UNIPOTENT ELEMENTS
IN MODULAR REPRESENTATIONS
OF THE CLASSICAL ALGEBRAIC
GROUPS
WITH LARGE HIGHEST WEIGHTS

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The main topic is the behaviour of unipotent elements in modular irreducible representations of the classical algebraic groups with large highest weights with respect to the ground field characteristic and a fixed element.

"Modular" = "in positive characteristic".
$K$ is an algebraically closed field of characteristic $p > 0$. $G$ is a simply connected simple algebraic group of a classical type over $K$.

Four types: $A_r$, $B_r$, $C_r$, $D_r$.

Assumptions. $p > 2$ unless otherwise stated; $r > 2$ for $G = B_r(K)$, $r > 1$ for $G = C_r(K)$, and $r > 3$ for $G = D_r(K)$.

$SL_{r+1}(K)$, $Spin_{2r+1}(K)$, $Sp_{2r}(K)$, $Spin_{2r}(K)$.

Unipotent elements: a unitriangular form. All eigenvalues are equal to 1 (in an arbitrary characteristic).

$|x| = p^l$.

The image of a unipotent element in every representation remains unipotent.
The Jordan block structure.

What elements can appear in images of different representations?

A special attention will be given to unipotent elements from subsystem subgroups of relatively small ranks.

\[ x \in SL_m(K) \subset SL_n(K), \quad m << n. \]
Principal results:

a) the minimal polynomials of the images of unipotent elements in irreducible representations of the classical algebraic groups in odd characteristic;

b) lower estimates for the number of the Jordan blocks of the maximal possible size in the images of these elements in such representations with large highest weights with respect to the characteristic (in terms of the group rank and the order of an element);

c) a fact on the behaviour of unipotent elements in irreducible representations of the special linear group with highest weight large enough with respect to the characteristic and a fixed unipotent element: the image of this element has at least two Jordan blocks of the size equal
to its order;

d) the dimensions of all Jordan blocks (without their multiplicities) for images of some small unipotent elements of order $p$ in such representations.

Small unipotent elements: $\dim(x - 1)V << \dim V$ for the standard $G$-module $V$.

**Motivation.** A more general program: investigating properties of unipotent elements in modular representations of semisimple algebraic groups and elaborating machinery for recognizing representations on this base.

To distinguish rare and typical situations.

Asymptotic regularities linked with the growth of the highest weight with respect to the ground field charac-
teristic, no analogs in characteristic 0.
Results can be easily transferred to representations of finite Chevalley groups in defining characteristic. Steinberg’s theorem on restrictions of representations of semisimple algebraic groups to finite Chevalley groups.

The structure of unipotent conjugacy classes does not depend upon the ground field characteristic if the latter is not too small.
On unipotent conjugacy classes

$F$ is an algebraically closed field of characteristic $p$ or 0, $\Gamma$ is a simply connected simple algebraic group of a classical type over $F$, the assumptions above on rank hold, $n$ is the dimension of the standard $\Gamma$-module.

We say that a positive integer $k$ is proper for $\Gamma$ if $\Gamma = A_r(F)$, if $k$ is odd for $\Gamma = B_r(F)$ or $D_r(F)$, and if $k$ is even for $\Gamma = C_r(F)$; otherwise we call $k$ improper.

For unipotent $u \in \Gamma$, if $d_1 \geq d_2 \geq \ldots \geq d_l$ is the complete collection of the Jordan block sizes of $u$ in the standard realization of $\Gamma$, set $J(u) = (d_1, d_2, \ldots, d_l)$ and $d(u) = d_1$. The same for $GL_n(F)$.

**Lemma 1** [9, Corollary 2.4 and Remark after Proposition 2.8]
i) Let $u \in \Gamma$ be a unipotent element. If $\Gamma \neq D_r(F)$ or some integer in $J(u)$ is odd, the conjugacy class of $u$ is uniquely determined by $J(u)$.

ii) Let $\Gamma = D_r(F)$ and all the integers in $J(u)$ be even. Then there are just two unipotent conjugacy classes $C_1$ and $C_2 \subset \Gamma$ with the same $J(u)$.

iii) Let $g \in GL_n(F)$ be unipotent. Then $g \in \Gamma$ if and only if each improper member of $J(g)$ appears in this sequence an even number of times.

**Lemma 2** [9, Lemma 2.12] Let $S \subset G$ be a finite classical group that is the group of fixed points of a Frobenius morphism of $G$. Assume that $u \in G$ is unipotent. Then the conjugacy class of $u$ has a nonempty intersection with $S$, except the case where
$S = \Omega_n^{-}(q)$ and all Jordan blocks of $u$ occur with even multiplicities.
Notation

$\mathbb{Z}$ the set of integers
$\mathbb{Z}^+$ the set of nonnegative integers
$\text{Irr}$ the set of irreducible rational representations of $G$
$n$ the dimension of the standard $G$-module
$X$ the set of weights of $G$
$\omega(\varphi)$ the highest weight of a representation $\varphi$
$\omega_i$, $1 \leq i \leq r$, the fundamental weights
$\alpha_i$, $1 \leq i \leq r$, the simple roots
$\langle \omega, \alpha \rangle$ the value of a weight $\omega$ on a root $\alpha$
$\omega(\varphi) = \sum_{i=1}^{r} a_i \omega_i$, $a_i \in \mathbb{Z}^+$
$\text{Irr}_p \subset \text{Irr}$ the subset of $p$-restricted representations
If $\varphi \in \text{Irr}$, then $\varphi \in \text{Irr}_p$ if and only if all $\alpha_i < p$.

A dominant weight $\mu$ of $G$ can be written in the form $\sum_{j=0}^{k} p^j \mu_j$ with $p$-restricted $\mu_j$.

In this case set $\overline{\mu} = \sum_{j=0}^{k} \mu_j$.

$n = r + 1$ for $G = A_r(K)$, $2r + 1$ for $G = B_r(K)$, $2r$ for $G = C_r(K)$ or $D_r(K)$.

Let $\Gamma = D_r(F)$, $u \in \Gamma$ be unipotent and all members of $J(u)$ be even. Then the following holds for the classes $C_1$ and $C_2$ in Item ii) of Lemma 1 that contain elements $x$ with $J(x) = J(u)$:

if $\delta_i^j$ is the label on the labelled Dynkin diagram of $C_j$ corresponding to the root $\alpha_i$ ($1 \leq i \leq r$, $j = 1, 2$), then $\delta_i^1 = \delta_i^2$ for $1 \leq i \leq r - 2$, $\delta_{r-1}^1 = \delta_r^2 = 0$, and $\delta_{r-1}^2 = \delta_r^1 = 2$. 

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This follows from [5, Propositions 3.3 and 3.5] and [11, Ch.IV, Exercise 2.15 and Item 2.27(ii)].
Theorem 3 [12] Let $q = p^r$ and $F$ be the Frobenius morphism of $G$ determined by raising elements of $K$ to the $q$th power or composing this map with a nontrivial graph morphism. Let $G^F$ be the group of fixed points of $F$. Assume that $\varphi \in \text{Irr}$ and $\omega(\varphi) = \sum_{i=1}^{r} a_i \omega_i$ with all $a_i < p$. Then $\varphi|G^F$ is irreducible, for different representations such restrictions are pairwise nonisomorphic and exhaust the isomorphism classes of irreducible representations of $G^F$ over $K$.

For $\varphi \in \text{Irr}$ and unipotent $x \in G$ let $d_\varphi(x)$ denote the degree of the minimal polynomial of $\varphi(x)$.

$\dim \varphi$ is unknown!

The famous Lusztig conjecture (1979) yields a quite involved algorithm for computing the characters of irre-
ducible representations in the case where the characteristic is large enough with respect to the group rank.

This algorithm does not allow one to estimate the dimension of the irreducible representation with a given highest weight. No conjectures in the general case.

We cannot find $J(\varphi(x))$.

A partial information useful for recognition purposes. $J(\varphi(x))$ is the set of Jordan block sizes for $\varphi(x)$ (without multiplicities).

d(\varphi(x)), J(\varphi(x))$, asymptotic estimates for the number of the biggest blocks.
Minimal polynomials

For $p$-elements in linear groups in characteristic $p > 0$ the minimal polynomials have the form $(t-1)^d$ and hence are completely determined by their degrees. For an element $x$ with $J(x) = (d_1, \ldots, d_l)$ this degree is equal to $d_1$.

The fundamental results of Hall and Higman (1956) [4, Theorem B]: finite irreducible $p$-solvable linear groups in characteristic $p$, if $|x| = p^s$, then $d(x) \geq (p-1)p^{s-1}$ for $p > 2$ and $\geq 3p^{s-2}$ for $p = 2$, $s > 1$. If $p$ is odd and is not a Fermat prime, then this degree is always $p^s$.

For groups which are not $p$-solvable the most famous result on the minimal polynomials of unipotent elements

Different methods for various types of groups, groups of Lie type: different methods for representations in defining and non-defining characteristics.
A.E. Zalesski (1986–2008): a series of results on minimal polynomials in characteristic 0 and non-defining characteristic for groups of Lie type, the complete solution of the problem in some important cases.

Some of these results were obtained jointly with Tiep and Di Martino.

Results on the minimal polynomials of certain semisimple elements in cross characteristic representations of finite classical groups: Guralnick, Magaard, Saxl, and Tiep.

Zalesski (1999)[22]: $p$-elements in $p$-modular representations of finite quasisimple groups with cyclic $p$-Sylow subgroups.

Kleshchev and Zalesski (2004) [7]: elements of order
$p$ in irreducible projective representations of symmetric and alternating groups in odd characteristic $p$ and elements of order 4 in such representations in characteristic 2.
The minimal polynomials of unipotent elements in representations of the classical groups in odd characteristic: reduction to characteristic 0.

\( \mathbb{C} \) is the complex field.

\( G_{\mathbb{C}} \) is the simply connected simple algebraic group over \( \mathbb{C} \) with the same root system as \( G \).

\( \text{Irr}_{\mathbb{C}} \) is the set of rational irreducible representations of \( G_{\mathbb{C}} \).

Let \( u \in G_{\mathbb{C}} \) be unipotent and \( J(u) = (d_1, \ldots, d_l) \). Set \( N(u) = (d_1 - 1, d_1 - 3, \ldots, 1 - d_1, \ldots, d_l - 1, \ldots, 1 - d_l) \).

For \( 1 \leq i \leq r \) define \( b_i(u) \) as follows: \( b_i(u) \) is the sum of \( i \) largest members of \( N(u) \) if \( G = A_r(K) \) or \( C_r(K) \), or \( G = B_r(K) \) and \( i < r \), or \( G = D_r(K) \) and \( i < r - 1 \); \( b_r(u) \) is the halfsum of \( r \) largest members of \( N(u) \) for
$G = B_r(K)$; if $G = D_r(K)$ and some of $d_j$ is odd, then $b_{r-1}(u) = b_r(u)$ and is such as above; let $G = D_r(K)$ and all $d_j$ be even; then there are 2 conjugacy classes $C_1$ and $C_2$ with the same $J(u)$; $b_r(C_1)$ is equal to the halfsum of the $r$ largest members of $N(u)$, $b_r(C_2) = b_r(C_1) - 1$, and $b_{r-1}(C_k) = b_r(C_m)$ for $\{k, m\} = \{1, 2\}$.

**Proposition 4** Let $\rho \in \text{Irr}_C$ and $\omega(\rho) = \sum_{i=1}^{r} a_i \omega_i$. Then $d_{\rho}(u) = 1 + \sum_{i=1}^{r} a_i b_i(u)$.

In characteristic 0 it is not difficult to prove this fact, but I have not met it in the literature. So I have to refer to this paper [13, Propositions 1.3 and 2.6 and Algorithm 1.4].

For $\varphi \in \text{Irr}$ the term $\varphi_{\mathbb{C}}$ denotes the representation in $\text{Irr}_C$ with highest weight $\omega(\varphi)$. 
**Theorem 5** Let $x \in G$ be a regular unipotent element, $|x| = p^{s+1}$, and let $z_i \in G_\mathbb{C}$, $0 \leq i \leq s$, be an element with the same canonical Jordan form as $x^{p^i}$. Let $\varphi \in \text{Irr}$. Then

$$d_{\varphi}(x) = \min\{p^{s+1}, p^i d_{\varphi_\mathbb{C}}(z_i) \mid 0 \leq i \leq s\}, \quad (1)$$

except the cases where $G = A_r(K)$ or $C_r(K)$, $n = p^s + p$ with $s > 0$, $\omega(\varphi) = \omega_p$ or $\omega_{n-p}$ for $G = A_r(K)$ and $\omega(\varphi) = \omega_p$ for $G = C_r(K)$. In the exceptional cases $d_{\varphi}(x) = p^{s+1} - p + 2$.

Regular unipotent elements: $J(x) = (n)$ for $G = A_r(K)$, $B_r(K)$, and $C_r(K)$; $J(x) = (n - 1, 1)$ for $G = D_r(K)$.

Theorem 5 does not hold for arbitrary unipotent ele-
ments, even for $G = A_r(K)$ and some elements of order $p^2$ with two Jordan blocks.

**Lemma 6** Let $n = 2p + l$ for $G = A_r(K)$ or $D_r(K)$ and $n = 3p + l$ for $G = B_r(K)$ or $C_r(K)$ with $1 < l < p - 1$, $l$ even for $G = B_r(K)$ or $D_r(K)$ and $l$ odd for $G = C_r(K)$. Assume that $J(x) = (p + l, p)$ for $G = A_r(K)$ or $D_r(K)$ and $J(x) = (p + l, p, p)$ for $G = B_r(K)$ or $C_r(K)$. Let $\varphi \in \text{Irr}$ and $\omega(\varphi) = \omega_l$. Then $d_\varphi(x) = lp + 1$ and is less than the value given by Formula (1).

For $G = D_r(K)$ the sequence $J(x)$ does not determine the conjugacy class of $x$ if all block sizes are even.

Use the labelled Dynkin diagram of $x^{p^i}$ and results of P. Bala and R.W. Carter (1976)[1] to define $z_i$.
With these changes Theorem 5 holds for elements of order $p$ (I.D. Suprunenko [13], 1996).
For some representations $\varphi \in \text{Irr}$ with large highest weights with respect to $p$ the degree $d_\varphi(x) = |x|$ for all unipotent elements $x$.

**Definition 7** A dominant weight $\nu \in X$ is called $p$-large if $\langle \nu, \beta \rangle \geq p$ for the maximal root $\beta$ of $G$. A representation $\varphi \in \text{Irr}$ is $p$-large whenever $\omega(\varphi)$ is such.

Let $\mu = \sum_{i=1}^{r} a_i \omega_i \in X$. Then

$$\langle \mu, \beta \rangle = \begin{cases} 
\sum_{i=1}^{r} a_i & \text{for } G = A_r(K) \\
a_1 + a_r + 2 \sum_{i=1}^{r-1} a_i & \text{for } G = B_r(K), \\
a_1 + a_{r-1} + a_r + 2 \sum_{i=1}^{r-2} a_i & \text{for } G = D_r(K).
\end{cases}$$
Theorem 8 Let $\varphi \in \text{Irr}$ be $p$-large. Then $d_{\varphi}(x) = |x|$ for each unipotent element $x \in G$.

The assumptions of Theorem 8 are threshold with respect to the degree of the minimal polynomial of a unipotent element.

Lemma 9 Let $n = p^s + 1$ for $G = A_r(K)$ and $C_r(K)$, $n = p^s + 2$ for $G = B_r(K)$, and $n = p^s + 3$ for $G = D_r(K)$; $s > 0$ in all cases. Assume that $\varphi \in \text{Irr}$ with $\omega(\varphi) = (p - 1)\omega_1$ for $G = A_r(K)$ and $C_r(K)$ and $\omega(\varphi) = \frac{1}{2}(p - 1)\omega_2$ for $G = B_r(K)$ and $D_r(K)$. Then $\langle \omega, \beta \rangle = p - 1$ and $d_{\varphi}(x) = (p - 1)p^s + 1 < |x|$ for a regular unipotent element $x \in G$. 
For some elements $d_\varphi(x) = |x|$ for all nontrivial representations.

**Theorem 10** Let $d_1 = p^{s+1}$. Then $d_\varphi(x) = p^{s+1}$ for every nontrivial representation $\varphi \in \text{Irr}$, except the cases where $G = B_r(K)$ or $D_r(K)$, $d_1 = 3$ or $5$, $d_2 = 1$, $\omega(\varphi) = p^j \omega_r$ for $G = B_r(K)$ and $p^j \omega_{r-1}$ or $p^j \omega_r$ for $G = D_r(K)$. 
The general case: \( d_1 < |x| = p^{s+1} > p \). Construct a semisimple subgroup \( S = S(x) \subset G \) such that \( x \) is regular in \( S \) and the standard \( G \)-module is a direct sum of such modules for the simple components of \( S \) and, may be, several copies of the trivial \( S \)-module. Let \( H^p \) and \( H \) be the simply connected groups with the same root system as \( S(x) \) over \( K \) and \( \mathbb{C} \). Write \( H^p = \prod_{j=1}^c H^p_j \) and \( H = \prod_{j=1}^c H_j \), \( H^p_j \) and \( H_j \) are simple and have the same root systems. Let \( x_{0j} \in H^p_j \) be a regular unipotent element, \( x_{ij} = x_{0j}^i \) for \( 1 \leq i \leq s \), \( h_{ij} \in H_j \), \( J(h_{ij}) = J(x_{ij}) \), \( 0 \leq i \leq s \), and \( h_i = \prod_{j=1}^c h_{ij} \). Let \( \theta : \mathbf{X} \to \mathbf{X}(H) \) be determined by the restriction of weights.

**Theorem 11** Let \( \varphi \in \text{Irr} \) and let \( \omega(\varphi) \) be not \( p \)-large.
Assume that \( x \in G \) and \( d_1 < p^{s+1} = |x| \). Denote by \( \psi \) the irreducible representation of \( H \) with highest weight \( \theta(\omega(\varphi)) \). Then
\[
d_\varphi(x) = \min\{p^{s+1}, \ p^f d_\psi(h_f) \mid 0 \leq f \leq s\},
\]
except the following cases:

1) \( G = A_r(K) \) or \( C_r(K) \), \( s > 0 \), \( d_1 = p^s + p \), \( x \) is a regular unipotent element or \( s > 1 \) and \( d_2 \leq p^s - p \), \( \omega(\varphi) = p^j \omega_p \) or \( p^j \omega_{r+1-p} \) for \( G = A_r(K) \) and \( p^j \omega_p \) for \( G = C_r(K) \);

2) \( G = D_{2p}(K) \), \( J(x) = (2p, 2p) \), and \( \omega(\varphi) = p^j \omega_{2p-1} \) or \( p^j \omega_{2p} \).

In the exceptional cases \( d_\varphi(x) = p^{s+1} - p + 2 \).

In Item 2 there are two conjugacy classes with the same
$J(x)$, each of them yields an exception for just one of relevant two groups of weights.
If $s > 0$ and $d_1 \geq p^s + p$, there is a rather large class of cases where the form of the highest weight guarantees that $d_\varphi(x) = |x|$ and no computations are needed.

**Theorem 12** Let $\varphi \in \text{Irr}$, $\omega(\varphi) = \sum_{i=1}^{r} a_i \omega_i$, and let $a_j \neq 0$ for some $j$ with $p \leq j \leq r + 1 - p$ for $G = A_r(K)$ and $p \leq j$ for other groups. Assume that $s > 0$, $p^s + p \leq d_1 \leq p^{s+1}$ and, moreover, for $G = B_r(K)$ either $d_1 \geq p^s + 2p$ or $j < r$ and for $G = D_r(K)$ either $d_1 \geq p^s + 2p$ or $j < r - 1$. Then either $d_\varphi(x) = |x| = p^{s+1}$ or the following holds:

$G = A_r(K)$ or $C_r(K)$, $s > 0$, $d_1 = p^s + p$, $x$ is a regular unipotent element or $s > 1$ and $d_2 \leq p^s - p$, $\omega(\varphi) = p^j \omega_p$ or $p^j \omega_{r+1-p}$ for $G = A_r(K)$ and $p^j \omega_p$ for $G = C_r(K)$, $d_\varphi(x) = p^{s+1} - p + 2$. 

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If $\omega(\varphi)$ is large enough with respect to $x$ and $p$, then $d_\varphi(x) = |x|$.  

**Proposition 13** In the assumptions of Theorem 11 let $d_\varphi \in (z_s) > p$. Then $p^f d_\psi(h_f) > p^{s+1}$ for all $f$ with $0 \leq f \leq s$ and $d_\varphi(x) = p^{s+1}$, unless one of the exceptional cases in Theorem 11 occurs and $d_\varphi(x) = p^{s+1} - p + 2$.

Write $\bar{\omega}(\varphi) = \sum_{i=1}^{r} \bar{a_i} \omega_i$. Set $b_i(x) = b_i(z_s)$ and $b(\varphi, x) = \sum_{i=1}^{r} \bar{a_i} b_i(x)$. Then $d_\varphi(\in (z_s) = 1 + b(\varphi, x)$.

The construction of $S$ and $\theta$ in Theorem 11 is rather involved, but explicit.

If $G = A_r(K)$ and $r_1 + 1, \ldots, r_c + 1$ are all members of $J(x)$ that are bigger than 1, then $S(x) \cong A_{r_1}(K) \times$
\[ \ldots \times A_{r_c}(K). \]

For other types \( J(x) \) can contain proper and improper integers. Each pair of equal improper members \( r_l + 1 > 1 \) of \( J(x) \) yields a component of \( S \) isomorphic to \( A_{r_l}(K) \), for \( G = B_r(K) \) or \( D_r(K) \) every proper integer \( 2r_l + 1 > 1 \) from \( J(x) \) yields a component isomorphic to \( B_{r_l}(K) \), for \( G = C_r(K) \) each proper integer \( 2r_l \) from \( J(x) \) yields a component isomorphic to \( C_{r_l}(K) \).

Minimal polynomials of elements of order \( > p \): I.D. Suprunenko [16].

If \( p = 2 \), then usually \( d_\varphi(x) = |x| \) for all unipotent elements \( x \) if \( \varphi \) is nontrivial and \( \varphi(G) \) does not coincide with the standard realization of \( G \). One can hope to determine all exceptions explicitly.
Here for $G \neq A_r(K)$ there is no bijection between the set of unipotent conjugacy classes for $G$ and such set for $G_{\mathbb{C}}$.

Exceptional groups: R. Lawther [8] (1995), the Jordan block structure of unipotent elements in the nontrivial representations of the minimal dimension and the adjoint representations is completely determined.

There is a striking difference between the behaviour of unipotent elements in irreducible representations of finite groups of Lie type in describing and non-describing characteristics.

In the first case there are many representations where some elements have the minimal polynomials of degrees much less than their orders.
Results on cross characteristic representations: usually the degree of the minimal polynomial is equal to the order of an element, and the exceptional cases are more or less explicitly described.
Some machinery

For elements of order $p$: embedding into subgroups of type $A_1$, analyzing the restrictions of representations to such subgroups.

Let $|x| = p^{s+1} > p$.

Induction on the group rank and the element order.

**Conjecture** $(r, s)$. Theorems 8, 10, 5, 12, and 11 are valid for classical groups of rank $< r$ and for unipotent elements in $G$ of order $\leq p^s$. If $G \neq A_r(K)$, we also assume that these results hold for unipotent elements of order $\leq p^{s+1}$ in $A_{n-1}(K)$.

Getting upper estimates by analyzing the characteristic 0 case.
$d_M(x)$ denotes the degree of the minimal polynomial of $x$ on a $G$-module $M$.

The following proposition plays a key role.

**Proposition 14** Let $M$ be a $G$-module, $x \in G$ be unipotent, and $|x| = p^{s+1} > p$.

a) Assume that $l \leq s$ and $z = x^{pl}$. Then

$$p^l(d_M(z) - 1) < d_M(x) \leq p^l d_M(z).$$

b) Set $y = x^{ps}$, $d_M(y) = a + 1$, $M_y = (y - 1)^a M$, and $d_{M_y}(x) = b$. Then $b \leq ps$, $d_M(x) = ap^s + b$ and

$$\dim(x - 1)^{ap^s+b-1} M = \dim(x - 1)^{b-1} M_y.$$

Naturally, $C_G(y)$ preserves $M_y$. 

In some cases we can find a vector \( m \in M_y \), a semisimple subgroup \( \Gamma \subset C_G(y) \), a regular unipotent element \( z \in \Gamma \) and show that \((x - 1)^km \neq 0\) if \((z - 1)^km \neq 0\).

We use Conjecture \((r, s)\) to estimate minimal \( k \) for which \((z - 1)^km = 0\).

Arguments connected with Zariski closures of conjugacy classes.

**Lemma 15** Let \( \Delta \) be a semisimple algebraic group, \( x, y \in \Delta \) be unipotent elements and \( y \) lies in the Zariski closure of the conjugacy class containing \( x \). Then \( d_M(y) \leq d_M(x) \) for each rational \( \Delta \)-module \( M \). Furthermore, if \( d_\varphi(y) = d_\varphi(x) = l \), then \( \dim(y - 1)^lM \leq \dim(x - 1)^lM \).

The second assertion of Lemma 15 is useful for finding
estimates for the number of blocks of the maximal size in images of representations with big highest weights.

Formulae for the Jordan block structure of tensor products of unipotent Jordan blocks.

Several important results on these formulae are contained in [3, Ch.6] (in Russian), some corollaries of them are given in [16].
Large weights

Estimates for the number of Jordan blocks of size $|x|$ for $\varphi(x)$.

In this section $p > 3$ for $G = B_r(K)$ or $D_r(K)$, $x \in G$ is a unipotent element, $|x| = p^{s+1}$, $s \geq 0$. Then $p^s < d(x) \leq p^{s+1}$.

a). $p$-large representations

For $G = B_r(K)$ or $D_r(K)$ we call $x$ special if it has a single Jordan block of size $> p^s$ in the standard realization of $G$.

**Theorem 16** Let $\varphi \in \text{Irr}$ be $p$-large. Then $\varphi(x)$ has at least two Jordan blocks of size $|x|$. For $G = B_r(K)$
or $D_r(K)$ the element $\varphi(x)$ has at least three such blocks and at least four if $x$ is not special.

For almost all $p$-large representations substantially stronger estimates are valid!
Theorem 17 Let \( \varphi \in \text{Irr} \) be \( p \)-large and \( \omega = \omega(\varphi) \).
Put \( l = p^s + 1 \) and \( l_1 = (p^s + 1)/2 \).

a) Let \( G = A_r(K), r \geq l, \) and \( \overline{\omega} \neq a_1\omega_1 + a_r\omega_r \) with \( a_1 + a_r = p \). Then \( \varphi(x) \) has at least \( r - l + 1 \) Jordan blocks of size \( |x| \).

b) Let \( G = B_r(K) \) and \( \overline{\omega} \neq a_1\omega_1 + a_2\omega_2 \) with \( a_1 + 2a_2 = p \). Assume that \( r > l \) for nonspecial \( x \) and \( r > l_1 + 1 \) for special \( x \). Set \( k_1(1) = 2, k_1(2) = 4, \) and \( k_1(n) = 2n + 1 \) for \( n > 2, k_2(2) = 2, k_2(3) = 4, \) and \( k_2(n) = 2n \) for \( n > 3 \). Then \( \varphi(x) \) has at least \( 3k_2(r - l_1) \) Jordan blocks of size \( |x| \) for special \( x \) and at least \( 4k_1(r - l) \) such blocks for nonspecial \( x \). Suppose that \( x \) is not special and \( d(x) > p^s + 1 \). Then \( \varphi(x) \) has at least \( \max\{4k_1(r - l), 3k_2(r - l_1)\} \) blocks of size
\[ |x|.

c) Let \( G = C_r(K) \), \( r > l_1 \), and \( \omega \neq p\omega_1 \). Then \( \varphi(x) \) has at least \( 2(r - l_1) \) Jordan blocks of size \( |x| \).

d) Let \( G = D_r(K) \) and \( \omega \) satisfy the assumptions of Item b). Suppose that \( r > l + 1 \) for nonspecial \( x \) and \( r > l_1 + 1 \) for special \( x \). Define the functions \( k_1 \) and \( k_2 \) as in Item b). Then \( \varphi(x) \) has at least \( 3k_1(r - l_1 - 1) \) Jordan blocks of size \( |x| \) for special \( x \) and at least \( 4k_2(r - l) \) such blocks for nonspecial \( x \). If \( x \) is nonspecial, \( d(x) > p^s + 1 \), and \( r > l + 1 \), then \( \varphi(x) \) has at least \( \max\{4k_2(r - l), 3k_1(r - l_1 - 1)\} \) blocks of size \( |x| \).
The exceptions in Theorem 17 do exist!

**Proposition 18** Assume that $s > 0$.

a) Let $G = A_r(K)$ or $C_r(K)$ and $J(x) = (p^s + 1, 1, 1, \ldots, 1)$. Suppose that $\varphi \in \text{Irr}$, $\omega(\varphi) = a_1\omega_1 + a_r\omega_r$ with $a_1 + a_r = p$ for $G = A_r(K)$ and $\omega(\varphi) = p\omega_1$ if $G = C_r(K)$. Then $\varphi(x)$ has exactly two Jordan blocks of size $|x|$.

b) Let $G = B_r(K)$ or $D_r(K)$, $x_1, x_2 \in G$, $J(x_1) = (p^s + 1, p^s + 1, 1, \ldots, 1)$, and $J(x_2) = (p^s + 2, 1, \ldots, 1)$. Assume that $\varphi \in \text{Irr}$ and $\omega(\varphi) = a_1\omega_1 + a_2\omega_2$ with $a_1 + 2a_2 = p$. There exists a constant $C = C(a_1, p)$ such that $\varphi(x_1)$ has at most $C$ Jordan blocks of size $|x_1|$. If $\omega(\varphi) = \omega_1 + \frac{p-1}{2}\omega_2$, then $\varphi(x_1)$ has just four blocks of size $p^{s+1}$ and $\varphi(x_2)$ has just three such blocks.
In Items a) and b) it can occur that the sequences $J(x)$ and $J(x_i)$ contain no ones.

Some assertions of Theorem 16 for the groups of types $B_r$ and $D_r$ do not hold in characteristic 3.

**Proposition 19** Let $F$ be an algebraically closed field of characteristic 3, $s > 0$, $t = (3^s + 1)/2$, and let $\Gamma = B_t(F)$ or $D_{t+1}(F)$. Assume that $u \in \Gamma$ is a regular unipotent element and $\varphi \in \text{Irr} \, \Gamma$ with $\omega(\varphi) = \omega_1 + \omega_2$. Then $\varphi(u)$ has at most two Jordan blocks of size $|u|$. 
b). Large weights for a given element

Elements from subsystem subgroups.

A **subsystem subgroup** in $G$ is a subgroup generated by its root subgroups associated with all roots from a subsystem of the root system. Standard embeddings: $SL_{l+1}(K) \subset SL_{r+1}(K)$ and so on. You can meet other definitions of subsystem subgroups.

If $x$ lies in a proper subsystem subgroup, then $J(x)$ determines the conjugacy class of $x$ and $x$ has blocks of size 1.

Let $|x| = p^{s+1}$ ($s$ may be zero).

**Recall:** $z_s \in G_{\mathbb{C}}$, $J(z_s) = J(x^{p^s})$. Write $\omega(\varphi) = \sum_{i=1}^{r} \overline{a_i} \omega_i$. Set $b_i(x) = b_i(z_s)$ for $1 \leq i \leq r$ and
\[ b(\varphi, x) = \sum_{i=1}^{r} \overline{a_i} b_i(x). \]

Assume that \( \omega(\varphi) \neq 0 \) and put \( t(\varphi) = \max\{i \mid i \leq (r + 1)/2, \ a_i + a_{r+1-i} \neq 0\} \) for \( G = A_r(K) \) and \( t(\varphi) = \max\{i \mid a_i \neq 0\} \) otherwise.
Theorem 20 Assume that a unipotent element \( x \in G \) of order \( p^{s+1} \) lies in a subsystem subgroup \( H \) of type \( A_{l} \) with \( l < r - 1 \) for \( G = A_{r}(K) \) of type \( B_{l} \) with \( 2 < l < r - 3 \) for \( G = B_{r}(K) \), of type \( C_{l} \) with \( 1 < l < r \) for \( G = C_{r}(K) \), and of type \( D_{l} \) with \( 3 < l < r - 3 \) for \( G = D_{r}(K) \). If \( G = D_{r}(K) \), suppose also that \( x \) has at least one Jordan block of size different from \( 2kp^{s} \) in the standard realization of \( H \). Let \( \varphi \in \text{Irr}_{p} \), \( \omega(\varphi) \neq 0 \), \( t = t(\varphi) \), \( b_{x}(\varphi) \geq p + b_{t}(x) - b_{t-1}(x) \) for \( t > 1 \) and \( b_{x}(\varphi) \geq p + b_{1}(x) \) for \( t = 1 \). Then \( \varphi(x) \) has at least \( r - l \) Jordan blocks of size \( |x| \) for \( G = A_{r}(K) \) and at least \( 2(r - l) \) such blocks for other groups, except the following cases:

a) \( G = A_{r}(K) \) or \( C_{r}(K) \), in the standard realization
of \( G \) the element \( x \) has one block of size \( p^s + p \), \( s > 1 \), other block sizes (if any) are at most \( p^s - p \), \( \omega(\varphi) = \omega_{p+1} \) or \( \omega_{r-p} \) for \( G = A_r(K) \) and \( \omega(\varphi) = \omega_{p+1} \) for \( G = C_r(K) \);

b) \( G = C_{p+1}(K), \ l = p, \ \omega(\varphi) = \omega_{p+1} \) and \( x \) is a regular element of \( H \).

In Case a) the number of Jordan blocks of size \( |x| \) for \( \varphi(x) \) is completely determined by \( p \) and \( s \) and hence does not depend upon \( r - l \). In Case b) the element \( \varphi(x) \) has just one block of size \( p^2 = |x| \).
In Theorem 20 for $G = D_r(K)$ some unipotent elements of $H$ are not considered. If $r - l$ is large enough, this theorem yields some information for such elements as well as they can be regarded as elements from a subsystem subgroup of type $D_{l+1}$.

$b_x(\varphi)$ and blocks of size $|x|$. 

1). $\varphi(x)$ has no blocks of size $|x|$ if $b_x(\varphi) < p - 1$. 
2). $\varphi(x)$ surely has such blocks if $b_x(\varphi) > p$. 
3). If $b_x(\varphi) = p - 1$, such blocks can present and can absent. 
4). If $b_x(\varphi) = p$, such blocks always present, except the cases indicated in Theorem 11.

In the assumptions of Theorem 20 it can occur that
$\varphi(x)$ has many blocks of size $|x|$, but for some $y \in G$ with $|y| = |x|$ the element $\varphi(y)$ has no such blocks. The number of such examples grows with the growth of $r$ and $l$ with respect to $|x|$.

Unipotent elements from subsystem subgroups play a key role in the proofs of results on estimates for $p$-large representations. The ”smallest” elements of a fixed order almost usually lie in proper subsystem subgroups.
Let $x \in G$, $J(x) = (d_1, \ldots, d_t)$ and $|x| = p^{s+1}$. Denote by $\text{cl}(x)$ the Zariski closure of the conjugacy class containing $x$. If $G = A_r(K)$ or $C_r(K)$, then $\text{cl}(x)$ contains an element $u$ with $J(u) = (p^s + 1, 1, \ldots, 1)$, $u^{p^s}$ is conjugate to a root element (long for $G = C_r(K)$). Let $G = B_r(K)$ or $D_r(K)$. If $d_1 > p^s + 1$, then $\text{cl}(x)$ contains an element $u$ with $J(u) = (p^s + 2, 1, \ldots, 1)$, $u^{p^s}$ is a root element (long for $G = B_r(K)$). If $d_2 > p^s$, then $\text{cl}(x)$ contains an element $v$ with $J(v) = (p^s + 1, p^s + 1, 1, \ldots, 1)$, $v^{p^s}$ is conjugate to $u^{p^s}$. Set $k = (p^s + 1)/2$. One can assume the following: $u$ lies in a subsystem subgroup of type $A_{p^s}$ for $G = A_r(K)$, in such subgroup of type $B_k$ or $C_k$ for $G = B_r(K)$ or $C_r(K)$, in such subgroup of type $D_{k+1}$ for $G = D_r(K)$; the element $v$ lies
in subsystem subgroups of type $A_p^s$ and $D_p^s$.

**Some machinery for studying the behaviour of elements from subsystem subgroups**

Let a unipotent element $x \in G$ lie in a subsystem subgroup of type $A_l$, $B_l$, $C_l$ or $D_l$ for $G = A_r(K)$, $B_r(K)$, $C_r(K)$, or $D_r(K)$, respectively. Embed $x$ into a subsystem subgroup $H = H_1 \times H_2$ with $H_1 \cong A_l(K)$, $B_l(K)$, $C_l(K)$, or $D_l(K)$, $H_2 \cong A_{r-l-1}(K)$, $D_{r-l}(K)$, $C_{r-l}(K)$, or $D_{r-l}(K)$. If $\varphi$ is large for $x$, find in the restriction $\varphi|_H$ a composition factor of the form $\varphi_1 \otimes \varphi_2$ with $\varphi_1$ large for $x$ ($p$-large if $\varphi$ is such) and nontrivial $\varphi_2$. 
c). Results on arbitrary elements. Two blocks of maximal dimension

What can be said on arbitrary unipotent elements?

**Theorem 21** Let $G = A_r(K)$, $\varphi \in \text{Irr}$, and $x \in G$ be a unipotent element of order $p^{s+1}$. Assume that $b_x(\varphi) \geq p$. Then $\varphi(x)$ has at least two Jordan blocks of size $|x|$, except some cases where $s > 0$, the degree of the minimal polynomial of $x$ in the standard realization of $G$ is $p^s + p$ and $\omega(\varphi) = \omega_p$ or $\omega_{r+1-p}$.

Theorem 11 shows that the exceptions in Theorem 21 really occur. Due to Proposition 18 the assumption of Theorem 20 on $b_x(\varphi)$ cannot be weakened, and the assertion of Theorem 21 cannot be strengthened without
reducing the class of representations considered.
A connection with recognition problems

Here $p \geq 2$.

Let $x \in GL_m(K)$ be a unipotent element with $J(x) = (d_1, d_2, \ldots, d_t)$. Set $\text{Cl}_m = \{SL_m(K), Sp_m(K), SO_m(K)\}$ for even $m$, $\text{Cl}_m = \{SL_m(K), SO_m(K)\}$ for $p \neq 2$ and odd $m$ and $\text{Cl}_m = \{SL_m(K)\}$ for $p = 2$ and odd $m$.

**Theorem 22** Let $S \subset GL_m(K)$ be an irreducible Zariski closed connected simple subgroup of rank greater than 1, and $S \notin \text{Cl}_m$. Assume that $S$ is not of type $G_2$. Let $x \in S$ and $|x| = p$. Then $d_1 - d_2 \leq 12$. If $S$ is a group of a classical type, then $d_1 - d_2 \leq 6$. Furthermore, $d_1 - d_2 \neq 9$ or $11$ and for groups of classical types $d_1 - d_2 \neq 5$. For groups of type $A_r$ the parameter $d_1 - d_2 \leq 4$. If $d_1 - d_2 > 4$, the group $S$ is
conjugate to the image of the fundamental representation associated with an extreme root on the relevant Dynkin diagram.

This can be deduced from results of [14] and [15]. So elements of order $p$ with $d_1 - d_2 > 12$ are rare in a certain sense.

Can Theorem 22 be extended to arbitrary unipotent elements?

May be, with another estimate for $d_1 - d_2$?

For elements of nonprime order the problem is more difficult since one cannot apply the representation theory of $SL_2(K)$.

Theorems 16, 20 and 21 are some steps in this direction.
Jordan block structure for some small elements

$J_{\varphi}(x)$ for elements $x$ lying in subsystem subgroups of small ranks.

**The basic Steinberg representation:** $\omega(\varphi) = \sum_{i=1}^{r}(p - 1)\omega_i$. In this case $J_{\varphi}(x) = \{|x|\}$ for every unipotent $x$.

Fr is the Frobenius morphism of $G$ associated with raising elements of $K$ to the $p$th power. Let $\varphi = \bigotimes_{j=0}^{k}\varphi_j \circ Fr^j$ with $\varphi_j \in \text{Irr}_p$ and some $\varphi_j$ is the basic Steinberg representation. Then $J_{\varphi}(x) = \{|x|\}$ for every unipotent $x$. 
Root elements

In this section \( x \in G \) is a nontrivial root element.

For \( p > 3 \) A.E. Zalesski and P.H. Tiep [19, Theorem 2.20] (2002) have classified \( p \)-restricted representations with \( J_{\varphi}(x) = \{1, p - 1, p\} \).

**Theorem 23** Let \( p > 3 \) and \( \Gamma \) be a simple algebraic group of rank \( r > 1 \). Assume that \( \varphi \in \text{Irr}_p \) is nontrivial and \( J_{\varphi}(x) = \{1, p - 1, p\} \) for all root elements \( x \in \Gamma \). Then \( \varphi \) is the basic Steinberg representation of \( \Gamma \), unless, possibly, \( p = 5 \) and \( \Gamma = G_2(K) \).

Theorem 23 was crucial for proving modulo \( p \) reducibility of many complex representations of finite Chevalley groups.
For some types the set $J_{\varphi}(x)$ can be found for every $\varphi$. If $x = x_\alpha(t)$, denote by $\alpha_{m,x}$ the maximal root of the same length as $\alpha$.

For $\varphi \in \text{Irr}$ set $m_{\varphi}(x) = \min\{p, 1 + \langle \omega(\varphi), \alpha_{m,x} \rangle \}$.

For integers $a \leq b$ denote by $\mathbb{N}^b_a$ the set of integers $i$ with $a \leq i \leq b$.

**Theorem 24 (M.V. Velichko [20])** Let $r > 2$ and $G = A_r(K)$, $D_r(K)$, or $E_r(K)$. Assume that $\varphi = \bigotimes_{j=0}^{k} \varphi_j \circ \text{Fr}^j$ with $\varphi_j \in \text{Irr}_p$ and each $\varphi_j$ is not the basic Steinberg representation for $0 \leq j \leq k$. Then $J_{\varphi}(x) = \mathbb{N}^m_{1}(x)$. The same holds for $G = B_r(K)$ and long root elements.

For symplectic groups the situation is more complicated.
Theorem 25 (M.V. Velichko [21]) Let $r > 2$ and $G = C_r(K)$. Assume that $\varphi \in \text{Irr}_p(K)$ and $\omega(\varphi) = \sum_{i=1}^{r} a_i \omega_i$ with $a_{r-1} < p - 1$. Set $a = \sum_{i=1}^{r} a_i$. Then

$$J_{\varphi}(x) = \begin{cases} N_{m_{\varphi}(x)}^{m_{p-a-1}} & \text{if } a + 3 \leq p < a_{r-1} + 2a_r + 3, \\ N_{m_{\varphi}(x)}^{m_1} & \text{otherwise} \end{cases}$$

for a long root element $x$. If $x$ is a short root element and $a_i < p - 1$ for some $i < r$, then $J_{\varphi}(x) = N_{m_{\varphi}(x)}^{m_1}$.

Particular cases where for some simple roots $\alpha_i$ and $\alpha_j$ connected on the Dynkin diagram a certain linear combination of $a_i$ and $a_j$ is small were settled by A.A. Osinovskaya and I.D. Suprunenko [10].
Other small elements

Quadratic elements with not too many blocks of size 2. $p > 2$ for $G \neq A_r(K)$, $\varphi \in \text{Irr}_p$, $\omega(\varphi) = \sum_{i=1}^r a_i \omega_i$.

Type $A_r$

**Theorem 26** (jointly with M.V. Velichko [17]) Let $G = A_r(K)$ and $u \in G$ be a unipotent element with $q > 1$ blocks of size 2 and other blocks of size 1 in the standard $G$-module. Assume that $\sum_{i=1}^r a_i \geq p - 1$, $r > \frac{5q+2}{2}$, and there exist $q$ indices $i_1, \ldots, i_q$ such that $a_{i_k} < p - 1$, $1 \leq k \leq q$, $1 \leq i_1 < \ldots < i_q \leq r$, and $i_s - i_{s-1} > 1$. Then $J_\varphi(u) = \mathbb{N}_1^p$. If all $a_i < p - 1$, it suffices to assume that $q \leq (r - 1)/2$. 
Theorem 27 Let $G = A_r(K)$ and $\sum_{i=1}^{r} a_i < p$. Assume that $u \in G$ is a unipotent element with $q > 1$ blocks of size 2 and other blocks of size 1 in the standard $G$-module and $q \leq r/2$. Set
\[ t = a_1 + 2a_2 + \ldots + (q-1)a_{q-1} + q(a_q + \ldots + a_{r-q+1}) + (q-1)a_{r-q+2} + \ldots \]
\[ l = \min\{p, t + 1\}. \]
Then $J_\varphi(u) = \mathbb{N}_1^l$. 

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Types $B_r$ and $D_r$

For these types all quadratic elements have even number of blocks of size 2 in the standard realization.

**Theorem 28** (jointly with M.V. Velichko) Let $G = B_r(K)$ or $D_r(K)$. Set $\delta(\varphi) = a_1 + 2a_2 + \ldots + 2a_{r-1} + a_r$ for $G = B_r(K)$ and $\delta(\varphi) = a_1 + 2a_2 + \ldots + 2a_{r-2} + a_{r-1} + a_r$ for $G = D_r(K)$. Let $u \in G$ be a unipotent element with $2q > 2$ blocks of size 2 and other blocks of size 1 in the standard $G$-module and $r > \frac{5q+4}{2}$. Assume that $\delta(\varphi) \geq p$ and there exist $q$ indices $i_1, \ldots, i_q$ such that for both types $a_{i_k} < p - 1$, $1 \leq k \leq q$, $1 \leq i_1 < \ldots < i_q \leq r$, $i_k - i_{k-1} > 1$ for $1 < k \leq q$, $i_q < r$ if $G = B_r(K)$ and $\{i_q, i_q-1\} \neq \{r, r - 2\}$ if $G = D_r(K)$. Then $J_\varphi(u) = \mathbb{N}^p_1$. If all $a_i < p - 1$, it
suffices to suppose that \( q \leq (r - 2)/2 \).

**Theorem 29** Let \( p \) and \( G \) be such as in Theorem 28, \( u \in G \) be a unipotent element with \( 2q > 2 \) blocks of size 2 and other blocks of size 1 in the standard \( G \)-module and \( r \geq 2q+1 \). Assume that \( \delta(\varphi) < p \). Set
\[
m = a_1 + 2a_2 + \ldots + (2q - 1)a_{2q-1} + 2q(a_{2q} + a_{2q+1} + \ldots + a_{r-1}) + qa_r
\]
for \( G = B_r(K) \) and
\[
m = a_1 + 2a_2 + \ldots + (2q - 1)a_{2q-1} + 2q(a_{2q} + a_{2q+1} + \ldots + a_{r-2}) + q(a_{r-1} + a_r)
\]
for \( G = D_r(K) \) (if for the latter group \( r = 2q + 1 \), set
\[
m = a_1 + 2a_2 + \ldots + (2q - 1)a_{2q-1} + q(a_{r-1} + a_r))\).

Put \( l = \min\{p, m + 1\} \). Then \( J_\varphi(u) = \mathbb{N}_{0}^l \).
Type $C_r$

**Theorem 30** (jointly with M.V. Velichko) Let $G = C_r(K)$ and $\sum_{i=1}^{r} a_i \geq p - 1$. Assume that $u \in G$ is a unipotent element with $2q + 1 > 1$ blocks of size 2 and other blocks of size 1 in the standard $G$-module. Suppose that one of the following conditions holds:

i) $r > \frac{5q+6}{2}$, $a_r < p - 1$, $\sum_{i=1}^{r-2} a_i \geq p - 1$ and there exist $q$ indices $i_1, \ldots, i_q$ such that $a_{i_j} < p - 1$, $1 \leq j \leq q$, $i_1 < \ldots < i_q < r - 1$, $i_s - i_{s-1} > 1$;

ii) $r \geq 2q + 3$, $a_j < p - 1$ for $j \neq r - 1$, $\sum_{i=1}^{r-2} a_i \geq p - 1$;

iii) $r > \frac{5q+10}{2}$, $a_{r-1} < p - 1$, $\sum_{i=1}^{r-4} a_i \geq p - 1$ and there exist $q$ indices $i_1, \ldots, i_q$ such that $a_{i_j} < p - 1$, $1 \leq j \leq q$, $i_1 < \ldots < i_q < r - 3$, $i_s - i_{s-1} > 1$;

iv) $r \geq 2q + 5$, $a_j < p - 1$ for $j \leq r - 4$ and for
\[ j = r - 1, \sum_{i=1}^{r-4} a_i \geq p - 1; \]
\[ v) \ r \geq 2q + 5, \ a_{r-1} < p - 1, \sum_{i=1}^{r-4} a_i < p - 1. \]

Then \( J_\varphi(u) = \mathbb{N}_1^p. \)

**Theorem 31** (jointly with M.V. Velichko) *Let* \( G = C_r(K) \) *and* \( a_1 + 2 \sum_{i=2}^{r} a_i \geq p. *Assume that* \( u \in G \) *is a unipotent element with* \( 2q > 2 \) *blocks of size* \( 2 \) *and other blocks of size* \( 1 \) *in the standard* \( G \)-*module. Suppose that* \( r > \frac{5q+4}{2} \) *and there exist* \( q \) *indices* \( i_1, \ldots, i_q \) *such that* \( a_{i_j} < p - 1, 1 \leq j \leq q, i_1 < \ldots < i_q < r, i_s - i_{s-1} > 1. \) *Then* \( J_\varphi(u) = \mathbb{N}_1^p. \)

The number of conjugacy classes of elements considered in Theorems 26–31 grows with the growth of \( r. \) Compare the behaviour of small elements in repre-
sentations from Theorems 26–31 and the basic Steinberg representation. A small number of coefficients that are $< p - 1$ changes the picture!
**Machinery:** finding special direct summands (Weyl modules, tilting modules) in the restrictions of representations to subsystem subgroups.
Example

On the proof of Lemma 6 for $G = A_r(K)$.

Let $S_1 \cong A_{p+l-1}(K)$, $S_2 \cong A_{p-1}(K)$, and $S = S_1 \times S_2 \subset G$ be a subsystem subgroup.

One can assume that $x \in S$, $x = x_1 x_2$ with $x_i \in S_i$. $\varphi$ can be realized in $\bigwedge^l V$ ($V$ is the standard module). Denote by $\varphi_i$ the representation in $\text{Irr } S$ of the form $\varphi(\omega_i) \otimes \varphi(\omega_{l-i})$, $\omega_0 = 0$. Then $\varphi|S \cong \bigoplus_{i=0}^l \varphi_i$. $|x^p| = p$, $x^p = x_1^p \in S_1$, $x^p$ has $l$ blocks of size 2 and other blocks of size 1. Let $\rho_i \in \text{Irr } S_1$ and $\omega(\rho_i) = \omega_i$, $0 \leq i \leq l$. Then $d_{\rho_i}(x^p) = d_{\varphi_i}(x^p) = i + 1$ (use Formula (1) for elements of order $p$). $d_{\varphi}(x) = \max\{d_{\varphi_i}(x)\}$, $d_{\varphi_i}(x) \leq p(i + 1)$, $d_{\varphi_i}(x) \leq lp$ for $i < l$. $d_{\varphi_l}(x) =$
\[ d_{\varphi_l}(x_1) = lp + 1 \text{ (use Theorem 5). Hence } d_{\varphi}(x) = lp + 1. \]

Formula (1): we need to find the sum \( S \) of \( l \) maximal members of the following sequence \((p + l - 1, p + l - 3, \ldots, -p - l + 3, -p - l + 1, p - 1, p - 3, \ldots, 3 - p, 1 - p)\).
Let $N$ consist of these members. 1. $l = 4t$, $N = (p + 4t - 1, p + 4t - 3, \ldots, p + 1, p - 1, p - 1, \ldots, p - 2t + 1, p - 2t + 1)$, $S = lp + 2t^2$. 2. $l = 4t + 1$, $N = (p + 4t, p+4t-2, \ldots, p+2, p, p-1, p-2, \ldots, p-2t+1, p-2t)$, $S = lp + 2t^2 + t$. 3. $l = 4t + 2$, $N = (p + 4t + 1, p + 4t - 1, \ldots, p + 1, p - 1, p - 1, \ldots, p - 2t + 1, p - 2t + 1, p - 2t - 1)$, $S = lp + 2t^2 + 2t$. 4. $l = 4t + 3$, $N = (p + 4t + 2, p + 4t, \ldots, p + 2, p, p - 1, p - 2, \ldots, p - 2t + 1, p - 2t, p - 2t - 1)$, $S = lp + 2t^2 + 3t + 1$. In all cases Formula (1) yields a bigger value $(S + 1)$ than $d_\varphi(x)$.

Some problems

1. To extend results on $J_\varphi(x)$ to other classes of elements.
2. Estimates for the number of blocks of the maximal size in representations with large highest weights found in terms of the dimension: if $|x| = p^s$ and $\varphi(x)$ has $l$ blocks of size $p^s$, then $lp^s / \dim \varphi \geq c$ (or $lp^s \geq \dim \varphi^k$ with $0 < k < 1$).

3. Estimates for the dimension of the fixed point space for $\varphi(x)$.

4. To find $J(\varphi(x))$ for representations of small dimensions ($< 100?$) and all unipotent $x$. 248 seems too much. A big list of examples.
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