Presentations

of

Finite Simple Groups

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Summary

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High rank groups
**Theorem** *Every finite group is finitely presented.*

There are many invariants associated with a given presentation like:

- number of generators
- number of relations
- total length of the relations

What are the minimal values these invariants?

Nice presentations have many applications in the computational group theory.
Easy Bounds

- If $G$ is finite then the number of relations is at least the number of generators.

- If $G$ is finite then the number of relations is at least the number of generators plus the minimal number of generators of the Shur multiplier.

- If $M$ is a $G$ module then the number of generators of a $G$ is at least $\dim H^1(G; M) / \dim M$.

- If $M$ is a $G$ module then the number of relations of a $G$ is at least $\dim H^2(G; M) / \dim M$.

- Some bounds for general groups can be obtained using bounds for the finite simple group.

**Question** Are these bounds exact?

**How to compute** $\sup_M \dim H^2(G; M) / \dim M$?
Interesting Questions

- What is the “smallest” presentations of a finite simple groups?
- Does there exist a presentation of $S$ with a bounded number of generators and relations, where the bound is independent on the finite simple group $S$?
- What is the “shortest” presentations of a finite simple groups?

Small presentations of FSG are used to obtain bounds of the subgroup growth group and other asymptotic invariants.
Main result

**Theorem** There exist a constant $C$ such that: Almost all non-abelian finite simple group $G$ of rank $n$ over a field with $q$ elements has a presentation with

- $C$ generators
- $2C$ relations
- length $C(\log n + \log q)$

We do not have a proof in the case of Ree groups $^2G_2$. In fact $C = 1000$ suffices.
Comments

If we do not insist of having “short” relation the bound can be significantly improved:

- The symmetric/alternating groups have presentations with 3 generators and 7 relations.
- The groups $\text{SL}_2(q)$ and $\text{PSL}_2(q)$ have presentations with 3 generators and 9 relations.
- The groups $\text{SL}_n(q)$ and $\text{PSL}_n(q)$ have presentations with 10 generators and 30 relations.
- Any non-abelian finite simple group $G$ which is not of Ree type ($^2G_2$) has presentation with 15 generators and 80 relations.
Cyclic groups

The theorem is not valid for simple groups of prime order:

- They have bounded presentations

\[ C_n = \langle x \mid x^n = 1 \rangle. \]

- They also have short presentations

\[ C_n = \langle x_0, \ldots, x_k \mid x_i^2 = x_{i+1}, \prod x_i^{a_i} = 1 \rangle \]

where \( n = \sum a_i 2^i \).

- However there are not bounded and short ones, because any presentation with \( k \) relations has length at least \( n^{1/k} \).
Holt’s Conjecture

Knowing presentations of a group $G$ with small number of relations one can obtain bounds for the cohomology groups $H^2(G; M)$:

**Theorem**  For every finite simple group $G$ every prime $p$ and every simple $\mathbb{F}_p[G]$-module $M$ we have

$$\dim H^2(G, M) \leq 1000 \dim M.$$  

This result is also valid for the Ree groups because they have a pro-finite presentations with small number of relations. Using co-homological arguments the constant 1000 can be improved to 20.  

Guralnick has conjectured that there exist a constant $C$ such that

$$\dim H^k(G, M) \leq C,$$

for any finite simple group $G$ and any irreducible module $M$. 
Standard Presentations

- Coxeter Presentation
- Burnside and CarMichael Presentations
- Gluing – Bernside Lemma
- Another presentation of \( \text{Sym}(n) \)
- Presentations of Rank 1 groups
- Curtis-Steinberg-Tits presentations of high rank groups
- The Baumslag Group

Transitive Groups

- Bounded Presentation for \( \text{Sym}(n) \) and \( \text{Alt}(n) \)
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Short Presentations

- \( \text{SL}_2(q) \)
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The Coxeter presentation of $\text{Sym}(n)$ has $n - 1$ generators and $(n^2 - n)/2$ relations:

$$\text{Sym}(n) = \langle t_1, t_2, \ldots, t_{n-1} | t_i^2, (t_i t_{i+1})^3, [t_i, t_j] \rangle$$

**Sketch of the proof:** Use induction on $n$ – any word in $t_i$-es can be re-written to one of the following:

$$w(t_1, \ldots, t_{n-2}) \quad \text{or} \quad t_i t_{i+1} \cdots t_{n-1} w(t_1, \ldots, t_{n-2}).$$

There is a variant of this presentation where the generating set consist of all transpositions $t_{ij}$, and “Coxeter” type relations.
Burnside and CarMichael Presentations

The Burnside presentation of $\text{Sym}(n)$ also has $n - 1$ generators (all transpositions with a common point) and $\sim n^3$ relations

$$\text{Sym}(n) = \langle s_1, s_2, \ldots, s_{n-1} | s_i^2, (s_i s_j)^3, (s_i s_j s_k)^2 \rangle$$

**Sketch of the proof:** Uses that any 3 generators generate $\text{Sym}(4)$ and reduce to the Coxeter presentation.

The Carmichael presentation for $\text{Alt}(n)$ has $n - 2$ generators (all 3-cycles with two common points) and $(n - 1)(n - 2)/2$ relations

$$\text{Alt}(n) = \langle c_1, c_2, \ldots, c_{n-2} | c_i^3, (c_i c_j)^2 \rangle$$
**Gluing – Bernside Lemma**

**Lemma** Let $G = \langle X \mid R \rangle$ be a presentation of the group $G$ and let $H = \langle Y \mid R' \rangle$ be a group acting on $G$ which fixes the generating set $X$. Then

$$\langle Y \cup X/H \mid R', R/H, R'' \rangle$$

is a presentation of the group $H \ltimes G$, where

- $X/H$ is a set of orbit representatives of $X$ under $H$;
- $R/H$ is a set of orbit representatives of $R$ under $H$, where we have replaced each generator by a conjugate of the orbit representative by $H$;
- $R''$ are the relations $[x, Stab_H(x)] = 1$ for each $x \in X/H$. 


Another presentation of $\text{Sym}(n)$

The Coxeter presentation of $\text{Sym}(n)$ is “almost” invariant under the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Using the Bernside Lemma we obtain the following presentation

$$\mathbb{Z}/n\mathbb{Z} \rtimes \text{Sym}(n) = \langle c, t \mid c^n, t^2, (tt^c)^3, [t, t^{ck}], (\prod t^{ci})^n \rangle$$

However we have

$$\mathbb{Z}/n\mathbb{Z} \rtimes \text{Sym}(n) = \mathbb{Z}/n\mathbb{Z} \times \text{Sym}(n),$$

which gives that

$$\text{Sym}(n) = \langle c, t \mid c^n, t^2, (tt^c)^3, [t, t^{ck}], \prod t^{ci} = c \rangle$$

$$\text{Sym}(n) = \langle c, t \mid c^n, t^2, (tt^c)^3, [t, t^{ck}], (ct)^{n-1} \rangle$$
Presentations of Rank 1 groups

A presentation of a rank 1 groups can be obtained by

- presentation of the Borel subgroup $B = T \rtimes U$;
- presentation of $N = Norm(T) \quad N = \mathbb{Z}/2\mathbb{Z} \rtimes T$;
- $|U| - 1$ relations of the form
  \[ u_0^t = u_1 h t u_2, \]
  for each $u_0 \in U \setminus \{1\}$.

It is sufficient to add one relation of the final type for each orbit of $T$ in $U \setminus \{1\}$. 


**Curtis-Steinberg-Tits presentations of high rank groups**

**Theorem**  *Up to a central extension a high rank Lie group has the presentation:*

\[ G = \langle G_\alpha | R \rangle, \]

*where \( G_\alpha \) the (rank 1) root subgroups and the relations \( R \) guarantee that any two of them generate the correct rank 2 subgroup.*

There is a “messier” form of the presentation – the generating subgroups are the torus \( T \) and root subgroups \( U_\alpha \) and the relations are

- \( T \) acts on each \( U_\alpha \),
- if \( \alpha \neq -\beta \) then the commutator \( [U_\alpha, U_\beta] \) can be expressed as product of elements in other root subgroups.

The relations of this type can be very complicated, e.g., in the case of \( ^2F_4 \).
The Baumslag Group

One of the main difficulties in obtaining bounded presentations of $SL_2(q)$ is the fact the root subgroups are elementary abelian and does not have bounded presentations.

Baumslag discovered a finitely presented group with an infinitely generated commutator subgroup:

**Lemma** The group defined by the presentation

$$G = \langle a, b, t \mid [a, b], tt^a = t^b, [t, t^a] \rangle$$

contains the group the infinitely generated abelian group $H = \langle t \rangle^G$ and $G/H \simeq \mathbb{Z}^2$.

**Sketch of the Proof:** Use induction to show that $[t, t^{a_i b_j}] = 1$. 