# The dual approach to the $K(\pi, 1)$ conjecture

Giovanni Paolini
(Caltech)

Berlin

August 31, 2021

# Artin groups and generalized configuration spaces

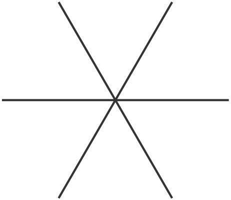
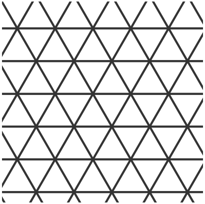
Let $W$ be a Coxeter group and $G_W$ the associated Artin group:

$$G_W = \langle\, S \mid \underbrace{sts\cdots}_{m_{s,t}\text{ factors}} = \underbrace{tst\cdots}_{m_{s,t}\text{ factors}} \quad \forall\, s \neq t \,\rangle.$$

$G_W$ is the fundamental group of a (generalized) configuration space $Y_W$.

If $W$ is finite or affine, $Y_W$ is given by:

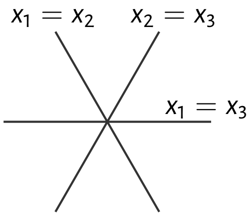
$$Y_W = \left( \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}_W} H_{\mathbb{C}} \right) / \, W.$$

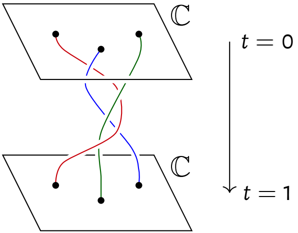
# Example: the braid group on 3 strands

Let $W$ be the symmetric group $\mathfrak{S}_3 = \langle a, b \mid a^2 = b^2 = 1,\ aba = bab \rangle$.

Its configuration space is $Y_W = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_i \neq x_j\}/\mathfrak{S}_3$.



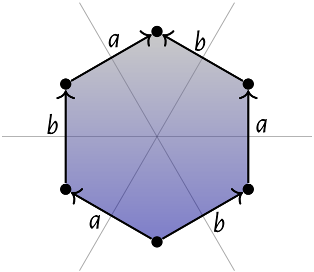

The (real) arrangement




Loops in $Y_W$ are "braids"

# The Salvetti complex

The configuration space $Y_W$ has the homotopy type of a CW complex $X_W$ with cells indexed by the standard parabolic subgroups of $W$.




The Salvetti complex for $W = \mathfrak{S}_3$

The Artin group presentation can be read off the 2-skeleton of the Salvetti complex:

$$G_W = \langle a, b \mid aba = bab \rangle.$$

## $K(\pi, 1)$ conjecture (Brieskorn, Arnol'd, Pham, Thom '60s)

The configuration space $Y_W$ is a classifying space for $G_W$:

$\pi_1(Y_W) = G_W$ and the higher homotopy groups are trivial (equivalently, the universal cover of $Y_W$ is contractible).

## $K(\pi, 1)$ conjecture (Brieskorn, Arnol'd, Pham, Thom '60s)

The configuration space $Y_W$ is a classifying space for $G_W$:

$\pi_1(Y_W) = G_W$ and the higher homotopy groups are trivial (equivalently, the universal cover of $Y_W$ is contractible).

Until recently, this conjecture was proved in the following cases:

- Spherical Artin groups (Brieskorn 1971, Deligne 1972)
- The affine Artin groups of type $\tilde{A}_n$, $\tilde{C}_n$ (Okonek 1979), and $\tilde{B}_n$ (Callegaro-Moroni-Salvetti 2010)
- Large-type Artin groups (Hendriks 1985)
- Artin groups of FC type (Charney-Davis 1995)
- 2-dimensional Artin groups (Charney-Davis 1995)
  (includes the affine Artin group $\tilde{G}_2$)

**Theorem (P.-Salvetti 2021)**

The $K(\pi, 1)$ conjecture holds for all affine Artin groups.

# Interval groups and Garside groups

$G$ group, $R$ generating set with $R = R^{-1}$, $g \in G$.

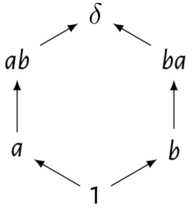
Let $[1, g]^G$ be the interval between 1 and $g$ in the (right) Cayley graph of $G$ (it is a poset, whose cover relations are labeled by some subset $R_0 \subseteq R$).

## Definition

The *interval group* $G_g$ is the group generated by $R_0$, with the relations visible in $[1, g]^G$. If $[1, g]^G$ is a balanced lattice, then $G_g$ is a *Garside group*.

## Example

If $G = W$ (a finite Coxeter group), $R = S$, and $g = \delta$ (the longest element), then $G_g$ is the spherical Artin group $G_W$.




$$W = \langle a, b \mid a^2 = b^2 = 1,\ aba = bab \rangle$$
$$\delta = aba = bab$$

# Interval groups and Garside groups

$G$ group, $R$ generating set with $R = R^{-1}$, $g \in G$.

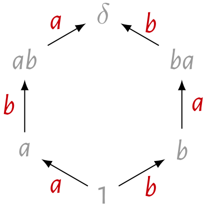
Let $[1, g]^G$ be the interval between 1 and $g$ in the (right) Cayley graph of $G$ (it is a poset, whose cover relations are labeled by some subset $R_0 \subseteq R$).

## Definition

The *interval group* $G_g$ is the group generated by $R_0$, with the relations visible in $[1, g]^G$. If $[1, g]^G$ is a balanced lattice, then $G_g$ is a *Garside group*.

## Example

If $G = W$ (a finite Coxeter group), $R = S$, and $g = \delta$ (the longest element), then $G_g$ is the spherical Artin group $G_W$.



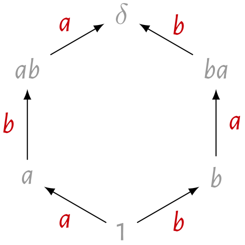

$$W = \langle a, b \mid a^2 = b^2 = 1,\ aba = bab \rangle$$
$$\delta = aba = bab$$

$$W_\delta = \langle a, b \mid aba = bab \rangle$$

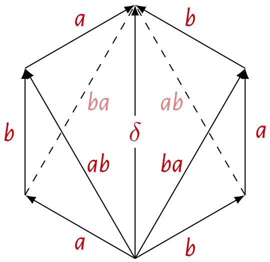
# Classifying space of Garside groups

**Theorem (Brady-Watt 2002, Charney-Meier-Whittlesey 2004)**
If $G_g$ is a Garside group, then the complex $K_G = \Delta([1, g]^G)/\text{labeling}$ is a classifying space for $G_g$.

We call $K_G$ the *interval complex* associated with $[1, g]^G$.




The balanced lattice $[1, \delta]^W$




The interval complex $K_W$

# Spherical Artin groups as Garside groups

Our favorite example: $W = \mathfrak{S}_3 = \langle a, b \mid a^2 = b^2 = 1,\ aba = bab \rangle$.

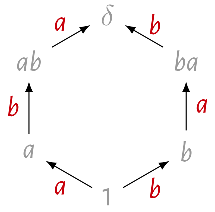
## Standard Garside structure

(Garside, Brieskorn-Saito, ...)

$R = S = \{a, b\}$ (simple system)

$g = \delta = aba$ (longest element)

$W_\delta = \langle a, b \mid aba = bab \rangle = G_W$




(weak Bruhat order)

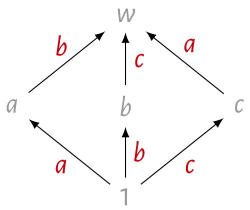
## Dual Garside structure

(Birman-Ko-Lee, Bessis, ...)

$R = \{\text{all reflections}\} = \{a, b, c\}$

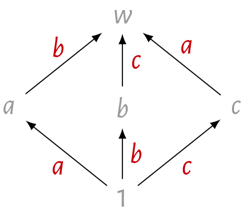
$g = w = ab$ (Coxeter element)

$W_w = \langle a, b, c \mid ab = bc = ca \rangle \cong G_W$

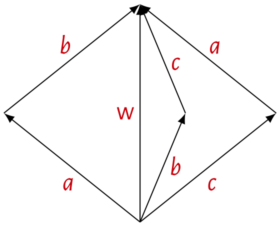



(noncrossing partition lattice)

# Example: the dual classifying space $K_W$ for $W = \mathfrak{S}_3$




The balanced lattice $[1, w]^W$




The interval complex $K_W$

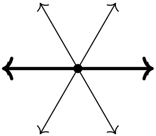
Simplices of $K_W$: $[\,]$, $[a], [b], [c], [w]$, $[a|b], [b|c], [c|a]$.

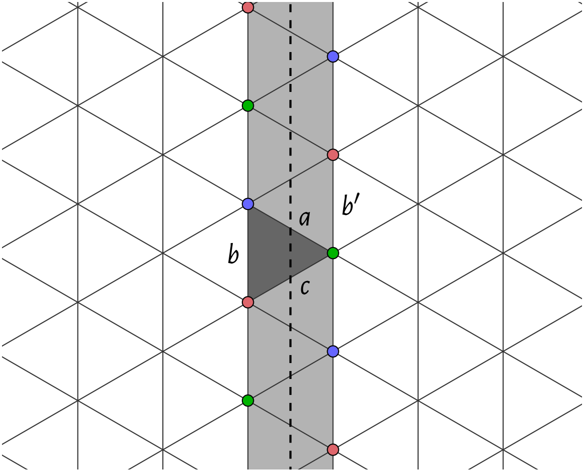
# The interval $[1, w]^W$ in affine Coxeter groups

**Example ($\tilde{A}_2$)**

$w = abc$ is a glide reflection w.r.t. the dashed line (axis)

$A_2$ root system:





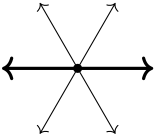

The minimal factorizations of $w$ can use any reflection that fixes a point on the axis (vertical). Among the remaining reflections (horizontal), only the ones closest to the axis ($b$ and $b'$).

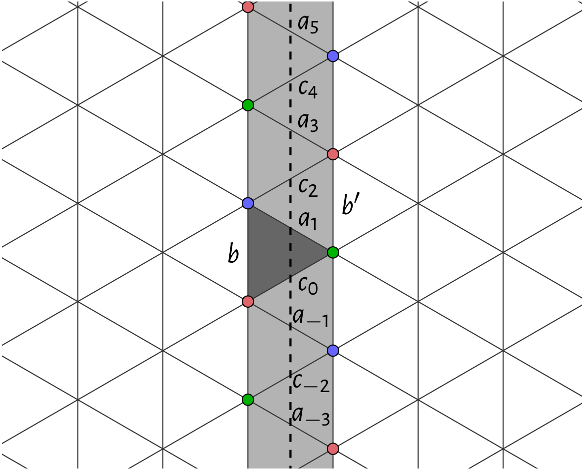
# The interval $[1, w]^W$ in affine Coxeter groups

## Example ($\tilde{A}_2$)

$w = abc$ is a glide reflection w.r.t. the dashed line (axis)

$A_2$ root system:






The length 2 elements of $[1, w]$ are:

- Rotations around colored vertices, e.g. $bc_0 = c_0 a_{-1} = a_{-1} b$;
- The two translations $a_1 a_{-1}$ and $c_2 c_0$.

# The interval $[1, w]^W$ in affine Coxeter groups

Theorem (P.-Salvetti 2021)

Any element $u \in [1, w]^W$ is a Coxeter element of the Coxeter subgroup generated by the elements $\leq u$.

# Failure of the lattice property

## Theorem (McCammond 2015)

Let $W$ be an irreducible affine Coxeter group. The interval $[1, w]^W$ is a lattice if and only if the horizontal root system is irreducible.

| Type | Horizontal root system |
|---|---|
| $\tilde{A}_n$ | $\Phi_{A_{p-1}} \sqcup \Phi_{A_{q-1}}$ |
| $\tilde{C}_n$ | $\Phi_{A_{n-1}}$ |
| $\tilde{B}_n$ | $\Phi_{A_1} \sqcup \Phi_{A_{n-2}}$ |
| $\tilde{D}_n$ | $\Phi_{A_1} \sqcup \Phi_{A_1} \sqcup \Phi_{A_{n-3}}$ |
| $\tilde{G}_2$ | $\Phi_{A_1}$ |
| $\tilde{F}_4$ | $\Phi_{A_1} \sqcup \Phi_{A_2}$ |
| $\tilde{E}_6$ | $\Phi_{A_1} \sqcup \Phi_{A_2} \sqcup \Phi_{A_2}$ |
| $\tilde{E}_7$ | $\Phi_{A_1} \sqcup \Phi_{A_2} \sqcup \Phi_{A_3}$ |
| $\tilde{E}_8$ | $\Phi_{A_1} \sqcup \Phi_{A_2} \sqcup \Phi_{A_4}$ |

# A new hope

**Theorem (McCammond-Sulway 2017)**

Let $W$ be an irreducible affine Coxeter group.

- Any dual Artin group $W_w$ is isomorphic to the Artin group $G_W$.
- $W_w$ can be embedded into a Garside group $C_w$.

  Idea: extend $W$ to $C$ by adding suitable translations so that $[1, w]^C$ is a lattice.

# Proof of the $K(\pi, 1)$ conjecture for affine Artin groups

1. The complex $K_W$ is a classifying space, even when $[1, w]^W$ is not a lattice.

# Proof of the $K(\pi, 1)$ conjecture for affine Artin groups

1. The complex $K_W$ is a classifying space, even when $[1, w]^W$ is not a lattice.
2. We construct a "dual" model $X'_W \subseteq K_W$ for the configuration space $Y_W$:

$$X'_W := \bigcup_{W_T \subseteq W \text{ finite}} K_{W_T} \simeq Y_W.$$

(done for an arbitrary Coxeter group $W$)

# Proof of the $K(\pi, 1)$ conjecture for affine Artin groups

1. The complex $K_W$ is a classifying space, even when $[1, w]^W$ is not a lattice.
2. We construct a "dual" model $X'_W \subseteq K_W$ for the configuration space $Y_W$:

$$X'_W := \bigcup_{W_T \subseteq W \text{ finite}} K_{W_T} \simeq Y_W.$$

   (done for an arbitrary Coxeter group $W$)
3. We construct a deformation retraction $K_W \searrow X'_W$, using discrete Morse theory.
   - The set of reflections $R_0$ can be totally ordered to make $[1, w]^W$ EL-shellable.

# Proof of the $K(\pi, 1)$ conjecture for affine Artin groups

1. The complex $K_W$ is a classifying space, even when $[1, w]^W$ is not a lattice.
2. We construct a "dual" model $X'_W \subseteq K_W$ for the configuration space $Y_W$:

$$X'_W := \bigcup_{W_T \subseteq W \text{ finite}} K_{W_T} \simeq Y_W.$$

   (done for an arbitrary Coxeter group $W$)
3. We construct a deformation retraction $K_W \searrow X'_W$, using discrete Morse theory.
   - The set of reflections $R_0$ can be totally ordered to make $[1, w]^W$ EL-shellable.

## Theorem (P.-Salvetti 2021)

Let $W$ be an affine Coxeter group.

- The configuration space $Y_W$ is a classifying space for $G_W$.
- Any dual Artin group $W_w$ is isomorphic to the Artin group $G_W$.

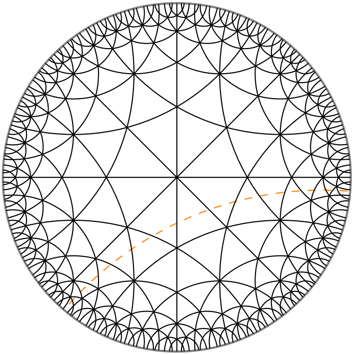
# The dual approach to the $K(\pi, 1)$ conjecture

Let $W$ be a Coxeter group with a fixed Coxeter element $w$. Can we prove the following?

- $K_W$ is a classifying space
  - Optionally because $[1, w]^W$ is a lattice (when?)
- $K_W$ deformation retracts onto $X'_W$
  - Optionally using an EL-labeling of $[1, w]^W$ (always?)

These imply the $K(\pi, 1)$ conjecture for $G_W$ and the natural isomorphism $W_w \cong G_W$.

# Next directions



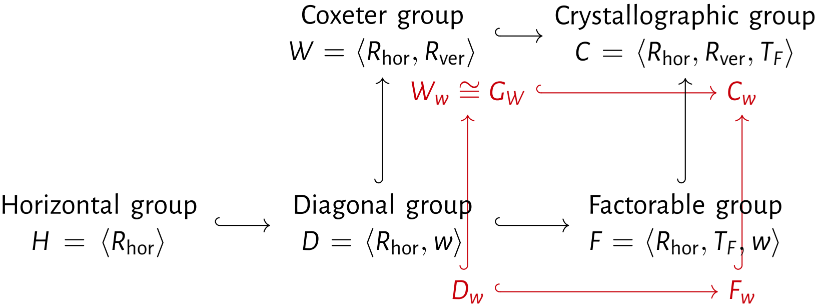
**Theorem (Delucchi-P.-Salvetti 2021+)**

Let $W$ be a Coxeter group of rank 3.

- $[1, w]$ is an EL-shellable lattice.
- $Y_W$ is $K(\pi, 1)$.
- $W_w \cong G_W$.
- The word problem for $G_W$ is solvable.

# Step 1: New groups (McCammond-Sulway 2017)

- $R_{\text{hor}} = \{\text{horizontal reflections}\}$
- $R_{\text{ver}} = \{\text{vertical reflections}\}$
- $T_F = \{\text{factored translations}\}$



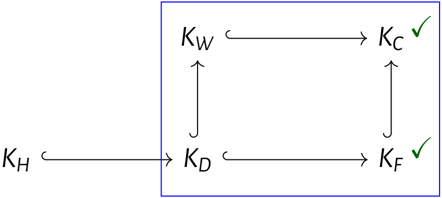
# Step 1: Looking for classifying spaces

We introduce the interval complex $K_G$ for $G = H, D, F, W, C$ (even though only $F_w$ and $C_w$ are Garside groups).

$$\begin{array}{ccccc}
 & & K_W & \hookrightarrow & K_C \ \checkmark \\
 & & \uparrow & & \uparrow \\
K_H & \hookrightarrow & K_D & \hookrightarrow & K_F \ \checkmark
\end{array}$$

# Step 1: Looking for classifying spaces

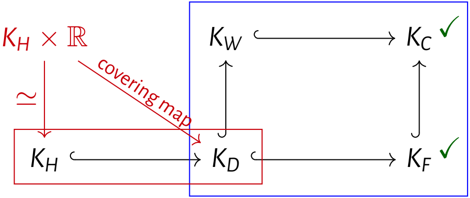
We introduce the interval complex $K_G$ for $G = H, D, F, W, C$ (even though only $F_w$ and $C_w$ are Garside groups).




if three are classifying spaces,
then the fourth also is
(Mayer-Vietoris exact sequence)

# Step 1: Looking for classifying spaces

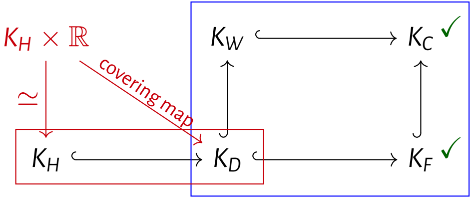
We introduce the interval complex $K_G$ for $G = H, D, F, W, C$ (even though only $F_w$ and $C_w$ are Garside groups).




if three are classifying spaces, then the fourth also is (Mayer-Vietoris exact sequence)

# Step 1: Looking for classifying spaces

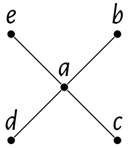
We introduce the interval complex $K_G$ for $G = H, D, F, W, C$ (even though only $F_w$ and $C_w$ are Garside groups).




$K_H = K_{m_1} \times \cdots \times K_{m_k}$,
where $\Phi = \Phi_{A_{m_1}} \sqcup \cdots \sqcup \Phi_{A_{m_k}}$

if three are classifying spaces,
then the fourth also is
(Mayer-Vietoris exact sequence)

# Example: $\tilde{D}_4$




$$a = \text{reflection w.r.t. } \{x_1 + x_2 + x_3 + x_4 = 1\}$$
$$b = \text{reflection w.r.t. } \{x_1 = 0\}$$
$$c = \text{reflection w.r.t. } \{x_2 = 0\}$$
$$d = \text{reflection w.r.t. } \{x_3 = 0\}$$
$$e = \text{reflection w.r.t. } \{x_4 = 0\}$$

Coxeter element: $w = abcde$ with axis $\langle 1, 1, 1, 1 \rangle$.

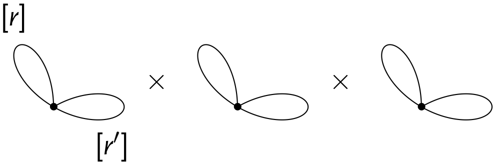
Jon says that the horizontal root system is $\Phi = \Phi_{A_1} \sqcup \Phi_{A_1} \sqcup \Phi_{A_1}$.

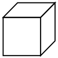
$$w = abcde = bc \cdot a^{bc} \cdot de = de \cdot a^{de} \cdot ab$$
$$a^{bc} = \text{reflection w.r.t. } \{x_1 + x_2 - x_3 - x_4 = -1\} =: r$$
$$a^{de} = \text{reflection w.r.t. } \{x_1 + x_2 - x_3 - x_4 = 1\} =: r'$$

The horizontal directions are: $\langle 1, 1, -1, -1 \rangle$, $\langle 1, -1, 1, -1 \rangle$, $\langle 1, -1, -1, 1 \rangle$.

# Example: $\tilde{D}_4$

We have $[1, w]^W \cap \langle r, r' \rangle = \{1, r, r'\}$, so $K_H$ is the product of three copies of $S^1 \vee S^1$:

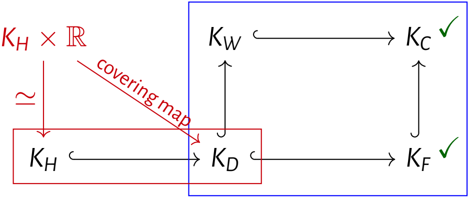
$K_H =$ 


The 6 horizontal reflections enclose a prism  $\times \mathbb{R}$.

The Coxeter element $w$ acts on this prism by central symmetry on the cube and translation along the $\mathbb{R}$ direction.

Therefore $K_D = K_H \times [0, 1]/\sim$, where $\sim$ identifies $K_H \times \{0\}$ and $K_H \times \{1\}$ by swapping the two $S^1$'s in each of the three components.

# Step 1: Looking for classifying spaces



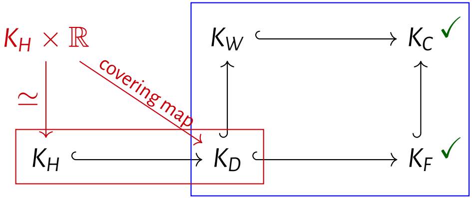

$K_H = K_{m_1} \times \cdots \times K_{m_k}$,
where $\Phi = \Phi_{A_{m_1}} \sqcup \cdots \sqcup \Phi_{A_{m_k}}$

if three are classifying spaces,
then the fourth also is
(Mayer-Vietoris exact sequence)

Each $K_m$ is (a variation of) the "dual" model $X'_{\tilde{A}_m}$ !

So the $K(\pi, 1)$ conjecture for the case $\tilde{A}_m$ implies that $K_H$ is a classifying space, so $K_D$ and $K_W$ also are classifying spaces.

# Step 1: Looking for classifying spaces




$K_H = K_{m_1} \times \cdots \times K_{m_k}$,
where $\Phi = \Phi_{A_{m_1}} \sqcup \cdots \sqcup \Phi_{A_{m_k}}$

if three are classifying spaces, then the fourth also is (Mayer-Vietoris exact sequence)

Without using the $K(\pi, 1)$ conjecture for $\tilde{A}_m$:

- If $k = 1$, then $[1, w]^W$ is a lattice. Therefore $K_W$, $K_D$, and $K_H = K_{m_1}$ are classifying spaces.
- For every $m \geq 1$, the complex $K_m$ only depends on $m$ and can appear alone (e.g. if $W$ is of type $\tilde{C}_{m+1}$).
- Therefore, for any irreducible affine Coxeter group $W$, $K_H = K_{m_1} \times \cdots \times K_{m_k}$ is a classifying space, so $K_D$ is, and $K_W$ also is.

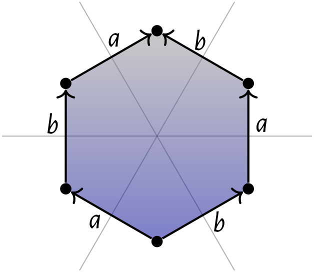
# Step 2: A “dual” model for the configuration space $Y_W$

The Salvetti complex $X_W$ has cells indexed by

$$\Delta_W = \{T \subseteq S \mid \text{the standard parabolic subgroup } W_T \text{ is finite}\}.$$

It is natural: $X_W = \bigcup_{T \in \Delta_W} X_{W_T}$.

Both $X_{W_T}$ and $K_{W_T}$ are classifying spaces for the Artin group $G_{W_T}$, so $X_{W_T} \simeq K_{W_T}$.




The Salvetti complex $X_W$ for $W = \mathfrak{S}_3$

# Step 2: A "dual" model for the configuration space $Y_W$

The Salvetti complex $X_W$ has cells indexed by

$$\Delta_W = \{T \subseteq S \mid \text{the standard parabolic subgroup } W_T \text{ is finite}\}.$$

It is natural: $X_W = \bigcup_{T \in \Delta_W} X_{W_T}$.

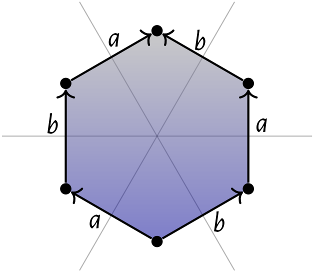
Both $X_{W_T}$ and $K_{W_T}$ are classifying spaces for the Artin group $G_{W_T}$, so $X_{W_T} \simeq K_{W_T}$.

## Definition (dual model)

$$X'_W = \bigcup_{T \in \Delta_W} K_{W_T}.$$

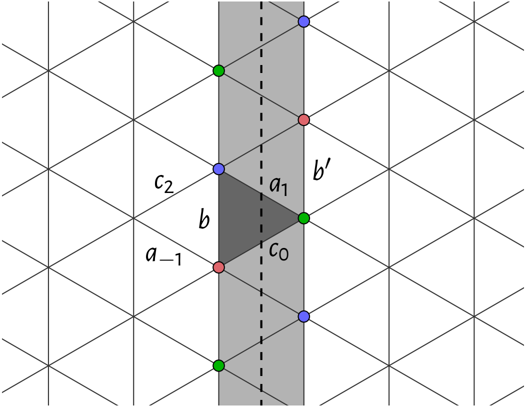
## Theorem

$X'_W \simeq X_W \simeq Y_W$.




The Salvetti complex $X_W$ for $W = \mathfrak{S}_3$

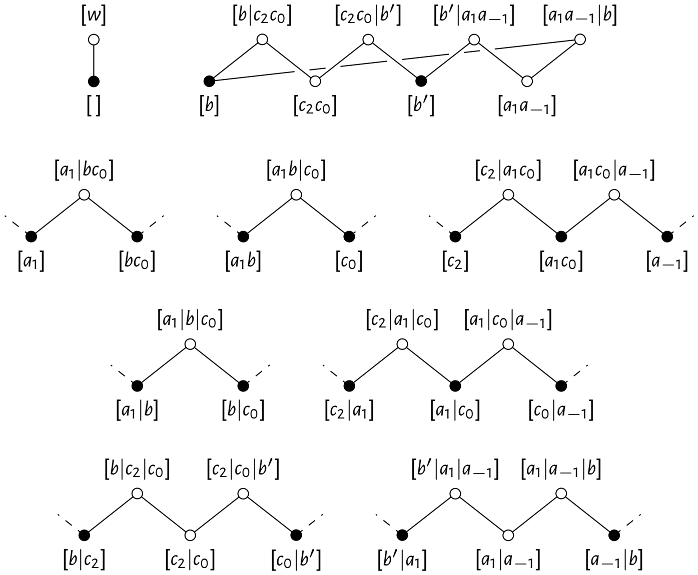
# Example: $\tilde{A}_2$




$X'_W$ is the union of three different copies of $K_{A_2}$ sitting inside $K_W$:

- $[\,]$, $[a_1]$, $[b]$, $[c_2]$, $[a_1 b]$, $[a_1|b]$, $[b|c_2]$, $[c_2|a_1]$
- $[\,]$, $[a_1]$, $[c_0]$, $[b']$, $[a_1 c_0]$, $[a_1|c_0]$, $[c_0|b']$, $[b'|a_1]$
- $[\,]$, $[b]$, $[c_0]$, $[a_{-1}]$, $[b c_0]$, $[b|c_0]$, $[c_0|a_{-1}]$, $[a_{-1}|b]$

# Step 3: Deformation retraction $K_W \searrow X'_W$



## Step 3: Deformation retraction $K_W \searrow X'_W$

We order the set of reflections $R_0$ so that:

1. each element $u \in [1, w]$ has a unique minimal factorization $u = r_1 r_2 \cdots r_k$ with $r_1 \prec r_2 \prec \cdots \prec r_k$;
2. the increasing factorization is the lexicographically smallest and co-lexicographically largest.

(this makes $[1, w]^W$ EL-shellable)

### Why? (How do we use this ordering?)

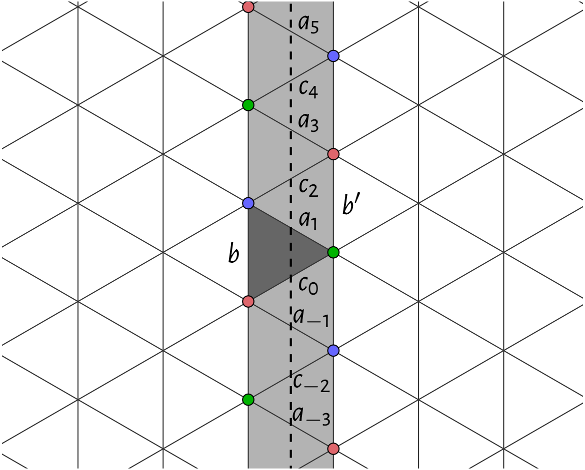
The remaining cells are collapsed following increasing factorizations greedily:

- $[w] \to [a_1 | b c_0]$ because $a_1 \prec b \prec c_0$;
- $[a_1 b | c_0] \to [a_1 | b | c_0]$ because $a_1 \prec b$;
- $\ldots$

# Step 3: Axial ordering of $R_0$

We order $R_0$ following the axis of $w$:

$$a_1 \prec c_2 \prec a_3 \prec \cdots \prec b \prec b' \prec \cdots \prec c_{-2} \prec a_{-1} \prec c_0.$$



Thanks!

paolini@caltech.edu