SUBGROUPS OF MAXIMAL RANK IN FINITE EXCEPTIONAL GROUPS OF LIE TYPE

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[Received 9 September 1990—Revised 6 September 1991]

Introduction

The purpose of this paper is to investigate a large and natural class of maximal subgroups of the finite exceptional groups of Lie type, which we call subgroups of maximal rank. These subgroups play a prominent role both in the classification of local maximal subgroups in [**9**, Theorem 1] and in the reduction theorem for subgroups in [**28**, Theorem 2]: in [**9**] it is shown that any local maximal subgroup of a finite exceptional group of Lie type is either a subgroup of maximal rank or one of a small list of exceptions; and in [**28**], using this result, it is proved that any maximal subgroup is either of maximal rank, or almost simple, or in a known list.

To describe these subgroups, we require the following notation. Let G be a simple adjoint algebraic group over an algebraically closed field K of characteristic $p>0$, and let σ be an endomorphism of G such that $L=(G_\sigma)'$ is a finite exceptional simple group of Lie type over F_q, where $q=p^a$. Let X be a group such that $F^*(X)=L$. The group $\operatorname{Aut} L$ is generated by G_σ, together with field and graph automorphisms (see [**5, 38**]), all of which extend to morphisms of the abstract group G commuting with σ. Thus there is a subgroup $\tilde{X}$ of $C_{\operatorname{Aut}(G)}(\sigma)$ such that $X=\tilde{X}/\langle\sigma\rangle$, and so X acts on the set of σ-stable subsets of G. For a σ-stable subset Y, we write $N_X(Y)$ for the stabilizer in X of Y.

If D is a σ-stable closed connected reductive subgroup of G containing a maximal torus T of G, and $M=N_X(D)$, we call M a *subgroup of maximal rank* in X. In this paper we determine the structure and conjugacy classes of those subgroups of maximal rank which are maximal in X.

THEOREM. *With the above notation, let M be a subgroup of maximal rank which is maximal in X. Take D to be maximal subject to the condition $M=N_X(D)$. Then $D_\sigma\neq 1$, $M=N_X(D_\sigma)$ and the group $N_{G_\sigma}(D_\sigma)=N_{G_\sigma}(D)$ is given in Table* 5.1 *(when $T<D$) and Table* 5.2 *(when $T=D$). (These tables can be found in* § 5 *at the end of the paper.)*

Each subgroup in Tables 5.1 *and* 5.2 *gives a maximal subgroup of the groups X satisfying the conditions specified in the tables, and each corresponds to a unique conjugacy class of subgroups in L, except for four cases with $G=F_4$ or G_2, $p=2$ or* 3, *indicated in Table* 5.1.

Section 1 of the paper contains some general results on the structure and conjugacy of subgroups of maximal rank. The constructions of the subgroups in Table 5.1 are based on standard calculations with root systems, and examples of these constructions are given in § 1. The proof of the Theorem falls naturally into

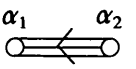
The third author acknowledges the support of an SERC Visiting Fellowship, and also an NSF Grant.

1991 *Mathematics Subject Classification*: 20G40, 20E28.

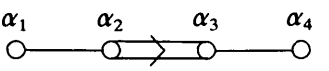
two parts, according as $T < D$ or $T = D$; these are treated in §§ 2 and 3, respectively. The beginning of § 3 contains a discussion of the groups T_σ in Table 5.2. Each T_σ has an associated diagram, as introduced by Carter in [**6**], which we call the *Carter diagram* of T_σ. Section 4 contains the proofs that the subgroups in the tables are indeed maximal subgroups, and this forms a considerable portion of the paper. The classification of finite simple groups is used in this section (but in no other). The tables can be found in the final § 5.

The maximal subgroups of the exceptional groups ${}^2B_2(q)$, ${}^3D_4(q)$ and their automorphism groups have been determined in [**39, 20**]. Thus in the proof of the Theorem, we may assume that G is of exceptional type G_2, F_4, E_6, E_7 or E_8. We use the standard labelling of these Dynkin diagrams with fundamental roots α_i as in [**4**, pp. 250–275]:

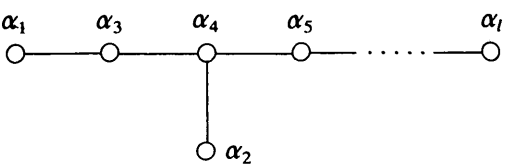
G_2



F_4



E_l $(l = 6, 7, 8)$



REMARK. In Table 5.1 we give a maximal subgroup $SU_3(2).2$ in ${}^2F_4(2)$ which does not appear in the lists given in [**11, 40**] (but does appear in [**29**]).

NOTATION IN THE TABLES. For a prime p and a positive integer n, we denote by n a cyclic group of that order, and by p^n an elementary abelian group of that order. If A, B are groups, $A.B$ denotes an extension of A by B and $A \circ B$ a central product of A and B. For $\varepsilon = \pm 1$, $E_6^\varepsilon(q)$ is $E_6(q)$ if $\varepsilon = +1$, ${}^2E_6(q)$ if $\varepsilon = -1$; similarly $L_n^\varepsilon(q)$ is $L_n(q)$ if $\varepsilon = +1$, $U_n(q)$ if $\varepsilon = -1$. Finally, when we write 'X contains a graph aut.' in the Remark column in Table 5.1, we mean that for $M = N_X(D)$ to be maximal in X, the group X must contain an element in the coset of a graph automorphism of L.

1. *Conjugacy and structure of subgroups of maximal rank*

In this section we prove some general results on subgroups of maximal rank which are needed in the proof of the Theorem. Some of this material can also be found in [**7**]. Let $L = (G_\sigma)'$ be a simple exceptional group of Lie type over F_q as in the Introduction, and let D be a σ-stable closed connected reductive subgroup of G containing a σ-stable maximal torus T of G. We temporarily exclude the cases where L is ${}^2F_4(q)'$ or ${}^2G_2(q)'$; Example 1.5 below contains a discussion of the first of these excluded cases. Thus L is either an untwisted group or ${}^2E_6(q)$. Hence G contains a σ-stable maximal torus S on which σ acts as $s \to s^q$ if $L \neq {}^2E_6(q)$, and as $s \to s^{-q}$ if $L = {}^2E_6(q)$. Let $W = W(G) = N_G(S)/S$, the Weyl group of G. Then σ commutes with the action of W on S.

There exists $g \in G$ such that $T = S^g$. Let $E = D^{g^{-1}}$, so that $S \leq E$. Define Φ and

Δ to be the root systems of G and E, respectively, relative to S. As σ induces ± 1 on Φ, E is σ-stable. Let $W(\Delta) = W(E)$ be the Weyl group of E (that is, the subgroup of W generated by the reflections in the roots in Δ), and set

$$W_\Delta = N_W(\Delta)/W(\Delta).$$

Then $N_G(E)/E \cong W_\Delta$. Since $D = E^g$ and $T = S^g$ are σ-stable, $g^\sigma g^{-1} \in N_G(S) \cap N_G(E)$, and hence maps to an element of W_Δ.

LEMMA 1.1. *Suppose E^g and E^h are σ-stable G-conjugates of E. Then E^g and E^h are G_σ-conjugate if and only if the images in W_Δ of $g^\sigma g^{-1}$ and $h^\sigma h^{-1}$ are W_Δ-conjugate.*

Proof. This follows from [7, Proposition 1] and the choice of S.

By Lemma 1.1, the G_σ-classes of σ-stable G-conjugates of E are in one-to-one correspondence with the conjugacy classes of W_Δ.

We have $D = E^g$, $T = S^g$. Write $w = g^\sigma g^{-1} \in N(S) \cap N(E)$, and write w also for the corresponding element of W. Then $(\sigma w)^g = \sigma$, $(G_{\sigma w})^g = G_\sigma$, $(S_{\sigma w})^g = T_\sigma$ and $(E_{\sigma w})^g = D_\sigma$. Once we have chosen w therefore, we may freely work with σw, $G_{\sigma w}$, $S_{\sigma w}$, $E_{\sigma w}$ instead of σ, G_σ, T_σ, D_σ, respectively.

LEMMA 1.2. *We have $N_{G_{\sigma w}}(E)/E_{\sigma w} \cong C_{W_\Delta}(wW(\Delta))$.*

Proof. Write $U = N_{G_{\sigma w}}(E)/E_{\sigma w}$, $V = C_{W_\Delta}(wW(\Delta))$. Since $N_G(E)/E \cong W_\Delta$, and since σ centralizes W, there is a monomorphism from U into V. Let $v \in V$, and pick $n \in N_G(E)$ mapping to v under this monomorphism. Then n stabilizes the coset $E\sigma w$. Hence by Lang's theorem, for some $e \in E$, ne centralizes σw. Then $neE_{\sigma w} \in U$ and maps to v. Thus $U \cong V$.

LEMMA 1.3. (i) *Suppose $L \neq F_4(2^a)$ or $G_2(3^a)$. Then* $\mathrm{Aut}(L) = LN_{\mathrm{Aut}(L)}(E_{\sigma w})$.

(ii) *Let $L = F_4(2^a)$ or $G_2(3^a)$, and let τ be a graph automorphism of L normalizing S. Let τ_0 be the permutation of Φ induced by τ, and assume that $\Delta\tau_0 = \Delta$. Assume also that τ fixes the W_Δ-class of $wW(\Delta)$. Then* $\mathrm{Aut}(L) = LN_{\mathrm{Aut}(L)}(E_{\sigma w})$.

Proof. Let $q = p^a$. We may choose a field automorphism ϕ of G_σ having order a if $L \neq {}^2E_6(q)$, and having order $2a$ if $L = {}^2E_6(q)$, such that ϕ normalizes S and centralizes W and σ. Then $w^\phi = ws$ for some $s \in S$. Choose $r \in S$ such that $s = rr^{-\phi}$, and $t \in T$ such that $r^g = t^\sigma t^{-1}$. Then ϕ centralizes wr, and $(gt)^\sigma(gt)^{-1} = g^\sigma g^{-1} r = wr$, so we may replace w by wr and take ϕ to centralize w. Thus ϕ normalizes $G_{\sigma w}$ and $E_{\sigma w}$. As ϕ is a field automorphism of G_σ, $\phi^{g^{-1}}$ is a field automorphism of $(G_\sigma)^{g^{-1}} = G_{\sigma w}$. We check that $\phi^{g^{-1}}\phi^{-1} \in G_{\sigma w}$, and it follows that ϕ induces a field automorphism of $G_{\sigma w}$.

This completes the proof except for the cases $L = E_6(q)$, $F_4(2^a)$ or $G_2(3^a)$, where L has a graph automorphism. Let τ be a graph automorphism of L in each of these cases, chosen to normalize S and centralize σ. Choose $w_0 \in N_{G_\sigma}(S)$ mapping to the longest element of W, and set $x = w_0\tau\sigma$ when $G = E_6$, $x = \tau$ when $G = G_2$ or F_4. Then x centralizes σ. By the hypotheses of (ii), x normalizes E, and x fixes the coset Sw. So $w^x = ws$ for some $s \in S$. As x induces a Frobenius map on G, we can adjust w as before to centralize x. Then x normalizes $G_{\sigma w}$ and $E_{\sigma w}$. As above we check that x induces a graph automorphism on $G_{\sigma w}$.

We now give two examples to illustrate the use of Lemmas 1.1 and 1.2 in calculating the precise structure of subgroups of maximal rank. Many further examples of such calculations can be found in [**13**; **14**; **15**, § 4; **34**, § 3]. One useful general principle to bear in mind is that $|E_{\sigma w}|$ is always a monic polynomial in q.

EXAMPLE 1.4. Suppose that $G = E_6$ and that Δ is the subsystem $3A_2$ of Φ with fundamental roots $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_2, -\alpha_0\}$, where α_0 is the longest root and the Dynkin diagram is labelled as in the Introduction. We calculate the precise structures of the subgroups $N_{G_{\sigma w}}(E_{\sigma w})$ which arise. Here $W_\Delta = S_3$ (see [**8**, Table 9]). Assume first that $(G_\sigma)' = E_6(q)$. By Lemma 1.1, the number of G_σ-classes of σ-stable conjugates of E is equal to the number of conjugacy classes in W_Δ, which is three. We may take class representatives w_1, w_2, w_3 in W_Δ to have the following actions on Δ:

$$w_1 = 1, \quad w_2 = (\alpha_1, \alpha_2)(\alpha_3, -\alpha_0)(\alpha_5, \alpha_6), \quad w_3 = (\alpha_1, \alpha_6, -\alpha_0)(\alpha_2, \alpha_3, \alpha_5)$$

(see [**14**, p. 50]). First consider $w = w_1$. Here $E_{\sigma w} = E_\sigma$, and $O^{p'}(E_\sigma)$ is a central product of three groups $\mathrm{SL}_3(q)$. Since $Z(E) \cong Z_{(3,q^2-1)}$ (see [**34**, § 3]), we have $E_\sigma = e.(L_3(q)^3).e^2$ where $e = (3, q-1)$; and $N_{G_\sigma}(E_\sigma) = E_\sigma.W_\Delta$ by Lemma 1.2. Next let $w = w_2$. In [**34**, § 3] it is calculated that $E_{\sigma w} = f.(L_3(q^2) \times U_3(q)).g$ where $f = (3, q+1)$, $g = (3, q^2-1)$; and by Lemma 1.2, $N_{G_{\sigma w}}(E_{\sigma w}) = E_{\sigma w}.C_{W_\Delta}(w) = E_{\sigma w}.2$. Finally, let $w = w_3$. Writing $C = \mathrm{SL}_3$ and $Z(C) = \langle z \rangle$, we have $E \cong C^3/Z$ where $Z = \langle (z, z^{-1}, 1), (1, z, z^{-1}) \rangle$. Since $(\sigma w_3)^3 = \sigma^3$, we may take it that

$$E_{\sigma w_3} \geqslant R = \{(c, c^\sigma, c^{\sigma^2}) \mid c \in C_{\sigma^3}\} Z/Z.$$

Now $C_{\sigma^3} \cong \mathrm{SL}_3(q^3)$. If $3 \mid q-1$ then $Z(R) = \langle (z, z, z) \rangle Z/Z = 1$. Hence in all cases $Z(R) = 1$, and so $O^{p'}(E_{\sigma w}) \cong L_3(q^3)$. Moreover, when $3 \mid q-1$, $Z(E_{\sigma w})$ contains $(z, 1, 1)Z$ and so $E_{\sigma w} \cong L_3(q^3) \times 3$. Consequently, in all cases, $N_{G_{\sigma w}}(E_{\sigma w}) = E_{\sigma w}.C_{W_\Delta}(w) \cong L_3(q^3).e.3$.

The case where $(G_\sigma)' = {}^2E_6(q)$ is dealt with similarly; the relevant Weyl group calculations can be found in [**14**, p. 50].

EXAMPLE 1.5. Suppose now that $G = F_4$, $G_\sigma = {}^2F_4(q)$ (so that $p = 2$). The question of the existence of σ-stable subgroups of G of maximal rank is slightly more subtle here. Fix a subgroup E of maximal rank containing a maximally split torus S, and suppose E has root system Δ relative to S. Let σ_0 be the permutation of the root system Φ of G induced by σ. If $D = E^g$ is a σ-stable conjugate of E containing σ-stable maximal torus $T = S^g$, then as above $g^\sigma g^{-1}$ maps to an element $w \in W$, and σ acts on the roots of D relative to T as $\sigma_0 w$ acts on the roots of E relative to S. Consequently $\sigma_0 w(\Delta) = \Delta$. We shall need to consider only the possibilities $A_2 + \tilde{A}_2$ and $B_2 + B_2$ for the subsystem Δ (where $\tilde{A}_2$ is a subsystem consisting of short roots).

Let $\Delta = A_2 + \tilde{A}_2$. We may take Δ to have fundamental system $\{-\alpha_0, \alpha_1, \alpha_3, \alpha_4\}$, where α_0 is the highest root. Moreover $W_\Delta = \langle w_1 \rangle$, where w_1 induces the permutation $(-\alpha_0, \alpha_1)(\alpha_3, \alpha_4)$. The element w may be chosen so that $\sigma_0 w(\Delta) = \Delta$ and $\sigma_0 w$ induces $(\alpha_0, \alpha_3, -\alpha_1, \alpha_4)$. Then $\sigma_0 w w_1$ induces $(\alpha_0, \alpha_4, -\alpha_1, \alpha_3)$. If $Z = Z(E) \cong Z_3$ then w_1 inverts Z, so either σw or $\sigma w w_1$ centralizes Z, while the other inverts Z. Consequently there are two G_σ-classes of σ-stable conjugates D of E, with fixed point groups $\mathrm{SU}_3(q)$ and $\mathrm{PGU}_3(q)$. Moreover, $N_{G_\sigma}(D) = D_\sigma.2$ in both cases.

When $\Delta = B_2 + B_2$, we can choose Δ so that $\sigma_0(\Delta) = \Delta$. Here $W_\Delta = \langle w_2 \rangle$, where w_2 interchanges the two B_2 factors. Hence there are again two G_σ-classes of subgroups D_σ, one containing subgroups $E_\sigma \cong {}^2B_2(q) \times {}^2B_2(q)$, the other containing subgroups isomorphic to $E_{\sigma w_2} \cong B_2(q)$.

2. *The case where $T < D$*

In this section we prove the Theorem in the case where $T < D$. Thus $F^*(X) = L = (G_\sigma)'$ as in the Introduction, $M = N_X(D)$ is a maximal subgroup of X of maximal rank, where D is a σ-stable closed connected reductive subgroup of G maximal subject to the condition $M = N_X(D)$, and $T < D < G$ with T a maximal torus of G. Clearly $D_\sigma \neq 1$, so by maximality of M we have $M = N_X(D_\sigma)$ also.

Let Φ be the root system of G relative to T, and Δ the root system of D. As $T < D$, Δ is non-empty. Since D is connected and reductive, $D = D'Z(D)^0$ with D' semisimple and $Z(D)^0$ a torus. Let $W = W(G)$, the Weyl group of G.

We define a *homogeneous factor* of D to be the product of all the simple, connected, normal subgroups of D in a single $\mathrm{Aut}(G)$-conjugacy class; if $Z(D)^0 \neq 1$, we also define $Z(D)^0$ to be a homogeneous factor. Thus

$$D = H_1(D) \dots H_m(D),$$

a commuting product of the homogeneous factors $H_i(D)$, and each $H_i(D)$ is normal in $N_{\mathrm{Aut}(G)}(D)$.

LEMMA 2.1. (i) *$M \leqslant N_{\mathrm{Aut}(G)}(H_i(D))$ for each homogeneous factor $H_i(D)$.*
(ii) *For each i with $H_i(D) \neq Z(D)^0$, we have*

$$C_G(H_i(D))^0 = \prod_{j \neq i} H_j(D).$$

Further, if $Z(D)^0 \neq 1$ then $C_G(Z(D)^0)^0 = D$.

Proof. Part (i) is clear from the definitions. For (ii), fix $H_i(D)$ and let $C = H_i(D)C_G(H_i(D))^0$. This group contains D and is normalized by M and by σ. Clearly it suffices to show that $C = D$. This will follow from the maximality of D, once we show that C is reductive. Suppose then that C is not reductive, and put $Y = R_u(C)$, the unipotent radical of C, so that $Y \neq 1$. Then $H_i(D)$ is not a torus, since otherwise $C_G(H_i(D))^0$ is reductive by [**37**, II, 4.1]. As $T \leqslant C$, T normalizes Y, and Y contains some T-root group U_α, with $\alpha \in \Phi$ (see [**33**, Lemma 3]). Then $[U_{\pm\beta}, U_\alpha] = 1$ for each T-root group $U_\beta \leqslant H_i(D)$, which implies that $[U_{\pm\beta}, U_{-\alpha}] = 1$ for each such U_β. Consequently $U_{-\alpha} \leqslant C$, which contradicts the fact that $U_\alpha \leqslant Y = R_u(C)$. Thus C is reductive, as required.

LEMMA 2.2. *One of the following holds:*
(i) *$Z(D)^0 = 1$ and D' has maximal rank;*
(ii) *the Dynkin diagram of Δ is a subdiagram of the Dynkin diagram of G; moreover, $D' = (C_G(z)^0)'$ for some $z \in Z(D)^0$.*

Proof. Suppose D is not semisimple, that is, (i) does not hold. Then $D = D'Z$, where $Z = Z(D)^0 \neq 1$. Since $C_G(Z)$ is a Levi factor, it will suffice to show that

there exists $z \in Z$ with $D' = (C_G(z)^0)'$. Now for any $x \in Z^\#$ (where $Z^\# = Z\backslash\{1\}$), the semisimple group $C = (C_G(x)^0)'$ contains D' and is T-invariant. There are only finitely many possibilities for C, since each corresponds uniquely to a subsystem of the T-root system of G containing Δ. If $C > D'$ then $Z \nleq C_G(C)$: for if not then $C \leq C_G(Z)$ and $C_G(Z) = D$ by Lemma 2.1, whence $C \leq D$ and $D' < C$, an impossibility. Hence if $C > D'$ then $C_G(C) \cap Z$ is a closed subgroup of Z of smaller dimension. For each semisimple $C > D'$, set

$$Z_C = \{x \in Z \mid C = (C_G(x)^0)'\}.$$

Then $\bigcup Z_C$ (a finite union) is a closed, proper subset of Z. Hence there is a dense open set of elements $z \in Z$ such that $(C_G(z)^0)' = D'$, as required.

We now list all the possibilities for the subsystem Δ satisfying Lemmas 2.1 and 2.2. For each Δ we let $W_\Delta = N_W(\Delta)/W(\Delta)$, as in § 1.

In Table A we list the possibilities for Δ of maximal rank (i.e. with D semisimple). To produce this list we use the algorithm of [**2**], which yields all maximal rank closed subsystems of Φ, when Δ is closed; and we use inspection when Δ is not closed (which occurs only for $G = G_2$ or F_4, $p = 3$ or 2). The corresponding groups W_Δ are obtained using [**8**, Tables 7–11].

In Table B we list all the possibilities for Δ which are not of maximal rank. For this we use Lemma 2.2(ii), together with inspection of extended Dynkin diagrams.

TABLE A

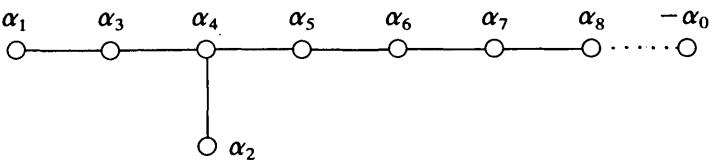
G	Δ	W_Δ
G_2	$A_1 + \tilde{A}_1$, A_2, $\tilde{A}_2$ ($p = 3$)	1, Z_2, Z_2
F_4	$A_1 + C_3$, B_4, $A_2 + \tilde{A}_2$,	1, 1, Z_2,
	$\tilde{A}_1 + A_3$, $2A_1 + B_2$, $4A_1$, D_4;	Z_2, Z_2, S_4, S_3
	with $p = 2$: these and their duals,	
	plus $B_2 + B_2$, $2A_1 + 2\tilde{A}_1$	Z_2, $(Z_2)^2$
E_6	$A_1 + A_5$, $3A_2$	1, S_3
E_7	$A_1 + D_6$, A_7, $A_2 + A_5$, $3A_1 + D_4$, $7A_1$	1, Z_2, Z_2, S_3, $L_3(2)$
E_8	D_8, $A_1 + E_7$, A_8, $A_2 + E_6$, $2A_4$,	1, 1, Z_2, Z_2, Z_4,
	$2A_1 + D_6$, $2D_4$, $A_3 + D_5$,	Z_2, $S_3 \times Z_2$, Z_2,
	$8A_1$, $4A_2$	$AGL_3(2)$, $GL_2(3)$

TABLE B

G	Δ	W_Δ
G_2	none	
F_4	B_3 (p odd)	Z_2
E_6	D_5, D_4, $4A_1$, $2A_1 + A_3$	1, S_3, S_4, Z_2
E_7	E_6, A_6, $3A_2$	Z_2, Z_2, $S_3 \times Z_2$
E_8	D_7, A_7, $2A_3$	Z_2, Z_2, Dih_8

In the tables, $\tilde{A}_i$ denotes a subsystem consisting of short roots, and Dih_8 a dihedral group of order 8.

Note that in Tables A and B, each possibility listed for Δ corresponds to a unique W-orbit of subsystems, and hence to a unique G-conjugacy class of subgroups of G. In particular, the possibilities $\Delta = A_7$ and $\Delta = 2A_3$ for $G = E_8$ can be taken as the subsystems with fundamental roots $\{-\alpha_0, \alpha_3, \ldots, \alpha_8\}$ and $\{\alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_8, -\alpha_0\}$, where the extended E_8-diagram is



We also remark that in all cases but one in the tables, W_Δ acts faithfully on the Dynkin diagram of Δ; in the exceptional case $(G, \Delta) = (F_4, B_3)$, and there is a root perpendicular to Δ, W_Δ being generated by the reflection in this root.

Now consider the case where $G_\sigma = {}^2F_4(q)$. As described in Example 1.5, there exists $w \in W$ such that $\sigma_0 w(\Delta) = \Delta$, where σ_0 is the permutation induced by σ on Φ. Inspecting Tables A and B, we see that the only possibilities for Δ are the systems $A_2 + \tilde{A}_2$, $B_2 + B_2$ and $2A_1 + 2\tilde{A}_1$. The last possibility does not arise: for, calculating as in Example 1.5, we find that when $\Delta = 2A_1 + 2\tilde{A}_1$, $N_X(D_\sigma) < N_X(B_2 + B_2)$. Thus D_σ is as described in Example 1.5, and is in Table 5.1. Note that $q > 2$ when $D_\sigma = {}^2B_2(q) \times {}^2B_2(q)$ or $\mathrm{PGU}_3(q)$, as when $q = 2$ these groups lie in $N(5^2)$ or $N(L_3(3))$, respectively (see **[40]**); and $X = {}^2F_4(2)$ when $D_\sigma = \mathrm{SU}_3(2)$ for similar reasons.

Next, note that when $G_\sigma = {}^2G_2(q)$, there is no σ-stable subgroup of type A_2; and for $\Delta = A_1 + \tilde{A}_1$ we obtain $N_{G_\sigma}(D_\sigma) = 2 \times L_2(q)$, as in Table 5.1.

Thus we may exclude for the rest of this section the cases where G_σ is ${}^2F_4(q)$ or ${}^2G_2(q)$. Then as described in § 1, we have $T = S^g$, $D = E^g$, where S is a maximal torus on which σ acts as $s \to s^{\pm q}$. As in § 1, we let $w = g^\sigma g^{-1}$ and work with σw, $G_{\sigma w}$, $S_{\sigma w}$, $E_{\sigma w}$ rather than σ, G_σ, T_σ, D_σ. Thus X is a subgroup of $\mathrm{Aut}((G_{\sigma w})')$ and $M = N_X(E) = N_X(E_{\sigma w})$. We also replace Φ, Δ by the root systems of G, E relative to S. The structure and conjugacy classes of subgroups $E_{\sigma w}$ and $N_{G_{\sigma w}}(E_{\sigma w})$ are given by Lemmas 1.1 and 1.2. We abuse notation slightly by writing also w for the image of w in the group $W_\Delta = N_W(\Delta)/W(\Delta)$.

LEMMA 2.3. *The following possibilities for Δ in Table A do not occur:*

$$G = F_4\colon\ \Delta = \tilde{A}_1 + A_3,\ 2A_1 + B_2,\ 4A_1,\ 2A_1 + 2\tilde{A}_1;$$

$$G = E_8\colon\ \Delta = 2A_1 + D_6,\ A_3 + D_5.$$

Proof. In each case we find a subsystem Σ of Φ such that $\Delta \subseteq \Sigma$ and $N_W(\Delta) \leq N_W(\Sigma)$: take $\Sigma = B_4$, B_4, D_4, $B_2 + B_2$ in the respective F_4 cases, and $\Sigma = D_8$ in both E_8 cases. Then if $\Sigma(K)$ denotes the subgroup generated by S and all the S-root groups U_α, with $\alpha \in \Sigma$, we have $N_{G_{\sigma w}}(E) < N_{G_{\sigma w}}(\Sigma(K))$ (see Lemma 1.2). By the uniqueness of the $G_{\sigma w}$-classes of these subgroups, it follows that $M = N_X(E) < N_X(\Sigma(K))$, contrary to the maximality of M.

LEMMA 2.4. *The following possibilities for Δ in Table B do not occur:*

$$G = F_4\colon\ \Delta = B_3;$$
$$G = E_6\colon\ \Delta = 4A_1,\ 2A_1 + A_3;$$
$$G = E_7\colon\ \Delta = A_6,\ 3A_2;$$
$$G = E_8\colon\ \Delta = D_7,\ A_7,\ 2A_3.$$

Proof. Define a subsystem Σ containing Δ as follows: when $G = F_4$, $\Sigma = B_4$; when $G = E_6$, $\Sigma = D_4$, D_5, respectively; when $G = E_7$, $\Sigma = A_7$, E_6, respectively; and when $G = E_8$, $\Sigma = D_8$ in all cases. Then $\Delta \subseteq \Sigma$, $N_W(\Delta) \leqslant N_W(\Sigma)$ and we see as in the proof of Lemma 2.3 that $N_X(E) < N_X(\Sigma(K))$.

For the remaining subsystems Δ in Tables A and B, we use the results of § 1, as illustrated in Example 1.4, to analyse all the subgroups $E_{\sigma w}$ arising.

LEMMA 2.5. *The group $N_{G_{\sigma w}}(E)$ is in Table 5.1. Each group in Table 5.1 corresponds to a unique L-conjugacy class of subgroups, except for the four cases with $G = F_4$ or G_2, $p = 2$ or 3, indicated in the table.*

Proof. Consider first $G = G_2$, so that Δ is $A_1 + \tilde{A}_1$, A_2 or $\tilde{A}_2$. We have $L = G_2(q)$, and D_σ is $(\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)).(2, q-1)$ or $\mathrm{SL}_3^\varepsilon(q)$. In all cases, $N_{G_{\sigma w}}(E)$ is in Table 5.1.

Next let $G = F_4$, so $L = F_4(q)$. By Lemmas 2.3 and 2.4, Δ is $A_1 + C_3$, B_4, $A_2 + \tilde{A}_2$, D_4 or $B_2 + B_2$ $(p = 2)$. All the possible subgroups $N(E_{\sigma w})$ appear in Table 5.1, except when $E'_{\sigma w} = {}^2D_4(q)$ (with $\Delta = D_4$). For this subgroup, $N_{G_{\sigma w}}(E)/E_{\sigma w} \cong Z_2$ by Lemma 1.2, and so $N_X(E) < N_X(B_4)$. Note also that q is odd when $\Delta = A_1 + C_3$, since when $p = 2$, $N(E_{\sigma w}) < N(B_4)$ here; and when $\Delta = B_2 + B_2$ $(p = 2)$, X must contain a graph automorphism of L, since otherwise $N(E_{\sigma w}) < N(B_4)$ again.

Now consider $G = E_6$. Let $L = E_6^\varepsilon(q)$, $\varepsilon = \pm$. Here Δ is $A_1 + A_5$, $3A_2$, D_4 or D_5. All the possible groups $N(E_{\sigma w})$ appear in Table 5.1, except when $E'_{\sigma w} = {}^2D_4(q)$, in which case $N_X(E) < N_X(D_5)$. Four further points should be noted, concerning the entries in Table 5.1 here. Firstly, if $\varepsilon = +$ and $\Delta = D_5$ then X must contain an element in the coset of a graph automorphism of L, as otherwise M lies in a D_5-parabolic subgroup. Secondly, when $E'_{\sigma w}$ is $D_5^\varepsilon(q)$, $D_4(q)$ or ${}^3D_4(q)$, $E'_{\sigma w}$ centralizes a torus of order $q - \varepsilon$, $(q - \varepsilon)^2$ or $q^2 + \varepsilon q + 1$, respectively, as indicated in Table 5.1—this can be calculated using [**7**, Proposition 8] (see [**14**, p. 50] for an illustration). Thirdly, when $L = {}^2E_6(2)$ and $E'_{\sigma w} = {}^3D_4(2)$, we have $N_L(E_{\sigma w}) = {}^3D_4(2) < F_4(2) < L$, so $X \geqslant G_{\sigma w} = L.3$, by maximality. Finally, $N(D_4(2))$ is non-maximal in $E_6(2)$ for the same reason.

Next let $G = E_7$, so that Δ is $A_1 + D_6$, A_7, $A_2 + D_5$, $3A_1 + D_4$, $7A_1$ or E_6. All possible groups $E_{\sigma w}$ appear in Table 5.1, except when Δ is $3A_1 + D_4$ or $7A_1$. For $\Delta = 3A_1 + D_4$, the subgroup $A_1(q^2) \circ A_1(q) \circ {}^2D_4(q)$ does not arise; for in this case $N_X(E) < N_X(A_1 D_6)$. Now consider $\Delta = 7A_1$. Here W_Δ acts 2-transitively as $L_3(2)$ on the seven factors A_1. If M normalizes a subproduct mA_1 $(1 \leqslant m \leqslant 6)$, then by the maximality of E, $C_G(mA_1) = (7-m)A_1$ and $C_G((7-m)A_1) = mA_1$. Hence m must be 3 or 4. Write Ω for the set of seven factors A_1 permuted by W_Δ. Then w fixes 0, 3, 4 or 7 points of Ω. No element of $L_3(2)$ fixes four points of Ω. And if w

fixes three points then $C_{L_3(2)}(w)$ is a group of order 8. Since $N_{W(A_1D_6)}(7A_1)/W(7A_1)$ has order divisible by 8, we deduce that $N_X(E) < N_X(A_1D_6)$ here. Thus w fixes no points or seven points. If $w=1$ then we obtain the group $N(A_1(q)^7)$ in Table 5.1 (note that $q>2$, as when $q=2$ we have $L_2(2)^7.L_3(2) < 3^7.W < E_7(2)$). Finally, if w fixes no points then w has order 7, giving the subgroup $N(A_1(q^7))$ in Table 5.1.

Lastly, consider $G=E_8$. Here Δ is D_8, A_1+E_7, A_8, A_2+E_6, $2A_4$, $2D_4$, $8A_1$ or $4A_2$. In the first five cases, all possible groups $E_{\sigma w}$ are in Table 5.1. When $\Delta = 2D_4$ and the element w of $W_\Delta = S_3 \times Z_2$ has order 2 and projects non-trivially to the factor S_3, Lemma 1.2 gives $N_{G_{\sigma w}}(E)/E_{\sigma w} \cong (Z_2)^2$, so $N_X(E) < N_X(D_8)$; thus only the cases where w has order 1, 3 or 6, or is the central involution, give rise to groups in Table 5.1.

Now consider $\Delta = 4A_2$. Here $w \in W_\Delta \cong \mathrm{GL}_2(3)$, permuting the four factors A_2 as S_4. The central involution in W_Δ induces a graph automorphism on each of the factors. Since the centralizer in G of each factor A_2 is a group E_6, w fixes 0, 2 or 4 factors. If $|\mathrm{fix}\, w| = 4$ then $N(E_{\sigma w})$ is $N(A_2^\varepsilon(q)^4)$, as in Table 5.1. If $|\mathrm{fix}\, w| = 2$ then $C_{W_\Delta}(w)$ fixes both factors A_2 fixed by w, so by Lemma 1.2, $N(E_{\sigma w}) < N(A_2) = N(A_2E_6)$. When $|\mathrm{fix}\, w| = 0$, w has order 4 or 8, and w^2 or w^4, respectively, is the central involution of W_Δ. Hence $E'_{\sigma w}$ is $({}^2A_2(q^2))^2$ or ${}^2A_2(q^4)$, as in Table 5.1.

Finally, let $\Delta = 8A_1$. Here $w \in W_\Delta \cong \mathrm{AGL}_3(2)$, acting 3-transitively on the eight factors A_1. As in the E_7 case, if M normalizes a subproduct mA_1, then $m=4$, and so $|\mathrm{fix}\, w|$ is 0, 4 or 8. Therefore w is a 2-element. If $w=1$ then $N(E_{\sigma w}) = N(A_1(q)^8)$, as in Table 5.1 (notice that $q>2$ here, as when $q=2$ we have $L_2(2)^8.\mathrm{AGL}_3(2) < 3^8.W(E_8) < E_8(2)$). Otherwise $C_{W_\Delta}(w)$ is either $2^3.S_4$ or a 2-group. Take a subsystem D_8 containing Δ. Then $N_{W(D_8)}(\Delta)/W(\Delta) \cong 2^3.S_4$, with a central involution. So from Lemma 1.2 we see that $N_X(E) < N_X(D_8)$ here.

To conclude, we justify the conjugacy statement in the lemma. It is immediate from Lemma 1.1 that each subgroup in Table 5.1 corresponds to a unique $G_{\sigma w}$-class; now Lemma 1.3 shows that the L-class is also unique, except in the four exceptional cases indicated in the table.

The proof of the Theorem for the case where $T < D$ is now complete, apart from the maximality assertion for the groups in Table 5.1. We defer the proof of this assertion until § 4.

3. *The case where $T = D$*

In this section we prove the Theorem in the case where $T = D$. Thus $L = F^*(X) = (G_\sigma)'$ as in the Introduction, and $M = N_X(T)$ where T is a σ-stable maximal torus of G. Moreover, M normalizes no σ-stable closed connected reductive subgroup K such that $T < K < G$.

We claim that $T_\sigma \neq 1$. For otherwise $q=2$ and σ acts on T as $t \to t^q$ $(t \in T)$. Hence $N_L(T) \cong W(G)$, the Weyl group. As M is maximal, $W(G)$ can have no central involution, and hence $G = E_6$. Thus $X = E_6(2)$, $M \cong O_6^-(2)$. But then M is non-maximal in X (see **[22]**). This proves the claim, and so we have

$$M = N_X(T) = N_X(T_\sigma).$$

We postpone dealing with the case where $L = {}^2F_4(q)'$ until Lemma 3.8. Also, the result for $L = {}^2G_2(q)'$ can be read off from **[19]**. Thus we assume until Lemma

3.8 that L is not ${}^2G_q(q)'$ or ${}^2F_4(q)'$, and recall from the Introduction that ${}^2B_2(q)$ and ${}^3D_4(q)$ are also excluded; so L is either an untwisted group or ${}^2E_6(q)$.

As in § 1 we have $T = S^g$, where σ acts on S as $s \to s^{\pm q}$, and $w = g^\sigma g^{-1}$ lies in $N_G(S)$. We write also w for the image of w in the Weyl group $W = W(G)$. Again as in § 1, we work with σw, $G_{\sigma w}$, $S_{\sigma w}$ rather than σ, G_σ, T_σ.

Write $l = \mathrm{rank}(G)$, and let V be a Euclidean space of dimension l on which W operates as a reflection group in the usual way. Denote by w_0 the longest element of W.

We associate a *Carter diagram* with the maximal torus $S_{\sigma w}$ as follows. As in [**6**], write

$$w = r_{a_1} \dots r_{a_h} r_{b_1} \dots r_{b_k},$$

where r_{a_i}, r_{b_j} are the reflections corresponding to the roots a_i, b_j in the root system Φ of G relative to S, the sets $\{a_1, \dots, a_h\}$ and $\{b_1, \dots, b_k\}$ consist of pairwise orthogonal roots, and $h + k$ is the rank of $w - 1$ on the space V. The *Carter diagram* Δ_w of T is the diagram with vertices $\{a_1, \dots, a_h, b_1, \dots, b_k\}$, and with a bond of strength $4(a_i, b_j)^2/(a_i, a_i)(b_j, b_j)$ joining a_i to b_j. The possible diagrams are listed in [**6**], where they are used to classify the conjugacy classes in $W(G)$. We shall use the properties of Δ_w given in [**6**].

Since $C_G(S_{\sigma w})^0$ is a σ-stable reductive subgroup containing S, we have $C_G(S_{\sigma w})^0 = S$ by the assumption in the first paragraph of this section. Hence by [**37**, II, 1.8],

$$N_{G_{\sigma w}}(S_{\sigma w})/S_{\sigma w} \cong C_W(\sigma w).$$

The orders of the groups $C_W(w)$ are given in [**6**], and their structures can be deduced from knowledge of centralizers in the Weyl group W. Also the order of $S_{\sigma w}$ can be read off from the Carter diagram Δ_w, using [**6**, Table 2]. The structure of $S_{\sigma w}$ is given by [**37**, II, 1.7].

Let Φ be the root system of G relative to S. For a closed subsystem Σ of Φ, we write $\Sigma(K)$ for the group generated by S and all the S-root groups U_α, with $\alpha \in \Sigma$.

LEMMA 3.1. (i) *The sum $h + k$ is either* 0 *or l.*

(ii) *Suppose that $G \neq E_6$. Then the rank of $w + 1$ on the Euclidean space V is either* 0 *or l.*

Proof. From the observations preceding the lemma, we have $M = N_X(S_{\sigma w}) = N_X(S)$. First consider Part (i). As σw induces the map $t \to t^{\pm q w}$ on S, w induces the map $s^{\pm q} \to s$ on $S_{\sigma w}$. Hence M fixes the coset Sw. Set $S_0 = C_S(w)^0$, an $M\langle \sigma w \rangle$-invariant subgroup of S. Then S_0 is either 1 or a torus, so $D_0 = C_G(S_0)$ is connected and S-invariant. Moreover, w lies in the Weyl group of D_0. If $w = 1$ then $h + k = 0$, so assume that $w \neq 1$. Then $S < D_0 \leq G$, and D_0 is σw-stable, connected, reductive and M-invariant. This forces $D_0 = G$ by the assumption in the first paragraph of this section. Hence $S_0 = 1$, that is, $C_S(w)^0 = 1$. If $h + k < l$ then w lies in the Weyl group of a subsystem of rank less than l, and hence centralizes a non-trivial torus. Thus $h + k = l$.

Finally, consider (ii). Since $G \neq E_6$, we have $w_0 \in Z(W)$. As before, we see that M also fixes the coset Sw_0w. Replacing $C_S(w)^0$ by $C_S(w_0w)^0$ in the above argument, we obtain the conclusion of (ii) (note that w_0w acts as $-w$ on V).

LEMMA 3.2. *Let Σ be a closed subsystem of Φ, and assume that $\Sigma(K)$ is σw-stable. Then $S_{\sigma w} \leq \Sigma(K)_{\sigma w}$. Further, if $C_W(\sigma w) \leq N_W(\Sigma)$ then $N_{G_{\sigma w}}(S_{\sigma w}) \leq N_{G_{\sigma w}}(\Sigma(K)_{\sigma w})$.*

Proof. The first part is clear. The last part follows from [**37**, II, 1.8] and Lemma 1.2.

LEMMA 3.3. *Let s be a prime divisor of $|S_{\sigma w}|$. Then*

$$C_G(\Omega_1(O_s(S_{\sigma w})))^0 = S$$

(where Ω_1 denotes the group generated by elements of order s).

Proof. The left-hand side is connected, reductive, $M\langle \sigma w\rangle$-invariant, and contains S. Hence it is equal to S by the assumption in the first paragraph of this section.

We now complete the proof of the case $T = D$ of the Theorem case by case. Write

$$L_1 = \text{Inndiagfield}(L),$$

the group generated by all inner, diagonal and field automorphisms of L. In the following lemmas we shall show that $S_{\sigma w}$ and $N(S_{\sigma w})$ are as in Table 5.2 (recall the above replacement of T_σ by $S_{\sigma w}$ and G_σ by $G_{\sigma w}$).

LEMMA 3.4. *If G is G_2 or F_4, then $S_{\sigma w}$ and $N(S_{\sigma w})$ are as in Table 5.2.*

Proof. By Lemma 3.1, $h + k$ is 0 or l, and either $w = w_0$ or w has no eigenvalues -1 in its action on the Euclidean space $V = \mathbb{R}^l$. Hence from [**6**] we deduce that the possible Carter diagrams Δ_w are as follows:

$$G = G_2\text{: diagrams } \varnothing,\ A_1 + \tilde{A}_1,\ A_2,\ G_2;$$

$$G = F_4\text{: diagrams } \varnothing,\ 4A_1,\ A_2 + \tilde{A}_2,\ D_4(a_1),\ B_4,\ F_4,\ F_4(a_1).$$

When X contains an element in the coset of a graph automorphism (that is, when $X \nleq L_1$), all these possibilities, except the diagram B_4, are included in Table 5.2. The possibility $\Delta_w = B_4$ with $G = F_4$ is excluded for the following reason. There is a subsystem $\Sigma = B_2 + B_2$ (unique up to W-conjugacy) and an element $u \in N_W(\Sigma)$ interchanging the factors B_2, such that $N_L(\Sigma(K)_{\sigma u}) \cong \mathrm{Sp}_4(q^2).2$, a subgroup in Table 5.1. Moreover, by the uniqueness of Σ and of the $G_{\sigma u}$-class of $\Sigma(K)_{\sigma u}$, this subgroup is normalized by a graph automorphism of L and also by a full group of field automorphisms. When $\Delta_w = B_4$, $N_L(S_{\sigma w}) = S_{\sigma w}.Z_8$ and $|S_{\sigma w}| = q^4 + 1$; moreover from [**6**] we see that any maximal torus of this order in $G_{\sigma w}$ is conjugate to $S_{\sigma w}$. As $\mathrm{Sp}_4(q^2).2$ has a torus of this order with Z_8 acting on it, and as $G_\sigma \cong G_{\sigma w} \cong G_{\sigma u}$, we may take it that $N_L(S_{\sigma w}) < N_L(\mathrm{Sp}_4(q^2))$. Then from the above remark on graph and field automorphisms, we conclude that $M = N_X(S_{\sigma w}) < N_X(\mathrm{Sp}_4(q^2))$, contrary to maximality.

Also, if $L = F_4(2)$ and Δ_w is $\varnothing$, $4A_1$, $D_4(a_1)$, F_4 or $F_4(a_1)$, then $N_X(S_{\sigma w})$ is non-maximal (see [**11**, p. 167] or [**30**]). And when $L = F_4(4)$ and $\Delta_w = \varnothing$, $N(S_{\sigma w})$ lies in $N(F_4(2))$ (as here $S_{\sigma w} = 3^4$ and $3^4.W < F_4(2)$).

Suppose now that $X \leqslant L_1$. Take Σ to be the subsystem of Φ consisting of all long roots; thus Σ is of type A_2 if $G = G_2$, and of type D_4 if $G = F_4$. Then $W = N_W(\Sigma)$. Hence certainly $C_W(\sigma w) \leqslant N_W(\Sigma)$, and so by Lemma 3.2, $N_L(S_{\sigma w}) \leqslant N_L(\Sigma(K)_{\sigma w})$. Since the subgroup $\Sigma(K)_{\sigma w}$ (of type $A_2^\varepsilon(q)$ or $D_4^\varepsilon(q)$) is normalized by a full group of field automorphisms, and $X \leqslant L_1$, which is generated by L and its field automorphisms, we deduce that $M < N_X(\Sigma(K)_{\sigma w})$, a contradiction.

Now suppose that G is E_6, E_7 or E_8. By Lemma 3.1 and **[6]**, the possibilities for the diagram Δ_w are those given in Table C. In Table C we also give the order of $S_{\sigma w}$ and of $N_{G_{\sigma w}}(S_{\sigma w})/S_{\sigma w}$, both taken from **[6]**.

In the ensuing arguments we shall often use certain primes q_i which divide $|S_{\sigma w}|$. Here q_i is defined to be a *primitive prime divisor* of $q^i - 1$: that is, a prime which divides $q^i - 1$ but does not divide $q^j - 1$ for $1 \leqslant j < i$. Such primes exist provided $i \geqslant 3$ and $(q, i) \neq (2, 6)$, by **[41]**.

TABLE C

L	Δ_w	$\lvert S_{\sigma w}\rvert$	$\lvert N_{G_{\sigma w}}(S_{\sigma w}) : S_{\sigma w}\rvert$
$E_6^\varepsilon(q)$	$\varnothing$	$(q-\varepsilon)^6$	$\lvert W\rvert$
	$3A_2$	$(q^2+\varepsilon q+1)^3$	648
	A_5+A_1	$(q+\varepsilon)(q^6-1)/(q-\varepsilon)$	36
	E_6	$(q^2+\varepsilon q+1)(q^4-q^2+1)$	12
	$E_6(a_1)$	$q^6+\varepsilon q^3+1$	9
	$E_6(a_2)$	$(q^2+\varepsilon q+1)(q^2-\varepsilon q+1)^2$	72
$E_7(q)$	$\varnothing$	$(q-1)^7$	$\lvert W\rvert$
	$7A_1$	$(q+1)^7$	$\lvert W\rvert$
$E_8(q)$	$\varnothing$	$(q-1)^8$	$\lvert W\rvert$
	$8A_1$	$(q+1)^8$	$\lvert W\rvert$
	A_8	$(q^9-1)/(q-1)$	54
	$2A_4$	$(q^5-1)^2/(q-1)^2$	600
	$4A_2$	$(q^2+q+1)^4$	155520
	$D_8(a_1)$	$(q^2+1)(q^6+1)$	72
	$D_8(a_3)$	$(q^4+1)^2$	192
	$2D_4(a_1)$	$(q^2+1)^4$	46080
	E_6+A_2	$(q^2+q+1)^2(q^4-q^2+1)$	288
	$E_6(a_2)+A_2$	$(q^2+q+1)^2(q^2-q+1)^2$	1728
	E_8	$q^8+q^7-q^5-q^4-q^3+q+1$	30
	$E_8(a_1)$	q^8-q^4+1	24
	$E_8(a_2)$	$q^8-q^6+q^4-q^2+1$	20
	$E_8(a_3)$	$(q^4-q^2+1)^2$	288
	$E_8(a_4)$	$(q^2-q+1)(q^6-q^3+1)$	54
	$E_8(a_5)$	$q^8-q^7+q^5-q^4+q^3-q+1$	30
	$E_8(a_6)$	$(q^4-q^3+q^2-q+1)^2$	600
	$E_8(a_7)$	$(q^2-q+1)^2(q^4-q^2+1)$	288
	$E_8(a_8)$	$(q^2-q+1)^4$	155520

LEMMA 3.5. *If $G = E_6$, then $S_{\sigma w}$ and $N(S_{\sigma w})$ are as in Table 5.2.*

Proof. When Δ_w is $\varnothing$ or $3A_2$, the possibility is included in Table 5.2. Note the conditions in the table that $q \geqslant 5$ when $L = E_6(q)$ and $\Delta_w = \varnothing$, and $q > 2$ when $L = {}^2E_6(q)$ and $\Delta_w = 3A_2$. These occur for the following reasons. First consider $L = E_6(q)$, $\Delta_w = \varnothing$. If $q = 2$ then $S_{\sigma w} = 1$, and if $q = 4$ then $N_{G_{\sigma w}}(S_{\sigma w}) = 3^6.W$ and $N(S_{\sigma w})$ lies in a subgroup $N({}^2E_6(2))$. When $q = 3$, $N_L(S_{\sigma w}) = 2^6.W$. This is centralized by γw_0, where γ is a graph automorphism of L. Since by [**9**, Proposition 2.7], $C_L(\gamma w_0)$ is $C_4(3)$ or $F_4(3)$, we conclude that $N(S_{\sigma w})$ is non-maximal here. Finally, when $L = {}^2E_6(2)$ and $\Delta_w = 3A_2$, $N(S_{\sigma w})$ is non-maximal (see [**11**, p. 191]).

Now consider the other possibilities for Δ_w given in Table C. Let $L = E_6^{\varepsilon}(q)$ with $\varepsilon = \pm$. First let $\Delta_w = A_5 + A_1$. Let s be a primitive prime divisor of $q^6 - 1$ if $q > 2$, and take $s = 7$ if $q = 2$. Thus s divides $|S_{\sigma w}|$. Define $P = \Omega_1(O_s(S_{\sigma w}))$. There is a subsystem $\Sigma_1 = A_5 + A_1$ of Φ such that $w \in W(\Sigma_1)$ and $O^{p'}(\Sigma_1(K)_{\sigma w}) = A_5^{\varepsilon}(q) \circ A_1(q)$. Moreover, P lies in the factor $A_5^{\varepsilon}(q)$. Hence $C_G(P)^0$ contains $A_1(K)$, which contradicts Lemma 3.3.

Next consider $\Delta_w = E_6$. Inspection of [**6**] shows that up to $G_{\sigma w}$-conjugacy, $S_{\sigma w}$ is the only maximal torus of its order in $G_{\sigma w}$. Let Σ_2 be the subsystem $D_4 = \langle \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle$ of Φ (labelling the Dynkin diagram as in the Introduction). The group $\Sigma_2(K)$ is centralized in G by a torus T_2 of rank 2. Now define

$$u = \begin{cases} w_0(D_5)w_0(E_6) & \text{if } \varepsilon = +, \\ w_0(D_5) & \text{if } \varepsilon = -, \end{cases}$$

where we take here $D_5 = \langle \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle$. Then σu induces a triality symmetry in W_{Σ_2}, and we have

$$(\Sigma_2(K)T_2)_{\sigma u} = {}^3D_4(q) \times (q^2 + \varepsilon q + 1).$$

Moreover (see [**20**]) there exist a torus T_4 of rank 4 in $\Sigma_2(K)$ and an element $v \in W_{\Sigma_2}$ of order 4 such that

$$|(T_4T_2)_{\sigma vu}| = (q^4 - q^2 + 1)(q^2 + \varepsilon q + 1).$$

By the uniqueness of the order of $S_{\sigma w}$ noted above, we may take $w = vu$, $S = T_4T_2$, so $S_{\sigma w} \leqslant {}^3D_4(q) \times (q^2 + \varepsilon q + 1)$. But then if s is a prime divisor of $q^2 + \varepsilon q + 1$ and $P = \Omega_1(O_s(S_{\sigma w}))$, we have $D_4(K) \leqslant C_G(P)^0$, contrary to Lemma 3.3.

A similar argument deals with $\Delta_w = E_6(a_2)$. (Note that here $|S_{\sigma w}|$ is unique only for sufficiently large q (in fact $q > 2$ will do), but this is enough to force $w = vu$ in the above argument.)

Finally, let $\Delta_w = E_6(a_1)$. Again, a glance at [**6**] shows that up to $G_{\sigma w}$-conjugacy, $S_{\sigma w}$ is the unique maximal torus of its order. Define $\Sigma_3 = 3A_2 = \langle \alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_2, -\alpha_0 \rangle$. When $\varepsilon = +$, the element u defined above induces the symmetry $(\alpha_1, -\alpha_0, \alpha_6)(\alpha_2, \alpha_5, \alpha_3) \in W_{\Sigma_3}$, and so $(\Sigma_3(K)_{\sigma u})' \cong A_2(q^3)$. Similarly, when $\varepsilon = -$, we have $(\Sigma_3(K)_{\sigma u})' \cong {}^2A_2(q^3)$. Thus there are a maximal torus $T_6 \leqslant \Sigma_3(K)$ and an element $v_1 \in W_{\Sigma_3}$ such that $|(T_6)_{\sigma v_1 u}| = q^6 + \varepsilon q^3 + 1$. Consequently, we may take $S = T_6$, $w = v_1 u$. Now $N_L(\Sigma_3(K)_{\sigma w}) \cong A_2^{\varepsilon}(q^3).3$. Since by Table C, $N_{G_{\sigma w}}(S_{\sigma w})/S_{\sigma w}$ has order 9, it follows that $M \cap G_{\sigma w} \leqslant N_{G_{\sigma w}}(A_2^{\varepsilon}(q^3))$. As this subgroup $A_2^{\varepsilon}(q^3)$ is normalized by a full group of field automorphisms of L as well as a graph automorphism, we conclude that $M < N_X(A_2^{\varepsilon}(q^3))$, which contradicts the maximality of M.

LEMMA 3.6. *If $G = E_7$ then $S_{\sigma w}$ and $N(S_{\sigma w})$ are as in Table* 5.2.

Proof. The possibilities $\varnothing$, $7A_1$ for Δ_w appear in Table 5.2, provided $q \geqslant 5$ when $\Delta_w = \varnothing$. The reasons for this condition on q are as follows. If $q = 2$ and $\Delta_w = \varnothing$ then $S_{\sigma w} = 1$, and if $q = 4$ then $N_L(S_{\sigma w}) = 3^7.W < E_7(2)$. Now suppose $q = 3$. Then $|S_{\sigma w} \cap L| = 2^6$, $N_L(S_{\sigma w}) = 2^6.(\mathrm{Sp}_6(2) \times \langle w_0 \rangle)$ and $N_{G_{\sigma w}}(S_{\sigma w}) = 2^7.(\mathrm{Sp}_6(2) \times \langle w_0 \rangle) = 2^8.\mathrm{Sp}_6(2)$ (see [**10**, Lemma $2(E_7)$(iii)]). The normal 2^8 contains three $\mathrm{Sp}_6(2)$-invariant subgroups 2^7. If any of these splits over $S_{\sigma w} \cap L$ then $N(S_{\sigma w}) = 2^8.\mathrm{Sp}_6(2)$ has a central involution, and hence M is non-maximal. So suppose that none of them splits. Write $Q = S_{\sigma w} \cap L$ and let $Q_1 = \langle Q, x \rangle$, $Q_2 = \langle Q, y \rangle$ be distinct $\mathrm{Sp}_6(2)$-invariant subgroups 2^7. By [**32**], $Q_1 \cong Q_2$ as $\mathrm{Sp}_6(2)$-modules; let ϕ: $Q_1 \to Q_2$ be an isomorphism. Then $\phi|Q$ is the identity, and so for $g \in \mathrm{Sp}_6(2)$, $x + x^g = (x + x^g)\phi$. Hence g fixes $x + x\phi$. But then the $\mathrm{Sp}_6(2)$-invariant subgroup $\langle Q, x + x\phi \rangle \cong 2^7$ splits over Q, a contradiction.

LEMMA 3.7. *If $G = E_8$ then $S_{\sigma w}$ and $N(S_{\sigma w})$ are as in Table* 5.2.

Proof. The possibilities $\varnothing$, $8A_1$, $2A_4$, $4A_2$, $2D_4(a_1)$, E_8 and $E_8(a_i)$ ($i = 3, 5, 6, 8$) for Δ_w appear in Table 5.2, provided $q \geqslant 5$ when $\Delta_w = \varnothing$ and $q > 2$ when $\Delta_w = E_8(a_8)$. These conditions on q occur for the following reasons. When $\Delta_w = \varnothing$ and $q = 2$ or 4 we have $S_{\sigma w} = 1$ or $N_L(S_{\sigma w}) = 3^8.W < E_8(2)$, respectively. And if $\Delta_w = \varnothing$ and $q = 3$ then $N(S_{\sigma w}) = 2^8.(2.O_8^+(2))$, and M centralizes an involution, since $H^1(O_8^+(2), 2^8) = 0$ by [**32**]. Finally, consider $\Delta_w = E_8(a_8)$ with $q = 2$. Here $N(S_{\sigma w})/S_{\sigma w} \cong 3 \times \mathrm{Sp}_4(3)$ and $N(S_{\sigma w}) \cong 3^4.(3 \times \mathrm{Sp}_4(3))$. As it has a central involution, the factor $\mathrm{Sp}_4(3)$ must fix an element of the coset $3^4\bar{w}$ (where $\bar{w}$ is a preimage of the central element of order 3 in $N(S_{\sigma w})/S_{\sigma w}$), and hence M centralizes a 3-element of L, contrary to the maximality of M.

Now consider $\Delta_w = E_6 + A_2$. There is a subsystem $\Sigma_1 = E_6 + A_2$ such that $w \in W(\Sigma_1)$ and $S_{\sigma w} \leqslant \Sigma_1(K)_{\sigma w} \leqslant N(E_6(q) \circ A_2(q))$. Let s be a primitive prime divisor of $q^{12} - 1$, and let $P = \Omega_1(O_s(S_{\sigma w}))$. Then $C_G(P)^0$ contains $A_2(K)$, contrary to Lemma 3.3. A similar argument rules out the diagram $E_6(a_2) + A_2$.

Next let $\Delta_w = A_8$. Choose a subsystem $\Sigma_2 = A_8$ such that $w \in W(\Sigma_2)$ and $S_{\sigma w} \leqslant \Sigma_2(K)_{\sigma w} \leqslant N(A_8(q))$. Then $S_{\sigma w}$ is generated by a Singer cycle in $N(A_8(q))$. Thus if s is a primitive prime divisor of $q^3 - 1$ and $P = \Omega_1(O_s(S_{\sigma w}))$ then $A_2(K) \leqslant C_{\Sigma_2(K)}(P)^0 \leqslant C_G(P)^0$, which contradicts Lemma 3.3. A similar argument excludes the diagram $D_8(a_1)$.

It remains to rule out the diagrams $D_8(a_3)$ and $E_8(a_i)$ for $i = 1, 2, 4, 7$. Let $\Delta_w = D_8(a_3)$. We may choose a subsystem $\Sigma_3 = 2D_4$, and take w to interchange the two components D_4 in such a way that $S_{\sigma w} \leqslant \Sigma_3(K)_{\sigma w} \leqslant N(D_4(q^2))$. In fact, defining $Y = N_{G_{\sigma w}}(D_4(q^2))$, we see from Table 5.1 that $Y = D_4(q^2).(S_3 \times Z_2)$. Thus (cf. [**21**]),

$$|N_Y(S_{\sigma w}) : S_{\sigma w}| = 2^4\,|S_3 \times Z_2| = 192.$$

This is equal to $|N_{G_{\sigma w}}(S_{\sigma w}) : S_{\sigma w}|$, so $M \cap G_{\sigma w} < Y$. Since the subgroup $D_4(q^2)$ is normalized by a full group of field automorphisms of L, it follows that $M < N_X(D_4(q^2))$. Similarly, if $\Delta_w = E_8(a_1)$ then we may take it that

$$S_{\sigma w} \leqslant \Sigma_3(K)_{\sigma w} \leqslant N_{G_{\sigma w}}({}^3D_4(q^2)) = {}^3D_4(q^2).6 = Y_1.$$

Then $|N_{Y_1}(S_{\sigma w}) : S_{\sigma w}| = 24$ (cf. [**20**]), so $M < N_X({}^3D_4(q^2))$ here.

Next, if $\Delta_w = E_8(a_2)$ then there is a subsystem $\Sigma_4 = 2A_4$ such that

$$S_{\sigma w} \leqslant \Sigma_4(K)_{\sigma w} \leqslant N_{G_{\sigma w}}({}^2A_4(q^2)) = {}^2A_4(q^2).4.$$

Hence as above, $M < N_X({}^2A_4(q^2))$.

Finally, if Δ_w is $E_8(a_4)$ or $E_8(a_7)$ then there is a subsystem $\Sigma_5 = E_6 + A_2$ such that $S_{\sigma w} \leqslant \Sigma_5(K)_{\sigma w} \leqslant N({}^2E_6(q) \circ {}^2A_2(q))$. Letting s be a primitive prime divisor of $q^{18}-1$ or $q^{12}-1$ in the respective cases, and $P = \Omega_1(O_s(S_{\sigma w}))$, we have $A_2(K) \leqslant C_G(P)^0$, contravening Lemma 3.3.

We now deal with the case where $L = {}^2F_4(q)'$, excluded by assumption at the beginning of this section. The maximal subgroups of ${}^2F_4(2)'$ and ${}^2F_4(2)$ are listed in [**11**, p. 74] and [**41**] (but recall the remark in the Introduction: $SU_3(2).2$ is a maximal subgroup of ${}^2F_4(2)$ which is omitted in [**11**, **40**]). So we suppose now that $L = {}^2F_4(q)$ with $q = 2^{2a+1} \geqslant 8$.

By [**36**, p. 8, 9], L has precisely eleven conjugacy classes of maximal tori. Let $S_1, \ldots, S_{11}$ be representatives of these classes. We list the possibilities for $|S_i|$ and $|N_L(S_i) : S_i|$ in Table D.

TABLE D

i	$\lvert S_i\rvert$	$\lvert N_L(S_i) : S_i\rvert$
1	$(q-1)^2$	16
2	q^2-1	4
3	$(q-1)(q-\sqrt{(2q)}+1)$	8
4	$(q-1)(q+\sqrt{(2q)}+1)$	8
5	q^2+1	16
6	$(q-\sqrt{(2q)}+1)^2$	96
7	$(q+\sqrt{(2q)}+1)^2$	96
8	$(q+1)^2$	48
9	q^2-q+1	6
10	$q^2-\sqrt{(2q^3)}+q-\sqrt{(2q)}+1$	12
11	$q^2+\sqrt{(2q^3)}+q+\sqrt{(2q)}+1$	12

LEMMA 3.8. *Let $M = N_X(S_i)$ be maximal in X. Then i is* 6, 7, 8, 10 *or* 11, *and S_i, $N_L(S_i)$ are as in Table* 5.2.

Proof. We must rule out the values $i \in \{1, 2, 3, 4, 5, 9\}$. Let Y_1, Y_2, Y_3 be the subgroups $SU_3(q).2$, ${}^2B_2(q) \times {}^2B_2(q)$, $Sp_4(q).2$, respectively, constructed in Example 1.5.

Now Y_3 contains a maximal torus of order $(q-1)^2$, which is hence conjugate to S_1. Thus we may take $S_1 < Y_3$. Moreover $|N_{Y_3}(S_1) : S_1| = 16 = |N_L(S_1) : S_1|$. Hence $N_X(S_1)$ is non-maximal.

Next, Y_1 contains conjugates of S_2 and S_9. Taking $S_i < Y_1$ for $i = 2, 9$, we have $|N_{Y_1}(S_i) : S_i| = |N_L(S_i) : S_i|$. Thus $N_X(S_i)$ is non-maximal for $i = 2, 9$.

Finally, Y_2 contains conjugates of S_3, S_4 and S_5 (note that

$$|S_5| = q^2+1 = (q+\sqrt{(2q)}+1)(q-\sqrt{(2q)}+1)).$$

Once again, taking $S_i < Y_2$ for $i = 3, 4, 5$, we have $|N_{Y_2}(S_i) : S_i| = |N_L(S_i) : S_i|$, so that $N_X(S_i)$ is non-maximal.

To conclude, note that when $q = 8$, $N_L(S_6) < {}^2F_4(2) < L$ (see [**11**, p. 74]); hence $q > 8$ when $i = 6$, as in Table 5.2.

Lemmas 3.4–3.8 complete the proof of the Theorem in the case where $T = D$, apart from the assertion that the subgroups in Table 5.2 give maximal subgroups. The maximality assertions for both the cases $T < D$ and $T = D$ of the Theorem will be proved in the next section.

4. *Proofs of maximality for the subgroups in Tables* 5.1, 5.2

In this section we complete the proof of the Theorem by proving the maximality assertions for the subgroups in Tables 5.1 and 5.2. All the subgroups contain a maximal torus of G'_σ, so for $p > 5$, $q > 11$, the problem of maximality could be solved using [**35**]. However the general case requires more argument.

When L is of type ${}^2B_2$, ${}^2G_2$, G_2, ${}^3D_4$ or ${}^2F_4$, the required maximality results follow from [**39, 19, 20, 29**], so we assume that L is not of one of these types. In particular, G is F_4, E_6, E_7 or E_8.

The section is divided into three subsections. In § 4A, we prove various preliminary results concerning the possible simple subgroups of the exceptional groups. In § 4B, we prove the maximality of subgroups in Table 5.2. Finally in § 4C we handle the subgroups in Table 5.1.

4A. *Simple subgroups of exceptional groups*

Let G be a simple adjoint algebraic group of type F_4, E_6, E_7 or E_8 in characteristic p, and let $\hat{G}$ be the simply connected cover of G.

LEMMA 4.1. *Suppose the alternating group* Alt_c *lies in* G. *Then* $c \leq c(G)$, *where* $c(G)$ *is as follows:*

G	E_8	E_7	E_6	F_4
$c(G)$	21	18	15	15

Proof (cf. [**25**, (3B), Step 1]). Let $G = E_8$. As $\mathrm{Alt}_c < G$, we have $\mathrm{Alt}_{c-3} \leq C_G(x)$ for some $x \in G$ of order 3. If $p = 3$ then $C_G(x)$ lies in a parabolic subgroup, so Alt_{c-3} is isomorphic to a subgroup of a Levi factor. If $p \neq 3$ then x is semisimple and the possible types for $C_G(x)^0$ are given by [**15**, 14.1]. In all cases we conclude that Alt_{c-3} lies in either E_7 or a classical group with natural module of dimension at most 14. In the latter case we have $c - 3 \leq 16$ by [**25**, 1.8C]. In the former case we apply induction. The other possibilities for G are handled similarly.

For a prime r and a group Z, define $\Gamma_r(Z)$ to be the smallest value of n such that Z is contained in $\mathrm{P\Gamma L}_n(F)$, where F is some field of characteristic r. Also let

$$\Gamma_r(Z) = \min\{\Gamma_s(Z) \mid s \text{ prime}, s \neq r\},$$
$$\Gamma(Z) = \min\{\Gamma_r(Z) \mid \text{all } r\}.$$

Lower bounds for $\Gamma(Z)$ with Z a sporadic group are given by [**24**, 2.3.2]. These, together with the well-known facts

$$\Gamma(F_4) = 26 \ (25 \text{ if } p = 3), \quad \Gamma(E_6) = 27, \quad \Gamma(E_7) = 56, \quad \Gamma(E_8) = 248,$$

give Parts (i), (ii) and (iii) of the following lemma. Part (iv) follows from the fact that by [**10**], the 2-rank of E_8 for p odd is only 9.

LEMMA 4.2. *Suppose Z is a sporadic simple group and $Z < G$.*

(i) *If $G = F_4$ or E_6, then $Z \neq J_4$,* Ru, ON, Fi_{23}, Fi_{24}', HN, Ly, Th, BM *or M.*

(ii) *If $G = E_7$, then $Z \neq J_4$,* Fi_{23}, Fi_{24}', Ly, BM *or M.*

(iii) *If $G = E_8$, then $Z \neq \mathrm{Fi}_{24}'$ or M.*

(iv) *If $p \neq 2$ then $Z \neq$* Co_1, Co_2, Fi_{22}, Fi_{23}, Fi_{24}', J_4, BM *or M.*

For groups Z of Lie type in characteristic r, strong lower bounds for $\Gamma_{r'}(Z)$ are given by [**23**]. These, together with the assertions in [**10**] on the abelian rank of G, and use of [**37**, II, 5.16], yield the next lemma. For notational convenience we denote by Λ the following collection of simple groups:

$L_d(s)$, with $d = 2$, $s \leqslant 61$, or $d = 3$, $s \leqslant 5$, or $d = 4$, $s \leqslant 3$, or $d = 5$, $s = 2$,

$U_d(s)$, with $d = 3$, $s \leqslant 5$, or $d = 4$, $s \leqslant 3$, or $d = 5$, $s = 2$,

$\mathrm{PSp}_d(s)$, with $d = 4$, $s \leqslant 7$, or $d = 6$, $s \leqslant 3$,

$\Omega_7(3)$, $\Omega_8^{\pm}(2)$, $G_2(3)$, ${}^3D_4(2)$, ${}^2F_4(2)'$, ${}^2B_2(8)$.

LEMMA 4.3. *Suppose Z is a simple group of Lie type in characteristic r with $r \neq p$, and $Z < G$.*

(i) *If $G = F_4$ or E_6, then Z lies in the collection Λ.*

(ii) *If $G = E_7$, then either $Z \in \Lambda$ or Z is one of $L_3(7)$, $U_3(7)$, $U_3(8)$, $U_4(4)$.*

(iii) *If $G = E_8$, then either Z is as in* (ii) *or Z is one of $L_3(s)$ $(s \leqslant 9)$, $L_4(s)$ $(s \leqslant 5)$, $L_5(3)$, $L_6(s)$ $(s \leqslant 3)$, $U_3(s)$ $(9 \leqslant s \leqslant 16)$, $U_4(5)$, $U_5(s)$ $(s \leqslant 4)$, $U_6(2)$, $\mathrm{P}\Omega_8^+(3)$, $G_2(5)$, ${}^3D_4(3)$, ${}^2B_2(32)$.*

For example, $L_2(s)$ contains a Frobenius group of order $s(s-1)/(2, s-1)$. If $L_2(s) < E_8$ then by [**37**, II, 5.16], this Frobenius group normalizes a torus, whence $W(E_8)$ contains an element of order $(s-1)/(2, s-1)$. This forces $s \leqslant 61$. As another example, $\mathrm{PSp}_4(s)$ $(s \geqslant 8)$, $\mathrm{PSp}_6(s)$ $(s \geqslant 4)$ and $\mathrm{P}\Omega_8^-(3)$ do not lie in E_8, as these groups contain $L_2(r)$ with $r > 61$. As a final example, $U_6(3)$, $\Omega_{10}^-(2)$ and $F_4(2)$ do not lie in E_8 as $U_6(3)$ has 3-rank at least 9 and the other two groups contain $\mathrm{Sp}_8(2)$, which has 2-rank at least 10 (consider appropriate parabolic subgroups).

We shall need to rule out four of the possibilities in Lemma 4.3:

LEMMA 4.4. (i) *Suppose $G = F_4$ and $p \neq 7$. Then* $\mathrm{PSp}_4(7) \nleqslant G$.

(ii) *Suppose $G = E_6$ and $p > 3$. Then $L_4(3) \nleqslant G$.*

(iii) *Neither* Ly *nor* BM *is a subgroup of E_8.*

Proof. (i) Suppose $\mathrm{PSp}_4(7) < G$. Now $\mathrm{PSp}_4(7)$ has a subgroup $7^3.L_2(7)$. The elementary abelian normal subgroup 7^3 of this lies in a torus of G by [**37**, II, 5.8], and hence the factor $L_2(7)$ must lie in $W(F_4)$, a contradiction.

(ii) Suppose $Z = L_4(3) < G$. Let V be an irreducible 27-dimensional module for $\hat{G}$ in characteristic p, and let χ be the character of $\hat{G}$ on V. For an involution $t \in \hat{G}$, we have $\chi(t) = 3$ if $C_G(t) = A_1A_5$ (as $V|A_5$ has composition factors V_6, V_6 and $\Lambda^2 V_6^*$, where V_6 is the usual module), and $\chi(t) = -5$ if $C_G(t) = T_1D_5$ (as $V|D_5$ is the sum of a trivial, a 10-dimensional and a spin module).

Consider the restriction $\chi|Z$. By [**31**], X has precisely two non-trivial p-modular irreducibles of degree 27 or less, say ψ_1 and ψ_2; both have degree 26 and take only values 6 and 2 on involutions. Since $\chi|Z$ must be $1 + \psi_1$ or $1 + \psi_2$, this is incompatible with the possibilities for $\chi(t)$.

(iii) Suppose $\mathrm{BM} < E_8$. Now BM has a subgroup 47.23. By [**37**, II, 5.16], provided $p \neq 47$, this subgroup normalizes a maximal torus of E_8, which conflicts with the fact that 23 does not divide $|W(E_8)|$; for $p = 47$ the subgroup 23.11 gives a similar contradiction. Now suppose $\mathrm{Ly} < E_8$. When $p \neq 67$ the subgroup 67.11 cannot be in E_8, so assume $p = 67$. From the character table of $G_2(5)$ in [**11**], we see that an involution in a subgroup $G_2(5)$ of Ly must have trace 8 or 128 on the E_8 Lie algebra; but involutions in E_8 have trace -8 or 24.

We shall also need the following result.

LEMMA 4.5. *For q even, $U_3(q^2) \nleq F_4(q)$.*

Proof. Suppose $Z < Y$ with $Z \cong U_3(q^2)$, $Y \cong F_4(q)$ and q even. Let $B = UT$ be a Borel subgroup of Z with $|U| = q^6$, $|T| = \frac{1}{3}(q^4 - 1)$, and embed B in a parabolic subgroup P of Y with $U \leqslant O_2(P)$. As $|T| = \frac{1}{3}(q^4 - 1)$, the Levi factor of P must be of type B_3 or C_3. In either case $O_2(P)'$ has order q (see [**12**, § 4]), whereas U' has order q^2, a contradiction.

Recall from § 3 the *primitive prime divisor* q_n; this is a prime which divides $q^n - 1$ but not $q^i - 1$ for $1 \leqslant i < n$, and exists for any positive integers q, n with $n \geqslant 3$, except when $(q, n) = (2, 6)$. Clearly $q_n \equiv 1 \pmod{n}$. The product of all the primitive prime divisors of $q^n - 1$ (counting multiplicities) is denoted by q_n^*. Thus for example, $2_{10}^* = 11$, $3_5^* = 11^2$.

4B. *Maximality of the subgroups in Table* 5.2

In this section we show that the torus normalizers in Table 5.2 are maximal subgroups. We may exclude the cases where $L = F_4(2)$, ${}^2E_6(2)$, as the maximal subgroups of these are given in [**11**]. We show that if L and T_σ are as in Table 5.2 (with L of type F_4, E_6^ε, E_7 or E_8), and

$$L_1 = \begin{cases} L\langle\tau\rangle, & \text{if } L = F_4(2^a) \ (\tau \text{ an element in the coset of a graph automorphism}), \\ L, & \text{if } L \neq F_4(2^a), \end{cases}$$

then $N_{L_1}(T_\sigma)$ is maximal in L_1. By Lemma 1.3, this suffices to show the maximality of $N_X(T_\sigma)$ in any group X satisfying the conditions in Table 5.2.

Let Δ be the Carter diagram of w, where $T_\sigma = (S_{\sigma w})^g$ as in §§ 1, 3, and define $W_\sigma = N_{G_\sigma}(T)/T_\sigma$, as given in Table 5.2. It follows from Lemma 4.6(ii) below that $W_\sigma = N_{G_\sigma}(T_\sigma)/T_\sigma$ also. Let l be the rank of G, and $q = p^a$.

From Table 5.2 we see that $|T_\sigma|$ is a power of a cyclotomic polynomial $f(q)$ in q (recall that L is not of type ${}^2B_2$, ${}^2G_2$ or ${}^2F_4$). Let n be the smallest positive integer such that $f(q)$ divides $q^n - 1$. Observe that $f(q)$ possesses an odd prime

divisor except when $|T_\sigma| = (q-\delta)^l$ $(\delta = \pm 1)$ with $q - \delta = 2^s$ (and $s \geqslant 2$ if $\delta = +1$ and $G = E_l$). In this exceptional case, the group $N_L(T_\sigma)$ contains a Sylow 2-subgroup of L, and so the maximality of $N_L(T_\sigma)$ follows from [**26**]. Thus we may assume that $f(q)$ has an odd prime divisor r such that r is a primitive prime divisor of $q^n - 1$ when $n > 1$. Choose r as large as possible. From Table 5.2, we have $r > 5$ except possibly if $|T_\sigma| = (q \pm 1)^l$ or $(q^2+1)^{l/2}$; and $r \geqslant 5$ unless $|T_\sigma| = (q \pm 1)^l$. Let

$$R = \Omega_1(O_r(T_\sigma)) \cap L.$$

LEMMA 4.6. (i) *W_σ acts irreducibly on R.*
(ii) $C_G(R)^0 = T$.
(iii) *If R is cyclic then $C_G(R) = T$.*

Proof. Let $|T_\sigma| = f(q)^s$. We first claim that R has rank s. Suppose $s > 1$. Provided the Carter diagram Δ is not $F_4(a_1)$ or $E_8(a_i)$ with $i = 3, 6, 8$, it follows from Lemma 3.2 that $T_\sigma \leqslant K_\sigma$, where K is a σ-stable semisimple subgroup of G of type Δ (of type lA_1 if $\Delta = \varnothing$ and $G \neq E_6$, of type $3A_2$ if $G = E_6$). When Δ is $F_4(a_1)$ or $E_8(a_i)$, we have the same conclusion with K_σ of type $({}^2A_2(q))^2$ $(\Delta = F_4(a_1))$, or of type $({}^2A_2(q^2))^2$, $({}^2A_4(q))^2$ or $({}^2A_2(q))^4$ $(\Delta = E_8(a_i)$, $i = 3$, 6 or 8, respectively). We can obtain the structure of T_σ by working in K_σ (of which the structure is given in Table 5.1), and in all cases we conclude that R has rank s, as claimed. On the other hand, if $s = 1$ then R is cyclic: for otherwise some $x \in R^\#$ would centralize a T-root subgroup of G, whence $C_G(x)$ would be reductive, and not a torus. But then $|T_\sigma|$ would not divide $|C_G(x)_\sigma|$, a contradiction. Thus R has rank s, as claimed.

If $s = 1$, there is nothing to prove for (i); also $r > 3$, whence $C_G(R)$ is connected and (iii) follows. Now assume $s > 1$. As above we have $R \leqslant K_\sigma$. From the structure of W_σ given in Table 5.2, and the known action of $N_{K_\sigma}(T_\sigma)$, we obtain (i). Suppose (ii) fails. Then $C_G(R)^0 = DZ$, where D is semisimple and Z a torus. Then $|(T \cap D)_\sigma|$ is a monic polynomial in q which must be a power of $f(q)$. Consequently $T \cap D$ contains an element of $R^\#$. By (i) this forces $R \leqslant Z(D)$, and this is easily seen to be impossible.

LEMMA 4.7. *Let $M = N_{L_1}(T_\sigma)$ be a group in Table* 5.2. *Then M is contained in no other subgroup in Table* 5.2, *and in no subgroup in Table* 5.1.

Proof. Suppose $M < N$, where $N = N_X(D_\sigma) = N_X(D)$ is a group in Table 5.1 or 5.2. Irreducibility of M on R forces $R \leqslant D$. If D is a torus then $D = T$ as $C_G(R)^0 = T$, and hence $M = N$, a contradiction. Hence D is not a torus and N is in Table 5.1.

We claim that R lies in a maximal torus of D. If $r > 5$, this is clear, by [**37**, II, 5.8], so suppose $r \leqslant 5$. Then $|T_\sigma| = (q \pm 1)^l$ or $(q^2+1)^4$. Consider the first possibility. Here $R \cong r^l$ and $r \geqslant 3$. If D has more than one homogeneous component then $D = EF$ with E, F homogeneous, and irreducibility of M on R gives $R \cap E = 1$; but then D/E contains r^l, which is impossible. Thus D has only one homogeneous component. Then $N_D(R)$ cannot induce $W(G)$ on R, except possibly when $G = F_4$, $D = A_2A_2$; but here $r \neq 3$ by the conditions on q in Table 5.2, so R lies in a torus of D. Now take $|T_\sigma| = (q^2+1)^4$, so that $r = 5$ and $G = E_8$.

Then r is a good prime for every factor of D, and only divides $|Z(D)|$ if $D = A_4A_4$; in this case $N_D(R)$ cannot induce $(4 \circ 2^{1+4}).\mathrm{Sp}_4(2)$ on R.

Hence R lies in a maximal torus of D, as claimed. As $C_G(R)^0 = T$, this forces $R \leqslant T \leqslant D$. But then $N(T) \cap N_G(D)$ does not induce W_σ on R, a contradiction.

Now let $M = N_{L_1}(T_\sigma)$ be any torus normalizer in Table 5.2 (with L of type F_4, E_6^ε, E_7 or E_8). Suppose that M is non-maximal in L_1, so that $M < H < L_1$ with H a maximal subgroup of L_1. Define

$$J = F^*(H).$$

LEMMA 4.8. *J is a non-abelian simple group. Also $R < J$ if R is non-cyclic.*

Proof. Suppose that the lemma is false. Then Theorem 2 of [**28**] applies to give the possibilities for H. By Lemma 4.7, H is not of maximal rank, and so H is as in (b)–(e) of [**28**, Theorem 2]; and Lagrange's theorem implies that H is as in (c), that is, H is the centralizer of a graph, field or graph-field automorphism of L of prime order. But then $F^*(H)$ is simple. The last statement follows from Lemma 4.6(i) and the structure of $\mathrm{Out}(J)$.

We now rule out the various possibilities for J. In doing this, the following lemma is useful. Define $W_1 = N_{L_1}(T_\sigma)/(L_1 \cap T_\sigma)$, and let m_R be the size of the smallest orbit on the non-trivial linear characters of R of any normal subgroup W_0 of W_1 with W_1/W_0 cyclic.

LEMMA 4.9. *One of the following holds:*

(i) $\Gamma_{r'}(JM) \geqslant m_R$;

(ii) *r divides the order of the multiplier of J, R has even rank, and* $\Gamma_{r'}(JM) \geqslant |R|^{\frac{1}{2}}$.

Proof. Suppose $JM < \mathrm{P\Gamma L}_n(F)$ with F an algebraically closed field of r'-characteristic. Let $M_0 = M \cap \mathrm{PGL}_n(F)$. Then $R \leqslant M_0$ (use Lemma 4.6(i), (iii)), and $M_0/M_0 \cap T_\sigma \cong W_0$, where $W_0 \lhd W_1$ and W_1/W_0 is cyclic. Let $\bar{R}$ be a minimal preimage of R in $\mathrm{GL}_n(F)$. Since W_0 acts irreducibly on R, either $\bar{R}$ is abelian, or it is an r-group with $\bar{R}' = Z(\bar{R}) = \Phi(\bar{R})$ cyclic. If $\bar{R}$ is abelian then the restriction of the underlying space $V_n(F)$ to $\bar{R}$ is a direct sum of $\bar{R}$-invariant 1-spaces, permuted by W_0, and hence $n \geqslant m_R$. And if $\bar{R}$ is non-abelian then r divides the order of the multiplier of J, and one can see, using [**18**, 2.31], that $n \geqslant |R|^{1/2}$.

LEMMA 4.10. *Lower bounds for m_R are as follows:*

$$\begin{aligned}
L = F_4(q)&: \quad 24 \ (\Delta \neq F_4),\\
&\quad 1 \ (\Delta = F_4);\\
L = E_6^\varepsilon(q)&: \quad 27;\\
L = E_7(q)&: \quad 28;\\
L = E_8(q)&: \quad 120 \ (\Delta = \varnothing \text{ or } 8A_1),\\
&\quad 80 \ (\Delta = 4A_2 \text{ or } E_8(a_8)),\\
&\quad 64 \ (\Delta = 2D_4(a_1)),\\
&\quad 24 \ (\Delta = 2A_4,\ E_8(a_3) \text{ or } E_8(a_6)),\\
&\quad 1 \ (\Delta = E_8 \text{ or } E_8(a_5)).
\end{aligned}$$

Proof. When $L = F_4(q)$ and $\Delta \neq F_4$, each subgroup W_0 described above contains a subgroup $Q_8.3$ acting fixed-point-freely on the non-trivial linear characters of R, giving the lower bound 24. If $L = E_6^\varepsilon(q)$ with $\Delta = \varnothing$, or $L = E_7(q)$, the lower bounds 27, 28 follow from the lower bounds for the degrees of permutation representations of W' (see [**11**]). And when $L = E_6^\varepsilon(q)$ with $\Delta = 3A_2$, the bound 27 follows from consideration of the representation of $3^{1+2}.Q_8$ on R: for if the stabilizer of a character contains a 3-element, then it has order at most 6. Finally, when $L = E_8(q)$, the lower bounds follow from elementary facts concerning the representation of W on R. We give details only for the most difficult case, in which $\Delta = 2D_4(a_1)$. Here $W_\sigma = (4 \circ 2^{1+4}).\mathrm{Sp}_4(2)$, and any normal subgroup with cyclic factor group contains $W_0 = (4 \circ 2^{1+4}).A_6$. Also $R \cong (Z_r)^4$ and $r \equiv 1 \pmod 4$. Let $N = O_2(W_0)$, and let $X(R)$ be the character group of R. The centre of N acts as scalars on $X(R)$, so if $\chi \in X(R)^\#$ then $|N_\chi| \leqslant 2^2$. If $(W_0)_\chi$ contains a 5-element w, then as w normalizes N_χ we must have $N_\chi = 1$, whence $|\chi^{W_0}| \geqslant 64$. Otherwise, $|\chi^{W_0}| \geqslant 5\,|\chi^N| \geqslant 5.16 = 80$. Thus $m_R \geqslant 64$ here, as required.

LEMMA 4.11. *J is not alternating or sporadic.*

Proof. Suppose $J = \mathrm{Alt}_c$. Then J has a faithful representation of degree $c-1$ over any field, so we conclude from the proof of Lemma 4.9 that $c - 1 \geqslant m_R$. By Lemmas 4.1 and 4.10 then, (G, Δ) must be one of (F_4, F_4), (E_8, E_8) and $(E_8, E_8(a_5))$. In the first case $q^4 - q^2 + 1$ divides $|\mathrm{Alt}_c|$, so the product q_{12}^* of the primitive prime divisors of $q^{12} - 1$ divides $c!$. Hence by Lemma 4.1, $q_{12}^* = 13$. This forces $q = 2$ by [**16**, 3.9], contrary to the fact that $q > 2$ for this case (see Table 5.2). Similar arguments exclude the other cases.

Now suppose that J is a sporadic group. The possibilities for J are restricted by Lemma 4.2. First assume that Δ is not $\varnothing$ or lA_1. Then from Table 5.2 we see that q_i^* divides $|T_\sigma|$ for some $i \geqslant 3$. For $L = F_4(q)$, in the respective cases $\Delta = A_2 + \tilde{A}_2$, $D_4(a_1)$, F_4 or $F_4(a_1)$, $|T_\sigma|$ is divisible by $(q_3^*)^2$, $(q_4^*)^2$, q_{12}^* or $(q_6^*)^2$. If $\Delta = A_2 + \tilde{A}_2$ then from the orders of the sporadic groups we see that q_3^* must be 7, and hence q is 2 or 4 by [**16**, 3.9]. Moreover by Lemma 4.2, J must be He or Co_1. However, $\Gamma_2(\mathrm{He}) \geqslant 48$ since He contains a subgroup $7^2.\mathrm{SL}_2(7)$ (see [**11**]), so $\mathrm{He} \nleqslant F_4(q)$; and $|\mathrm{Co}_1|$ does not divide $|F_4(4)|$. If Δ is $D_4(a_1)$, F_4 or $F_4(a_1)$, then, in the respective cases, $q_4^* = 5$, $q_{12}^* = 13$ or $q_6^* = 7$; this conflicts with [**16**, 3.9] in each case. Entirely similar arguments deal with the other possibilities for L; the only case which does not immediately succumb occurs when $L = E_8(q)$ and $\Delta = 2D_4(a_1)$. Here $(q_4^*)^4$ divides $|T_\sigma|$ so $q_4^* = 5$, q is 2 or 3 by [**16**, 3.9], and J is Ly, Co_1, HN or BM. Since $|E_8(2)|_5 = |E_8(3)|_5 = 5^5$, in fact $J = \mathrm{Co}_1$. But Co_1 does not have a subgroup $(Z_5)^4$ (see [**11**]).

To complete the proof, we must rule out the cases where Δ is $\varnothing$ or lA_1. When $L = F_4(q)$, $q = 2^c \geqslant 4$ and either $(2_c^*)^4$ or $(2_{2c}^*)^4$ divides $|T_\sigma|$. We now obtain a contradiction using the above arguments. Similar reasoning applies to all cases when q is even. So suppose q is odd. By an earlier reduction, r is odd, and so $q > 3$. The fact that $|N(T_\sigma)|$ divides $|\mathrm{Aut}\,J|$ forces $r = 3$. Thus 3^9 divides $|J|$, so J is Co_1, Fi_{22}, Fi_{23}, Fi_{24}', Th, BM or M. Now Lemma 4.2(iv) forces $J = \mathrm{Th}$. Then by Lemma 4.2(i), G is E_7 or E_8, and hence 3^{11} divides $|N(T_\sigma)|$, whereas 3^{11} does not divide $|\mathrm{Th}|$.

LEMMA 4.12. *J is not a group of Lie type.*

Proof. Suppose that the lemma is false. Consider first the case where J is of Lie type in characteristic r. Then Lemma 4.6(i), (iii) implies that $R < J$, and so by [**3**, 3.12], $N_L(T_\sigma)$ lies in a parabolic subgroup of JM. Let $L = F_4(q)$. When $\Delta = \varnothing$ or $4A_1$, $r^4.W$ lies in a parabolic subgroup of JM, and either $r \geqslant 5$ or $r = 3$, $q = 8$ and $\Delta = 4A_1$. In any case there are no possibilities for J, by Lemma 4.3. If $\Delta = A_2 + \tilde{A}_2$ or $F_4(a_1)$ then $r \geqslant 7$ and $r^2.[72]$ lies in a parabolic, forcing $J = \mathrm{PSp}_4(7)$ by Lemma 4.3 (here [72] denotes a group of order 72). This conflicts with Lemma 4.4(i). Next consider $\Delta = D_4(a_1)$. Here $r \geqslant 5$ and $r^2.[96] < JM$, forcing $J = \mathrm{PSp}_4(5)$ by Lemma 4.3. Then $N_L(T_\sigma) \nleqslant JM$ as $q > 2$. Finally, if $\Delta = F_4$ then $r \geqslant 13$ and $T_\sigma \cong Z_{q^4-q^2+1}$, so $J = L_2(r)$ with $r = q^4 - q^2 + 1$ by Lemma 4.3. But then $|N_L(T_\sigma)|$ does not divide $|N_{JM}(T_\sigma)|$ (note that $q > 2$ from Table 5.2). This deals with the case where $L = F_4(q)$.

Next consider $L = E_6^\varepsilon(q)$. If $\Delta = \varnothing$ then $r^6.W$ (or $3^5.W$ if $r = 3$ and $(9, q - \varepsilon) = 3$) is contained in JM, forcing J to be $\Omega_7(3)$ by Lemma 4.3. Then we must have $N_L(T_\sigma) = 3^5.W$, so that $L = {}^2E_6(2)$ from Table 5.2, a case already excluded. When $\Delta = 3A_2$, we have $r^3.[648] < \mathrm{Aut}\, J$ and $r \geqslant 7$, giving a contradiction by Lemma 4.3 again. Finally, the cases where L is $E_7(q)$ or $E_8(q)$ yield to entirely similar arguments.

Thus J is of Lie type in r'-characteristic. Now Lemmas 4.9 and 4.10 give lower bounds for $\Gamma_{r'}(JM)$. First let $L = F_4(q)$. Then either $\Delta = F_4$, or $\Gamma_{r'}(JM) \geqslant 24$, or r divides $|M(J)|$ (where $M(J)$ is the multiplier of J) and $\Gamma_{r'}(JM) \geqslant |R|^{\frac{1}{2}}$. If $|R|^{\frac{1}{2}} < 24$ then from Table 5.2 we must have either $r \geqslant 7$ or $|T_\sigma| = (q+1)^4$ with $q = 4$ or 8. It is easily checked that there are no possibilities for J with r dividing $|M(J)|$, $R < J$ and $|J|$ dividing $|L|$. Thus either $\Delta = F_4$ or $\Gamma_{r'}(JM) \geqslant 24$. Suppose the latter holds. Then J is either an exceptional group of type F_4^ε, E_6^ε, E_7 or E_8, or of type D_4. Moreover J must be of characteristic 2, otherwise $\Gamma_2(J) > 26 = \Gamma_2(L)$. If J is exceptional then J must be $F_4^\varepsilon(2^b)$, and we check that then $N(T_\sigma) \nleqslant \mathrm{Aut}\, J$. And if $J = D_4(2^b)$, then the fact that $N(T_\sigma) \leqslant \mathrm{Aut}\, J$ forces $J = D_4(q)$. The conjugacy class of J in L is identified by [**25**, 7.3]; but this implies that J is not normalized by an element in the coset of a graph automorphism of L, so $N_{L_1}(T_\sigma) \nleqslant N_{L_1}(J)$, a contradiction.

In the case where $\Delta = F_4$ we have $(q^4 - q^2 + 1).12 < JM$. In particular, q_{12}^* divides $|\mathrm{Aut}\, J|$. Arguing as in [**17**, p. 520], we see that this forces J to be $L_2(q^6)$, $L_3(q^4)$, $U_3(q^2)$, $\mathrm{Sp}_4(q^3)$, $G_2(q^2)$, ${}^3D_4(q)$ or ${}^2F_4(q)$. Now $L_2(q^6)$, $L_3(q^4)$, $\mathrm{Sp}_4(q^3)$ and $G_2(q^2)$ are not subgroups of L, by consideration of semisimple elements of large order: for example, $G_2(q^2)$ has an element of order $q^4 + q^2 + 1$ (in a subgroup $\mathrm{SL}_3(q^2)$), which cannot lie in $F_4(q)$ as $q^4 + q^2 + 1$ does not divide the order of any torus of $F_4(q)$ (see [**6**]). Also $\mathrm{Aut}({}^2F_4(q))$ contains no element of order $q^4 - q^2 + 1$, so cannot contain $N(T_\sigma)$. If $J = {}^3D_4(q)$ then J is not normalized by an element in the coset of a graph automorphism of L, by [**25**, 7.3]. Finally $J \neq U_3(q^2)$ by Lemma 4.5. This settles the case where $L = F_4(q)$.

Next let $L = E_6^\varepsilon(q)$. If Lemma 4.9(ii) holds then $\Delta = \varnothing$ and $|R|^{\frac{1}{2}} \geqslant 27$. Thus $\Gamma_{r'}(JM) \geqslant 27$ by Lemma 4.10. Consequently J must be of type E_6^δ, E_7 or E_8. As $\Gamma_p(J) \leqslant 27 = \Gamma_p(L)$, we have $J = E_6^\delta(p^b)$. Then $N(T_\sigma) < \mathrm{Aut}\, J$ forces $p^b = q$, a contradiction. The cases where L is $E_7(q)$ or $E_8(q)$ are handled with similar arguments.

This completes the maximality proofs for the groups in Table 5.2.

4C. *Maximality of the subgroups in Table* 5.1

We complete the proof of the Theorem by proving the maximality of the subgroups in Table 5.1. Thus let L be a simple group of type F_4, E_6^ε, E_7 or E_8 over $\mathbf{F}_q$. We show that each subgroup $N_X(D)$ given in Table 5.1 is maximal in X, where $F^*(X) = L$ and X satisfies the conditions in the table. As in § 4B, we may exclude $L = F_4(2)$ or ${}^2E_6(2)$ by [**11**]. Also $L \neq E_6(2)$ by [**22**].

Let $M = N_X(D)$ be a subgroup in Table 5.1, and suppose M is non-maximal in X, so that $M < H < X$ for some maximal subgroup H of X. Let $J = F^*(H)$. Arguing as in the proof of Lemma 4.8, we have

LEMMA 4.13. *J is a non-abelian simple group.*

We now consider separately two cases:

(A) every factor of $(D')_\sigma$ is quasisimple, and

(B) some factor of $(D')_\sigma$ is $A_1(2)$, $A_1(3)$ or ${}^2A_2(2)$.

Case (A). Assume every factor of $(D')_\sigma$ is quasisimple, and write $D_1 = D_\sigma^{(\infty)}$, $D_0 = D_\sigma^{(\infty)}/Z(D_\sigma^{(\infty)})$. Then D_0 is not $L_2(q)'$ with q odd, $G = E_l$ and $l = 7$ or 8; for otherwise, D_1 is a commuting product of fundamental subgroups $\mathrm{SL}_2(q)$, and hence $N(D_\sigma)$ is maximal by [**1**, Theorems 5, 6].

LEMMA 4.14. *J is not alternating or sporadic.*

Proof. Suppose first that $J = \mathrm{Alt}_c$. Then $c \leqslant c(G)$ with $c(G)$ as in Lemma 4.1. However none of the groups D_σ in Table 5.1 can lie in Sym_c with $c \leqslant c(G)$, as can be seen from the lower bounds for the degrees of permutation representations of groups of Lie type given by [**23**].

Now suppose J is a sporadic group. Using Table 5.1 and Lemma 4.2, we check that the only possibility with $|M|$ dividing $|\mathrm{Aut}\, J|$ and every factor of D_σ quasisimple is

$$L = F_4(4), \quad M \cap L = 3.(L_3(4) \times L_3(4)).S_3, \quad J = \mathrm{Co}_1.$$

However $|\mathrm{Co}_1|$ does not divide $|F_4(4)|$.

LEMMA 4.15. *J is not of Lie type in p'-characteristic.*

Proof. Suppose that the lemma is false. The possibilities for J are given by Lemma 4.3. Now $\Gamma_{p'}(J) \geqslant \Gamma_{p'}(D_1)$, and from [**23**] we calculate that $\Gamma_{p'}(D_1) \geqslant r(G)$, where $r(G) = 8$ if $G = F_4$, 12 if $G = E_6$, 14 if $G = E_7$, and 16 if $G = E_8$. (Here, the lower bounds $r(G)$ for $G = F_4$, E_6 or E_8 arise when D_0 is a direct product of groups $L_3(4)$; and for $G = E_7$ the lower bound arises when $q = 4$ and $D_0 = L_2(4)^7$.) If J is a classical group then its natural module has dimension at least $r(G)$, which is easily seen, using [**23**], to conflict with the fact that $\Gamma_{p'}(J) \leqslant \Gamma(G)$ (which is at most 26, 27, 56 or 248, according as G is F_4, E_6, E_7 or E_8). And if J is an exceptional group then the facts that $\Gamma_{p'}(J) \geqslant r(G)$ and $\Gamma_p(J) \leqslant \Gamma(G)$ force $J = F_4(2)$, ${}^2F_4(2)'$ or ${}^3D_4(2)$ (with $G = E_7$ or E_8 if $J = F_4(2)$). But then J contains no subgroup isomorphic to D_1.

LEMMA 4.16. *J is not of Lie type in characteristic p.*

Proof. Suppose that the lemma is false. By [**27**, § 5, Fact 1], the Lie rank of J is at most that of L. Using the primitive prime divisors q_i dividing $|D_1|$, and the method of [**25**, (3B)], we can quickly list all possibilities for J such that $|D_1|$ divides $|J|$ and $|J|$ divides $|L|$. Also, if D_0 has more than one simple factor then the centralizer of a unipotent element in a factor lies in a parabolic subgroup of J and contains the other factors. Excluding various other possible J in the list which clearly do not contain suitable groups D_1, we reduce to the list of possibilities shown in Table E. Let $L = F_4(q)$. In the first case, $C_4(q)$ (q odd) is not a subgroup of L, since the centralizer of an involution of type $C_2 \times C_2$ in $C_4(q)$ does not lie in an involution centralizer in L (which is of type $A_1 \times C_3$ or B_4). In the other two cases, the conjugacy class of J in L is identified in [**25**, 7.1], from which we see that J is not normalized by a graph automorphism of L; hence $N_X(D) \nleq N_X(J)$, a contradiction.

TABLE E

L	D_0	J
$F_4(q)$	$L_2(q) \times \mathrm{PSp}_6(q)$ (q odd)	$\mathrm{PSp}_8(q)$
	$\mathrm{Sp}_4(q) \times \mathrm{Sp}_4(q)$ (q even)	$\mathrm{PSp}_8(q)$
	$\mathrm{Sp}_4(q^2)$ (q even)	$\mathrm{PSp}_8(q)$
$E_6^\varepsilon(q)$	$L_3^\varepsilon(q^3)$	$L_4^\varepsilon(q^3)$
	${}^3D_4(q)$	$F_4(q)$
	$D_4(q)$	$B_4(q)$, $F_4(q)$, $D_5^\varepsilon(q)$
$E_8(q)$	$D_4(q^2)$	$D_8(q)$
	$D_4(q)^2$	$D_8(q)$
	$U_3(q^4)$	$G_2(q^4)$

Next consider $L = E_6^\varepsilon(q)$. In the first case $L_4^\varepsilon(q^3)$ is not a subgroup of L, as is shown in [**25**, (5B) and (6B)]. If $D_0 = D_4(q)$ and $J \neq F_4(q)$ then $\mathrm{Aut}\, J$ cannot induce a triality automorphism on D_0, so $N(D) \nleq N(J)$. And in the remaining cases $N(D) \nleq N(J)$ since $C_L(D_1) \neq 1$, while $C_{N(J)}(D_1) = C_L(J) = 1$ (see [**25**, 5.3, 6.3, 7.2, 7.3]).

Finally, suppose $L = E_8(q)$. Since $D_8(q)$ does not induce a triality automorphism on any subgroup $D_4(q)$ or $D_4(q^2)$, we must have $J = G_2(q^4)$. This group has a cyclic maximal torus of order $q^8 + q^4 + 1$ (in a subgroup $\mathrm{SL}_3(q^4)$). When $q > 2$, $E_8(q)$ has no torus of this order (see [**6**]). And when $q = 2$, there is an element $x \in J$ of order 3 such that $C_J(x) = \mathrm{SL}_3(16)$. Then $C_L(x)^{(\infty)}$ must be ${}^2E_6(2)$ or $\mathrm{SU}_9(2)$, neither of which has an element of order $2^8 + 2^4 + 1$.

Case (B). Now suppose some factor of $(D')_\sigma$ is $A_1(2)$, $A_1(3)$ or ${}^2A_2(2)$ (and q is 2, 3 or 2, respectively). By [**25**], the normalizer of a fundamental subgroup $\mathrm{SL}_2(q)$ is maximal in L; also for $L = E_l(3)$ with $l = 7$ or 8, the normalizer of a commuting product of l fundamental subgroups $\mathrm{SL}_2(3)$ is maximal by [**1**, Theorems 5, 6]. And when $L = E_7(3)$ and $N(D_\sigma) = N(2^2.(L_2(3)^3 \times P\Omega_8^+(3))$, this subgroup is maximal by [**26**]. Since we have already excluded $L = F_4(2)$, $E_6^\varepsilon(2)$, the cases

remaining to be investigated are as follows:

$$L = E_7(2): \quad N_L(D_\sigma) = 3.(U_3(2) \times U_6(2)).S_3 \text{ or } (L_2(2)^3 \times \Omega_8^+(2)).S_3,$$
$$L = E_8(2): \quad N_L(D_\sigma) = 3.(U_3(2) \times {}^2E_6(2)).S_3 \text{ or } 3^2.(U_3(2)^4).3^2.2.S_4.$$

Assume first that D_σ has a factor $SU_3(2)$, and let R be the product of all the normal subgroups $SU_3(2)$ in D_σ. Let $S \leqslant R$ with $S/O_3(R) = \Omega_1(R/O_3(R))$. Then S is generated by long root subgroups of L. Arguing as in the proof of [**34**, (2.8)], we find that $O_3(R) \leqslant J$ and, as JS is generated by long root elements, either J is of Lie type over $\mathbf{F}_2$, or $J = P\Omega_n^\varepsilon(3)$, Alt_n, Fi_{22}, Fi_{23} or Fi_{24}'. Using Lemmas 4.2, 4.3 and knowledge of centralizers of 3-elements in J from [**11**], we obtain a contradiction. Finally, let $N(D_\sigma) = (L_2(2)^3 \times \Omega_8^+(2)).S_3$ with $L = E_7(2)$, and set $V = (D_\sigma)^{(\infty)} \cong \Omega_8^+(2)$. Then $V \leqslant J$. Using [**34**, (2.8)] as above, we see that J is of Lie type over $\mathbf{F}_2$. Moreover, J must induce a triality automorphism on V, and it follows that J is $F_4(2)$ or $E_6^\varepsilon(2)$. But then $N_L(D_\sigma) \nleqslant N_L(J)$, a contradiction.

This completes the proof of the maximality of the subgroups in Table 5.1, finishing the proof of the Theorem.

5. *The tables of maximal subgroups of maximal rank*

This final section consists of Tables 5.1 and 5.2, giving the maximal subgroups of maximal rank in the Theorem.

TABLE 5.1. *Maximal subgroups $N_{G_\sigma}(D)$ with $T < D$*
Notation: $d = (2, q-1)$, $\varepsilon = \pm 1$, $e = (3, q-\varepsilon)$

L	$N_{G_\sigma}(D)$	Remarks
${}^2B_2(q)$	none	
${}^2G_2(q)$, $q = 3^{2a+1} > 3$	$2 \times L_2(q)$	
${}^3D_4(q)$	$d.(L_2(q) \times L_2(q^3).d$	
	$(\mathrm{SL}_3^\varepsilon(q) \circ (q^2 + \varepsilon q + 1)).f.2$	$\varepsilon = \pm 1$, $f = (3, q^2 + \varepsilon q + 1)$
$G_2(q)$, $q > 2$	$d.(L_2(q) \times L_2(q)).d$	
	$\mathrm{SL}_3^\varepsilon(q).2$	$\varepsilon = \pm 1$, $1 + \delta_{p,3}$ classes (fused by graph aut. if $p = 3$)
${}^2F_4(q)'$, $q = 2^{2a+1}$	$\mathrm{SU}_3(q).2$	$X = {}^2F_4(2)$ when $q = 2$
	$\mathrm{PGU}_3(q).2$	$q > 2$
	$({}^2B_2(q) \times {}^2B_2(q)).2$	$q > 2$
	$\mathrm{Sp}_4(q).2$	$X = {}^2F_4(2)'$ when $q = 2$

TABLE 5.1. (*Continued*)

L	$N_{G_\sigma}(D)$	Remarks
$F_4(q)$	$2.(L_2(q) \times \mathrm{PSp}_6(q)).2$	q odd
	$d.\Omega_9(q)$	$(2, p)$ classes (fused by graph aut. if $p = 2$)
	$d^2.\mathrm{P}\Omega_8^+(q).S_3$	$(2, p)$ classes
	${}^3D_4(q).3$	$(2, p)$ classes
	$e.(L_3^\varepsilon(q) \times L_3^\varepsilon(q)).e.2$	$\varepsilon = \pm 1$
	$(\mathrm{Sp}_4(q) \times \mathrm{Sp}_4(q)).2$	$p = 2$, X contains graph aut.
	$\mathrm{Sp}_4(q^2).2$	$p = 2$, X contains graph aut.
$E_6^\varepsilon(q)$, $\varepsilon = \pm 1$	$d.(L_2(q) \times L_6^\varepsilon(q)).de$	
	$e.(L_3^\varepsilon(q))^3.e^2.S_3$	
	$f.(L_3(q^2) \times L_3^{-\varepsilon}(q)).g.2$	$f = (3, q + \varepsilon)$, $g = (3, q^2 - 1)$
	$L_3^\varepsilon(q^3).(e \times 3)$	
	$d^2.(P\Omega_8^+(q) \times (q - \varepsilon/d)^2).d^2.S_3$	$q > 2$ if $\varepsilon = +1$
	$({}^3D_4(q) \times (q^2 + \varepsilon q + 1)).3$	$X \geqslant G_o$ if $(q, \varepsilon) = (2, -1)$
	$h.(P\Omega_{10}^\varepsilon(q) \times (q - \varepsilon/h)).h$	$h = (4, q - \varepsilon)$, X contains graph aut. if $\varepsilon = +1$
$E_7(q)$	$d.(L_2(q) \times P\Omega_{12}^+(q)).d$	
	$f.L_8^\varepsilon(q).g.(2 \times (2/f))$	$\varepsilon = \pm 1$, $f = (4, q - \varepsilon)/d$, $g = (8, q - \varepsilon)/d$
	$e.(L_3^\varepsilon(q) \times L_6^\varepsilon(q)).de.2$	$\varepsilon = \pm 1$
	$d^2.(L_2(q)^3 \times P\Omega_8^+(q)).d^3.S_3$	
	$(L_2(q^3) \times {}^3D_4(q)).3d$	
	$d^3.(L_2(q)^7).d^4.L_3(2)$	$q > 2$
	$L_2(q^7).7d$	
	$e.(E_6^\varepsilon(q) \times (q - \varepsilon/e)).e.2$	$\varepsilon = \pm 1$
$E_8(q)$	$d.P\Omega_{16}^+(q).d$	
	$d.(L_2(q) \times E_7(q)).d$	
	$f.L_9^\varepsilon(q).e.2$	$\varepsilon = \pm 1$, $f = (9, q - \varepsilon)/e$
	$e.(L_3^\varepsilon(q) \times E_6^\varepsilon(q)).e.2$	$\varepsilon = \pm 1$
	$g.(L_5^\varepsilon(q))^2.g.4$	$\varepsilon = \pm 1$, $g = (5, q - \varepsilon)$
	$\mathrm{SU}_5(q^2).4$	
	$\mathrm{PGU}_5(q^2).4$	
	$d^2.(P\Omega_8^+(q))^2.d^2.(S_3 \times 2)$	
	$d^2.P\Omega_8^+(q^2).(S_3 \times 2)$	
	$({}^3D_4(q))^2.6$	
	${}^3D_4(q^2).6$	
	$e^2.(L_3^\varepsilon(q))^4.e^2.\mathrm{GL}_2(3)$	$\varepsilon = \pm 1$
	$(U_3(q^2))^2.8$	
	$U_3(q^4).8$	
	$d^4.(L_2(q))^8.d^4.\mathrm{AGL}_3(2)$	$q > 2$

TABLE 5.2. *Maximal subgroups* $N_{G_\sigma}(T)$

L	Carter diagram	$\lvert T_\sigma \rvert$	$N_{G_\sigma}(T_\sigma)/T_\sigma$	Conditions
${}^2B_2(q)$,		$q-1$	Z_2	
$q=2^{2a+1}>2$		$q+\sqrt{(2q)}+1$	Z_4	
		$q-\sqrt{(2q)}+1$	Z_4	
${}^2G_2(q)$,		$q+1$	Z_6	
$q=3^{2a+1}>3$		$q+\sqrt{(3q)}+1$	Z_6	
		$q-\sqrt{(3q)}+1$	Z_6	
${}^3D_4(q)$		$(q^2+q+1)^2$	$\mathrm{SL}_2(3)$	
		$(q^2-q+1)^2$	$\mathrm{SL}_2(3)$	
		q^4-q^2+1	Z_4	
$G_2(q)$, $q=3^a$	$\varnothing$	$(q-1)^2$	D_{12}	$q>3$
X contains	$A_1+\tilde{A}_1$	$(q+1)^2$	D_{12}	$q>3$
graph aut.	A_2	q^2+q+1	Z_6	$q>3$
	G_2	q^2-q+1	Z_6	$q>3$
${}^2F_4(q)'$,		$(q+1)^2$	$\mathrm{GL}_2(3)$	$q>2$
$q=2^{2a+1}$		$(q+\sqrt{(2q)}+1)^2$	$Z_4\circ\mathrm{GL}_2(3)$	
		$(q-\sqrt{(2q)}+1)^2$	$Z_4\circ\mathrm{GL}_2(3)$	$q>8$
		$q^2+\sqrt{(2q^3)}+q+\sqrt{(2q)}+1$	Z_{12}	$X=L.2$ if $q=2$
		$q^2-\sqrt{(2q^3)}+q-\sqrt{(2q)}+1$	Z_{12}	$q>2$
$F_4(q)$, $q=2^a$,	$\varnothing$	$(q-1)^4$	$W(F_4)$	$q\neq 2, 4$
X contains	$4A_1$	$(q+1)^4$	$W(F_4)$	$q>2$
graph aut.	$A_2+\tilde{A}_2$	$(q^2+q+1)^2$	$3\times\mathrm{SL}_2(3)$	
	$D_4(a_1)$	$(q^2+1)^2$	$Z_4\circ\mathrm{GL}_2(3)$	$q>2$
	F_4	q^4-q^2+1	Z_{12}	$q>2$
	$F_4(a_1)$	$(q^2-q+1)^2$	$3\times\mathrm{SL}_2(3)$	$q>2$
$E_6^\varepsilon(q)$, $\varepsilon=\pm1$	$\varnothing$	$(q-\varepsilon)^6$	$W(E_6)$	$q\geqslant 5$ if $\varepsilon=+1$, $X\geqslant G_\sigma$ if $(\varepsilon, q)=(-1, 2)$
	$3A_2$	$(q^2+\varepsilon q+1)^3$	$3^{1+2}.\mathrm{SL}_2(3)$	$(\varepsilon, q)\neq(-1, 2)$
$E_7(q)$	$\varnothing$	$(q-1)^7$	$W(E_7)$	$q\geqslant 5$
	$7A_1$	$(q+1)^7$	$W(E_7)$	
$E_8(q)$	$\varnothing$	$(q-1)^8$	$W(E_8)$	$q\geqslant 5$
	$8A_1$	$(q+1)^8$	$W(E_8)$	
	$2A_4$	$(q^4+q^3+q^2+q+1)^2$	$5\times\mathrm{SL}_2(5)$	
	$4A_2$	$(q^2+q+1)^4$	$2.(3\times U_4(2))$	
	$2D_4(a_1)$	$(q^2+1)^4$	$(4\circ 2^{1+4}).A_6.2$	
	E_8	$q^8+q^7-q^5-q^4-q^3+q+1$	Z_{30}	
	$E_8(a_3)$	$(q^4-q^2+1)^2$	$Z_{12}\circ\mathrm{GL}_2(3)$	
	$E_8(a_5)$	$q^8-q^7+q^5-q^4+q^3-q+1$	Z_{30}	
	$E_8(a_6)$	$(q^4-q^3+q^2-q+1)^2$	$5\times\mathrm{SL}_2(5)$	
	$E_8(a_8)$	$(q^2-q+1)^4$	$2.(3\times U_4(2))$	$q>2$

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