it contains all numbers from 1 to r so that they increase from left to right along each row and form top to bottom along each column. There is a unique standard λ-tableau such that any entry in the $(i+1)$-th column is greater than any entry in the i-th column for all i. This is called the leading standard λ-tableau.

A λ-tableau is said to be *semistandard* if its numbers (which are not necessarily distinct) increase strictly from top to bottom along each column and increase in the wide sense from left to right along each row.

A λ-tableau is said to be of type λ', $\lambda'=(\lambda'_1, \lambda'_2, \ldots)$ (λ'_i not necessarily decreasing) if it contains the number i precisely λ'_i times for all $i \geqq 1$.

Let $\mathscr{T}(\lambda, \lambda')$ be the set of all λ-tableaux of type λ' and let $\mathscr{T}_0(\lambda, \lambda')$ be the set of all semistandard λ-tableaux of type λ'.

There is a unique tableau in $\mathscr{T}_0(\lambda, \lambda)$; it is called the leading semistandard λ-tableau. It has the number i in each square of the i-th row, for all i.

Define the k-th position in a partition diagram to be the square in which the leading standard tableau has the number k ($1 \leqq k \leqq r$).

Let σ be any permutation of $\{1, 2, \ldots, r\}$ and let T be any λ-tableau. Suppose that in the k-th position T has the entry $T(k)$. Let $\sigma(T)$ be the λ-tableau which in the k-th position has the entry $T(\sigma^{-1}(k))$ ($1 \leqq k \leqq r$). In this way, the symmetric group S_r acts on the set $\mathscr{T}(\lambda, \lambda')$.

We consider functions f: $\mathscr{T}(\lambda, \lambda') \to G$ with values in some abelian group G satisfying properties (35), (36), (37) below:

(35) *$f(T)=0$ if T has equal entries in two distinct squares in the same column.*

(36) *Let $T, T' \in \mathscr{T}(\lambda, \lambda')$ such that T' is obtained from T by a transposition interchanging two squares in the same column. Then*

$$f(T)+f(T')=0.$$

(37) *Let $I_1, I_2, \ldots, I_s, J \subset I_{h+1}, \mathscr{G}'(J)$ be as in* 3.2, *so that I_h can be regarded as the set of squares in the h-th column. Then*

$$\sum_{\sigma \in \mathscr{G}'(J)} \varepsilon(\sigma) f(\sigma T) = (-1)^{|J|} f(T).$$

Note that if $\mu_h = \mu_{h+1}$, (37) implies

(38) *$f(\sigma T) = f(T)$ where σ is defined as in* (34).

Lemma. *Let f: $\mathscr{T}(\lambda, \lambda') \to G$ be any function satisfying* (35), (36), (37) *and hence also* (38). *Then the image of f lies in the subgroup of G generated by $f(\mathscr{T}_0(\lambda, \lambda'))$.*

Proof. The result is obvious for partition diagrams with 1 square. We assume the result for all partition diagrams with $r-1$ squares. Let $[\lambda]$ be a partition diagram with r squares.

Let G' be the subgroup of G generated by $f(\mathscr{T}_0(\lambda, \lambda'))$. We want to prove that $f(T) \in G'$ for all $T \in \mathscr{T}(\lambda, \lambda')$. Let $l(T) \geqq 1$ be the smallest integer such that the set of entries of T in the last $l(T)$ columns of $[\lambda]$ includes one of the maximal entries of T.

Assume that $f(T) \in G'$ for all T such that $l(T) \leqq l_0$ ($l_0 \geqq 1$). Let T be such that $l(T) = l_0 + 1$. Then the maximal entry of T, say N, must occur in the (l_0+1)-th column C (counted from right to left). We can assume that this column is strictly longer than l_0-th column C (counted from right to left). Indeed if C and C' have the same length we could apply (38) and get that $f(T)=f(T')$ where $l(T')=l_0$ and the induction hypothesis would show that $f(T)\in G'$.

If C is strictly longer than C' then it ends in a corner square Q of $[\lambda]$. Using (36) we can assume that N actually occurs in Q.

By removing the corner square Q from $[\lambda]$ we get a new partition diagram $[\bar{\lambda}]$ with $r-1$ squares. Let $\bar{\lambda}'$ have the same components as λ' excent for the N-th component:

$$\bar{\lambda}'_N=\lambda'_N-1.$$

There is a natural map $\imath\colon \mathscr{T}(\bar{\lambda},\bar{\lambda}')\to\mathscr{T}(\lambda,\lambda')$ obtained by adding to a $\bar{\lambda}$-tableau T the corner square Q with the entry N.

Moreover the image of any semistandard tableau under i is either semistandard or has two entries equal to N in the same column (in which case it annihilates f, cf. (35)).

Consider the function $f\circ\imath\colon \mathscr{T}(\bar{\lambda},\bar{\lambda}')\to G$. This function clearly satisfies (35) and (36). It does not satisfy (37). However, a close look at the definitions shows that $f\circ\imath$ satisfies (37) modulo terms of the form $\pm f(T')$ with $l(T')\leqq l_0$. It follows from the inductive hypothesis with respect to $l(T')$ that $f(T')\in G'$ if $l(T')\leqq l_0$. We conclude that the function $\Pi\circ f\circ\imath$: $\mathscr{T}(\bar{\lambda},\bar{\lambda}')\to G/G'$ where $\Pi\colon G\to G/G'$ is the canonical projection, satisfies (35), (36), (37). Note that $\Pi\circ f\circ\imath(\mathscr{T}_0(\bar{\lambda},\bar{\lambda}'))=0$. If $\bar{T}=\imath^{-1}(T)$ the inductive hypothesis with respect to r implies that $\Pi\circ f\circ\imath(\bar{T})=0$ hence $f(T)\in G'$. This proves the validity of the induction step from l_0 to l_0+1. The first step of the induction $(l_0=1)$ is proved in a completely similar manner. This completes the proof of the Lemma.

3.4. Now let $\mathscr{T}(\lambda)$ be the set of all λ-tableaux whose entries are all the numbers $1,2,\ldots,r$ without repetition. Let $\mathscr{T}_0(\lambda)$ be the subset of $\mathscr{T}(\lambda)$ consisting of standard tableaux.

Consider functions $F\colon\mathscr{T}(\lambda)\to G$ with values in some abelian group G satisfying properties (39) and (40) below:

(39) *Let $T, T'\in\mathscr{T}(\lambda)$ be such that T' is obtained from T by a transposition interchanging two squares in the same row. Then $F(T)=F(T')$.*

Let $\tilde{I}_h$ be the set of entries in the h-th row of the leading standard λ-tableau $(1\leqq h\leqq u)$. Let $\tilde{J}$ be some non empty subset of $\tilde{I}_{h+1}$, h fixed, $1\leqq h\leqq u-1$. Let $\tilde{\mathscr{G}}'(\tilde{J})$ be the set of permutations of $(1,2,\ldots,r)$ which are the identity outside $\tilde{I}_h\cup\tilde{J}$, satisfy $\sigma(i)<\sigma(j)$ for $i<j$ in $\tilde{I}_h$ and for $i<j$ in $\tilde{J}$, and are such that $\sigma(\tilde{J})\subset\tilde{I}_h$. Then

(40) $$\sum_{\sigma\in\tilde{\mathscr{G}}'(\tilde{J})} F(\sigma(T))=(-1)^{|\tilde{J}|}F(T)\qquad \textit{for any}\ \ T\in\mathscr{T}(\lambda).$$

Note that if $\lambda_h = \lambda_{h+1}$, (40) implies

$$F(\sigma(T)) = (-1)^{|\tilde{J}|} F(T),$$

where σ is the unique element of $\tilde{\mathscr{G}}'(\tilde{I}_{h+1})$.

Let $\tilde{\mathscr{G}}(\tilde{J})$ be the set of permutations defined in the same way as $\tilde{\mathscr{G}}'(\tilde{J})$ except that the condition $\sigma(\tilde{J}) \subset \tilde{I}_h$ is dropped. The condition (40) is then equivalent to

$$(41) \qquad \sum_{\sigma \in \tilde{\mathscr{G}}(\tilde{J})} F(\sigma(T)) = 0, \qquad T \in \mathscr{T}(\lambda)$$

provided it is known that F satisfies (39). The equivalence of (40) and (41) is proved in exactly the same way as the equivalence of (29) and (31) (see 3.2).

Lemma. *Let $F\colon \mathscr{T}(\lambda) \to G$ be any function satisfying* (39) *and* (40). *Then the image of F lies in the subgroup of G generated by $F(\mathscr{T}_0(\lambda))$.*

The proof is completely similar to the proof of Lemma 3.3 (use double induction on r and on $\tilde{l}(T)$, where $\tilde{l}(T)$ is the smallest integer ≥ 1 such that the maximal entry of T occurs in one of the last $\tilde{l}(T)$ rows).

3.5. We have already remarked (3.2) that $\bar{V}^\mu = 0$ if some part of μ is strictly greater than $n = \operatorname{rank} \bar{V}$. (We use the notation of 3.2.)

On the other hand if all parts of μ are $\leq n$, we have $\bar{V}^\mu \neq 0$. In fact let $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n$ be an A-basis of $\bar{V}$. The choice of this basis amounts to a choice of an isomorphism $\bar{V} \cong V \otimes_{\mathbb{Z}} A$ where V is the free $\mathbb{Z}$-module with basis $X_1, X_2, \ldots, X_n$; we have $\bar{X}_i = X_i \otimes 1$.

Then

$$(42) \qquad \begin{aligned} \bar{\Phi}^\mu = \sum_{\sigma_1 \in S_{\mu_1}} & \varepsilon(\sigma_1)\, \bar{X}_{\sigma_1(1)} \bar{X}_{\sigma_1(2)} \cdot \ldots \cdot \bar{X}_{\sigma_1(\mu_1)} \\ & \cdot \sum_{\sigma_2 \in S_{\mu_2}} \varepsilon(\sigma_2)\, \bar{X}_{\sigma_2(1)} \bar{X}_{\sigma_2(2)} \cdot \ldots \cdot \bar{X}_{\sigma_2(\mu_2)} \\ & \cdot \ldots \cdot \sum_{\sigma_s \in S_{\mu_s}} \varepsilon(\sigma_s)\, \bar{X}_{\sigma_s(1)} \bar{X}_{\sigma_s(2)} \cdot \ldots \cdot \bar{X}_{\sigma_s(\mu_s)}, \end{aligned}$$

satisfies (28), (29) (by elementary properties of determinants) hence it lies in $\bar{V}^\mu$; it is clear that $\bar{\Phi}^\mu \neq 0$. Note that $\bar{\Phi}^\mu$ is not defined if $\mu_h > n$ for some h, $1 \leq h \leq s$.

Since $V^{\otimes r}$ is a $\mathscr{U}_{\mathbb{Z}}$-module (see 3.1) we get a natural $\mathscr{U}_A = \mathscr{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} A$-module structure on $\bar{V}^{\otimes r} = V^{\otimes r} \otimes_{\mathbb{Z}} A$ by extension of scalars. We now claim that $\bar{V}^\mu$ is a $\mathscr{U}_A$-submodule of $\bar{V}^{\otimes r}$. In fact since $\mathscr{U}_A$ commutes with the place permutations (i.e. S^r on $\bar{V}^{\otimes r}$) the set of tensors $\bar{X} \in \bar{V}^{\otimes r}$ satisfying (29) is $\mathscr{U}_A$-invariant (the condition (29) involves an element in $\mathbb{Z}[S_r]$).

We prove now that the set of tensors $\bar{X} \in \bar{V}^{\otimes r}$ satisfying (28) is also $\mathscr{U}_A$-invariant. Consider the set $S_{i,j}(\bar{V}^{\otimes r})$ of all tensors $\bar{X} \in \bar{V}^{\otimes r}$ satisfying

$$\langle \bar{X}, \bar{v}_1' \otimes \bar{v}_2' \otimes \cdots \otimes \bar{v}_r' \rangle = 0 \qquad \text{whenever } \bar{v}_1', \bar{v}_2', \ldots, \bar{v}_n' \in \bar{V}^*$$

and $\bar{v}_i' = \bar{v}_j'$ ($i<j$ fixed). Let $S_{i,j}(V^{\otimes r})$ be the similar set constructed from V instead of $\bar{V}$.

It is easy to see that $S_{i,j}(\bar{V}^{\otimes r}) \cong \bar{V}^{\otimes(r-2)} \underset{A}{\otimes} \Lambda^2 \bar{V}$ hence $S_{i,j}(\bar{V}^{\otimes r}) = S_{i,j}(V^{\otimes r}) \underset{\mathbb{Z}}{\otimes} A$. Hence it is enough to prove that $S_{i,j}(V^{\otimes r})$ is $\mathscr{U}_{\mathbb{Z}}$-invariant. But since 2 is not a zero divisor in $\mathbb{Z}$ we have

$$S_{i,j}(V^{\otimes r}) = \{X \in V^{\otimes r}, \sigma X = -X\}$$

where σ is the transposition $(i\,j)$. Now this again involves an element of $\mathbb{Z}[S_r]$ and the claim follows.

The elements $\theta^{(N)} \bar{\Phi}^{\mu}$ (see (2)) must then belong to $\bar{V}^{\mu}$ for all matrices $N = (N_j^i)$, $N_j^i \geq 0$ $N_j^i \in \mathbb{Z}$.

It is clear from (42) that

$$\frac{(\theta_j^i)^{N_j^i}}{(N_j^i)!} \bar{\Phi}^{\mu} = 0, \qquad n \geqq i > j \geqq 1, \; N_j^i \geqq 1 \tag{43}$$

and

$$\binom{\theta_i^i - \lambda_i}{N_i^i} \bar{\Phi}^{\mu} = 0, \qquad 1 \leqq i \leqq n, \qquad N_i^i \geqq 1. \tag{44}$$

(Recall that λ is the partition dual to μ.) Next we study the effect of applying elements of $\mathscr{U}_A^-$ to $\bar{\Phi}^{\mu}$. Let T be a λ-tableau with entries $T(1)$, $T(2), \ldots, T(r)$ in the positions $1, 2, \ldots, r$, such that $T(i) \leqq n$ for all i. Consider the element

$$\begin{aligned} \bar{X}_{(T)} = & \sum_{\sigma_1} \varepsilon(\sigma_1) \bar{X}_{T(\sigma_1(1))} \bar{X}_{T(\sigma_1(2))} \cdot \ldots \cdot \bar{X}_{T(\sigma_1(\mu_1))} \\ & \cdot \sum_{\sigma_2} \varepsilon(\sigma_2) \bar{X}_{T(\sigma_2(\mu_1+1))} \cdot \ldots \cdot \bar{X}_{T(\sigma_2(\mu_1+\mu_2))} \\ & \cdot \ldots \cdot \sum_{\sigma_s} \varepsilon(\sigma_s) \bar{X}_{T(\sigma_s(\mu_1+\cdots+\mu_{s-1}+1))} \cdot \ldots \cdot \bar{X}_{T(\sigma_s(\mu_1+\cdots+\mu_{s-1}+\mu_s))} \end{aligned} \tag{45}$$

where σ_h runs through the group of all permutations of the μ_h numbers $\mu_1+\mu_2+\cdots+\mu_{h-1}+1, \ldots, \mu_1+\mu_2+\cdots+\mu_h$.

Then $\bar{\Phi}^{\mu} = \bar{X}_{(T)}$ where T is the unique semistandard λ-tableau of type λ (see 3.3 and (42)).

The following formula follows by applying repeatedly the definition of $\theta_j^i \colon \bar{V}^{\otimes r} \to \bar{V}^{\otimes r}$:

$$\prod_{1 \leqq i < j \leqq n} \frac{(\theta_j^i)^{N_j^i}}{(N_j^i)!} \bar{\Phi}^{\mu} = \sum_T \bar{X}_{(T)} \tag{46}$$

where the sum is over all λ-tableaux T such that T has N_j^i entries equal to j $(1 \leqq i < j \leqq n)$ and $(\lambda_i - \sum_{j, j > i} N_j^i)$ entries equal to i in the i-th row. (The factors in the product are taken in lexicographic order (see 2.1).)

Since $\bar{V}^\mu$ is a $\mathcal{U}_A$ submodule of $\bar{V}^{\otimes r}$ we see that the left hand side of (46) must lie in $\bar{V}^\mu$.

Theorem. *Let $\mu = (\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_s > 0)$ be a partition of $r = \mu_1 + \mu_2 + \cdots + \mu_s$ such that $\mu_h \leqq n$ $(1 \leqq h \leqq s)$, $n = \operatorname{rank} \bar{V}$, and let λ be the partition dual to μ.*

Then the elements (46) *in* 1-1 *correspondence with the set of integral matrices N_j^i $(1 \leqq i < j \leqq n)$ such that $N_j^i \geqq 0$ and*

$$N_j^i + (N_{j+1}^i - N_{j+1}^{i+1}) + (N_{j+2}^i - N_{j+2}^{i+1}) + \cdots + (N_n^i - N_n^{i+1}) \leqq \lambda_i - \lambda_{i+1} \quad (1 \leqq i < j \leqq n) \tag{47}$$

form an A-basis of $\bar{V}^\mu$.

These basis elements are also in 1-1 *correspondence with the set of semistandard λ-tableaux.*

Proof. We first prove that the elements described in the theorem are linearly independent.

Let T, T' be two λ-tableaux. We say that T is equivalent to T' if and only if the sum of entries in the i-th column of T equals the sum of entries in the i-th column of T', for all i. Let $[T]$ denote the equivalence class of T. On the other hand we say that $T \geqq T'$ if and only if the sum of entries in the first i columns of T is greater or equal to the sum of entries in the first i columns of T' for all i. Clearly $T \geqq T'$ and $T' \geqq T$ imply $[T] = [T']$. We get then a partial order on the set of equivalence classes of λ-tableaux.

Let $N = (N_j^i)_{1 \leqq i < j \leqq n}$ be an integral matrix with $N_j^i \geqq 0$. Let $\theta^{(N)} \bar{\Phi}^\mu$ be the left hand side of (46) and let $\mathcal{S}_N$ be the set of λ-tableaux T occurring in the right hand side of (46). It is clear that $\mathcal{S}_N$ is non-empty if and only if

$$N_{i+1}^i + N_{i+2}^i + \cdots + N_n^i \leqq \lambda_i \qquad \text{for all } i, \ 1 \leqq i \leqq n-1.$$

Assume that $\mathcal{S}_N$ is non-empty. There is a unique tableau T_N in $\mathcal{S}_N$ whose equivalence class is strictly less than the equivalence class of any other tableau in $\mathcal{S}_N$. T_N is characterized by the fact that its entries are increasing in the wide sense along each row from left to right. It is easy to see that T_N is semistandard if and only if N satisfies (47). Note that the condition $N_{i+1}^i + N_{i+2}^i + \cdots + N_n^i \leqq \lambda_i$ $(1 \leqq 1 \leqq n-1)$ is a consequence of (47). We have hence a 1-1 correspondence between the set of matrices N satisfying (47) and the set of semistandard λ-tableaux with entries from 1 to n.

Assume now that

$$\sum_N a_N \cdot \theta^{(N)} \bar{\Phi}^\mu = 0 \tag{48}$$

where the sum is over all N satisfying (47), and $a_N \in A$. Replacing $\theta^{(N)} \bar{\Phi}^\mu$ by $\sum_{T \in \mathcal{S}_N} \bar{X}_{(T)}$, we find $\sum_N (a_N \sum_{T \in \mathcal{S}_N} \bar{X}_{(T)}) = 0$. Now this sum can be decomposed in sums over T in a fixed equivalence class and each of these sums must be zero. Consider some minimal equivalence class among the equivalence classes $[T]$, $T \in \bigcup_N \mathcal{S}_N$. The tableaux $T \in \bigcup_N \mathcal{S}_N$ which belong to this minimal equivalence class must be all of the form T_N for some N (i.e. they are semistandard). We find hence a relation $\sum_N a_N \bar{X}_{(T_N)} = 0$ where N takes all values such that T_N is in the minimal equivalence class considered. Since the elements $\{\bar{X}_{(T_N)} \mid T_N \text{ semistandard}\}$ are clearly linearly independent we conclude that $a_N = 0$ for at least one N. Introduce this in (48); we can then repeat the same reasoning and find successively that all a_N are equal to 0. This proves that the elements $\theta^{(N)} \bar{\Phi}^\mu$ where N satisfies (47) are linearly independent. For any λ-tableaux T with entries from 1 to n define $\bar{X}_T = \bar{X}_{T(1)} \cdot \bar{X}_{T(2)} \cdot \ldots \cdot \bar{X}_{T(r)} \in \bar{V}^{\otimes r}$ where $T(i)$ is the entry on the i-th position of T. Then an arbitrary tensor $\bar{X}$ in $\bar{V}^{\otimes r}$ can be written uniquely in the form

$$\bar{X} = \sum_T a_T \bar{X}_T, \qquad a_T \in A$$

where T runs over the set of all λ-tableaux with entries from 1 to n. Hence $\bar{X}$ can be regarded as an A-valued function $f_{\bar{X}}$ on the set $\bigcup_{\lambda'} \mathcal{T}(\lambda, \lambda')$ where λ' runs over all sequences $(\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ of integers such that $\lambda'_i \geqq 0$ $(i \geqq 1)$ and $\lambda'_1 + \lambda'_2 + \cdots + \lambda'_n = r$ (see (3.3)). We have $f_{\bar{X}}(T) = a_T$. Moreover $\bar{X}$ belongs to $\bar{V}^\mu$ if and only if $f_{\bar{X}}$ restricted to $\mathcal{T}(\lambda, \lambda')$ satisfies (35), (36) and (37). This shows that $\bar{V}^\mu$ can be considered as the kernel of $L^1 \otimes A \xrightarrow{u \otimes 1} L_2 \otimes A$ where $u: L_1 \to L_2$ is a homomorphism independent of A between two free $\mathbb{Z}$-modules of finite rank L_1, L_2. It follows that $\bar{V}^\mu$ is a free A-module with basis as in the theorem for arbitrary A if and only if this is true whenever A is a field. [Here we have used the following general fact: let $0 \to L_0 \xrightarrow{v} L_1 \xrightarrow{u} L_2$ be a sequence of free $\mathbb{Z}$-modules of finite rank and homomorphisms u, v such that $u \circ v = 0$. Then this sequence is exact if and only if $0 \to L_0 \otimes A \xrightarrow{v \otimes 1} L_1 \otimes A \xrightarrow{u \otimes 1} L_2 \otimes A$ is exact for any field A or if and only if the latter sequence is exact for any ring A. We take for L_0 the abstract free $\mathbb{Z}$-module with basis in 1-1 correspondence with the set of semistandard λ-tableaux.]

Assume now that A is a field. The Lemma in 3.3 applied for $G = A$ shows that $\dim_A \bar{V}^\mu \leqq \sum_{\lambda'} |\mathcal{T}_0(\lambda, \lambda')| =$ number of semistandard λ-tableaux with entries from 1 to $n = d(\lambda)$ (the last equality is a definition). But we know that the $d(\lambda)$ elements $\theta^{(N)} \bar{\Phi}^\mu$ (N satisfying (47)) are linearly independent; they must hence form a basis for $\bar{V}^\mu$ and the theorem is proved.

Corollary. *Let V be a free $\mathbb{Z}$-module of rank n; and let $\bar{V}=V\otimes_{\mathbb{Z}} A$. There is a canonical isomorphism of $GL(\bar{V})$-modules $V^\mu\otimes A\cong \bar{V}^\mu$.*

3.6. We now describe some S_r-modules inside $\bar{V}^{\otimes r}$. Let $\bar{\mathscr{V}}_\lambda$ be the set of all tensors $\bar{X}\in\bar{V}^{\otimes r}$ satisfying properties (49) and (50) below:

$$\text{(49)}\qquad \frac{(\theta_i^{i+1})^{N_i^{i+1}}}{(N_i^{i+1})!}\bar{X}=0,\qquad 1\leqq i\leqq n-1,\ N_i^{i+1}\geqq 1,$$

$$\text{(50)}\qquad \binom{\theta_i^i-\lambda_i}{N_i^i}\bar{X}=0,\qquad 1\leqq i\leqq n,\ N_i^i\geqq 1.$$

Recall that $\lambda=(\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_u>0)$ is a partition of $\lambda_1+\lambda_2+\cdots+\lambda_u=r$. We assume that λ has at most n parts i.e. $u\leqq n$. Note that (49) is equivalent to the following apparently stronger condition (cf. (22)):

$$\text{(51)}\qquad \frac{(\theta_j^i)^{N_j^i}}{(N_j^i)!}\bar{X}=0,\qquad n\geqq i>j\geqq 1,\ N_j^i\geqq 1.$$

It is clear that $\bar{\mathscr{V}}_\lambda$ is an $A[S_r]$-submodule of $\bar{V}^{\otimes r}$. Note that $\bar{\mathscr{V}}_\lambda$ depends on the choice of basis in $\bar{V}$.

Let

$$\bar{X}=\sum_{(i_1,\ldots,i_r)} a_{i_1,\ldots,i_r}\bar{X}_{i_1}\bar{X}_{i_2}\cdot\ldots\cdot\bar{X}_{i_r}.$$

Let $\mathscr{I}_\lambda$ be the set of multiindices $(i_1,i_2,\ldots,i_r)$ which contain h λ_h times $(1\leqq h\leqq n)$. Then $\bar{X}$ satisfies (50) if and only if $a_{i_1,\ldots,i_r}=0$ unless $(i_1,i_2,\ldots,i_r)\in\mathscr{I}_\lambda$. Assuming that $\bar{X}$ satisfies (50), it is easy to see that $\bar{X}$ satisfies (49) if and only if the condition (52) below is satisfied:

(52) *For any $(i_1,i_2,\ldots,i_r)\in\mathscr{I}_\lambda$, any $1\leqq h\leqq n-1$, and any non-empty subset J of the set $\{k\mid 1\leqq k\leqq r,\ i_k=h+1\}$ we must have:*

$$\sum a_{i'_1,i'_2,\ldots,i'_r}=0$$

where the sum is over all multiindices $(i'_1,i'_2,\ldots,i'_r)\in\mathscr{I}_\lambda$ such that $i'_k=i_k$ if $i_k\neq h,\ h+1$ or if $i_k=h+1$ but $k\notin J$ $(1\leqq k\leqq r)$.

Note that if A is a field of characteristic zero, we can assume $N_i^{i+1}=1$ in (49) $N_i^i=1$ in (50) and $|J|=1$ in (52) and the conditions are not changed.

Returning to the general case we see from (43) and (44) that $\bar{\Phi}^\mu\in\bar{\mathscr{V}}_\lambda$. More generally for any $\sigma\in S_r$ we have $\sigma\bar{\Phi}^\mu\in\bar{\mathscr{V}}_\lambda$.

Theorem. *The elements $\sigma\bar{\Phi}^\mu$, in 1-1 correspondence with the set of permutations σ of $(1,2,\ldots,r)$ such that σ applied to the leading standard λ-tableau gives another standard tableau, form an A-basis for $\bar{\mathscr{V}}_\lambda$.*

Proof. As in the proof of the Theorem in 3.5 there is no loss of generality if we assume that A is a field. Let $F\colon\mathscr{T}(\lambda)\to A$ be a function defined on the set of all λ-tableaux with entries $1, 2, \ldots, r$ without repetition. Assume that F satisfies condition (39). We associate to F a tensor $\bar{X}_F \in \bar{V}^{\otimes r}$ defined by

$$\bar{X}_F = \sum_{(i_1, \ldots, i_r) \in \mathscr{I}_\lambda} F(T_{i_1, \ldots, i_r}) \bar{X}_{i_1} \cdot \ldots \cdot \bar{X}_{i_r}$$

where $T = T_{i_1, \ldots, i_r}$ is any tableau in $\mathscr{T}(\lambda)$ with the property that the entry k occurs in the i_k-th row of $T (1 \leqq k \leqq r)$; note that $F(T)$ is independent of the choice of T, cf. (39).

The correspondence $F \to \bar{X}_F$ induces an isomorphism ι between $\bar{\mathscr{V}}_\lambda$ and the A-vector space $\mathscr{F}_\lambda$ consisting of all functions $F: \mathscr{T}(\lambda) \to A$ satisfying (39) and (41); this isomorphism describes $\bar{\mathscr{V}}_\lambda$ in a way extremely similar to the definition of $\bar{V}^\mu$ (see 3.2); (39) is analogous to (28) and (41) is analogous to (29). Note that (41) corresponds to (52) under ι. We have observed in 3.4 that in the presence of (39) the conditions (40) and (41) are equivalent. It follows then from Lemma 3.4 that $\dim_A \mathscr{F}_\lambda$ is not greater than the number of standard λ-tableaux.

We can identify the set of functions $F: \mathscr{T}(\lambda) \to A$ with the A-vector space with basis $\{T \mid T \in \mathscr{T}(\lambda)\}$. Then any element in $\mathscr{F}_\lambda$ can be written in the form $F = \sum_{T \in \mathscr{T}(\lambda)} a_T \cdot T$. Given any $T \in \mathscr{T}(\lambda)$ define $m(T) = \sum T'$ where the sum is over all T' in $\mathscr{T}(\lambda)$ which are obtained from T by row preserving permutations. Define now $M(T) = \sum_{\sigma \in K_\lambda} \varepsilon(\sigma) m(\sigma T)$ where the sum is over the set K_λ of all column preserving permutations. It is easy to see that the elements $M(T)$ with T standard correspond under ι precisely to the elements $\sigma \bar{\Phi}^\mu$ described in the Theorem. In particular $M(T) \in \mathscr{F}_\lambda$. It remains to be shown that the elements $M(T)$ with T standard are linearly independent. This would imply that they form a basis by our earlier remarks.

The proof of this fact is very similar to the independence proof in Theorem 3.5. We say that $T, T' \in \mathscr{T}(\lambda)$ are equivalent if the sum of entries in the i-th row of T equals the sum of entries in the i-th row of T', for all i. Let $[T]$ denote the equivalence class of T. We say that $T \geqq T'$ if and only if the sum of entries in the first i rows of T is greater or equal to the sum of entries in the first i rows of T' for all i. Clearly $T \geqq T'$ and $T' \geqq T$ imply $[T] = [T']$. We get then a partial order on the set of equivalence classes of tableaux in $\mathscr{T}(\lambda)$. Assume now that $\sum_{T \in \mathscr{T}_0(\lambda)} a_T M(T) = 0$. We have then also

$$\sum_{T \in \mathscr{T}_0(\lambda)} a_T \Big(\sum_{\sigma \in K_\lambda} m(\sigma T) \Big) = 0. \tag{53}$$

This sum can be decomposed in sums over tableaux in a fixed equivalence class and each of these sums must be zero. Consider some minimal equivalence class among the equivalence classes $[T']$, $T' = \sigma T$, $\sigma \in K_\lambda$,

$T\in\mathscr{T}_0(\lambda)$. The tableaux in this equivalence class must be all standard, since $[T]<[\sigma T]$ for $T\in\mathscr{T}_0(\lambda)$, $\sigma\in K_\lambda$, $\sigma\neq 1$. We find hence a relation $\sum a_T m(T)=0$ where T takes all values in the minimal equivalence class considered. Since the elements $\{m(T)|T \text{ standard}\}$ are clearly linearly independent we conclude that $a_T=0$ for at least one $T\in\mathscr{T}_0(\lambda)$. Introduce this in (53); we can then repeat the same procedure and find successively that all a_T are equal to 0. This completes the proof of the Theorem.

Corollary. *Let V be the free $\mathbb{Z}$-module with basis $X_1, X_2, \ldots, X_n$ and $\bar{V}=V\underset{\mathbb{Z}}{\otimes} A$. Then $\bar{\mathscr{V}}_\lambda\cong\mathscr{V}_\lambda\underset{\mathbb{Z}}{\otimes} A$ as $A[S_r]$-modules where $\mathscr{V}_\lambda$ is constructed from V the same way as $\bar{\mathscr{V}}_\lambda$ is constructed from $\bar{V}$.*

Example. Take $\lambda=(2\geqq 2)$; $\bar{\mathscr{V}}_\lambda$ can be described in this case as the set of all functions F on the set of all standard λ-tableaux with values in A such that

$$F\begin{pmatrix} i & j\\ k & l\end{pmatrix}=F\begin{pmatrix} j & i\\ k & l\end{pmatrix}=F\begin{pmatrix} i & j\\ l & k\end{pmatrix}$$

$$F\begin{pmatrix} i & j\\ k & l\end{pmatrix}+F\begin{pmatrix} k & i\\ j & l\end{pmatrix}+F\begin{pmatrix} j & k\\ i & l\end{pmatrix}=0$$

$$F\begin{pmatrix} i & j\\ k & l\end{pmatrix}=F\begin{pmatrix} k & l\\ i & j\end{pmatrix}$$

for any standard λ-tableau $\begin{pmatrix} i & j\\ k & l\end{pmatrix}$. Note that the last symmetry condition follows from the other conditions if 2 is invertible in A. This is a free A-module of rank 2.

3.7. It is useful to change our notation slightly. We shall write $\bar{V}^\mu=\bar{V}_\lambda$, $\bar{\Phi}^\mu=\bar{\Phi}_\lambda$ where μ and λ are dual partitions of r (λ has at most n parts). Note that $\bar{V}_\lambda$ is the $\mathscr{U}_A$-submodule of $\bar{V}^{\otimes r}$ generated by $\bar{\Phi}_\lambda$ (cf. Theorem 3.5) and similarly $\bar{\mathscr{V}}_\lambda$ is the $A[S_r]$-submodule of $\bar{V}^{\otimes r}$ generated by $\bar{\Phi}_\lambda$ (cf. Theorem 3.6). In particular we have $\bar{\Phi}_\lambda\in\bar{V}_\lambda\cap\bar{\mathscr{V}}_\lambda$. It is an easy consequence of Theorem 3.5 that in fact $\bar{V}_\lambda\cap\bar{\mathscr{V}}_\lambda$ is the free A-module on one generator $\bar{\Phi}_\lambda$.

Theorem. *Let λ, λ' be two partitions of r into at most n parts. There exist natural A-homomorphisms*

$$\mathrm{Hom}_{\mathscr{U}_A}(\bar{V}_{\lambda'}, \bar{V}_\lambda)\xleftarrow[P]{} \bar{V}_\lambda\cap\bar{\mathscr{V}}_{\lambda'}\xrightarrow[Q]{}\mathrm{Hom}_{A[S_r]}(\bar{\mathscr{V}}_\lambda, \bar{\mathscr{V}}_{\lambda'})$$

such that

(i) *P is an isomorphism,*

(ii) *Q is injective,*

(iii) *If 2 is not a zero divisor in A then Q is an isomorphism.*

Proof. Let $\overline{X}\in\overline{V}_\lambda\cap\overline{\mathscr{V}}_{\lambda'}$. We must have

$$\overline{X}=s\cdot\overline{\Phi}_{\lambda'}=u\cdot\overline{\Phi}_\lambda \qquad \text{for some } s\in A[S_r],\, u\in\mathscr{U}_A.$$

Define $P\overline{X}\in\mathrm{Hom}_{\mathscr{U}_A}(\overline{V}_{\lambda'},\overline{V}_\lambda)$ by $(P\overline{X})(\overline{X}_1)=s\overline{X}_1$, $\overline{X}_1\in\overline{V}_{\lambda'}$. We must have $\overline{X}_1=u_1\overline{\Phi}_{\lambda'}$, $u_1\in\mathscr{U}_A$ hence

$$s\overline{X}_1=s u_1\overline{\Phi}_{\lambda'}=u_1 s\overline{\Phi}_{\lambda'}=u_1\overline{X}\in u_1\overline{V}_\lambda\subset\overline{V}_\lambda.$$

It is clear that $P\overline{X}$ is a $\mathscr{U}_A$-homomorphism from $\overline{V}_{\lambda'}$ to $\overline{V}_\lambda$. $P\overline{X}$ is independent of the choice of s: assume $\overline{X}=s_1\overline{\Phi}_{\lambda'}$ $s_1\in A[S_r]$: then $s\overline{X}_1-s_1\overline{X}_1=u_1\overline{X}-u_1\overline{X}=0$. Hence P is well defined.

We shall define an inverse P' to P. Let $d\in\mathrm{Hom}_{\mathscr{U}_A}(\overline{V}_{\lambda'},\overline{V}_\lambda)$; define $P'(d)=d(\overline{\Phi}_{\lambda'})$. It is clear that $P'(d)\in\overline{V}_\lambda$. Actually $P'(d)$ must also lie in $\overline{\mathscr{V}}_{\lambda'}$.

In fact, $\overline{\Phi}_{\lambda'}$ satisfies (49) and (50) with λ replaced by λ' (see (43) and (44)) and d commutes with elements of $\mathscr{U}_A$, hence $d(\overline{\Phi}_{\lambda'})$ must satisfy (49) and (50). This proves that P': $\mathrm{Hom}_{\mathscr{U}_A}(\overline{V}_{\lambda'},\overline{V}_\lambda)\to\overline{V}_\lambda\cap\overline{\mathscr{V}}_{\lambda'}$ is well defined. One checks immediately that $PP'=1$, $P'P=1$ and (i) is proved. Define now $Q\overline{X}\in\mathrm{Hom}_{A[S_r]}(\overline{\mathscr{V}}_\lambda,\overline{\mathscr{V}}_{\lambda'})$ by $(Q\overline{X})(\overline{X}_2)=u\overline{X}_2$, $\overline{X}_2\in\overline{\mathscr{V}}_\lambda$. Then Q is well defined (the proof is completely similar to the case of P). Assuming that 2 is not a zero divisor in A we shall construct an inverse Q' to Q. In fact under this assumption the conditions (28) and (29) defining $\overline{V}_\lambda$ involve only elements in the group algebra of S_r (this is not the case with (28) if for example $2=0$ in A). Then the same proof as in the case of P' shows that $\delta\to Q'(\delta)=\delta(\overline{\Phi}_\lambda)$ defines a map

$$Q'\colon\ \mathrm{Hom}_{A[S_r]}(\overline{\mathscr{V}}_\lambda,\overline{\mathscr{V}}_{\lambda'})\to\overline{V}_\lambda\cap\overline{\mathscr{V}}_{\lambda'}$$

which is the inverse of Q. This proves (iii).

Returning to the general case we prove that Q is injective. In fact assume $Q\overline{X}=0$. Then in particular $0=Q\overline{X}(\overline{\Phi}_\lambda)=u\overline{\Phi}_\lambda=\overline{X}$. Hence Q is injective and the Theorem is proved.

Remark. We shall give an example to show that Q is not necessarily an isomorphism in general.

Take $A=$ field with 2 elements, $\overline{V}=A$-vector space with basis $\overline{X}_1,\overline{X}_2$, $\lambda=(1,1)$, $\lambda'=(2)$. Then with respect to the basis $\overline{X}_1\overline{X}_1,\overline{X}_1\overline{X}_2,\overline{X}_2\overline{X}_1,\overline{X}_2\overline{X}_2$ of $\overline{V}^{\otimes 2}$ we have: $\overline{V}_\lambda=\overline{\mathscr{V}}_\lambda$ has a single basis element $\overline{X}_1\overline{X}_2+\overline{X}_2\overline{X}_1=\overline{\Phi}_\lambda$, $\overline{\mathscr{V}}_{\lambda'}$ has a single basis element $\overline{X}_1\overline{X}_1=\overline{\Phi}_{\lambda'}$, $\overline{V}_{\lambda'}$ has a basis

$$\{\overline{X}_1\overline{X}_1,\overline{X}_2\overline{X}_2,\overline{X}_1\overline{X}_2+\overline{X}_2\overline{X}_1\}.$$

Then $\overline{V}_\lambda\cap\overline{\mathscr{V}}_{\lambda'}=0$; however $\mathrm{Hom}_{A[S_2]}(\overline{\mathscr{V}}_\lambda,\overline{\mathscr{V}}_{\lambda'})$ is one dimensional.

3.8. We have the following

Theorem. *Let λ, λ' be two partitions of r into at most n parts. Assume that* $\mathrm{Hom}_{\mathscr{U}_A}(\overline{V}_{\lambda'},\overline{V}_\lambda)\neq 0$ *and that A is a field.*

Then there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that

$$\lambda_i - i = \lambda'_{\sigma(i)} - \sigma(i) \quad \text{in } A \qquad (1 \leqq i \leqq n). \tag{54}$$

(We recall that if $\lambda = (\lambda_1 \geqq \cdots \geqq \lambda_u > 0)$ with $u < n$, we put $\lambda_i = 0$ for $u < i \leqq n$.)

Proof. According to 2.2, the elements C_k $(1 \leqq k \leqq n)$ defined in the Corollary 2.2 can be considered as elements in the centre of $\mathscr{U}_A$.

Expanding the determinant C_k we get

$$C_k = \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} (\theta_{i_1}^{i_1} - i_1)(\theta_{i_2}^{i_2} - i_2) \cdot \ldots \cdot (\theta_{i_k}^{i_k} - i_k)$$
$$+ \text{left } \mathscr{U}_A\text{-multiples of elements of form } \theta_j^i \qquad (i > j).$$

Let $\lambda(C_k) = k$-th elementary symmetric function in

$$\lambda_1 - 1, \ \lambda_2 - 2, \ldots, \lambda_n - n.$$

It follows that

$$C_k(\bar{\Phi}_\lambda) = \lambda(C_k)\, \bar{\Phi}_\lambda.$$

More generally for any $\bar{X} \in \bar{V}_\lambda$ we must have

$$C_k(\bar{X}) = \lambda(C_k)\, \bar{X}.$$

This follows from the fact that $\bar{X} = u\, \bar{\Phi}_\lambda$ for some $u \in \mathscr{U}_A$ and from $u \cdot C_k = C_k \cdot u$.

Now let $d\colon \bar{V}_{\lambda'} \to \bar{V}_\lambda$ be a non-zero $\mathscr{U}_A$-homomorphism. Since $d(\bar{\Phi}_{\lambda'}) \in \bar{V}_\lambda$ we must have $C_k\, d(\bar{\Phi}_{\lambda'}) = \lambda(C_k)\, d(\bar{\Phi}_{\lambda'})$. On the other hand

$$C_k\, d(\bar{\Phi}_{\lambda'}) = d(C_k\, \bar{\Phi}_{\lambda'}) = d\big(\lambda'(C_k)\, \bar{\Phi}_{\lambda'}\big) = \lambda'(C_k)\, d(\bar{\Phi}_{\lambda'}).$$

It follows that $\big(\lambda(C_k) - \lambda'(C_k)\big)\, d(\bar{\Phi}_{\lambda'}) = 0$. Since $d \neq 0$ we must have $d(\bar{\Phi}_{\lambda'}) \neq 0$ and hence $\lambda(C_k) = \lambda'(C_k)$ since A has no zero divisors by assumption.

It follows that

$$\prod_{i=1}^{n} (t + \lambda_i - i) = \prod_{i=1}^{n} (t + \lambda'_i - i) \tag{55}$$

in the polynomial ring $A[t]$. Since the roots of a polynomial with coefficients in a field must be unique up to permutation the Theorem follows.

Remarks. 1) In certain situations one can prove that the conclusion of the Theorem holds even if A has zero divisors. For example, take $A = \mathbb{Z}/p^h\mathbb{Z}$ $(h \geqq 2)$ and let $\bar{A} = \mathbb{Z}/p\mathbb{Z}$ be the quotient of A modulo its unique maximal ideal. Let $\bar{V}$ be a free A-module with basis $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n$ and let $d\colon \bar{V}_{\lambda'} \to \bar{V}_\lambda$ be some $\mathscr{U}_A$-homomorphism such that $d \underset{A}{\otimes} 1_{\bar{A}}$ is non-zero. Then the method of proof of the Theorem shows that (55) must hold in $A[t]$. Making now the genericity assumption $\lambda_i - i \not\equiv \lambda'_j - j \pmod{p}$ for $i \neq j$, it follows from the Hensel lemma that we must have $\lambda_i - i \equiv \lambda'_{\sigma(i)} - \sigma(i) \pmod{p^h}$ $1 \leqq i \leqq n$ for a unique permutation σ.

2) It is easy to prove that if λ, λ' are two partitions of r (resp. s) into at most n parts and if $\mathrm{Hom}_{\mathscr{U}_A}(\overline{V}_{\lambda'}, \overline{V}_\lambda) \neq 0$ (A a field) then we must have $r = s$. In fact, the element $\binom{C_1}{m}$ is in the centre of $\mathscr{U}_A$ for any integer m, $m \geqq 1$. It follows as in the proof of the theorem that

$$\binom{\sum_i (\lambda_i - i)}{m} = \binom{\sum_i (\lambda'_i - i)}{m}$$

in A for all $m \in \mathbb{Z}$, $m \geqq 1$. This clearly implies $\sum_i (\lambda_i - i) = \sum_i (\lambda'_i - i)$ in $\mathbb{Z}$ and hence we must have $\sum_i \lambda_i = \sum_i \lambda'_i$ in $\mathbb{Z}$.

3.9. We introduce a partial order in the set of partitions of r. Given two partitions λ, λ' of r we say that $\lambda \geqq \lambda'$ if and only if $\lambda_1 \geqq \lambda'_1$, $\lambda_1 + \lambda_2 \geqq \lambda'_1 + \lambda'_2$, and so on. It is well known that $\lambda \geqq \lambda'$ if and only if $[\lambda]$ can be obtained from $[\lambda']$ by a sequence of elementary steps, each step consisting in raising the last square of the j-th row (say) to the end of the i-th row ($j > i$) of some partition diagram so that the result is still a partition diagram.

We say that an element $\overline{X} \in \overline{V}_\lambda$ is a weight vector of weight

$$v = (v_1, v_2, \dots, v_n) \quad \text{if} \quad \binom{\theta^i_i - v_i}{N^i_i} \overline{X} = 0, \qquad 1 \leqq i \leqq n, \; N^i_i \geqq 1.$$

For example the general element of the basis of $\overline{V}_\lambda$ described in the Theorem 3.5 is a weight vector of weight v given by

$$v_i = \lambda_i + (N^1_i + N^2_i + \cdots + N^{i-1}_i) - (N^i_{i+1} + N^i_{i+2} + \cdots + N^i_n).$$

It follows easily that

$$v_1 + v_2 + \cdots + v_k = \lambda_1 + \lambda_2 + \cdots + \lambda_k - \sum_{\substack{(i,j) \\ 1 \leqq i \leqq k < j \leqq n}} N^i_j.$$

In particular $v_1 + v_2 + \cdots + v_k \leqq \lambda_1 + \lambda_2 + \cdots + \lambda_k$ for $k = 1, 2, \dots, n-1$ and this becomes equality for $k = n$. Now let $d: \overline{V}_{\lambda'} \to \overline{V}_\lambda$ be a non-zero $\mathscr{U}_A$-homomorphism. Then $d(\overline{\Phi}_{\lambda'})$ must be a weight vector of weight λ' hence it must be a linear combination of standard basis elements in $\overline{V}_\lambda$ of weight λ'. It follows that we must have $\lambda' \leqq \lambda$.

3.10. From now on we shall assume that A is an infinite field. Let T be the subgroup of $GL_n(A)$ consisting of diagonal matrices and let U

be the subgroup of $GL_n(A)$ consisting of matrices (g^i_j) with $g^i_j=0$ for $i>j$ and $g^i_i=1$ for all i. Let $\hat{T}$ be the group of all rational homomorphisms from T to A^*. Then $\hat{T}$ is a free abelian group of rank n. The elements of $\hat{T}$ will be called the weights of $GL_n(A)$. An arbitrary element $\chi\in\hat{T}$ is of the form $\chi(g)=(g^1_1)^{\lambda_1}(g^2_2)^{\lambda_2}\cdot\ldots\cdot(g^n_n)^{\lambda_n}$ where $g\in T$ and $\lambda_1,\lambda_2,\ldots,\lambda_n$ are integers, so we can identify χ with the vector $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)\in\mathbb{Z}^n$ and consider λ itself as a weight in $\hat{T}$. By definition the set of dominant weights in $\hat{T}$ is the semigroup $\hat{T}_+\subset\hat{T}$ consisting of all $\lambda\in\hat{T}$ such that $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n\geqq 0$. Clearly $\hat{T}_+$ is in 1-1 correspondence with the set of partitions with at most n parts.

3.11. A polynomial representation of $GL_n(A)$ is a homomorphism from $GL_n(A)$ into the group of automorphisms of some finite dimensional A-vector space whose components are given by polynomial functions on $GL_n(A)$. Any invariant subspace or quotient space of a polynomial $GL_n(A)$ representation is again a polynomial $GL_n(A)$ representation. Note that the dual of a polynomial $GL_n(A)$ representation is not in general polynomial. Any polynomial $GL_n(A)$ representation restricted to the torus T splits into one dimensional representations of T and hence gives rise to a family of weights. If M is an irreducible polynomial $GL_n(A)$ representation, there is up to a scalar a unique vector $v_0\in M$ which is fixed by U; v_0 is a weight vector corresponding to the highest weight of M with respect to the partial order of partitions introduced in 3.9. The torus T leaves invariant the line generated by v_0. We get thus a weight defined by $g\in T\to\dfrac{g\,v_0}{v_0}$, which must necessarily lie in $\hat{T}_+$. We get thus a 1-1 correspondence between the set of isomorphism classes of irreducible polynomial $GL_n(A)$ representations and $\hat{T}_+$. We shall denote by M_λ the representation associated to $\lambda\in\hat{T}_+$ under this correspondence (λ is called the highest weight of M_λ). It is obvious that $\bar{V}^{\otimes r}$ is a polynomial $GL_n(A)$ representation hence so must be $\bar{V}_\lambda\subset\bar{V}^{\otimes r}$ for $\lambda\in\hat{T}_+$.

$\bar{V}_\lambda$ has a unique U-invariant vector of weight λ (up to a scalar); this is $\bar{\Phi}_\lambda\cdot\bar{\Phi}_\lambda$ generates $\bar{V}_\lambda$ as a $GL_n(A)$-module (since it generates $\bar{V}_\lambda$ as a $\mathcal{U}_A$-module and A is an infinite field). It follows that $\bar{V}_\lambda$ contains a unique maximal $GL_n(A)$-submodule, and the quotient by this must be isomorphic to M_λ. If A has characteristic zero we have actually $\bar{V}_\lambda=M_\lambda$. In fact since $\bar{V}_\lambda\cap\bar{\mathscr{V}}_\lambda$ is clearly 1 dimensional we see from Theorem 3.7 that $\mathrm{Hom}_{\mathcal{U}_A}(\bar{V}_\lambda,\bar{V}_\lambda)=\mathrm{Hom}_{GL_n(A)}(\bar{V}_\lambda,\bar{V}_\lambda)$ is one dimensional. Since $\bar{V}_\lambda$ must be completely reducible this implies that $\bar{V}_\lambda$ is irreducible. A similar proof shows that $\bar{\mathscr{V}}_\lambda$ is an irreducible S_r-module if A is a field of characteristic zero. Note that in our case (A any infinite field) $\bar{\mathscr{V}}_\lambda$ can be regarded as the set of all U-invariant tensors of weight λ in $\bar{V}^{\otimes r}$.

If A has characteristic $p>0$, the corresponding $\bar{V}_\lambda$ can be regarded as the reduction mod p of the $\bar{V}_\lambda$ in characteristic zero using a "minimal admissible lattice", see [2] or [7]. Note that in characteristic $p>0$, $\overline{V}_\lambda$ is in general reducible (see Corollary 2 in 4.1).

3.12. Let λ, λ' be two partitions of r into at most n parts. If A is a field of characteristic zero and $\overline{V}_\lambda \cap \overline{\mathscr{V}}_{\lambda'} \neq 0$ we must have $\lambda = \lambda'$ because $\overline{V}_\lambda$ is irreducible.

We have the following

Theorem. *Let A be a field of characteristic $p>0$. Assume that λ and λ' are related by the formula*

$$\lambda'_i = \lambda_i - d, \quad \lambda'_j = \lambda_j + d, \quad \lambda'_h = \lambda_h, \qquad 1 \leqq h \leqq n, \ h \neq i, j$$

for some $1 \leqq i < j \leqq n$. We assume further that

$$(\lambda_i - i) - (\lambda_j - j) \equiv d \,(\operatorname{mod} p), \qquad 0 < d < p, \tag{56}$$

$$(\lambda_i - i) - (\lambda_h - h) \not\equiv 0, 1, 2, \ldots, d-1 \,(\operatorname{mod} p) \tag{57}$$

for all h, $i < h < j$.

Then $\overline{V}_\lambda \cap \overline{\mathscr{V}}_{\lambda'} \neq 0$.

Proof. Let

$$\overline{X}_{\lambda, \lambda'} = \frac{1}{d!} \, T^i_j(\lambda_i - i - (d-1)) \cdot \ldots \cdot T^i_j(\lambda_i - i - 1) \, T^i_j(\lambda_i - i) \, \overline{\Phi}_\lambda$$

$$= \frac{1}{d!} \, (S^i_j)^d \, \overline{\Phi}_\lambda$$

where the operators T^i_j are defined by (5), and $\frac{1}{d!} (S^i_j)^d$ is defined in 2.9. It is clear that $\overline{X}_{\lambda, \lambda'} \in \overline{V}_\lambda$ and it follows from Theorem 2.7 and from (56) that $\frac{(\theta^a_b)^l}{l!} \overline{X}_{\lambda, \lambda'} = 0$ for any $n \geqq a > b \geqq 1$ and any $l \geqq 1$.

It is easy to see that $\binom{\theta^i_i - \lambda'_i}{l} \overline{X}_{\lambda, \lambda'} = 0$ for any $1 \leqq i \leqq n$ and any $l \geqq 1$.

This shows that $\overline{X}_{\lambda, \lambda'} \in \overline{V}_\lambda \cap \overline{\mathscr{V}}_{\lambda'}$.

Next we show that under the assumption (57) we have $\overline{X}_{\lambda, \lambda'} \neq 0$. Using (17) we can write

$$\begin{aligned} \overline{X}_{\lambda, \lambda'} = \sum_{(N)} \prod_{i<c<j} & \Big[\big(\sum_a N^a_c\big)! \, \big(d - \sum_a N^a_c\big)!\Big] \\ & \cdot \prod_{i \leqq a < b \leqq j} \frac{(\theta^a_b)^{N^a_b}}{(N^a_b)!} \prod_{i<c<j} \binom{(\lambda_i - i) - (\lambda_c - c)}{d - \sum_a N^a_c} \overline{\Phi}_\lambda \end{aligned} \tag{58}$$

where (N) runs over a certain set described in (17). To prove that (58) is non-zero we introduce the following notion. We say that a λ-tableau of type λ' is distinguished if its entries are increasing strictly along columns from top to bottom. Let $\mathcal{T}_d(\lambda, \lambda')$ be the set of all distinguished λ-tableaux of type λ'. The elements $\{\bar{X}_{(T)} \mid T \in \mathcal{T}_d(\lambda, \lambda')\}$ are clearly linearly independent tensors in $\bar{V}^{\otimes r}$ (see (45) for the notation $\bar{X}_{(T)}$). Let $T_1 \in \mathcal{T}_d(\lambda, \lambda')$ be the unique semistandard λ-tableau whose entries in the h-th row are all equal to h $(h \neq i)$ and which has d entries equal to j, $(\lambda_i - d)$ entries equal to i in the i-th row. Using (46) and appropriate column-preserving permutations we can clearly write

$$\bar{X}_{\lambda, \lambda'} = \sum_{T \in \mathcal{T}_d(\lambda, \lambda')} a_T \bar{X}_{(T)}$$

where $a_T \in A$ are uniquely determined (note that column preserving permutations change $\bar{X}_{(T)}$ at most in sign).

Moreover it is easy to see that $\bar{X}_{(T_1)}$ can only come from the term

$$(d!)^{j-i-1} \frac{(\theta_j^i)^d}{d!} \prod_{i<c<j} \binom{(\lambda_i - i) - (\lambda_c - c)}{d} \bar{\Phi}_\lambda$$

corresponding to taking $N_b^a = 0$ unless $a = i$ and $b = j$ in the sum (58). It follows that we must have

$$a_{T_1} = \prod_{i<c<j} \Big(\prod_{\substack{e \\ 0 \leq e \leq d-1}} ((\lambda_i - i) - (\lambda_c - c) - e) \Big).$$

Now the assumption (57) implies that $a_{T_1} \neq 0$ in A. It follows that $\bar{X}_{\lambda, \lambda'} \neq 0$ and the theorem is proved.

Remark. There exists an alternative way to describe $\bar{X}_{\lambda, \lambda'}$. According to Theorem 3.6 we can write $\bar{X}_{\lambda, \lambda'} = \sum_\sigma a_\sigma \sigma \bar{\Phi}_\lambda$ where σ runs through the set of all permutations of $\{1, 2, \ldots, r\}$ such that σ applied to the leading standard λ-tableau gives a standard λ-tableau, and $a_\sigma \in A$ are uniquely determined. The precise values of a_σ can be determined (in principle) from (58) but this is complicated in practice. They are known in case $d = 1$ (M. Beetham, not yet published).

4. The Lattice of Weights and the Affine Weyl Group

4.1. We shall now place our results in a geometric framework. We shall assume that A is a field of characteristic $p > 0$. Recall that T denotes the diagonal subgroup of $GL_n(A)$ and $\hat{T}$ its group of rational characters (see 3.10). We can identify $\hat{T}$ with the set of all sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of integers.

The real vector space $\hat{T} \otimes \mathbb{R}$ has a natural euclidian structure defined by $\langle \lambda, \lambda' \rangle = 2 \sum_{1 \leq i \leq n} \lambda_i \lambda'_i$.

Let $\tilde{x}_i\colon \hat{T}\otimes\mathbb{R}\to\mathbb{R}$ be the affine-linear function defined by $\tilde{x}_i(\lambda)=\lambda_i-i$ (i fixed, $1\leqq i\leqq n$).

Let W_a be the group of all affine transformations w of $\hat{T}\otimes\mathbb{R}$ such that there exists a permutation σ of $\{1,2,\dots,r\}$ such that

$$\tilde{x}_i(w(\lambda))=\tilde{x}_{\sigma(i)}(\lambda)+k_i\cdot p \qquad (1\leqq i\leqq n)$$

where k_i ($1\leqq i\leqq n$) are integers such that $k_1+k_2+\dots+k_n=0$. W_a is called the affine Weyl group. It is clear that $W_a(\hat{T})=\hat{T}$, and that all elements of W_a are distance preserving. Consider the affine hyperplane

$$L^i_j(k)=\{\lambda\in\hat{T}\otimes\mathbb{R}\,|\,\tilde{x}_i(\lambda)-\tilde{x}_j(\lambda)=kp\} \qquad (1\leqq i<j\leqq n,\ k\in Z).$$

The orthogonal reflection with respect to $L^i_j(k)$ is given by the element $w=s^i_j(k)\in W_a$ such that

$$\begin{aligned}&\tilde{x}_i(w(\lambda))=\tilde{x}_j(\lambda)+kp\\&\tilde{x}_j(w(\lambda))=\tilde{x}_i(\lambda)-kp\\&\tilde{x}_h(w(\lambda))=\tilde{x}_h(\lambda), \qquad h\neq i,j.\end{aligned}$$

The reflections $s^i_j(k)$ generate W_a as a group (actually W_a is already generated by $s^1_2(0), s^2_3(0),\dots,s^{n-1}_n(0), s^1_n(1)$ and is in fact a Coxeter group on these generators). An element $\lambda\in\hat{T}\otimes\mathbb{R}$ is said to be p-singular if and only if λ belongs to at least one of the hyperplanes $L^i_j(k)$. An element $\lambda\in\hat{T}\otimes\mathbb{R}$ is p-regular if and only if it is not p-singular. The set of all p-regular elements in $\hat{T}\otimes\mathbb{R}$ is a disconnected open set; its connected components are called alcoves. For example

$$C_0=\{\lambda\in\hat{T}\otimes\mathbb{R}\,|\,\tilde{x}_1(\lambda)>\tilde{x}_2(\lambda)>\dots>\tilde{x}_n(\lambda),\ \tilde{x}_1(\lambda)-\tilde{x}_n(\lambda)<p\}$$

is an alcove (the fundamental alcove). Its closure $\bar{C}_0$ is a fundamental domain for the action of W_a on $\hat{T}\otimes\mathbb{R}$. W_a acts transitively on the set of alcoves with trivial isotropy. In particular any alcove is of the form wC_0 for a unique $w\in W_a$. Note that any alcove is the cartesian product of an open $(n-1)$ simplex and the real line.

We introduce now a relation $\upuparrows$ on the set of dominant weights. Given λ',λ in $\hat{T}_+$ we say that $\lambda'\upuparrows\lambda$ if and only if $\lambda'\leqq\lambda$ and the condition (59) below is satisfied.

(59) *There exists a reflection $s^i_j(k)\in W_a$ such that $s^i_j(k)\lambda'=\lambda$. Moreover, if d denotes the distance from λ'* (or λ) *to the reflecting hyperplane $L^i_j(k)$, we must have $0<d<p$.*

Note that the reflection $s^i_j(k)$ in (59) is uniquely determined.

If $\lambda' \upuparrows \lambda$ we must clearly have

$$(60)\qquad \begin{aligned} \tilde{x}_i(\lambda') &= \tilde{x}_i(\lambda) - d \\ \tilde{x}_j(\lambda') &= \tilde{x}_j(\lambda) + d \\ \tilde{x}_h(\lambda') &= \tilde{x}_h(\lambda), \qquad h \neq i, j,\ i<j \\ \tilde{x}_i(\lambda) - \tilde{x}_j(\lambda) &= kp + d, \qquad 0<d<p \end{aligned}$$

and conversely (60) implies $\lambda' \upuparrows \lambda$. Given λ', λ in $\hat{T}_+$ we say that $\lambda' \uparrow \lambda$ if and only if $\lambda' \upuparrows \lambda$, but we cannot find a sequence $\lambda_1, \lambda_2, \ldots, \lambda_k$ $(k \geqq 1)$ of elements in $\hat{T}_+$ such that $\lambda' \upuparrows \lambda_1, \lambda_1 \upuparrows \lambda_2, \ldots, \lambda_{k-1} \upuparrows \lambda_k, \lambda_k \upuparrows \lambda$.

Lemma. *Let λ', λ in $\hat{T}_+$ be such that* (60) *holds and $\lambda' \uparrow \lambda$. Assume that $\tilde{x}_i(\lambda) - \tilde{x}_h(\lambda) \neq 0 \pmod p$ for all h, $i<h<j$. Then we have also $\tilde{x}_i(\lambda) - \tilde{x}_h(\lambda) \neq 1, 2, \ldots, d \pmod p$ for all h, $i<h<j$.*

Proof. Assume that (60) holds and that for some h, $i<h<j$ we have $\tilde{x}_i(\lambda) - \tilde{x}_h(\lambda) = lp + d'$ for some $l \in \mathbb{Z}$ and some d', $0<d' \leqq d$. We can, of course, assume that h is minimal with this property.

It follows that

$$\tilde{x}_i(\lambda) - \tilde{x}_{h-1}(\lambda) = mp + d'', \qquad m \in \mathbb{Z},\ d<d''<p,$$

provided $i+1<h<j$.

Define $\lambda^{(1)}, \lambda^{(2)} \in \hat{T}$ by

$$\tilde{x}_i(\lambda^{(1)}) = \tilde{x}_i(\lambda) - d', \qquad \tilde{x}_h(\lambda^{(1)}) = \tilde{x}_h(\lambda) + d', \qquad \tilde{x}_s(\lambda^{(1)}) = \tilde{x}_s(\lambda)$$

for $s \neq i, h$.

$$\begin{aligned} &\tilde{x}_i(\lambda^{(2)}) = \tilde{x}_i(\lambda^{(1)}) - (d-d'), \qquad \tilde{x}_j(\lambda^{(2)}) = x_j(\lambda^{(1)}) + (d-d') \\ &\tilde{x}_s(\lambda^{(2)}) = \tilde{x}_s(\lambda^{(1)}) \qquad \text{for } s \neq i, j. \end{aligned}$$

Then we have (cf. (60)):

$$\tilde{x}_h(\lambda') = \tilde{x}_h(\lambda^{(2)}) - d', \qquad \tilde{x}_j(\lambda') = \tilde{x}_j(\lambda^{(2)}) + d', \qquad \tilde{x}_s(\lambda') = \tilde{x}_s(\lambda^{(2)})$$

for $s \neq h, j$.

It follows that

$$\lambda^{(1)} = s_h^i(l)\,\lambda, \qquad \lambda^{(2)} = s_j^i(k)\,\lambda^{(1)}, \qquad \lambda' = s_j^h(k-l)\,\lambda^{(2)}.$$

It is clear that $\lambda, \lambda^{(1)}$ are at distance d' from $L_h^i(l)$; $\lambda^{(1)}, \lambda^{(2)}$ are at distance $(d-d')$ from $L_j^i(k)$ and $\lambda^{(2)}, \lambda'$ are at distance d' from $L_j^h(k-l)$.

We now prove that $\lambda^{(1)}, \lambda^{(2)}$ are dominant. For this it is enough to see that

$$\begin{aligned} \tilde{x}_{h-1}(\lambda) - \tilde{x}_h(\lambda) &> d', \qquad h>i+1 \\ \tilde{x}_i(\lambda) - \tilde{x}_{i+1}(\lambda) &> d+d', \qquad h=i+1. \end{aligned}$$

Assume first $h>i+1$. We have

$$\tilde{x}_{h-1}(\lambda)-\tilde{x}_h(\lambda)=(lp+d')-(mp+d'')=(l-m-1)\,p+(p-d''+d').$$

This must be >0 since λ is dominant; on the other hand $0<p-d''+d'<p$. It follows that $l-m-1\geqq 0$ and hence

$$\tilde{x}_{h-1}(\lambda)-\tilde{x}_h(\lambda)\geqq p-d''+d'>d' \qquad \text{since } p-d''>0.$$

Assume now that $h=i+1$. From the fact that λ' is dominant, it follows that

$$\tilde{x}_i(\lambda)-\tilde{x}_{i+1}(\lambda)>d.$$

We have

$$\tilde{x}_i(\lambda)-\tilde{x}_{i+1}(\lambda)=lp+d'>d.$$

Hence $(l-1)\,p+(p-d+d')>0$. Using $0<p-d+d'\leqq p$, it follows that $l\geqq 1$ hence

$$\tilde{x}_i(\lambda)-\tilde{x}_{i+1}(\lambda)\geqq p+d'>d+d'.$$

We have proved that $\lambda^{(1)}\uparrow\uparrow\lambda$, $\lambda^{(2)}\uparrow\uparrow\lambda^{(1)}$, $\lambda'\uparrow\uparrow\lambda^{(2)}$ which contradicts the hypothesis $\lambda'\uparrow\lambda$. The conclusion of the Lemma follows.

We can now reformulate our results from 3.8, 3.9 and 3.12 as follows:

Theorem. *Let* $\lambda, \lambda'\in\hat{T}_+$.

(i) *If* $\operatorname{Hom}_{GL_n(A)}(\overline{V}_{\lambda'}, \overline{V}_\lambda)\neq 0$ *we must have* $\lambda=w(\lambda')$ *for some* $w\in W_a$ *and* $\lambda'\leqq\lambda$.

(ii) *If* λ, λ' *are p-regular and* $\lambda'\uparrow\lambda$ *then* $\operatorname{Hom}_{GL_n(A)}(\overline{V}_{\lambda'}, \overline{V}_\lambda)\neq 0$.

Proof. The inequality $\lambda'\leqq\lambda$ follows from 3.9. According to Theorem 3.8 the hypothesis of (i) implies the existence of a permutation σ of $\{1,2,\dots,n\}$ such that

$$\lambda_i-i=\lambda'_{\sigma(i)}-\sigma(i)+k_i p, \qquad i=1,2,\dots,n,\ k_i\in\mathbb{Z}.$$

According to the Remark 2 in 3.8 we must have $k_1+k_2+\dots+k_n=0$ and the assertion (i) follows. (ii) follows from Theorem 3.12 and the Lemma.

Remarks. 1. The conclusion of (ii) remains valid if instead of assuming that λ, λ' are p-regular we assume only that $\tilde{x}_i(\lambda)-\tilde{x}_h(\lambda)\not\equiv 0 \pmod p$ for all h, $i<h<j$. (According to the Lemma this implies $\tilde{x}_h(\lambda)-\tilde{x}_j(\lambda)\not\equiv 0$ $(\operatorname{mod} p)$.)

2. It is rather plausible that with the hypothesis of (ii) we actually have

$$\dim_A \operatorname{Hom}_{GL_n(A)}(\overline{V}_{\lambda'}, \overline{V}_\lambda)=1.^{2}$$

[2] Dr. J.C. Jantzen has informed us that he is able to prove this when λ' is maximal (for $\leqq$) among the λ'' with $\lambda''\uparrow\lambda$.

3. The following questions are not yet decided: Let $\lambda \neq \lambda' \in \hat{T}_+$ be p-regular. Assume that $\mathrm{Hom}_{GL_n(A)}(\bar{V}_{\lambda'}, \bar{V}_\lambda) \neq 0$; is it true that we must have $\lambda' \uparrow \lambda^{(1)} \uparrow \lambda^{(2)} \uparrow \cdots \uparrow \lambda^{(k)} \uparrow \lambda$ for some $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$ in $\hat{T}_+$?

Conversely assume that such a sequence $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$ exists. When is it true that $\mathrm{Hom}_{GL_n(A)}(\bar{V}_{\lambda'}, \bar{V}_\lambda) = 0$? A special case of this last question is: Assuming $\lambda' \uparrow\uparrow \lambda$, is it true that $\mathrm{Hom}_{GL_n(A)}(\bar{V}_{\lambda'}, \bar{V}_\lambda) \neq 0$?

4. It is amusing to note that whenever $\mathrm{Hom}_{GL_n(A)}(\bar{V}_{\lambda'}, \bar{V}_\lambda) \neq 0$ we must have $\dim \bar{V}_{\lambda'} \equiv \pm \dim \bar{V}_\lambda \pmod p$, provided $p \geqq n$. In fact the function $d\colon \hat{T}_+ \to \mathbb{Z}$ defined by Weyl's dimension formula

$$d(\lambda) = \dim \bar{V}_\lambda = \frac{\prod\limits_{1 \leqq i < j \leqq n} (\tilde{x}_i(\lambda) - \tilde{x}_j(\lambda))}{\prod\limits_{1 \leqq i < j \leqq n} (j-i)}$$

clearly satisfies the property

$$d(w\lambda) \equiv \det(w)\, d(\lambda) \pmod p, \qquad w \in W_a,\ \lambda \in \hat{T}_+,$$

(note that $\prod\limits_{1 \leqq i < j \leqq n} (j-i) \not\equiv 0 \pmod p$ for $p \geqq n$), and our claim follows from part (i) of the Theorem.

We also note that in case $p \geqq n$ a weight $\lambda \in \hat{T}_+$ is p-regular if and only if $\dim \bar{V}_\lambda \not\equiv 0 \pmod p$. On the other hand no weight in $\hat{T}$ is p-regular in case $p < n$.

Corollary 1. *Let λ, λ' be two partitions of r into at most n parts. We can regard λ, λ' as elements of $\hat{T}_+$.*

(i) *If* $\mathrm{Hom}_{A[S_r]}(\bar{\mathscr{V}}_\lambda, \bar{\mathscr{V}}_{\lambda'}) \neq 0$, *and* $\operatorname{char} A \neq 2$ *we must have* $\lambda = w(\lambda')$ *for some* $w \in W_a$ *and* $\lambda' \leqq \lambda$.

(ii) *If λ, λ' are p-regular and $\lambda' \uparrow \lambda$ then* $\mathrm{Hom}_{A[S_r]}(\bar{\mathscr{V}}_\lambda, \bar{\mathscr{V}}_{\lambda'}) \neq 0$.

Proof. Use 3.7.

Corollary 2. *Let $\lambda \in \hat{T}_+$ be a p-regular weight. Then $\bar{V}_\lambda$ is an irreducible $GL_n(A)$-module if and only if λ lies in the alcove C_0.*

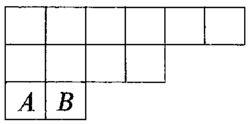
Proof. Assume first that $\lambda \in C_0$, i.e. $(\lambda_1 - 1) - (\lambda_n - n) < p$. In order to prove that $\bar{V}_\lambda$ is irreducible it is sufficient to prove that there is no non-zero U-invariant vector in $\bar{V}_\lambda$ of weight λ', $\lambda' \neq \lambda$ or, equivalently, that $\mathrm{Hom}_{GL_n(A)}(\bar{V}_{\lambda'}, \bar{V}_\lambda) = 0$ for $\lambda' \neq \lambda$. But if $\mathrm{Hom}_{GL_n(A)}(\bar{V}_{\lambda'}, \bar{V}_\lambda) \neq 0$ we must have $\lambda' = w(\lambda)$, $w \in W_a$, $\lambda' < \lambda$, $\lambda' \in \hat{T}_+$, $(\lambda_1 - 1) - (\lambda_n - n) < p$. It is easy to see that these conditions are incompatible, and the irreducibility of $\bar{V}_\lambda$ follows. Assume now that $\lambda \notin C_0$. Suppose λ belongs to the alcove $C \neq C_0$. Then we can find an alcove C' such that C' and C have a wall in common, and such that, if λ' denotes the unique element of C in the W_a-orbit of λ, we have $\lambda' \in \hat{T}_+$ and $\lambda' \uparrow \lambda$. It now follows easily from part (ii) of the Theorem that $\bar{V}_\lambda$ is reducible and the Corollary is proved.

Remark. If $\lambda \in \hat{T}_+$ is not necessarily p-regular the problem of deciding when $\bar{V}_\lambda$ is irreducible is not solved apart from low cases. The fact that $\bar{V}_\lambda$ is irreducible for $\lambda \in C_0$ (and even for $\lambda \in \bar{C}_0$) was first proved by Verma [14], using a result of Humphreys [6].

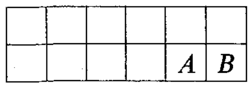
4.2. There is a natural interpretation of the relation $\lambda' \uparrow \lambda$ in terms of partition diagrams.

First, $\lambda' \uparrow\uparrow \lambda$ means that the partition diagram $[\lambda]$ is obtained from the partition diagram $[\lambda']$ by raising the last d squares in the j-th row of $[\lambda']$ to the end of the i-th row of $[\lambda']$ $(1 \leqq i < j \leqq n, 0 < d < p)$ so that each of the raised squares moves through a number of squares equal to a multiple of p.

For example, take

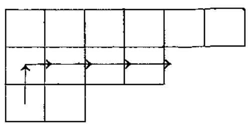


$\lambda' = (6, 4, 2)$

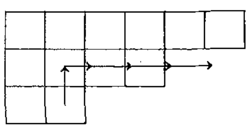


$\lambda = (6, 6, 0)$

$(n = 3, \ p = 5)$. The movement of the square A from the old to the new position can be described by the diagram



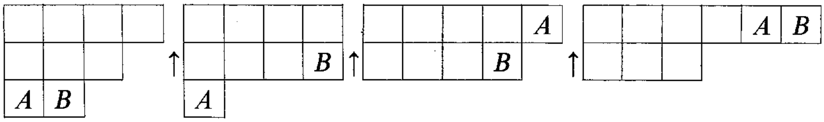
similarly the movement of the square B from the old to the new position is described by the diagram



Both processes clearly involve 5 steps. The process of raising squares is of course very old; it was already present in the work of Young on the symmetric group. The idea of raising squares through a number of steps divisible by p in order to obtain information about the p-modular representations of the symmetric group appears in the book of Robinson [11] and also in more recent work of Kerber [8]. The process of raising several squares at the same time has not, to our knowledge, been previously considered.

Clearly, $\lambda' \uparrow \lambda$ means that $[\lambda]$ can be obtained from $[\lambda']$ by raising squares as described above but not by a composition of such processes.

This is true in the example considered above, but not in the example



λ' λ

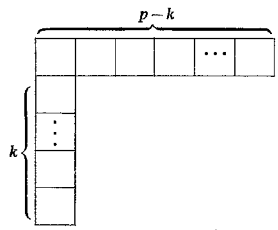
($n=3$, $p=3$) where we have $\lambda' \uparrow\uparrow \lambda$ but not $\lambda' \uparrow \lambda$.

4.3. As an application consider the weight

$$\lambda(k) = (p-k, \underbrace{1, 1, \ldots, 1}_{k}, \underbrace{0, 0, \ldots, 0}_{n-k-1}) \in \hat{T}_+,$$

$0 \leqq k \leqq \min(p-1, n-1)$, $p = \operatorname{char}(A)$.

The partition diagram corresponding to $\lambda(k)$ is a p-hook:



Note that $\lambda(k) \uparrow \lambda(k-1)$, $1 \leqq k \leqq \min(p-1, n-1)$. In fact, $[\lambda(k-1)]$ is obtained from $[\lambda(k)]$ by raising one square from the $(k+1)$-th row to the first row, or equivalently $\lambda(k-1) = s^1_{k+1}(1)\,\lambda(k)$ in the notation of 3.14. Note that $\lambda(k)$ is p-regular if and only if $p \geqq n$. However, without any assumption on p and n we have

$$(\lambda(k-1)_1 - 1) - (\lambda(k-1)_{k+1} - (k+1)) = p+1$$

and

$$(\lambda(k-1)_1 - 1) - (\lambda(k-1)_h - h) \not\equiv 0, \quad \bmod p, \ 1 < h < k+1,$$

which is a somewhat weaker condition than p-regularity. We can apply Theorem 3.12 and conclude that there exists a non-zero $GL_n(A)$-homomorphism $d: \overline{V}_{\lambda(k)} \to \overline{V}_{\lambda(k-1)}$.

According to a theorem of Thrall [13], $\overline{V}_{\lambda(k)}$ has either one or two irreducible composition factors. It follows that d must be unique up to a non-zero scalar.

We could, of course, describe d in terms of the element $T^1_{k+1}(t)$ used in the proof of Theorem 3.12. However in this case there is a more attractive way to define d in terms of the symmetric group. It follows from 3.2 that $\overline{V}_{\lambda(k)}$ can be regarded as the set of all tensors $\overline{X} \in \overline{V}^{\otimes p}$ such that $\langle \overline{X}, \overline{v}'_1 \otimes \overline{v}'_2 \otimes \cdots \otimes \overline{v}'_p \rangle$ is alternating in the variables $\overline{v}'_1, \overline{v}'_2, \ldots, \overline{v}'_{k+1}$, symmetric in the variables $\bar{v}'_{k+2}, \ldots, \bar{v}'_{p-1}, \bar{v}'_p$ and satisfies the condition

$$\sum_\sigma \varepsilon(\sigma)\langle \bar{X}, \bar{v}'_{\sigma(1)} \otimes \bar{v}'_{\sigma(2)} \otimes \cdots \otimes \bar{v}'_{\sigma(k+2)} \otimes \bar{v}'_{k+3} \otimes \cdots \otimes \bar{v}_p \rangle = 0$$

where σ runs through all cyclic permutations of $(1, 2, \ldots, k+2)$ and $\bar{v}'_1, \bar{v}'_2, \ldots, \bar{v}'_p$ are arbitrary elements in $\bar{V}^*$.

Define $d: \bar{V}_{\lambda(k)} \to \bar{V}_{\lambda(k-1)}$ by the formula

$$\begin{aligned}&\langle d\bar{X}, \bar{v}'_1 \otimes \bar{v}'_2 \otimes \cdots \otimes \bar{v}'_p \rangle \\ &\quad = \sum_\sigma \langle \bar{X}, \bar{v}'_1 \otimes \bar{v}'_2 \otimes \cdots \otimes \bar{v}'_k \otimes \bar{v}'_{\sigma(k+1)} \otimes \bar{v}'_{\sigma(k+2)} \otimes \cdots \otimes \bar{v}'_{\sigma(p)} \rangle\end{aligned}$$

where σ runs through all cyclic permutations of $(k+1, k+2, \ldots, p)$. One checks easily that $d\bar{X} \in \bar{V}_{\lambda(k-1)}$ if $\bar{X} \in \bar{V}_{\lambda(k)}$ and that $d(\bar{\Phi}_{\lambda(k)}) \neq 0$ $(1 \leqq k \leqq \min(p-1, s-1))$.

It is obvious that d is a $GL_n(A)$-homomorphism hence it must be the same (up to a scalar) as the one given by Theorem 3.12. Let $s = \min(p-1, n-1)$. We have the sequence

$$(61)\quad 0 \to \bar{V}_{\lambda(s)} \xrightarrow{d} \bar{V}_{\lambda(s-1)} \xrightarrow{d} \cdots \xrightarrow{d} \bar{V}_{\lambda(2)} \xrightarrow{d} \bar{V}_{\lambda(1)} \xrightarrow{d} \bar{V}_{\lambda(0)} \xrightarrow{\varepsilon} M_{\lambda(0)} \to 0$$

where ε is the natural projection of $\bar{V}_{\lambda(0)}$ onto its unique irreducible quotient $M_{\lambda(0)}$.

Note that $\bar{V}_{\lambda(s)}$ is irreducible. In fact, if $p \geqq n$ we have

$$\lambda(s) = \lambda(n-1) = (p-n+1, \underbrace{1, 1, \ldots, 1}_{n-1})$$

and as this lies in the alcove C_0, $\bar{V}_{\lambda(n-1)}$ is irreducible by Corollary 2 in 4.1. On the other hand, if $p < n$, we have

$$\lambda(s) = \lambda(p-1) = (\underbrace{1, 1, \ldots, 1}_{p}, \underbrace{0, \ldots, 0}_{n-p})$$

and in this case $\bar{V}_{\lambda(p-1)}$ is just the p-th exterior power of $\bar{V}$ and this is again clearly irreducible.

On the other hand according to the result of Thrall mentioned above $\bar{V}_{\lambda(i)}$ has two irreducible composition factors. This implies easily that the sequence (61) must be exact. Note that $M_{\lambda(0)}$ is just the natural representation $\bar{V}$, twisted by the Frobenius automorphism of A.

Remarks. 1) If char $A = 0$, the sequence (61) cannot be defined. However, it has been pointed out to us by M. Atiyah that the identity

$$\sum_{0 \leqq i \leqq s} (-1)^i \bar{V}_{\lambda(i)} = \psi^p(\bar{V})$$

holds in the representation ring of $GL_n(A)$; here ψ^p denotes the Adams operation (see [1]).

2) For the symmetric group S_p there is an exact sequence of $A[S_p]$-modules, similar to (61) and due to Peel [10]:

$$(62) \qquad 0 \leftarrow \overline{\mathscr{V}}_{\lambda(p-1)} \leftarrow \overline{\mathscr{V}}_{\lambda(p-2)} \leftarrow \cdots \leftarrow \overline{\mathscr{V}}_{\lambda(2)} \leftarrow \overline{\mathscr{V}}_{\lambda(1)} \leftarrow \overline{\mathscr{V}}_{\lambda(0)} \leftarrow 0$$

(we assume here $n=p$). Note that $\overline{\mathscr{V}}_{\lambda(p-1)} = \overline{\mathscr{V}}_{(1,1,\ldots,1)}$ is the sign representation of S_p (trivial if $p=2$) and $\overline{\mathscr{V}}_{\lambda(0)}$ is the trivial one dimensional representation of S_p.

3) It would be very interesting to generalize (61) as follows. Let $\lambda \in \hat{T}_+$ be a p-regular weight satisfying $\tilde{x}_i(\lambda) - \tilde{x}_{i+1}(\lambda) \leqq p$ $(1 \leqq i \leqq n-1)$ or equivalently

$$(63) \qquad \lambda_i - \lambda_{i+1} \leqq p-1 \qquad (1 \leqq i \leqq n).$$

Let $S_{\lambda,k}$ be the set of all $\lambda' \in \hat{T}_+$ such that there exists a sequence

$$\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k-1)} \in \hat{T}_+$$

such that

$$\lambda' \uparrow \lambda^{(1)} \uparrow \lambda^{(2)} \uparrow \cdots \uparrow \lambda^{(k-1)} \uparrow \lambda.$$

Note that $S_{\lambda,k}$ is empty for k sufficiently large, and finite for all k.

Let $R_\lambda(k) = \bigoplus_{\lambda' \in S_{\lambda,k}} \overline{V}_{\lambda'}$ $(k \geqq 1)$ and $R_\lambda(0) = \overline{V}_\lambda$.

Define a map D: $R_\lambda(k+1) \to R_\lambda(k)$ by a matrix of homomorphisms

$$\varphi_{\mu,\nu}\colon \overline{V}_\mu \to \overline{V}_\nu, \qquad u \in S_{\lambda,k+1}, \ \nu \in S_{\lambda,k}$$

where $\varphi_{\mu,\nu}$ is the homomorphism constructed in 3.12 in the case $\mu \uparrow \nu$ and is zero otherwise.

Form the sequence

$$(64) \qquad \begin{gathered} 0 \to R_\lambda(s) \xrightarrow{D} \cdots \xrightarrow{D} R_\lambda(2) \xrightarrow{D} R_\lambda(1) \xrightarrow{D} R_\lambda(0) \to M_\lambda \to 0, \\ s = \max\{k \mid R_\lambda(k) \neq 0\}. \end{gathered}$$

We speculate that (64) might be an exact sequence. This would give in particular a formula for the character of M_λ as an alternating sum of characters of Weyl modules. (A somewhat analogous exact sequence in the infinite dimensional case has been proved recently by I.I. Bernstein, I.M. Gelfand and S.I. Gelfand see [5].) The significance of the condition (63) is that the set of weights $\lambda \in \hat{T}$ satisfying (63) is in 1-1 correspondence with the set of irreducible p-modular representations of $GL_n(\mathbb{Z}/p\mathbb{Z})$.

4) An exact sequence similar to (64) might also exist in the case of S_r.

Let $\lambda \in \hat{T}_+$ be as in Remark 3 and such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = r$. Assume $n \leqq p \leqq r$. (If $p < n$, λ cannot be p-regular, and if $p > r$, all $A[S_r]$ modules are completely reducible.) Let $\mathscr{S}_{\lambda,k}$ be the set of all $\lambda' \in \hat{T}_+$ such that there exists a sequence $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k-1)} \in \hat{T}_+$ such that

$$\lambda \uparrow \lambda^{(1)} \uparrow \lambda^{(2)} \uparrow \cdots \uparrow \lambda^{(k-1)} \uparrow \lambda'.$$

Note that $\mathscr{S}_{\lambda,k}$ is empty for k sufficiently large and finite for all k.

Let $\mathscr{R}_\lambda(k) = \bigoplus_{\lambda' \in \mathscr{S}_{\lambda,k}} \overline{\mathscr{V}_{\lambda'}}$ $(k \geq 1)$ and $\mathscr{R}_\lambda(0) = \overline{\mathscr{V}_\lambda}$.

Define a map $\mathscr{D}: \mathscr{R}_\lambda(k+1) \to \mathscr{R}_\lambda(k)$ by a matrix of homomorphisms

$$\pm \psi_{\mu,\nu}: \overline{\mathscr{V}_\mu} \to \overline{\mathscr{V}_\nu}, \qquad \mu \in \mathscr{S}_{\lambda,k+1}, \ \nu \in \mathscr{S}_{\lambda,k}$$

where $\psi_{\mu,\nu}$ is the homomorphism constructed in 3.12 (see also 3.7) in the case $\nu \uparrow \mu$ and is zero otherwise.

Form the sequence

$$\begin{aligned} &0 \to \mathscr{R}_\lambda(s') \xrightarrow{\mathscr{D}} \cdots \xrightarrow{\mathscr{D}} \mathscr{R}_\lambda(2) \xrightarrow{\mathscr{D}} \mathscr{R}_\lambda(1) \xrightarrow{\mathscr{D}} \mathscr{R}_\lambda(0) \to \mathscr{M}_\lambda \to 0, \\ &s' = \max\{k \mid \mathscr{R}_\lambda(k)\} \neq 0. \end{aligned} \tag{65}$$

Here $\mathscr{M}_\lambda$ is defined as the cokernel of $\mathscr{R}_\lambda(1) \to \mathscr{R}_\lambda(0)$. It might be conjectured that (65) is exact and that $\mathscr{M}_\lambda$ is an irreducible $A[S_r]$-module. This would imply a formula for the character of $\mathscr{M}_\lambda$ as an alternating sum of characters of $\overline{\mathscr{V}_\mu}$ s.

5. Generalisation to Algebraic Groups of Other Types

Our results on the existence of non-trivial homomorphisms may be expressed in terms of $SL_n(A)$-modules instead of $GL_n(A)$-modules. Let $T' = T \cap SL_n(A)$. By restricting characters of T to T' we obtain a surjective map $\pi: \hat{T} \to \hat{T}'$. We note that $\pi(\mu) = \pi(\nu)$ if and only if $\mu_i = \nu_i + c$, $c \in \mathbb{Z}$, where c is independent of i. The affine Weyl group W_a operates effectively in $\hat{T}'$ by the rule $w(\pi(\lambda)) = \pi(w(\lambda))$ for $\lambda \in \hat{T}$, $w \in W_a$. In this way W_a can be regarded as a subgroup of the affine orthogonal group of $\hat{T}' \otimes \mathbb{R}$, which inherits a Euclidean structure from $\hat{T} \otimes \mathbb{R}$ via π. As such W_a is generated by the reflections in the hyperplanes $L_j'^i(k) = \pi(L_j^i(k)) \subset \hat{T}'$, and is the affine Weyl group of SL_n. The images under π of the alcoves in $\hat{T}$ are alcoves in $\hat{T}'$ with respect to the hyperplanes $L_j'^i(k)$ and are $(n-1)$-simplices. Note that the map $\pi: \hat{T} \to \hat{T}'$ corresponds to restricting $GL_n(A)$-modules to $SL_n(A)$-modules.

Let $\hat{T}'_+ = \pi(\hat{T}_+)$. The elements in $\hat{T}'_+$ are the dominant weights of the irreducible rational $SL_n(A)$-modules and we have an example of the situation encountered in the theory of semi-simple algebraic groups. The $L_j'^i(k)$ are affine hyperplanes orthogonal to the roots of $SL_n(A)$. The elements of $\hat{T}'_+$ are the non-negative integral combination of the fundamental weights $q_1, q_2, \ldots, q_{n-1}$ of $SL_n(A)$ given by

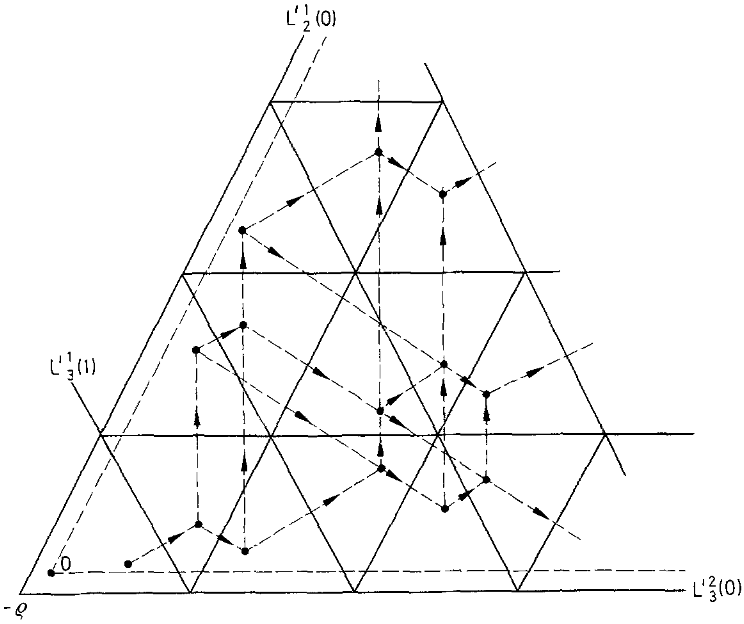
$$\begin{aligned}\pi(1,0,\ldots,0) &= q_1\\ \pi(1,1,0,\ldots,0) &= q_2\\ &\vdots\\ \pi(1,1,\ldots,1,0) &= q_{n-1}.\end{aligned}$$

Let $\rho = q_1 + \cdots + q_{n-1}$ be the sum of the fundamental weights. If we denote the weight $k_1 q_1 + \cdots + k_{n-1} q_{n-1} \in \hat{T}'_+$ by $[k_1, k_2, \ldots, k_{n-1}]$, we have

$$\pi(\lambda_1, \lambda_2, \ldots, \lambda_n) = [\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n].$$

We observe that the affine hyperplanes $L_j'^i(0)$ all pass through the weight $-\rho$.

We represent in the figure the case $n=3$. We have shown only alcoves in $\hat{T}'$ whose closures have non-empty intersection with $\hat{T}'_+$. Pairs of points $\pi(\lambda)$, $\pi(\lambda')$ such that $\lambda' \uparrow \lambda$ are joined by a dotted line oriented towards λ. The dominant weights (i.e. those in $\hat{T}'_+$) lie in the closed acute cone with vertex 0.



It is intriguing to consider how the result on the existence of non-trivial homomorphisms between Weyl modules for $SL_n(A)$ might generalise to other simple algebraic groups.

Let G be a simple simply-connected algebraic group over the complex field, let T be a maximal torus of G and $\hat{T}$ be the group of rational characters of T. Let $\hat{T}_+$ be the set of dominant weights with respect to a suitable ordering on $\hat{T}$. For each $\lambda \in \hat{T}_+$ let V_λ be the irreducible G-module with highest weight λ. Let $V_{\lambda,\mathbb{Z}}$ be a minimal admissible lattice in V_λ (see Humphreys [7] p. 159) and $\bar{V}_\lambda = V_{\lambda,\mathbb{Z}} \otimes K$ where K is an algebraically closed field of characteristic p. Let $G(K)$ be the simple algebraic group of type G over the field K. Then $\bar{V}_\lambda$ is a rational $G(K)$-module, which is however not in general irreducible.

Given $\lambda, \lambda' \in \hat{T}_+$ we consider under what circumstances

$$\mathrm{Hom}_{G(K)}(\bar{V}_{\lambda'}, \bar{V}_\lambda) \neq 0.$$

The real vector space $\hat{T} \otimes R$ admits a positive definite scalar product invariant under the action of the Weyl group, and we can decompose the resulting Euclidean space into alcoves in the following way. For each root r let $h_r = \dfrac{2r}{(r,r)}$ be the corresponding coroot. For each $k \in \mathbb{Z}$ let $L_r(k)$ be the set of $\lambda \in \hat{T} \otimes R$ satisfying the condition

$$(h_r, \lambda + \rho) = kp$$

where ρ is the sum of the fundamental weights. Observe that this relation is obtained by equating to an integral multiple of p one of the factors in the numerator of Weyl's dimension formula. An element $\lambda \in \hat{T} \otimes \mathbf{R}$ is called p-singular if λ belongs to some affine hyperplane $L_r(k)$ and p-regular otherwise. The set of p-regular elements in $\hat{T} \otimes R$ is a disconnected open set, whose connected components are called alcoves. Let $s_r(k)$ be the reflection in $L_r(k)$, and let W_a be the group of affine transformations of $\hat{T} \otimes R$ generated by $s_r(k)$ for all r, k.

Given two p-regular elements λ, λ' of $\hat{T}_+$ we define $\lambda' \uparrow\uparrow \lambda$ if $\lambda' \leqq \lambda$ and $s_r(k)\,\lambda' = \lambda$ for some r, k such that $L_r(k)$ intersects the closure of the alcove containing λ. We define $\lambda' \uparrow \lambda$ to mean that $\lambda' \uparrow\uparrow \lambda$ but there do not exist $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$ $(k \geqq 1) \in \hat{T}_+$ such that

$$\lambda' \uparrow\uparrow \lambda_1, \quad \lambda_1 \uparrow\uparrow \lambda_2, \ldots, \lambda_{k-1} \uparrow\uparrow \lambda_k, \quad \lambda_k \uparrow\uparrow \lambda.$$

A natural generalisation of our theorem in 3.14 would be given by the following conjecture:

Let $\lambda, \lambda' \in \hat{T}_+$. If $\mathrm{Hom}_{G(K)}(\bar{V}_{\lambda'}, \bar{V}_\lambda) \neq 0$ then $\lambda = w(\lambda')$ for some $w \in W_a$, and $\lambda' \leqq \lambda$. If λ, λ' are p-regular and $\lambda' \uparrow \lambda$ then $\mathrm{Hom}_{G(K)}(\bar{V}_{\lambda'}, \bar{V}_\lambda) \neq 0$.

The former of the two statements was conjectured by Verma [14] and proved by Humphreys when p is greater than the Coxeter number of G.

It appears from the results of the present paper that the existence of such non-trivial homomorphisms could be established by first proving the existence of certain elements in $\mathscr{U}_{\mathbb{Z}}$, the Konstant $\mathbb{Z}$-form of the enveloping algebra $\mathscr{U}$ of the Lie algebra $\mathfrak{g}$ of G, which have favourable commutation properties. One such element $S_{r,d}$ is needed for each positive root r and each positive integer d.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and e_r, f_r be root vectors in $\mathfrak{g}$ corresponding to roots r, $-r$ respectively such that $[e_r, f_r] = h_r \in h$. By analogy with the results of 2.9 we seek elements $S_{r,d} \in \mathscr{U}_{\mathbb{Z}}$ satisfying the following conditions (i), (ii).

(i) $S_{r,d} = \sum c_{r_1,\dots,r_k} f_{r_1} f_{r_2} \cdots f_{r_k} u_{r_1,\dots,r_k}$ where $c_{r_1,\dots,r_k} \in \mathbb{Z}$, $u_{r_1,\dots,r_k}$ lies in the $\mathbb{Z}$-form $\mathscr{U}_{\mathbb{Z}}(\mathfrak{h})$ of the enveloping algebra $\mathscr{U}(\mathfrak{h})$ of $\mathfrak{h}$, and the sum extends over all partitions of dr into a sum of positive roots. Moreover in the leading term (in which each r_i is a fundamental root) we have $u_{r_1,\dots,r_k} = 1$ and $c_{r_1,\dots,r_k}$ is some suitable normalizing factor in $\mathbb{Z}$.

(ii) For all positive integers l and all fundamental roots r_i, $\frac{e_{r_i}^l}{l!} S_{r,d}$ lies in the left ideal of $\mathscr{U}_{\mathbb{Z}}$ generated by $\frac{e_s^k}{k!}$ for all positive roots s and all integers $k \geqq 1$, and by $h_r + (h_r, \rho) - d$.

We have proved the existence of such elements $S_{r,d}$ when $G = SL_n$ is a simple algebraic group of type A_{n-1}. In this case we have one positive root r_{ij} for each pair of integers i, j such that $0 < i < j \leqq n$. Let $e_{ij} = e_{r_{ij}}$, $f_{ij} = f_{r_{ij}}$, $h_{ij} = h_{r_{ij}}$. Identifying with our previous notation we have $e_{ij} = \theta_i^j$, $f_{ij} = \theta_j^i$, $h_{ij} = \theta_i^i - \theta_j^j$ $(i<j)$. The element $S_r = S_{r,1}$ of the enveloping algebra $\mathscr{U}_{\mathbb{Z}}$ is then given by

$$S_{r_{ij}} = \sum_{i < i_1 < \cdots < i_k < j} f_{i i_1} f_{i_1 i_2} \cdot \ldots \cdot f_{i_k j} (h_{i j_1} + j_1 - i)(h_{i j_2} + j_2 - i) \cdot \ldots \cdot (h_{i j_t} + j_t - i)$$

summed over all subsets $\{i_1, \dots, i_k\}$ of $\{i+1, \dots, j-1\}$, where $\{j_1, \dots, j_t\}$ is the complementary subset of $\{i_1, \dots, i_k\}$ in $\{i+1, \dots, j-1\}$. Moreover the element $S_{r,d}$ is given by $S_{r,d} = S_r^d / d!$

We observe that an element $S_r \in \mathscr{U}_{\mathbb{Z}}$ satisfying conditions (i), (ii) for $d=1$ is not uniquely determined since the terms $u_{r_1,\dots,r_k} \in \mathscr{U}_{\mathbb{Z}}(\mathfrak{h})$ can always be modified by adding multiples of $h_r + (h_r, \rho) - 1$. Apart from this ambiguity, however, we have observed that in the simple groups of rank 2, viz A_2, B_2, G_2, conditions (i), (ii) for $d=1$ and $l=1$ are sufficient to determine S_r to within a scalar multiple. We conclude by describing these elements S_r for each positive root r in a system of type A_2, B_2 or G_2. We write $[e_r, e_s] = N_{r,s} e_{r+s}$ and $M_{r,s,i} = \frac{1}{i!} N_{r,s} N_{r,r+s} \cdot \ldots \cdot N_{r,(i-1)r+s}$ and recall that in a Chevalley basis of g both $N_{r,s}$ and $M_{r,s,i}$ are rational integers. (See [4], p. 62.) We also write $\tilde{h}_r = h_r + (h_r, \rho) - 1 \in \mathscr{U}_{\mathbb{Z}}(\mathfrak{h})$.

Type A_2. Root system $\pm\{a, b, a+b\}$.

$$S_a = f_a \qquad \tilde{h}_a = h_a$$
$$S_b = f_b \qquad \tilde{h}_b = h_b$$
$$S_{a+b} = f_a f_b - N_{a,b} f_{a+b} h_b \qquad \tilde{h}_{a+b} = h_a + h_b + 1.$$

Type B_2. Root system $\pm\{a, b, a+b, 2a+b\}$.

$$S_a = f_a \qquad \tilde{h}_a = h_a$$
$$S_b = f_b \qquad \tilde{h}_b = h_b$$
$$S_{a+b} = f_a f_b - N_{a,b} f_{a+b} h_b \qquad \tilde{h}_{a+b} = h_a + 2h_b + 2$$
$$S_{2a+b} = f_a f_a f_b - N_{a,b} f_a f_{a+b} h_b + M_{a,b,2} f_{2a+b} h_b(h_b+1) \qquad \tilde{h}_{2a+b} = h_a + h_b + 1.$$

Type G_2. Root system $\pm\{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$.

$$S_a = f_a \qquad \tilde{h}_a = h_a$$
$$S_b = f_b \qquad \tilde{h}_b = h_b$$
$$S_{a+b} = f_a f_b - N_{a,b} f_{a+b} h_b \qquad \tilde{h}_{a+b} = h_a + 3h_b + 3$$
$$S_{2a+b} = f_a f_a f_b - N_{a,b} f_a f_{a+b} h_b + M_{a,b,2} f_{2a+b} \frac{h_b}{2} \cdot \left(\frac{3h_b+2}{2}\right) \qquad \tilde{h}_{2a+b} = 2h_a + 3h_b + 4$$

or, alternatively (with the elements in $\mathscr{U}(\mathfrak{h})$ in integral form):

$$S_{2a+b} = f_a f_a f_b - N_{a,b} f_a f_{a+b} h_b + M_{a,b,2} f_{2a+b}(h_{3a+2b}+2)(3h_{3a+2b}+7)$$
$$S_{3a+b} = f_a f_a f_a f_b - N_{a,b} f_a f_a f_{a+b} h_b + M_{a,b,2} f_a f_{2a+b} h_b(h_b+1) - M_{a,b,3} f_{3a+b} h_b(h_b+1)(h_b+2)$$
$$\tilde{h}_{3a+b} = h_a + h_b + 1$$
$$\begin{aligned} S_{3a+2b} = {} & f_a f_a f_a f_b f_b - 2N_{a,b} f_a f_a f_b f_{a+b}(h_b-1) \\ & + f_a f_{a+b} f_{a+b} h_b(h_b-1) + M_{a,b,2} f_a f_b f_{2a+b}(h_b-1)(h_b-2) \\ & - \frac{N_{a,a+b}}{2} f_{a+b} f_{2a+b} h_b h_b(h_b-1) + M_{a,b,3} f_b f_{3a+b} h_b(h_b-1)(h_b+4) \\ & + M_{a,b,3} N_{b,3a+b} f_{3a+2b}(h_b+4)(h_b+1) h_b(h_b-1) \end{aligned}$$
$$\tilde{h}_{3a+2b} = h_a + 2h_b + 2.$$

(In fact we have $N_{r,s} = \pm 1$ in all the above formulae, except for $N_{a,a+b} = \pm 2$ in B_2 and G_2. We also have $M_{r,s,i} = \pm 1$ in these formulae.)

In order to prove that these elements of $\mathscr{U}_{\mathbb{Z}}$ give rise to homomorphisms of the required type it would also be necessary to verify that they satisfy the commutation condition (ii) for arbitrary values of l.

Since completing this work we have learned of a recent paper of Shapovalov [12] in which the existence and uniqueness of elements $S_{r,d}$ of the above type is proved in quite a different context. However in this paper analogues of the conditions (i), (ii) are used involving $\mathscr{U}$ rather than $\mathscr{U}_{\mathbb{Z}}$. It is likely that, by showing that Shapovalov's elements satisfy the commutation formula (ii) over $\mathscr{U}_{\mathbb{Z}}$, the existence of the conjectured homomorphisms could be proved for any semi-simple group. We note, however, that the leading term in $S_{r,d}$ which is used in this paper to prove the non-triviality of the homomorphisms is the term involving f_r^d, rather than the leading term $f_{r_1}^d f_{r_2}^d \dots f_{r_k}^d$ given by Shapovalov, where $r = r_1 + \dots + r_k$ is the decomposition of r into a sum of fundamental roots.

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