# Artin groups and associated spaces
# Part II

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## Lemma

Let $\Gamma$ be a Coxeter graph. Let $\preceq$ be the relation on $W[\Gamma] \times \mathcal{S}_{\rm sph}$ defined by:

$$(u, X) \preceq (v, Y) \iff [X \subset Y,\ v^{-1}u \in W[\Gamma_Y] \text{ and } v^{-1}u \text{ is } X\text{-minimal}]$$

Then $\preceq$ is an order relation.

**Definition.** A *simplicial complex* is a pair $\Upsilon = (S, A)$, where $S$ is a set, called the *set of vertices* of $\Upsilon$, and $A$ is a set of subsets of $S$, called the *set of simplices* of $\Upsilon$, satisfying the following properties:

(a) $\emptyset$ is not a simplex and every simplex is finite;

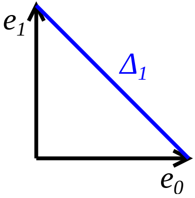
(b) each singleton is a simplex;

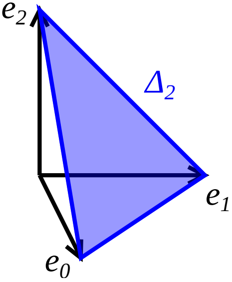
(c) a non-empty subset of a simplex is a simplex.

**Definition.** Let $\Upsilon = (S, A)$ be a simplicial complex. Let $B = \{e_S \mid s \in S\}$ be a set in one-to-one correspondence with $S$ and let $V$ be the real vector space having $B$ as a basis. For each simplex $\Delta = \{s_0, s_1, \dots, s_p\}$ in $A$ we set:

$$|\Delta| = \left\{ \sum_{i=0}^{p} t_i e_{s_i} \mid 0 \le t_0, t_1, \dots, t_p \le 1 \text{ and } \sum_{i=1}^{p} t_i = 1 \right\} .$$

Note that $|\Delta|$ is a geometric simplex of dimension $p$.



The *geometric realization* of $\Upsilon$ is the following subset of $V$:

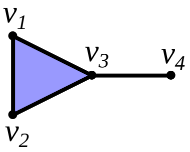
$$|\Upsilon| = \bigcup_{\Delta \in A} |\Delta|\,,$$

which we endow with the so-called "weak topology".

**Example.** Let $S = \{v_1, v_2, v_3, v_4\}$ and let:

$$A = \{\{v_1\}\,,\ \{v_2\}\,,\ \{v_3\}\,,\ \{v_4\}\,,\ \{v_1, v_2\}\,,\ \{v_1, v_3\}\,,\ \{v_2, v_3\}\,,\ \{v_3 v_4\}\,,\ \{v_1, v_2, v_3\}\}\,.$$

Then $\Upsilon = (S, A)$ is a simplicial complex whose geometric realization is:



**Definition.** If $(E, \leq)$ is an ordered set, then the set of non-empty chains of $(E, \leq)$ forms a simplicial complex, called the *derived complex* of $(E, \leq)$, denoted by $E'$ or by $(E, \leq)'$.

**Definition.** Let $\Gamma$ be a Coxeter graph. The *Salvetti complex* of $\Gamma$, denoted $\mathrm{Sal}[\Gamma]$, is the geometric realization of the derived complex of $(W[\Gamma] \times \mathcal{S}_{\mathrm{sph}}, \preceq)$. Note that the action of $W[\Gamma]$ on $(W[\Gamma], \mathcal{S}_{\mathrm{sph}})$ defined by $w \cdot (u, X) = (wu, X)$ preserves the order, hence it induces an action of $W[\Gamma]$ on $\mathrm{Sal}[\Gamma]$.
Attention: the quotient $\mathrm{Sal}[\Gamma]/W[\Gamma]$ is not a simplicial complex. It is a CW-complex or, more precisely, a "$\Delta$-complex".

**Theorem (Salvetti [1994], Charney–Davis [1995])**

Let $\Gamma$ be a Coxeter graph. There exists a homotopy equivalence $\mathrm{Sal}[\Gamma] \to M[\Gamma]$ which is equivariant under the action of $W[\Gamma]$ and which induces a homotopy equivalence $\mathrm{Sal}[\Gamma]/W[\Gamma] \to M[\Gamma]/W[\Gamma] = N[\Gamma]$.

**Definition.** Let $\Gamma$ be a Coxeter graph. For each $(u, X) \in W[\Gamma] \times \mathcal{S}_{\rm sph}$ we set:

$$C(u, X) = \{(v, Y) \in W[\Gamma] \times \mathcal{S}_{\rm sph} \mid (v, Y) \preceq (u, X)\},$$

and we denote by $\mathbb{B}(u, X)$ the geometric realization of $C(u, X)$, that is, the simplicial subcomplex of ${\rm Sal}[\Gamma]$ spanned by $C(u, X)$.

## Proposition

Let $\Gamma$ be a Coxeter graph. Let $(u, X) \in W[\Gamma] \times \mathcal{S}_{\rm sph}$. Then $\mathbb{B}(u, X)$ is homeomorphic to a ball of dimension $|X|$.

## Corollary

Let $\Gamma$ be a Coxeter graph. Then ${\rm Sal}[\Gamma]$ has a cellular decomposition described as follows. For each $w \in W$ we have a vertex $x(w)$ corresponding to $\mathbb{B}(w, \emptyset)$. The 0-skeleton of ${\rm Sal}[\Gamma]$ is

$\{x(w) \mid w \in W[\Gamma]\}$. More generally, for $p \in \mathbb{N}$, the set of $p$-cells of $\mathrm{Sal}[\Gamma]$ is $\{\mathbb{B}(u,X) \mid u \in W[\Gamma]\,,\ X \in \mathcal{S}_{\mathrm{sph}}\,,\ |X| = p\}$, and the $p$-skeleton is the union of these cells.

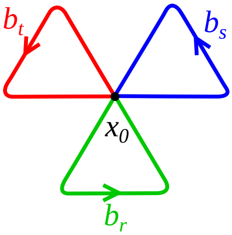
We set $\overline{\mathrm{Sal}}[\Gamma] = \mathrm{Sal}[\Gamma]/W[\Gamma]$. For $w \in W[\Gamma]$ and $(u,X) \in W[\Gamma] \times \mathcal{S}_{\mathrm{sph}}$ we have $w \cdot \mathbb{B}(u,X) = \mathbb{B}(wu,X)$. Hence, the the action of $W[\Gamma]$ on $\mathrm{Sal}[\Gamma]$ is combinatorial, and therefore the cellular decomposition of $\mathrm{Sal}[\Gamma]$ induces a cellular decomposition of $\overline{\mathrm{Sal}}[\Gamma]$. For each $X \in \mathcal{S}_{\mathrm{sph}}$, the orbit of $\mathbb{B}(1,X)$ under the action of $W[\Gamma]$ is $\{\mathbb{B}(u,X) \mid u \in W[\Gamma]\}$. With this orbit we associate a cell $\overline{\mathbb{B}}(X)$ of $\overline{\mathrm{Sal}}[\Gamma]$, and every cell of $\overline{\mathrm{Sal}}[\Gamma]$ is of this form. In particular, for $p \in \mathbb{N}$, the set of cells of $\overline{\mathrm{Sal}}[\Gamma]$ of dimension $p$ is $\{\overline{\mathbb{B}}(X) \mid X \in \mathcal{S}_{\mathrm{sph}}\,,\ |X| = p\}$.

**Example.** Let $\Gamma$ be:

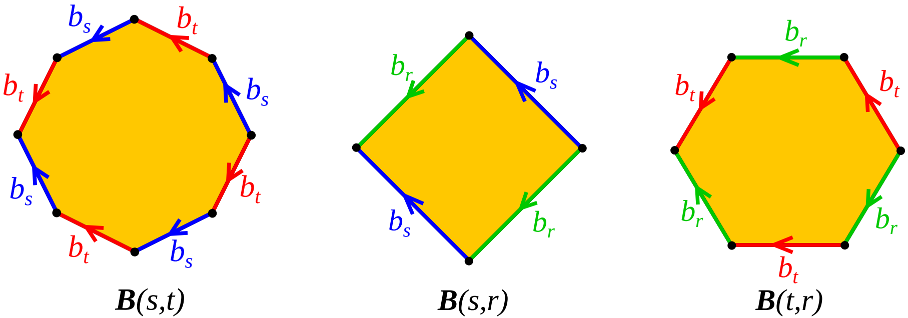
Recall that:

$$A[\Gamma] = \langle s, t, r \mid stst = tsts\,,\ sr = rs\,,\ trt = rtr \rangle\,.$$

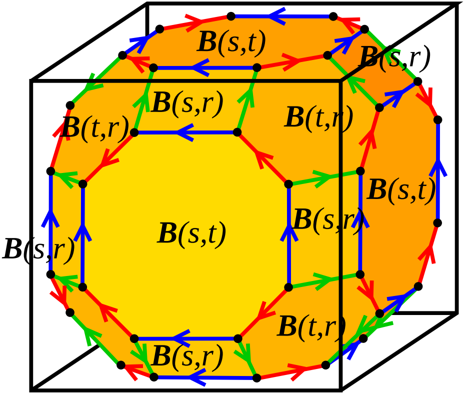
The 0-skeleton of $\overline{\mathrm{Sal}}[\Gamma]$ is a single point, $x_0$. The 1-skeleton of $\overline{\mathrm{Sal}}[\Gamma]$ is formed by three (oriented) edges, $b_s$, $b_t$ and $b_r$.



The 2-skeleton of $\overline{\mathrm{Sal}}[\Gamma]$ is formed by three cells, $\overline{\mathbb{B}}(s,t)$, $\overline{\mathbb{B}}(s,r)$ and $\overline{\mathbb{B}}(t,r)$, whose boundary are:



The 3-skeleton is formed by a unique cell whose boundary is:




Note that a straightforward consequence of this description is:

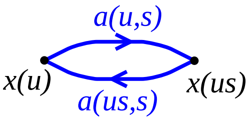
$$\pi_1(\overline{\mathrm{Sal}}[\Gamma]) = \langle s, t, r \mid stst = tsts\,,\ sr = rs\,,\ trt = rtr \rangle = A[\Gamma]\,.$$

Since $\overline{\mathrm{Sal}}[\Gamma] \simeq_h N[\Gamma]$, we deduce that $\pi_1(N[\Gamma]) = A[\Gamma]$.

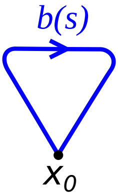
More generally, the $p$-skeleton of $\mathrm{Sal}[\Gamma]$ and $\overline{\mathrm{Sal}}[\Gamma]$ for $p = 0, 1, 2$ are described as follows.

**0-skeleton:** The 0-skeleton of $\mathrm{Sal}[\Gamma]$ is a set $\{x(w) \mid w \in W[\Gamma]\}$ in one-to-one correspondence with $W[\Gamma]$. The 0-skeleton of $\overline{\mathrm{Sal}}[\Gamma]$ is reduced to a single point denoted by $x_0$.

**1-skeleton:** With each pair $(w, s) \in W[\Gamma] \times S$ we associate an edge $\mathbb{B}(u, \{s\})$ of $\mathrm{Sal}[\Gamma]$ whose extremities are $x(u)$ and $x(us)$. We denote this edge by $a(u, s)$ and we assume it is oriented from $x(u)$ towards $x(us)$. So, for $u, v \in W[\Gamma]$, if $v$ is of the form $v = us$, then there is an edge $a(u, s)$ going from $x(u)$ to $x(v)$ and there is an edge $a(v, s)$ going from $x(v)$ to $x(u)$.



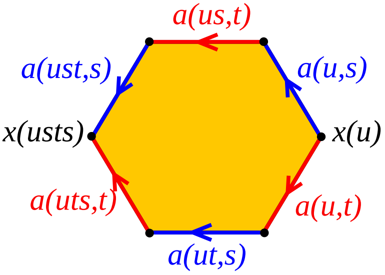
On the other hand, there is no edge between $x(u)$ and $x(v)$ if $v$ is not of the form $v = us$ with $s \in S$. With each $s \in S$ we associate an edge $b(s) = \mathbb{B}(\{s\})$ of $\overline{\mathrm{Sal}}[\Gamma]$ whose both extremities are $x_0$.




Observe that the action of $W[\Gamma]$ on $\{a(u,s) \mid u \in W[\Gamma]\}$ preserves the orientation, hence it induces an orientation on $b(s)$. Thus, we can suppose that $b(s)$ is endowed with this orientation.

**2-skeleton:** Let $s, t \in S$, $s \neq t$. Note that $\{s, t\} \in \mathcal{S}_{\mathrm{sph}}$ if and only if $m_{s,t} \neq \infty$. Assume that $m_{s,t} \neq \infty$. With each $u \in W[\Gamma]$ we associate a 2-cell $\mathbb{B}(u, \{s, t\})$ of $\mathrm{Sal}[\Gamma]$ whose boundary is:

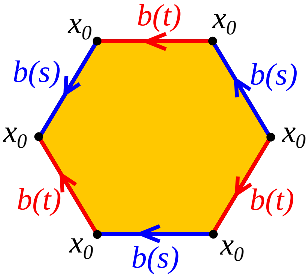
$$(a(u, s)\, a(us, t)\, a(ust, s) \cdots)(a(u, t)\, a(ut, s)\, a(uts, t) \cdots)^{-1}$$




***B***(*u*,{*s*,*t*})

The $W$-orbit $\{\mathbb{B}(u,\{s,t\}) \mid u \in W[\Gamma]\}$ determines a 2-cell $\overline{\mathbb{B}}(\{s,t\})$ of $\overline{\mathrm{Sal}}[\Gamma]$ whose boundary is:

$$(b(s)\, b(t)\, b(s) \cdots)(b(t)\, b(s)\, b(t) \cdots)^{-1} =$$
$$\mathrm{Prod}(b(s), b(t), m_{s,t})\, \mathrm{Prod}(b(t), b(s), m_{s,t})^{-1}$$




$\overline{\boldsymbol{B}}(\{s,t\})$

A straightforward consequence of this description is the following.

## Theorem (Van der Lek [1983])

Let $\Gamma$ be a Coxeter graph. Then $\pi_1(N[\Gamma]) = \pi_1(\overline{\mathrm{Sal}}[\Gamma]) = A[\Gamma]$.

Let $\Gamma$ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. Let $X \subset S$. Recall that $M_X = (m_{s,t})_{s,t \in X}$, that $\Gamma_X$ is the Coxeter graph of $M_X$, and that $W[\Gamma_X]$ is the subgroup of $W[\Gamma]$ generated by $X$. We denote by $\mathcal{S}_{\mathrm{sph}}(X)$ the set of subsets $Y \subset X$ such that $\Gamma_Y$ is of spherical type. The inclusion map $(W[\Gamma_X] \times \mathcal{S}_{\mathrm{sph}}(X)) \hookrightarrow (W[\Gamma] \times \mathcal{S}_{\mathrm{sph}})$ preserves $\preceq$, hence it induces an embedding $\iota_X : \mathrm{Sal}[\Gamma_X] \hookrightarrow \mathrm{Sal}[\Gamma]$.

## Theorem (Godelle–Paris [2012])

Let $\Gamma$ be a Coxeter graph, let $S$ be its set of vertices, and let $X$ be a subset of $S$. Then the embedding $\iota_X : \mathrm{Sal}[\Gamma_X] \to \mathrm{Sal}[\Gamma]$ admits a retraction $\pi_X : \mathrm{Sal}[\Gamma] \to \mathrm{Sal}[\Gamma_X]$.

**Note on the proof.** The map $\pi_X : \mathrm{Sal}[\Gamma] \to \mathrm{Sal}[\Gamma_X]$ is constructed combinatorially in the sense that we define a map $\hat{\pi}_X : (W[\Gamma] \times \mathcal{S}_{\mathrm{sph}}) \to (W[\Gamma_X] \times \mathcal{S}_{\mathrm{sph}}(X))$ and we show that this map preserves the order $\preceq$. Hence it induces a continuous map $\pi_X : \mathrm{Sal}[\Gamma] \to \mathrm{Sal}[\Gamma_X]$.

Let $(u, Y) \in W[\Gamma] \times \mathcal{S}_{\mathrm{sph}}$. We know that the coset $W[\Gamma_X]u$ has a unique element of minimal length, $u_1$. Let:

$$u_0 = uu_1^{-1} \in W_X\,,\ \ Y_0 = X \cap (u_1 Y u_1^{-1}) \in \mathcal{S}_{\mathrm{sph}}(X)\,.$$

Then:

$$\hat{\pi}_X(u, Y) = (u_0, Y_0)\,. \quad \square$$

**Example.** We return to the example where $\Gamma$ is:

We have $S = \{s, t, r\}$. We set $X = \{s, t\}$. The elements $u_1 \in W[\Gamma]$ that are of minimal length in their cosets $W[\Gamma_X]u_1$ are:

$$1\,,\ r\,,\ rt\,,\ rts\,,\ rtst\,,\ rtstr\,.$$

Take one of these elements, say $u_1 = rts$. We choose any element $u_0 \in W[\Gamma_X]$, say $u_0 = st$, and we set $u = u_0 u_1 = strts$. We have:

$$u_1 s u_1^{-1} = rtstr \notin S\,,\ u_1 t u_1^{-1} = strts \notin S\,,\ u_1 r u_1^{-1} = t\,.$$

So:

$$\hat{\pi}_X(strts, \emptyset) = \hat{\pi}_X(strts, \{s\}) =$$
$$\hat{\pi}_X(strts, \{t\}) = \hat{\pi}_X(strts, \{s, t\}) = (st, \emptyset)\,,$$
$$\hat{\pi}_X(strts, \{r\}) = \hat{\pi}_X(strts, \{s, r\}) =$$
$$\hat{\pi}_X(strts, \{t, r\}) = \hat{\pi}_X(strts, S) = (st, \{t\})\,.$$

We can use $\iota_X$ and $\pi_X$ to prove the following results.

**Theorem (Van der Lek [1983])**

Let $\Gamma$ be a Coxeter graph, let $S$ be its set of vertices, and let $X$ be a subset of $S$. Then the homomorphism $A[\Gamma_X] \to A[\Gamma]$ induced by the inclusion map $X \hookrightarrow S$ is injective.

**Theorem**

Let $\Gamma$ be a Coxeter graph, let $S$ be its set of vertices, and let $X$ be a subset of $S$. If $A[\Gamma]$ satisfies the $K(\pi, 1)$ conjecture, then $A[\Gamma_X]$ satisfies the $K(\pi, 1)$ conjecture.

**Theorem (Godelle–Paris [2012])**

Let $\Gamma$ be a Coxeter graph, let $S$ be its set of vertices, and let $X$ be a subset of $S$. Suppose that $A[\Gamma]$ has a solution to the word problem.

There is an algorithm which, for a given $g \in A[\Gamma]$, decides whether $g$ lies in $A[\Gamma_X]$ or not.

Let $\Gamma$ be a Coxeter graph and let $S$ be its set of vertices. We denote by $\lg : A[\Gamma] \to \mathbb{N}$ the word length with respect to $S$. A word $w = s_1^{\varepsilon_1} \dots s_\ell^{\varepsilon_\ell}$ on $S \cup S^{-1}$ is *reduced* (or *geodesic*) if $\ell = \lg(g)$, where $g$ is the element of $A[\Gamma]$ represented by $w$.

### Theorem (Charney–Paris [2014])

Let $\Gamma$ be a Coxeter graph, let $S$ be its set of vertices, and let $X$ be a subset of $S$. Let $g \in A[\Gamma]$ and let $w = s_1^{\varepsilon_1} \dots s_\ell^{\varepsilon_\ell}$ be a reduced word which represents $g$. If $g \in A[\Gamma_X]$, then $s_i \in X$ for all $i \in \{1, \dots, \ell\}$.

**Thank you for your attention!**