Representations of algebraic groups and their Lie algebras

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Lecture II

Set-up. Let K be again an algebraically closed field and G a connected and simply connected semi-simple group over K. Recall the crucial notations from Lecture I:

- $X = \mathrm{Hom}_{\mathrm{alg.gr}}(T, K^\times)$, the character group of T
- $X^\vee = \mathrm{Hom}_{\mathrm{alg.gr}}(K^\times, T)$, the cocharacter group of T
- $\langle\ ,\ \rangle$, the pairing $X \times X^\vee \to \mathbf{Z}$ such that $\lambda(\varphi(t)) = t^{\langle \lambda, \varphi \rangle}$ for all $t \in K^\times$
- $\Phi \subset X$, the root system of G with respect to T
- $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\} \subset X^\vee$, the corresponding dual roots
- Φ^+, a chosen system of positive roots
- B, the Borel subgroup of G containing T that corresponds to $-\Phi^+$
- U, the unipotent radical of B. So we have a semi-direct decomposition $B = T\,U$ as algebraic group.
- X_+, the set of dominant characters with respect to Φ^+
- $L(\lambda)$, the unique simple submodule of $\mathrm{ind}_B^G K_\lambda$, for any $\lambda \in X_+$

Here are two additional notations that will soon play an important role:

- W, the Weyl group of G with respect to T
- ρ, half the sum of the positive roots. Under our assumption we have $\rho \in X$.

Induction and sheaves. Our first examples for induced modules led to infinite dimensional modules: $\mathrm{ind}_{\{1\}}^G K = K[G]$ and in the SL_2 case $\mathrm{ind}_U^G K \simeq S(V^*)$. But then we got again for SL_2 finite dimensional spaces: $\mathrm{ind}_B^G K_n \simeq S^n(V^*)$.

The reason for this different behaviour is the fact that G/B is a projective variety whereas in the earlier examples G/H was not projective. Take for example for general (semi-simple etc.) G the trivial B–module $K = K_0$. Then

$$\mathrm{ind}_B^G K = \{\, f \in K[G] \mid f(gb) = f(g) \text{ for all } g \in G \text{ and } b \in B \,\}$$

can be identified with the algebra of regular functions on G/B: Map any f as above to $\overline{f}$ with $\overline{f}(gB) = f(g)$. Now G/B is projective and connected, so algebraic geometry tells us that the only regular functions on G/B are the constant functions. So we get

$$\mathrm{ind}_B^G K = K$$

and hence $L(0) = K$.

In order to see that all $\operatorname{ind}_B^G K_\lambda$ are finite dimensional we introduce certain sheaves. Let $\pi\colon G \to G/B$ denote the canonical map with $\pi(g) = gB$. For any open subset Z in G/B set (for any $\lambda \in X$)

$$\mathcal{L}_\lambda(Z) = \{\, f \in K[G] \mid f(gb) = \lambda(b)^{-1} f(g) \text{ for all } g \in \pi^{-1}Z \text{ and } b \in B \,\}.$$

Then $\mathcal{L}_\lambda$ is a sheaf on G/B, in fact a module over the sheaf $\mathcal{O}_{G/B}$ of regular functions on G/B: For any $f_1 \in \mathcal{O}_{G/B}(Z)$ and $f_2 \in \mathcal{L}_\lambda(Z)$ the product $f_1 f_2$ is the function $g \mapsto f_1(gB)\, f_2(g)$ on $\pi^{-1}Z$. It is clear by definition that $\operatorname{ind}_B^G K_\lambda$ is just the space of global sections of $\mathcal{L}_\lambda$:

$$\operatorname{ind}_B^G K_\lambda = \mathcal{L}_\lambda(G/B) = H^0(G/B, \mathcal{L}_\lambda).$$

The sheaf $\mathcal{L}_\lambda$ is actually a locally free module of rank 1 over $\mathcal{O}_{G/B}$: The map π is locally trivial. This means that G/B has a covering by open subsets Z such that there exists a section $\sigma\colon Z \to \pi^{-1}Z$ with $\sigma(z)\,B = z$ for all $z \in Z$ such that the map $(z,b) \mapsto \sigma(z)\,b$ is an isomorphisms of varieties $Z \times B \xrightarrow{\sim} \pi^{-1}Z$. For such Z we have an isomorphism

$$\mathcal{O}_{G/B}(Z) \xrightarrow{\sim} \mathcal{L}_\lambda(Z), \qquad f \mapsto \widehat{f} \quad \text{with } \widehat{f}(\sigma(z)\,b) = \lambda(b)^{-1} f(z).$$

So $\mathcal{L}_\lambda$ is indeed locally free of rank 1 over $\mathcal{O}_{G/B}$.

Now general results from algebraic geometry say that the space of global sections of $\mathcal{L}_\lambda$ is finite dimensional, i.e., that

$$\dim \operatorname{ind}_B^G K_\lambda < \infty \qquad \text{for all } \lambda \in X.$$

Derived functors of induction and sheaf cohomology. The results from algebraic geometry used a moment ago say not only that the global sections of our sheaves are finite dimensional. They imply the same for all cohomology groups:

$$\dim H^i(G/B, \mathcal{L}_\lambda) < \infty \qquad \text{for all } i \in \mathbf{N}$$

and they imply also that only finitely many of these groups can be non-zero:

$$H^i(G/B, \mathcal{L}_\lambda) = 0 \qquad \text{for all } i > N := \dim G/B = |\Phi^+|.$$

It turns out that all $H^i(G/B, \mathcal{L}_\lambda)$ have a natural structure as a G–module, not only for $i = 0$ where $H^0(G/B, \mathcal{L}_\lambda) \simeq \operatorname{ind}_B^G K_\lambda$. The $H^i(G/B, \ )$ are the derived functors of the global section functor $H^0(G/B, \ )$. In a similar way ind_B^G has derived functors, and they yield the G–structures on the $H^i(G/B, \mathcal{L}_\lambda)$.

For this to make sense we have to observe for any algebraic group H that the category of H–modules contains enough injective objects. Using the adjointness of induction and restriction one checks for any vector space M over K that $\operatorname{ind}_{\{1\}}^H M$ is injective in the category of all H–modules. If V is an H–module then there is a homomorphism $\varphi\colon V \to \operatorname{ind}_{\{1\}}^H V$ that corresponds to the identity map $V \to V$ under adjunction. One then checks that φ is injective. So every H–module is isomorphic to a submodule of an injective H–module.

We apply this to $H = B$ and get for each B–module M an injective resolution

$$0 \to M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

in the category of B–modules. Applying ind_B^G we get a complex of G–module

$$\operatorname{ind}_B^G I_0 \longrightarrow \operatorname{ind}_B^G I_1 \longrightarrow \operatorname{ind}_B^G I_2 \longrightarrow \cdots$$

and the i–th cohomology module of the complex is the value $R^i \operatorname{ind}_B^G M$ of the i–th derived functor of ind_B^G. By general arguments these modules are independent of the chosen resolution and the $R^i \operatorname{ind}_B^G$ are indeed functors. One gets that $R^0 \operatorname{ind}_B^G = \operatorname{ind}_B^G$; here one has to check that ind_B^G is left exact. And finally one shows that we have isomorphisms as vector spaces

$$R^i \operatorname{ind}_B^G K_\lambda \simeq H^i(G/B, \mathcal{L}_\lambda) \qquad \text{for all } \lambda \text{ and all } i.$$

We can now restate the results above on the cohomology groups as

$$\dim R^i \operatorname{ind}_B^G K_\lambda < \infty \qquad \text{for all } i \in \mathbf{N}$$

and

$$R^i \operatorname{ind}_B^G K_\lambda = 0 \qquad \text{for all } i > N = |\Phi^+|.$$

One can also check that Serre duality is compatible with the action of G. This means that we have isomorphisms of G–modules

$$(R^i \operatorname{ind}_B^G K_\lambda)^* \;\simeq\; R^{N-i} \operatorname{ind}_B^G K_{-\lambda-2\rho}$$

for all $\lambda \in X$ and all i, $0 \le i \le N$.

Formal characters. Any G–module V decomposes under the action of T into a direct sum of *weight spaces*:

$$V = \bigoplus_{\lambda \in X} V_\lambda \qquad \text{where } \; V_\lambda = \{\, v \in V \mid t\,v = \lambda(t)\,v \text{ for all } t \in T\,\}.$$

If $\dim V < \infty$, then we associate to V its *formal character*:

$$\operatorname{ch} V = \sum_{\lambda \in X} \dim(V_\lambda)\, e(\lambda) \in \mathbf{Z}[X]$$

where $(e(\lambda) \mid \lambda \in X)$ is the standard basis of the group ring $\mathbf{Z}[X]$ of X.

If now V' is a submodule of V, then $\operatorname{ch} V = \operatorname{ch} V' + \operatorname{ch} V/V'$. If also M is a finite dimensional G–module, then $\operatorname{ch} V \otimes M = \operatorname{ch} V \cdot \operatorname{ch} M$. In case $\operatorname{char} K = p > 0$ one gets the character of a Frobenius twist using the rule

$$\operatorname{ch} V = \sum_\lambda a_\lambda e(\lambda) \implies \operatorname{ch} V^{(1)} = \sum_\lambda a_\lambda e(p\,\lambda).$$

The formal character of V determines the usual character of V: Every semi-simple $g \in G$ is conjugate in G to some $t \in T$; then the trace of g acting on V is equal to $\sum_{\lambda \in X} \dim(V_\lambda)\lambda(t)$. For an arbitrary element h in G its trace on V is equal to the trace of the semi-simple component of h under Jordan decomposition.

Suppose for the moment that $\operatorname{char} K = p > 0$. If Γ is a finite subgroup of G — for example, a group of type $G(\mathbf{F}_q)$ — then the formal character $\operatorname{ch} V$ determines also the Brauer character of V as a $K\Gamma$–module: Each p–regular element $g \in \Gamma$ is semi-simple, hence conjugate in G (not necessarily in Γ) to some $t \in T$. Then the eigenvalues of g acting on V are the $\lambda(t)$ with $V_\lambda \neq 0$; each $\lambda \in X$ contributes $\dim V_\lambda$ eigenvalues $\lambda(t)$.

Euler characteristics. The main problem is to find the formal characters of the simple modules $L(\lambda)$, $\lambda \in X$. One tries to do this with the help of the formal characters of the induced modules $\operatorname{ind}_B^G K_\lambda$ which makes sense by the finite dimensionality results above. In fact, these results show that all $R^i \operatorname{ind}_B^G K_\lambda$ with $\lambda \in X$ and $i \in \mathbf{N}$ have a formal character. It turns out to look at the Euler characteristic

$$\chi(\lambda) := \sum_{i=0}^{N} (-1)^i \operatorname{ch} R^i \operatorname{ind}_B^G K_\lambda.$$

These alternating sums are independent of K whereas the individual summands can depend on $\operatorname{char} K$.

Theorem: *One has for all* $\lambda \in X$

$$\chi(\lambda) = \frac{\sum_{w \in W} \det(w)\, e(w\,(\lambda + \rho))}{\sum_{w \in W} \det(w)\, e(w\,\rho)} \;.$$

For $\lambda \in X_+$ the right hand side is *Weyl's character formula* for simple G–modules in case $K = \mathbf{C}$. In characteristic 0 the theorem follows from Weyl's work together with the results presented below in the next subsection. One can then deduce them for any K by showing a priori that $\chi(\lambda)$ is independent of K. Direct proofs of the theorem were given by Donkin and Andersen. They use Kempf's vanishing theorem quoted below.

Characteristic 0. In case $K = \mathbf{C}$ Borel & Weil and Bott have determined all $R^i \operatorname{ind}_B^G K_\lambda$. In order to formulate their results we use the "dot action"

$$w{\bullet}\lambda = w(\lambda + \rho) - \rho$$

that later on will be used over any field.

Theorem: *Suppose that* $\operatorname{char} K = 0$. *Let* $\lambda \in X$.

(a) *If* $\lambda \in X_+$, *then* $L(\lambda) = \operatorname{ind}_B^G K_\lambda$.

(b) *If there exists a root* $\alpha \in \Phi$ *with* $\langle \lambda + \rho, \alpha^\vee \rangle = 0$, *then* $R^i \operatorname{ind}_B^G K_\lambda = 0$ *for all* $i \in \mathbf{N}$.

(c) *If* $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ *for all* $\alpha \in \Phi$, *then there exists a unique* $w \in W$ *with* $w{\bullet}\lambda \in X_+$. *We have then* $R^{l(w)} \operatorname{ind}_B^G K_\lambda \simeq L(w{\bullet}\lambda)$ *and* $R^i \operatorname{ind}_B^G K_\lambda = 0$ *for all* $i \neq l(w)$.

Here $l(w)$ is the length of w with respect to the generating system of W consisting of the reflections with respect to the simple roots.

Note that (a) and (c) imply in characteristic 0 that $\operatorname{ch} L(\lambda) = \chi(\lambda)$ for all $\lambda \in X_+$. So this theorem together with the preceding one implies Weyl's character formula.

Prime characteristic. Suppose from now on that $\operatorname{char} K = p > 0$. We have seen examples for SL_2 where $L(\lambda) \neq \operatorname{ind}_B^G K_\lambda$. One can find for SL_3 characters λ where $R^i \operatorname{ind}_B^G K_\lambda \neq 0$ for more than one i. So the characteristic 0 results definitely do not extend to K. However, we have at least:

Kempf's vanishing theorem: *If* $\lambda \in X_+$*, then* $R^i \operatorname{ind}_B^G K_\lambda = 0$ *for all* $i > 0$*.*

This implies for all $\lambda \in X_+$ that $\operatorname{ch} \operatorname{ind}_B^G K_\lambda = \chi(\lambda)$. It follows that

$$\dim L(\lambda)_\lambda = \dim (\operatorname{ind}_B^G K_\lambda)_\lambda = 1$$

and for all $\mu \in X$ that

$$L(\lambda)_\mu \neq 0 \implies (\operatorname{ind}_B^G K_\lambda)_\mu \neq 0 \implies \mu \leq \lambda$$

where the (partial) ordering $\leq$ on X is defined by

$$\mu \leq \nu \iff \nu - \mu \in \mathbf{N}\Phi^+ .$$

Therefore $L(\lambda)$ is called the simple G–module with *highest weight* λ.

These results imply that $L(\lambda)$ is a composition factor of $\operatorname{ind}_B^G K_\lambda$ with multiplicity 1; any other composition factor $L(\mu)$ of $\operatorname{ind}_B^G K_\lambda$ satisfies $\mu < \lambda$. If we denote by $b_{\lambda\mu}$ the multiplicity of $L(\mu)$ as a composition factor of $\operatorname{ind}_B^G K_\lambda$, then we get

$$\operatorname{ch} L(\lambda) \;=\; \chi(\lambda) - \sum_{\mu<\lambda} b_{\lambda\mu} \operatorname{ch} L(\mu).$$

It follows inductively that there are integers $a_{\lambda\mu}$ such that

$$\operatorname{ch} L(\lambda) \;=\; \sum_{\mu\leq\lambda} a_{\lambda\mu}\chi(\mu)$$

with $a_{\lambda\lambda} = 1$.

Examples for SL_3. Let us illustrate these general facts by some simple examples in case $G = \mathrm{SL}_3(K)$. Let ϖ_1, ϖ_2 denote the two fundamental weights in this case; we write $(r, s) = r\varpi_1 + s\varpi_2$. Note that $\rho = (1, 1)$. Let $\lambda \in X_+$. It will convenient to write $\lambda + \rho = (r, s)$ where now $r, s > 0$.

One finds that

$$r + s \leq p \implies L(\lambda) = \operatorname{ind}_B^G K_\lambda \tag{1}$$

and hence $\operatorname{ch} L(\lambda) = \chi(\lambda)$ in this case.

If $r + s > p$, but $r, s < p$, then one gets a short exact sequence

$$0 \to L(\lambda) \longrightarrow \operatorname{ind}_B^G K_\lambda \longrightarrow L(\lambda') \to 0$$

where $\lambda' + \rho = (p-s, p-r)$. It follows that

$$\operatorname{ch} L(\lambda) = \chi(\lambda) - \operatorname{ch} L(\lambda') = \chi(\lambda) - \chi(\lambda') \tag{2}$$

where the second equality follows from (1) applied to λ'.

If $r = p$ and $s \leq p$ or if $s = p$ and $r \leq p$, then one gets $L(\lambda) = \operatorname{ind}_B^G K_\lambda$ and thus $\operatorname{ch} L(\lambda) = \chi(\lambda)$.

The examples so far cover all $\lambda \in X_p$ for SL_3. Together with Steinberg's tensor product theorem they allow the determination of the formal characters for all simple modules.

Let us, however, add one more explicit example. Suppose that $r > p$ and $r + s < 2p$. Then there is a short exact sequence

$$0 \to L(\lambda) \longrightarrow \operatorname{ind}_B^G K_\lambda \longrightarrow L(\lambda_1) \to 0$$

where $\lambda_1 + \rho = (2p - r, r + s - p)$. We can apply (2) to λ_1 and get

$$\operatorname{ch} L(\lambda_1) = \chi(\lambda_1) - \chi(\lambda_2) \qquad \text{where } \lambda_2 + \rho = (2p - (r+s), r - p),$$

hence

$$\operatorname{ch} L(\lambda) = \chi(\lambda) - \operatorname{ch} L(\lambda_1) = \chi(\lambda) - \chi(\lambda_1) + \chi(\lambda_2). \tag{3}$$

Linkage principle. For any root $\alpha \in \Phi$ denote by $s_\alpha \in W$ the corresponding reflection. If we denote the two simple roots in the SL_3 case by α_1 and α_2, then we have above in (3)

$$\lambda_1 + \rho = s_{\alpha_1}(\lambda + \rho) + p\alpha_1 \qquad \text{and} \qquad \lambda_2 + \rho = s_{\alpha_1+\alpha_2}(\lambda_1 + \rho) + p(\alpha_1 + \alpha_2)$$

or, using the dot action

$$\lambda_1 = s_{\alpha_1} \bullet \lambda + p\alpha_1 \qquad \text{and} \qquad \lambda_2 = s_{\alpha_1+\alpha_2} \bullet \lambda_1 + p(\alpha_1 + \alpha_2).$$

These formulae illustrate a general fact. For arbitrary G denote by W_p the group acting on X (or on $X \otimes_{\mathbf{Z}} \mathbf{R}$) generated by W and by the translations with all $p\alpha$, $\alpha \in \Phi$. We call W_p the *affine Weyl group*. It isomorphic to what is otherwise called the affine Weyl group of the dual root system $\Phi^\vee$.

Now the linkage principle says:

Theorem: *If $b_{\lambda\mu} \neq 0$ or $a_{\lambda\mu} \neq 0$, then $\mu \in W_p \bullet \lambda$.*

There is also a *strong linkage principle* that adds further restrictions on μ.

This principle was first proved by Humphreys for large p (greater that the Coxeter number of Φ) and then by Kac and Weisfeiler for all good primes. They worked with the centre of the enveloping algebra of $\operatorname{Lie} G$. A general proof by Andersen uses the derived functors of induction. Using Serre duality one sees that $L(\lambda)$ is not only a submodule of $\operatorname{ind}_B^G K_\lambda$, but also a homomorphic image of $R^N \operatorname{ind}_B^G K_{w_0\bullet\lambda}$ where $w_0 \in W$ is the unique element with $w_0(\Phi^+) = -\Phi^+$. So there is a homomorphism of G–modules

$$R^N \operatorname{ind}_B^G K_{w_0\bullet\lambda} \longrightarrow \operatorname{ind}_B^G K_\lambda \qquad \text{with image } L(\lambda).$$

It turns out that this map factors

$$R^N \operatorname{ind}_B^G K_{w_0\bullet\lambda} \to R^{N-1} \operatorname{ind}_B^G K_{w_1\bullet\lambda} \to \cdots \to R^1 \operatorname{ind}_B^G K_{w_{N-1}\bullet\lambda} \to \operatorname{ind}_B^G K_\lambda$$

with suitable $w_i \in W$, $1 \leq i < N$. A careful analysis of the maps

$$R^{i+1} \operatorname{ind}_B^G K_{w_{N-i-1}\bullet\lambda} \longrightarrow R^i \operatorname{ind}_B^G K_{w_{N-i}\bullet\lambda}$$

leads to a proof not only of the linkage principle, but also of other useful results.

Translation principle. The affine Weyl group W_p acts as a reflection group on X. A fundamental domain (for the dot action) is the "closed first alcove"

$$C_p \;=\; \{\, \lambda \in X \mid 0 \leq \langle \lambda+\rho, \alpha^\vee \rangle \leq p \text{ for all } \alpha \in \Phi^+ \,\}.$$

The group W_p is a Coxeter group with the set S_p of reflections with respect to the walls of C_p as Coxeter generators.

Any weight in X_+ can be written in the form $w\bullet\lambda$ with $w \in W_p$ and (unique) $\lambda \in C_p$. The linkage principle implies then that character formula has the form

$$\operatorname{ch} L(w\bullet\lambda) \;=\; \sum_{w'} a_{w,w'}^{(\lambda)} \chi(w'\bullet\lambda) \tag{4}$$

where we sum over $w' \in W_p$ with $w'\bullet\lambda \in X_+$ and $w'\bullet\lambda \leq w\bullet\lambda$. Now w' is in general not uniquely determined by $w'\bullet\lambda$ as λ may have a non-trivial stabiliser in W_p. However, for each $\lambda' \in W_p\bullet\lambda$ there is a unique $w' \in W_p$ with $\lambda' = w'\bullet\lambda$ such that $l(w')$ is minimal for this property. (Here $l(w')$ is the length of w' with respect to the system S_p of generators.) We assume from now on that we sum in (4) only over such w' of minimal length.

The stabiliser of $\lambda \in C_p$ in W_p (always for the dot action) is determined by the two sets

$$\Phi_\lambda^0 = \{\, \alpha \in \Phi \mid \langle \lambda+\rho, \alpha^\vee \rangle = 0 \,\} \quad \text{and} \quad \Phi_\lambda^1 = \{\, \alpha \in \Phi \mid \langle \lambda+\rho, \alpha^\vee \rangle = p \,\}.$$

The translation principle (or rather a weak version of it) says now:

Theorem: *Let $\lambda, \mu \in C_p$ with $\operatorname{Stab}_{W_p}\lambda = \operatorname{Stab}_{W_p}\mu$. Then $a_{w,w'}^{(\lambda)} = a_{w,w'}^{(\mu)}$ for all w and w'.*

Note that our SL_3 examples illustrate the result.

Lusztig's conjecture. A character $\lambda \in C_p$ satisfies

$$\operatorname{Stab}_{W_p}\lambda = \{1\} \iff 0 < \langle \lambda+\rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Phi^+.$$

Such a character exists if and only if p is greater than or equal to the Coxeter number h of Φ. (For example, if $G = \mathrm{SL}_n(K)$, then $h = n$.)

Assume that $p \geq h$ and choose $\lambda \in C_p$ with $\operatorname{Stab}_{W_p}\lambda = \{1\}$; set $a_{w,w'} = a_{w,w'}^{(\lambda)}$. The translation principle as stated above says that $a_{w,w'} = a_{w,w'}^{(\mu)}$ for all $\mu \in C_p$ with $\operatorname{Stab}_{W_p}\mu = \{1\}$. A stronger version of the translation principle says that the $a_{w,w'}$ determine all $a_{w,w'}^{(\mu)}$ with $\mu \in C_p$.

About 30 years ago Lusztig stated the following

Conjecture: *Let $w \in W_p$ with $w{\bullet}\lambda \in X_+$ and $\langle \lambda+\rho, \alpha^\vee \rangle \le p(p-h+2)$ for all $\alpha \in \Phi^+$. Then each $a_{w,w'}$ is the value at -1 of a certain Kazhdan-Lusztig polynomial for W_p.*

These Kazhdan-Lusztig polynomials are constructed combinatorially within the Hecke algebra of W_p and can be computed recursively.

For $p \ge 2h-3$ the conjecture covers all $w \in W_p$ with $w{\bullet}\lambda \in X_p$. Combined with a stronger translation principle it would yield for such p all $\operatorname{ch} L(\mu)$ with $\mu \in X_p$, hence using Steinberg's tensor product theorem all $\operatorname{ch} L(\mu)$ with $\mu \in X_+$.

Any $\mu \in X_+$ can be written uniquely $\mu = \mu_0 + p\mu'$ with $\mu_0 \in X_p$ and $\mu' \in X_+$. The bound $p(p-h+2)$ in the conjecture has been chosen such that it covers only $\mu = w{\bullet}\lambda$ with $\mu' \in C_p$ which then implies $L(\mu') = \operatorname{ind}_B^G K_{\mu'}$. Steinberg's tensor product theorem implies that the conjecture cannot extend to cases where $L(\mu') \ne \operatorname{ind}_B^G K_{\mu'}$.

Since this is the reason for that bound, one may optimistically hope that the claim in the conjecture holds for all $w \in W_p$ with $w{\bullet}\lambda \in X_p$ also for $h \le p < 2h-3$. However an example by Andersen and me for $G = \mathrm{SL}_{p+3}$ shows that the natural extension of the conjecture to smaller p will fail in general.

About 15 years ago, Andersen, Soergel, and I proved that the conjecture holds for all p greater than an unknown bound depending on the type of Φ. We used representations of the Lie algebra of G (see Lecture III) to construct a "combinatorial category" that allowed a comparison with representations of quantum groups where a similar conjecture (again due to Lusztig) already had been proved.

Thus our work relied first of all on old work by Kazhdan and Lusztig that showed that the coefficients of their polynomials (in the case of W_p) are dimensions of intersection cohomology groups for Schubert varieties in the flag variety of the affine Kac-Moody group associated to the dual root system $\Phi^\vee$. This result was then used by Kashiwara and Tanisaki to prove character formulae for affine Kac-Moody algebras. Finally Kazhdan and Lusztig showed how to relate these affine Kac-Moody algebras to quantum groups.

Recently Peter Fiebig has found another approach to Lusztig's conjecture. He relates the intersection cohomology of those Schubert varieties directly our combinatorial category. His approach allows the computation of an explicit bound on p. However, his bounds are much larger that what one expects: In type A_8 he gets a 40 digits number where $p > 9$ is hoped to suffice.

References. You can find some background on Lusztig's conjecture and related topics in several of the lectures of an introductory workshop at the Newton Institute from 1997, edited by R. W. Carter and M. Geck and published in 1998 under the title *Representations of Reductive Groups* by Cambridge University Press. Fiebig's work mentioned above has not yet appeared in print, but can found on the `arkiv`.