Modular representation theory

The algebra $\mathbb{F}_\ell W$ is not semi-simple when $\ell \mid |W|$. It is a product of indecomposable algebras (blocks), corresponding to primitive central idempotents. These blocks can also be studied in characteristic 0.

Choose an extension $K$ of $\mathbb{Z}_\ell$ such that $KW$ splits, and let $\mathcal{O}$ be the corresponding ring of integers (the integral closure of $\mathbb{Z}_\ell$ in $K$); it is a complete discrete valuation ring, whose quotient by the maximal ideal is a finite field $\mathbb{F}$, an extension of $\mathbb{F}_\ell$. The idempotents of $\mathbb{F}W$ lift to $\mathcal{O}W$, and thus the blocks of $\mathcal{O}W$ are in bijection with those of $\mathbb{F}W$. The $K$-representations of $W$ admit invariant lattices and give rise to representations of $\mathcal{O}W$; they can thus be grouped under blocks.

Broué’s conjecture for the principal block

The problem is to understand the structure of the blocks. Sometimes a block can be shown to be

- isomorphic to some simpler block (occurring in a smaller group),
- or Morita equivalent (the module categories are equivalent)
- or derived equivalent.

(Broué conjecture for the principal block)

If the $\ell$-Sylow $S$ of $G$ is abelian, then the principal block (the one containing the trivial representation) of $G$ is derived equivalent to the principal block of $N_G(S)$.

This says in particular that there should be a bijection between the characters in the two blocks.

For finite reductive groups, this conjecture has been proved for $\text{GL}_n$ by Chuang and Rouquier. It is open in general, but gives a particularly nice description of the principal block since $\mathcal{O}(N_G(S))$ is the algebra of a complex reflection group acting on a power of $(\mathbb{Z}/\Phi_d(q))^r$. 

Bimodule induction, Harish-Chandra induction

If $G$ and $L$ are finite groups and $M$ is a $G$-module-$L$, given $E$ an $L$-module we define the induced via $M$ as $M \otimes_{OL} E$.
If $L < G$ and $M = OG$, we get the usual induction. We get also restriction the same way, by considering the dual module.

The idea of Harish-Chandra induction is to build representations of $G^F$ by starting with a smaller group of the “same type”, that is a Levi subgroup; we start with an $F$-stable Levi decomposition $P = V \ltimes L$. Harish-Chandra induction $R^G_L$ takes a representation of $L^F$, inflates it (extends it trivially) to $P^F$ and then induces. The first idea would be to consider $\text{Ind}^{G^F}_L$ but this has too many components.
It is equivalently bimodule induction through the module $O(G^F/V^F)$. One can show that it does not depend on the parabolic chosen.
Harish-Chandra restriction $^*R^G_L$ is defined through the dual module.

Cuspidal representations

A representation $\rho$ of $OG^F$ is cuspidal if every proper Harish-Chandra restriction of $\rho$ is trivial.
The main theorem about Harish-Chandra induction is

Theorem

Let $\gamma$ be an irreducible $OG^F$-module. Then

- there is, up to $G^F$-conjugacy, a unique pair $(L, \lambda)$, where $L$ comes from an $F$-stable Levi decomposition of a parabolic subgroup, and $\lambda$ is a simple $L^F$-module, such that $\gamma$ is a composition factor of the head of $R^G_L(\lambda)$.

- (over $K$) the components of $R^G_L(\lambda)$ correspond to $\text{Irr}(W_G(L, \lambda))$ with multiplicities the corresponding dimensions, where $W_G(L, \lambda) = \{ g \in N_G(L) \mid g \lambda = \lambda \}/L^F$.

Over $O$, the same remains true if one replaces $W_G(L, \lambda)$ by the corresponding Hecke algebra.
Morita equivalence

If $A$ and $B$ are Morita equivalent, there exists an $A$-module-$B$ say $M$, such that the equivalence $\text{mod } B \rightarrow \text{mod } A$ is given by $X \mapsto M \otimes_B X$. The typical example of Morita equivalence is between $\text{Mat}_n(A)$ and $A$.

Choose $\ell|q - 1$ and $\ell \nmid |W|$. Then the $\ell$-Sylow is a subgroup of the $\Phi_1$-Sylow, which is a subtorus of $T$ where $B = U \times T$ is an $F$-stable decomposition.

Theorem (Puig)

*In the above situation, the Harish-Chandra induction $R^G_T$ induces a Morita equivalence between the principal $\ell$-block of $G^F$ and that of $N_{G^F}(T)$.***

The principal $\ell$-block of $T^F$ consists of characters of order a power of $\ell$. What is not obvious is that $\text{End}_{G^F}(O(G^F/U^F)) \simeq O(N_{G^F}(T))$. The action of $T^F$ is clear but $W^F$ acts through an isomorphism of the Hecke algebra with the algebra of $W$.

Derived equivalence

Similarly to the result for a Morita equivalence, Rickard’s theorem says that if $\text{mod } A$ and $\text{mod } B$ are derived equivalent there exists then a *tilting complex* $T$, a complex $T$ in $D^b(A)$ of finitely generated and projective $A$-modules, such that

- $\text{Hom}_{D^b(A)}(T, T[k]) = 0$ for $k \neq 0$.
- $\text{End}_{D^b(A)}(T) \simeq B$.
- $T$ “generates” the derived category, that is for any other complex $X$ there exists $i$ such that $\text{Hom}_{D^b(A)}(T, X[i]) \neq 0$.

The equivalence is then given by $X \mapsto T \otimes_B X$.

For Broué’s conjecture, we take an abelian $\ell$-Sylow $S_\ell$, set $A = \text{principal block of } G^F$ and $B = \text{principal block of } N_{G^F}(S_\ell) = N_{G^F}(S)$ where $S$ is the unique $\Phi_d$-Sylow containing $S_\ell$.

The conjectural construction of $T$ uses $\ell$-adic cohomology to define an induction from $L^F$ where $L = C_G(S)$ to $G^F$. 
Characters from $\ell$-adic cohomology
Assume that the finite group $G$ acts on the variety $X$, the action commuting to that of $F$. Then for $g \in G$ and any $n$, the endomorphism $gF^n$ of $X$ is the Frobenius for an $\mathbb{F}_{q^n}$-structure.

The virtual character of $G$ given by $g \mapsto \sum_i (-1)^i \text{Trace}(g | H^i_c(X, \mathcal{O}))$ is given by $\lim_{t \to \infty} -\sum_{n=1}^{\infty} |X^{gF^n}| t^n$.

Indeed, consider a basis of $H^i_c$ where $F$ is triangular and $g$ diagonal. Given an eigenvalue $\lambda$ of $F$ in $H^i$, let $\lambda_g$ be the corresponding eigenvalue of $g$ and set $\epsilon_{\lambda} = (-1)^i$. We then get

$$-\sum_{n=1}^{\infty} |X^{gF^n}| t^n = -\sum_{n=1}^{\infty} \sum_i (-1)^i \text{Trace}(gF^n | H^i_c(X, \mathcal{O})) =$$

$$-\sum_{\lambda} \epsilon_{\lambda} \lambda_g (\sum_n \lambda t)^n = \sum_{\lambda} \epsilon_{\lambda} \lambda_g \frac{-\lambda t}{1 - \lambda t}$$

Deligne-Lusztig induction

This generalizes Harish-Chandra induction to the case of an $F$-stable Levi $L$ which is in no $F$-stable parabolic subgroup. Let $P = V \times L$ be the Levi decomposition of $P$. The Deligne-Lusztig variety

$$Y_V = \{ gV \in G/V \mid gV \cap F(gV) \neq \emptyset \}$$

has a left action of $G^F$ and a right action of $L^F$.
If $F(V) = V$ it reduces to the discrete variety $G^F/V^F$ since the equation for $g$ is $g^{-1}F(g) \in V$ and we apply Lang’s theorem in the connected group $V$ to write $g^{-1}F(g) = v^{-1}F(v)$ and find an $F$-stable $g v^{-1}$.

$$\sum_i (-1)^i H^i_c(Y_V, \mathcal{O})$$ is the (virtual) $G^F$-module-$L^F$ defining
Deligne-Lusztig induction $R^G_{L^F}$.
If $F(V) = V$ it reduces to $\mathcal{O}[G^F/V^F]$, giving Harish-Chandra induction.
**Broué conjectures**

We have a similar setup to the Puig case when Lusztig-inducing from the centralizer of a $\Phi_d$-Sylow:

Take $\ell \nmid |W|$, $\ell \nmid |\Phi_d(q)|$. Let $L = C_G(S)$, where $S$ is a $\Phi_d$-Sylow; recall that $W_{GF}(L) = N_{GF}(L)/L = N_{GF}(S)/S$ is a complex reflection group. Let $\lambda$ be a linear character of $L$.

Then the constituents of the Deligne-Lusztig induced $R^G_L(\lambda)$ correspond to $\text{Irr}(W_{GF}(L))$ with multiplicities equal up to sign to the corresponding dimensions (this should result from the existence of a representation of a “cyclotomic” Hecke algebra for $W_{GF}(L)$ on the cohomology, but has been checked case by case).

The principal $\ell$-block of $L$ consists of the linear characters of order a power of $\ell$. In this context the Broué conjecture more precisely states that the $\ell$-adic cohomology complex giving rise to $R^G_L$, cut by the idempotent corresponding to the $\ell$-characters of $L$, is a tilting complex giving rise to a derived equivalence.

**$d$-Harish-Chandra induction**

Define a $d$-split Levi as the centralizer of some $\Phi_d$-subgroup of $G$; when $d = 1$ this is thus the centralizer of a split torus, the same as a Levi of an $F$-stable parabolic subgroup.

Say that an irreducible representation $\gamma$ of $\mathbb{Q}_\ell G^F$ is $d$-cuspidal if $\rho(1)_{\Phi_d(q)} = |(G')^F|_{\Phi_d(q)}$.

**Theorem (d-Harish-Chandra induction)**

Let $\gamma$ be an irreducible $\mathbb{Q}_\ell G^F$-module. Then

- there is, up to $G^F$-conjugacy, a unique pair $(L, \lambda)$, where $L$ is a $d$-split Levi and $\lambda$ a $d$-cuspidal irreducible representation of $\mathbb{Q}_\ell L^F$, such that $\langle \gamma, R^G_L(\lambda) \rangle_{GF} \neq 0$.

- The components of $R^G_L(\lambda)$ are in bijection with $\text{Irr}(W_G(L, \lambda))$ with multiplicities the corresponding dimensions up to sign, where $W_G(L, \lambda) = \{g \in N_{GF}(L) \mid g\lambda = \lambda\}/L^F$.

- $W_G(L, \lambda)$ is a complex reflection group.