Garside groups and the Yang-Baxter equation

Summer school: The dual approach to Coxeter and Artin groups, Garside theory and applications, Berlin 2021.

Fabienne Chouraqui

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Definition of a Garside monoid [P. Dehornoy, L. Paris 1999]

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A Garside group is the group of fractions of a Garside monoid.

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- G has finite homological dimension [P.Dehornoy and Y.Lafont 2003][R.Charney, J. Meier and K. Whittlesey 2004]

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Examples of Garside groups

- Braid groups [Garside]
- Artin groups of finite type [Deligne, Brieskorn-Saito]
- Torus link groups [Picantin]

Some questions about the Garside gps

Do Garside groups admit a finite quotient that plays the same role S_n plays for B_n or the Coxeter groups for finite-type Artin groups?

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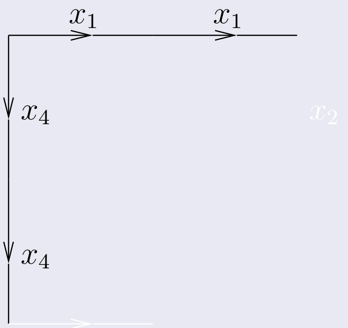
Are all the Garside groups left-orderable?

question raised in book *Ordering braids* by P.Dehornoy, I.Dynnikov, D.Rolfsen, B.Wiest.

Right reversing method

lcm of x_1^2 and x_4^2



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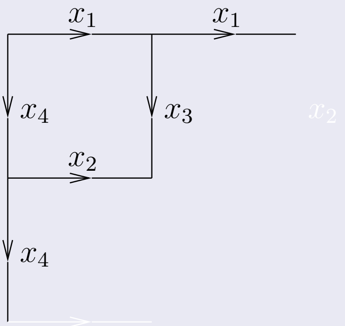

In M

$x_1x_3 = x_4x_2$

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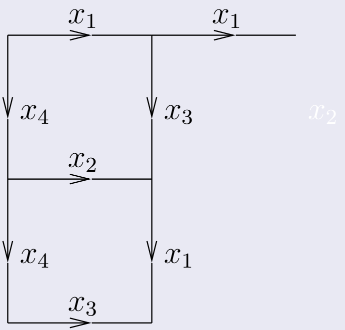

In M

$$x_1x_3 = x_4x_2$$
$$x_2x_1 = x_4x_3$$

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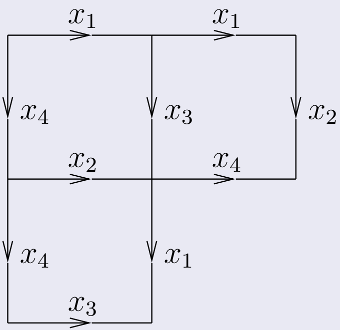

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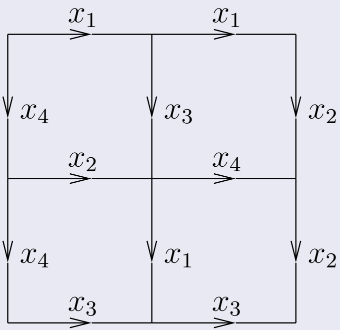

In M

$$
\begin{aligned}
x_1x_3 &= x_4x_2\\
x_2x_1 &= x_4x_3\\
x_1x_2 &= x_3x_4\\
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$$

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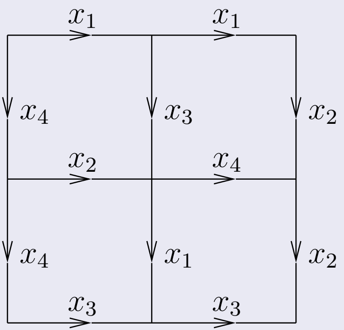

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The lcm is:

$$x_1^2x_2^2 = x_1^4 = x_4^2x_3^2 = x_4^4 = ..$$

The quantum Yang-Baxter equation - QYBE

Let $R : V \otimes V \to V \otimes V$ be a linear operator, where V is a vector space.
The QYBE is the equality $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ of linear transformations on $V \otimes V \otimes V$, where R^{ij} means R acting on the $i-$th and $j-$th components.

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A set-theoretical solution (X, S) of this equation [Drinfeld]

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A set-theoretical solution (X, S) of this equation [Drinfeld]

- V is a vector space spanned by a set X.
- R is the linear operator induced by a mapping $S : X \times X \to X \times X$, that satisfies $S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23}$.

Properties of a solution (X, S)

Let $X = \{x_1, ..., x_n\}$ and let S be defined in the following way: $S(i, j) = (g_i(j), f_j(i))$, where $f_i, g_i : X \to X$.

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- (X, S) is non-degenerate $\Leftrightarrow$ f_i and g_i are bijective, $1 \leq i \leq n$.

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- (X, S) is braided $\Leftrightarrow$ $S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23}$

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- (X, S) is non-degenerate $\Leftrightarrow$ f_i and g_i are bijective, $1 \leq i \leq n$.
- (X, S) is involutive $\Leftrightarrow$ $g_{g_i(j)} f_j(i) = i$ and $f_{f_j(i)} g_i(j) = j$, $1 \leq i, j \leq n$.
- (X, S) is braided $\Leftrightarrow$ $g_i g_j = g_{g_i(j)} g_{f_j(i)}$ and $f_j f_i = f_{f_j(i)} f_{g_i(j)}$ and $f_{g_{f_j(i)}(k)} g_i(j) = g_{f_{g_j(k)}(i)} f_k(j)$, $1 \leq i, j, k \leq n$.

The QYBE group: the structure group of (X, S)

Assumption: The pair (X, S) is a non-degenerate, involutive and braided. We call it a non-degenerate, involutive set-solution.

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There are exactly $\frac{n(n-1)}{2}$ defining relations.

Example 1

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$.

The functions that define S

Let $f_1 = g_1 = (1, 2, 3, 4)(5)$
$f_2 = g_2 = (1, 4, 3, 2)(5)$
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The defining relations in G and in M (the monoid with the same pres.)

$$\begin{array}{ccccc} x_1^2 = x_2^2 & x_1x_2 = x_3x_4 & x_1x_3 = x_4x_2 & x_1x_5 = x_5x_1 & x_4x_5 = x_5x_4 \\ x_3^2 = x_4^2 & x_2x_1 = x_4x_3 & x_2x_4 = x_3x_1 & x_2x_5 = x_5x_2 & x_3x_5 = x_5x_3 \end{array}$$

The correspondence between QYBE groups and Garside groups

Theorem (F.C. 2009)

Let (X, S) be a non-degenerate, involutive set-solution with structure group G. Then G is Garside.

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Assume that $\mathrm{Mon}\langle X \mid R\rangle$ is a **Garside monoid** such that:
- the cardinality of R is $n(n-1)/2$
- each side of a relation in R has length 2.
- if the word $x_i x_j$ appears in R, then it appears only once.

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Then $G = \text{Gp}\langle X \mid R\rangle$ is the structure group of a non-degenerate, involutive set-solution (X, S), with $\mid X \mid = n$.

Correspondence between the right complement and the functions defining the solution

right complement $\Leftrightarrow$ functions

Expressing $x_i \setminus x_j$ in terms of the functions g_i:
Let x_i, x_j be different elements in X.
Then $x_i \setminus x_j = g_i^{-1}(j)$.

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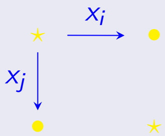

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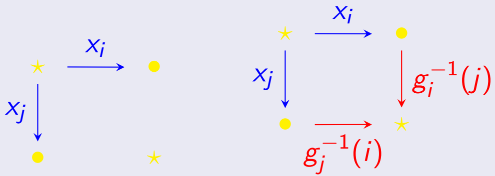

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- *The Garside element Δ has length n.*
- *The (co)homological dimension of the structure group G is n. [P.Dehornoy, Y.Laffont 2003] [R.Charney, J.Meier, K.Whittlesey 2004] [J. McCammond]*

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- The length of Δ is equal to $|X|$.

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- The length of s is equal to $|X_l| = |X_r|$.
- The length of Δ is equal to $|X|$.

The set of simples is equal to $\overline{X}^{\vee} \cup \{1\}$

Decomposability of a solution (X, S)

Let (X, S) be a non-degenerate, involutive set-solution.

Definition

(X, S) is *decomposable* if it is the union of two nonempty disjoint non-degenerate invariant subsets. Otherwise, (X, S) is *indecomposable*.

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Theorem (Etingof,Schedler,Soloviev)

(X, S) is indecomposable if and only if G acts transitively on X, where $x_i \to g_i^{-1}$ is a right action of G on X.

Example 1

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and S as before.

(X, S) is a decomposable solution

- $X = \{x_1, x_2, x_3, x_4\} \cup \{x_5\}$.
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The defining relations in G and in M

$$x_1^2 = x_2^2 \qquad x_3^2 = x_4^2 \qquad (x_5x_5 = x_5x_5)$$

$$x_1x_2 = x_3x_4 \qquad x_1x_5 = x_5x_1$$

$$x_1x_3 = x_4x_2 \qquad x_2x_5 = x_5x_2$$

$$x_2x_4 = x_3x_1 \qquad x_3x_5 = x_5x_3$$

$$x_2x_1 = x_4x_3 \qquad x_4x_5 = x_5x_4$$

Example 2

Let $X = \{x_0, x_1, x_2, x_3\}$.

$$\begin{array}{ll} g_0 = (0)(1)(2,3) & g_1 = (1,2,0,3) \\ g_2 = (2)(3)(0,1) & g_3 = (1,3,0,2) \end{array} \quad (1)$$

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The solution is indecomposable with defining relations:

$$\begin{array}{ll} x_1 x_1 = x_2 x_0 & x_1 x_0 = x_3 x_2 \\ x_0 x_3 = x_2 x_1 & x_1 x_2 = x_0 x_1 \\ x_2 x_3 = x_3 x_0 & x_3^2 = x_0 x_2 \end{array} \tag{2}$$

$\Delta-$pure Garside monoids [Picantin 2001]

Definition of a $\Delta-$pure Garside monoid

Let M be a Garside monoid. Then M is $\Delta-$*pure* if for every x, y in X, it holds that $\Delta_x = \Delta_y$,
where $\Delta_x = \vee(M \setminus x) = \vee\{w \setminus x; w \in M\}$.

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Theorem (Picantin 2001)

If M is a $\Delta-$pure Garside monoid, Δ is its Garside element and G its group of fractions. Then the center of M (resp. of G) is the infinite cyclic submonoid (resp. subgroup) generated by Δ^e, where e is a natural number (the order of the conjugation automorphism by Δ).

Which structure monoids are $\Delta-$pure Garside ?

Theorem (F.C. 2009)

Let (X, S) be a non-degenerate, involutive set-solution of the quantum Yang-Baxter equation with structure group G. Then (X, S) is indecomposable if and only if G is $\Delta-$pure Garside.

Which structure monoids are $\Delta-$pure Garside ?

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A consequence

If (X, S) is indecomposable then the center of G is cyclic, generated by some exponent of Δ.

Example 2

Let $X = \{x_0, x_1, x_2, x_3\}$.

$$g_0 = (0)(1)(2,3) \qquad g_1 = (1,2,0,3)$$
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The solution is indecomposable with defining relations:

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The center of G is generated by $\Delta = (x_0 x_1)^2 = (x_2 x_3)^2$, $e = 1$.

The BRAID group B_n

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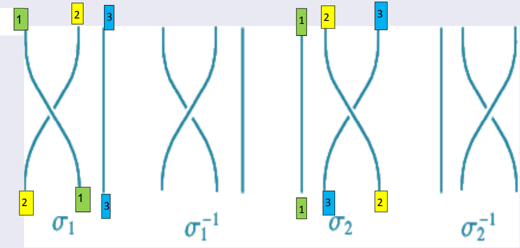
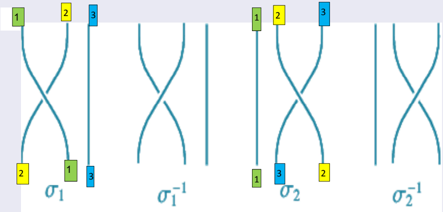
The BRAID group $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$



The original Coxeter group construction

$\exists$ epimorphism $B_3 \to S_3$:
$\sigma_1 \mapsto (1,2)$; $\sigma_2 \mapsto (2,3)$



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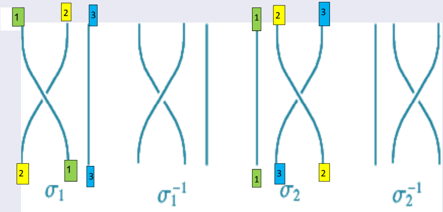

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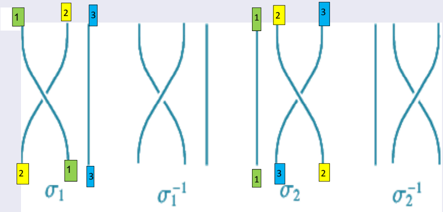

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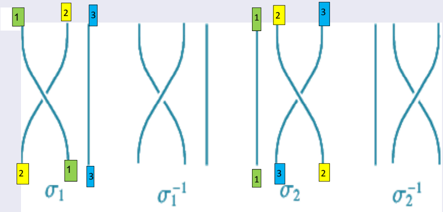

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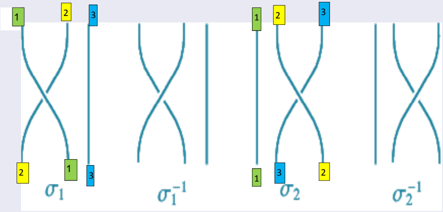

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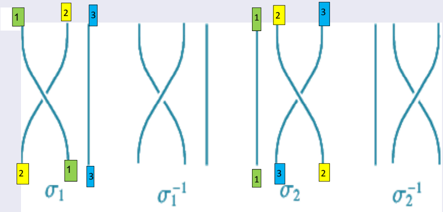

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Dehornoy's extension 2014: condition (C) can be relaxed

QYBE groups with condition (C) admit Coxeter-like quotient groups

Theorem (F.C and E.Godelle 2013)

Let (X, S) be a non-degenerate, involutive set-solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X, S) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$ satisfying

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What is condition (C)?

Let $x_i, x_j \in X$. If $S(i,j) = (i,j)$, then $f_i f_j = g_i g_j = Id_X$.

Coxeter-like group for Example 2

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There are 4 trivial relations:
$x_0x_0 = x_0x_0,\ x_1x_3 = x_1x_3,\ x_2x_2 = x_2x_2,\ x_3x_1 = x_3x_1$

The solution satisfies (C): $g_o^2 = g_1g_3 = g_3g_1 = g_2^2 = Id_X$

$N = \langle x_0x_0, x_1x_3, x_2x_2, x_3x_1 \rangle \simeq \mathbb{Z}^4$, $N \lhd G$, and $W \simeq G/N$

Coxeter-like group for other examples: $X = \{x_0, x_1, x_2\}$

A square-free solution

$g_0 = f_0 = g_1 = f_1 = Id$,
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Example from Dehornoy's paper

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Orderability of groups

A group G is *left-orderable*

if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication:
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Some more definitions

- A subgroup N of a left-orderable group G is called *convex* (w.r. $\prec$), if for any $x, y, z \in G$ such that $x, z \in N$ and $x \prec y \prec z$, we have $y \in N$.

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- The set $LO(G)$ cannot be countably infinite (P. Linnell). If G is a countable left-orderable group, $LO(G)$ is either finite, or homeomorphic to the Cantor set, or homeomorphic to a subspace of the Cantor space with isolated points.

So what if a group is left-orderable?

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Unique product $\Rightarrow$ Kaplansky's Unit conjecture satisfied $\Rightarrow$ Kaplansky's Zero-divisor conjecture satisfied $\Rightarrow$ Kaplansky's Idempotent conjecture satisfied: there are no non-trivial idempotents in the group algebra

Are all the Garside groups left-orderable? book of P. Dehornoy, I. Dynnikov, D. Rolfsen, B. Wiest

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Short answer is: Not necessarily!! Detailed answer [F.C. 2016]:

- There exist Garside groups that are locally indicable:
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Conditions that ensure a Garside group has a left order

D.Arcis, L.Paris 2018

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Thank you!