Representations of algebraic groups and their Lie algebras

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Lecture I

Set-up. Let $K$ be an algebraically closed field. By convention all our algebraic groups will be linear algebraic groups over $K$. For such a group $G$ let $K[G]$ denote the algebra of regular functions on $G$. For the background on algebraic groups I refer to the books *Linear Algebraic Groups* by J. E. Humphreys, T. A. Springer, and A. Borel.

Modules. Let $G$ be an algebraic group. A $G$–module is by definition a module $V$ over the group algebra $KG$ of $G$ as an abstract group such that $V$ satisfies two additional conditions:

- $V$ is locally finite: each element is contained in a finite dimensional $KG$–submodule.
- For each $v \in V$ and $\varphi \in V^*$ the matrix coefficient $c_{\varphi,v}$ given by $c_{\varphi,v}(g) = \varphi(gv)$ for all $g \in G$ is a regular function: $c_{\varphi,v} \in K[G]$.

For example, any finite dimensional vector space $V$ is a module under the natural action for the general linear group $\text{GL}(V)$ as well as for any closed subgroup of $\text{GL}(V)$. For any algebraic group $G$ the algebra of regular functions $K[G]$ is a $G$–module under the (left) regular representation given by $(gf)(g') = f(g^{-1}g')$.

It is easy to check that the category of all $G$–modules is closed under:

- submodules
- factor modules
- direct sums
- tensor products
- symmetric and exterior powers.

If a $G$–module $V$ is finite dimensional, then also its dual space $V^*$ is a $G$–module.

Frobenius twist. Assume $\text{char}(K) = p > 0$. In this case we have another method to construct new $G$–modules from known ones. If $V$ is a vector space over $K$ we denote by $V^{(1)}$ the vector space over $K$ that coincides with $V$ as an additive group and where the scalar multiplication is given by

$$a \cdot v = p^{\frac{1}{p}} a v$$

for all $a \in K$, $v \in V$

where the left hand side is the new multiplication and the right hand side the old one. (Since $K$ is algebraically closed, the Frobenius morphism $a \mapsto a^p$ is a bijection $K \to K$, so any element in $K$ has a unique $p$–th root.)
If $V$ is a $G$–module, then $V^{(1)}$ is a $G$–module using the given action of any $g \in G$ on the additive group $V^{(1)} = V$. The action is again $K$–linear. A finite dimensional $G$–stable subspace of $V$ is also a finite dimensional $G$–stable subspace of $V^{(1)}$ (of the same dimension); so also $V^{(1)}$ is locally finite. For any $\varphi \in V^*$ the map

$\varphi^{(1)}: V^{(1)} \to K, \quad v \mapsto \varphi(v)^p$

is $K$–linear. One checks that $(V^{(1)})^* = \{ (\varphi^{(1)} \mid \varphi \in V^* \}$ and gets finally for all $g \in G$

$c_{\varphi^{(1)},v}(g) = \varphi^{(1)}(gv) = \varphi(gv)^p = c_{\varphi,v}(g)^p$

hence $c_{\varphi^{(1)},v} = c_{\varphi,v}^p \in K[G].$

One defines higher Frobenius twists inductively: Set $V^{(r+1)} = (V^{(r)})^{(1)}$.

**Exercise:** Consider $K[G]$ as a $G$–module under the (left) regular representation. Show that $f \mapsto f^p$ is a homomorphism $K[G]^{(1)} \to K[G]$ of $G$–modules.

**Induction.** Let $H$ be a closed subgroup of an algebraic group $G$. Any $G$–module $V$ is by restriction an $H$–module since the restriction to $H$ of any function in $K[G]$ belongs to $K[H]$. This restriction of modules is clearly a functor from $G$–modules to $H$–modules. We can define an induction functor $\text{ind}^G_H$ in the opposite direction, from $H$–modules to $G$–modules. For any $H$–module $M$ set

$$\text{ind}^G_H M = \{ f: G \to M \text{ regular} \mid f(gh) = h^{-1} f(g) \text{ for all } g \in G, h \in H \}.$$  

Here a function $f: G \to M$ is called regular if there exists $m_1, m_2, \ldots, m_r \in M$ and $f_1, f_2, \ldots, f_r \in K[G]$ such that $f(g) = \sum_{i=1}^r f_i(g) m_i$ for all $g \in G$. All regular functions $G \to M$ form a vector space over $K$ and $\text{ind}^G_H M$ is then a subspace. We get a $G$–action on this subspace by setting $(gf)(g') = f(g^{-1} g')$. In order to show that this $KG$–module is a $G$–module one uses the fact that $K[G]$ is a $G$–module.

A trivial example for an induced module is $\text{ind}^G_{\{1\}} = K[G]$. We’ll soon see more interesting examples for $G = \text{SL}_2(K)$.

The functor $\text{ind}^G_H$ is right adjoint to the restriction functor from $G$–modules to $H$–modules. This means that we have functorial isomorphisms

$$\text{Hom}_G(V, \text{ind}^G_H M) \xrightarrow{\sim} \text{Hom}_H(V, M)$$

Here any $G$–homomorphism $\varphi: V \to \text{ind}^G_H M$ is sent to $\overline{\varphi}: V \to M$ with $\overline{\varphi}(v) = \varphi(v) (1)$. Conversely any $H$–homomorphism $\psi: V \to M$ goes to $\overline{\psi}: V \to \text{ind}^G_H M$ such that $\overline{\psi}(v) (g) = \psi(g^{-1} v)$.

**Example:** $\text{SL}_2(K)$. Consider $G = \text{SL}_2(K)$ acting on its natural module $V = K^2$ with its standard basis $e_1, e_2$. Then $G$ acts also on $V^*$ and on the symmetric algebra $S(V^*)$ that we can identify with the algebra $K[X_1, X_2]$ of polynomial functions on $V$. Here $X_1, X_2$ is the basis of $V^*$ dual to $e_1, e_2$.  

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The action of $G$ on $S(V^*)$ is given by $(gf)(v) = f(g^{-1}v)$. Each symmetric power $S^n(V^*)$ is a submodule. We choose as basis all $v_i = (-1)^i X_1^i X_2^{n-i}$, $0 \leq i \leq n$, and get (e.g.) for all $a \in K$

$$
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix} v_i = \sum_{j=0}^{i} \binom{i}{j} a^{i-j} v_j \quad \text{and} \quad \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} v_i = \sum_{j=i}^{n} \binom{n-i}{n-j} a^{j-i} v_j
$$

(1)

and for all $t \in K^\times$

$$
\begin{pmatrix}
t & 0 \\
0 & t^{-1}
\end{pmatrix} v_i = t^{n-2i} v_i.
$$

(2)

Consider the closed subgroups in $G$

$$
U = \{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mid a \in K \} \quad \text{and} \quad B = \{ \begin{pmatrix} t & 0 \\ a & t \end{pmatrix} \mid a \in K, t \in K^\times \}.
$$

Let us first observe that we have an isomorphism of $G$–modules

$$
S(V^*) \overset{\sim}{\to} \text{ind}_U^G K
$$

where $K$ is the trivial $U$–module such that

$$
\text{ind}_U^G K = \{ f \in K[G] \mid f(gu) = f(g) \text{ for all } g \in G, u \in U \}.
$$

Well, any $\psi \in S(V^*)$ defines $\hat{\psi} : G \to K$ setting $\hat{\psi}(g) = \psi(ge_2)$. One checks that $\hat{\psi} \in \text{ind}_U^G K$ and that $\psi \mapsto \hat{\psi}$ is a morphism of $G$–modules. On the other hand, any $f \in \text{ind}_U^G K$ defines $\overline{f} : V \setminus \{0\} \to K$: For any $v \in V$, $v \neq 0$ there exists $g \in G$ with $v = ge_2$; we then set $\overline{f}(v) = f(g)$. This is well defined as $ge_2 = g'e_2$ implies $g' \in gU$, hence $f(g) = f(g')$ since $f \in \text{ind}_U^G K$. One checks that $\overline{f}$ is a regular function on $V \setminus \{0\}$. Algebraic geometry tells you that $\overline{f}$ has a (unique) extension to a regular function on $V$. So there exists $\hat{\psi} \in S(V^*)$ with $\overline{f}(v) = \psi(v)$ for all $v \neq 0$, hence with $f(g) = \psi(ge_2)$ for all $g \in G$, i.e., with $f = \hat{\psi}$.

For each $n \in \mathbb{Z}$ let $K_n$ denote $K$ considered as a $B$–module such that

$$
\begin{pmatrix} t & 0 \\ a & t^{-1} \end{pmatrix} v = t^n v \quad \text{for all } a, t, v \in K, t \neq 0.
$$

Since any $u \in U$ acts trivially on any $K_n$, it is clear that $\text{ind}_B^G K_n \subset \text{ind}_U^G K$. The inverse image of $\text{ind}_B^G K_n$ under the isomorphism $S(V^*) \simeq \text{ind}_U^G K$ from above consists of all $\psi \in S(V^*)$ with $\psi(tv) = t^n \psi(v)$ for all $t \in K^\times$ and $v \in V$. This shows that we get isomorphisms

$$
\text{ind}_B^G K_n \overset{\sim}{\to} \begin{cases} S^n(V^*) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}
$$

**Simple modules for $\text{SL}_2(K)$.** Consider again $G = \text{SL}_2(K)$. If $\text{char } K = 0$, then one can show that all $S^n(V^*)$ with $n \in \mathbb{N}$ are simple $G$–modules. Using (2) above one checks that
each non-zero submodule contains some \(v_i\). Then (1) shows that the submodule contains all \(v_j\). Here one uses that all binomial coefficients are non-zero in \(K\). Furthermore, each simple \(G\)-module is isomorphic to \(S^n(V^*)\) for a unique \(n\).

If \(\text{char } K = p > 0\), then the same argument shows that all \(S^n(V^*)\) with \(0 \leq n < p\) are simple \(G\)-modules. But one can quickly see that \(S^p(V^*)\) is not simple: The map

\[
\varphi: V^* \longrightarrow S^p(V^*), \quad f \mapsto f^p
\]

is semi-linear (i.e., satisfies \(\varphi(f + f') = \varphi(f) + \varphi(f')\) and \(\varphi(a f) = a^p \varphi(f)\) for all \(a \in K\)) and \(G\)-equivariant. So its image is a \(G\)-submodule of \(S^p(V^*)\) that is non-zero and not the whole module.

Going back to the definition of a Frobenius twist one checks that \(\varphi\) is an isomorphism of \(G\)-modules

\[
(V^*)^{(1)} \sim \longrightarrow KG X_2^p
\]

onto the submodule of \(S^p(V^*)\) generated by \(X_2^p\).

For general \(n\) the characteristic 0 argument shows first that each non-zero submodule of \(S^n(V^*)\) contains some \(v_i\). Then the first formula in (1) implies that the submodule contains \(v_0 = X_2^n\). This shows that the submodule

\[
L(n) := KG v_0 = KG X_2^n
\]

of \(S^n(V^*)\) generated by \(X_2^n\) is simple, in fact, it is the only simple submodule of \(S^n(V^*)\).

It then turns out that each simple \(G\)-module is isomorphic to \(L(n)\) for exactly one \(n\).

One can describe \(L(n)\) alternatively as follows. Write \(n\) in \(p\)-adic expansion:

\[
n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r \quad \text{with } 0 \leq n_i < p \text{ for all } i.
\]

Then the map

\[
f_0 \otimes f_1 \otimes f_2 \otimes \cdots \otimes f_r \mapsto f_0 f_1^p f_2^{p^2} \cdots f_r^{p^r}
\]

is an isomorphism of \(G\)-modules

\[
S^{n_0}(V^*) \otimes S^{n_1}(V^*)^{(1)} \otimes S^{n_2}(V^*)^{(2)} \otimes \cdots \otimes S^{n_r}(V^*)^{(r)} \sim \longrightarrow L(n).
\]

**Notations.** Let \(G\) be a connected reductive group over \(K\). Choose a maximal torus \(T\) in \(G\). We want to generalise the \(\text{SL}_2\)-example. This involves some notation:

- \(X = \text{Hom}_{\text{alg.gr}}(T, K^\times)\), the character group of \(T\)
- \(X^\vee = \text{Hom}_{\text{alg.gr}}(K^\times, T)\), the cocharacter group of \(T\)
- \(\langle , \rangle\), the pairing \(X \times X^\vee \rightarrow \mathbb{Z}\) such that \(\lambda(\varphi(t)) = t(\lambda, \varphi)\) for all \(t \in K^\times\)
- \(\Phi \subset X\), the root system of \(G\) with respect to \(T\)
- \(\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\} \subset X^\vee\) the corresponding dual roots
- \(\Phi^+\) a chosen system of positive roots
• $B$ and $B^+$ the Borel subgroups of $G$ containing $T$ that correspond to $-\Phi^+$ and $\Phi^+$ respectively

• $U$ and $U^+$ the unipotent radicals of these Borel subgroups. So we have semi-direct decompositions $B = TU$ and $B^+ = TU^+$ as algebraic groups.

Each $\lambda \in X$ can be extended to a character $\lambda: B \rightarrow K^\times$ such that $\lambda(tu) = \lambda(t)$ for all $t \in T$ and $u \in U$. Denote by $K_\lambda$ the corresponding one dimensional $B$–module. The general theory of connected soluble algebraic groups yields:

**Proposition:** Each simple $B$–module is isomorphic to $K_\lambda$ for a unique $\lambda \in X$.

A corresponding statement holds for $B^+$ instead of $B$.

**Simple $G$–modules.** Any $G$–module is by definition locally finite, i.e., a union of finite dimensional submodules. Therefore any simple $G$–module $V$ is finite dimensional. Considered as a $B$–module it then has a composition series. This implies in particular that there is a surjective homomorphism of $B$–modules from $V$ onto a simple $B$–module, hence by the proposition above onto some $K_\lambda$ with $\lambda \in X$. The basic property of induction implies then

$$\text{Hom}_G(V, \text{ind}^G_B K_\lambda) \simeq \text{Hom}_B(V, K_\lambda) \neq 0.$$ 

Using the simplicity of $V$ we get now:

**Claim 1:** Each simple $G$–module is isomorphic to a simple submodule of some $\text{ind}^G_B K_\lambda$ with $\lambda \in X$.

In the case of $\text{SL}_2(K)$ this implies that any simple module is (as claimed above) isomorphic to some $L(n)$.

The next step towards a classification of the simple modules requires more subtle arguments that I cannot discuss here. They yield:

**Claim 2:** $\text{ind}^G_B K_\lambda \neq 0 \iff \lambda$ dominant

where the set $X_+$ of dominant characters is defined by

$$X_+ = \{ \lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+ \}.$$ 

As a third step we want to see that each $\text{ind}^G_B K_\lambda$ with $\lambda$ dominant contains a unique simple submodule. This is done by looking at fixed points for the subgroup $U^+$. For any non-zero $U^+$–module $M$ the space of fixed points

$$M^{U^+} = \{ x \in M \mid u x = x \text{ for all } u \in U^+ \}$$

is non-zero as $U^+$ is connected and unipotent.

The definition of $\text{ind}^G_B K_\lambda$ and of the $G$–action on this space shows for any $f \in (\text{ind}^G_B K_\lambda)^{U^+}$ that

$$f(u_1 tu_2) = \lambda(t)^{-1} f(1) \quad \text{for all } u_1 \in U^+, \, t \in T, \, u_2 \in U.$$
So the restriction of $f$ to the subset $U^+TU$ of $G$ is determined by $f(1)$. As $U^+TU$ is dense in $G$, it follows that $f$ itself is determined by $f(1)$. This implies that 

$$(\text{ind}^G_K K\lambda)^{U^+} = K f_\lambda$$

where $f_\lambda \in K[G]$ is the unique regular function on $G$ with 

$$f_\lambda(u_1 t u_2) = \lambda(t)^{-1} \quad \text{for all } u_1 \in U^+, \ t \in T, \ u_2 \in U.$$ 

Now every non-zero $G$-submodule $M$ of $\text{ind}^G_K K\lambda$ satisfies $M^{U^+} \neq 0$, hence $f_\lambda \in M$ and $KG f_\lambda \subset M$. It follows that 

$$L(\lambda) := KG f_\lambda$$

is the unique simple submodule of $\text{ind}^G_K K\lambda$.

**Theorem:** Each simple $G$-module is isomorphic to $L(\lambda)$ for a unique $\lambda \in X_+$. 

In order to get uniqueness one notes that $L(\lambda)^{U^+} = K f_\lambda$ and that $T$ acts on this subspace via $\lambda$.

In case $\text{char } K = 0$ one can show that $L(\lambda) = \text{ind}^G_K K\lambda$ for all $\lambda \in X_+$. The examples above show that this fails in prime characteristic already for $G = \text{SL}_2(K)$.

**Steinberg’s tensor product theorem.** In order to simplify notation assume now that $G$ is semi-simple and simply connected. Our choice of positive roots $\Phi^+$ determines a system of simple roots. The corresponding dual roots form under our assumption a basis for the free abelian group $X^\vee$ and $X$ has then a dual basis consisting of fundamental weight. For any integer $m > 0$ the set 

$$X_m = \{ \lambda \in X \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < m \text{ for all simple roots } \alpha \} \subset X_+$$

consists of those characters whose coefficients with respect to the basis of fundamental weights belong to the interval from 0 to $m - 1$.

Suppose now that $\text{char } K = p > 0$. Any $\lambda \in X_+$ has a $p$-adic expansion 

$$\lambda = \lambda_0 + p \lambda_1 + p^2 \lambda_2 + \cdots + p^r \lambda_r \quad \text{with all } \lambda_i \in X_{p^i}.$$ 

Now Steinberg’s tensor product theorem says (generalising the result for $\text{SL}_2$) that we have an isomorphism 

$$L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes L(\lambda_2)^{(2)} \otimes \cdots \otimes L(\lambda_r)^{(r)} \sim L(\lambda).$$

Note that each $L(\lambda_i) \subset \text{ind}^G_K K\lambda_i \subset K[G]$ is a submodule of $K[G]$; similarly for $L(\lambda)$. The isomorphism is then simply given by 

$$f_0 \otimes f_1 \otimes f_2 \otimes \cdots \otimes f_r \mapsto f_0 f_1^p f_2^{p^2} \cdots f_r^{p^r}$$

**Application to finite groups of Lie type.** Assume again that $G$ is semi-simple and simply connected. Suppose that $\text{char } K = p > 0$ and that $G$ is defined over a finite subfield $F_q$ of $K$. (So $q = p^n$ for some $n > 0$.) The group $G(F_q)$ of points of $G$ over $F_q$ is then a finite group of Lie type. We can describe $G(F_q)$ as the fixed points of a suitable Frobenius map on $G$. 

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In this situation a theorem due to Curtis for $q = p$ and to Steinberg in general says:

**Theorem:** Each $L(\lambda)$ with $\lambda \in X_q$ is still simple when considered as a $K G(F_q)$–module. Any simple $K G(F_q)$–module is isomorphic to $L(\lambda)$ for a unique $\lambda \in X_q$.

Note that the Ree and Suzuki groups are finite groups of Lie type, which are not of the form $G(F_q)$. However, a modification of this theorem yields also a classification of their irreducible representations over an algebraically closed field of their defining characteristic.

**References.** For a more detailed survey of the material in Lectures I and II see my article *Modular representations of reductive groups* in the proceedings of a conference in Beijing published under the title *Group Theory, Beijing 1984* as volume 1185 of the Springer Lecture Notes in Mathematics.