Complex reflection groups, braid groups, Hecke algebras (II)

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Some motivation

W Weyl group $\rightsquigarrow$ groups of Lie type $G = G(q)$ with Weyl group W.

Theorem

Constituents of $R_{T_0}^G(1_{T_0})$ are in bijection with $\mathrm{Irr}(W)$;
$R_{T_0}^G(1_{T_0})$ decomposes as regular character of W.

Explanation: *Hecke algebra* $\mathcal{H} := \mathrm{End}_{\mathbb{C}G}(R_{T_0}^G(1_{T_0}))$ isomorphic to $\mathbb{C}[W]$.

With S the Coxeter generators of W,

$$\mathcal{H} = \big\langle \boldsymbol{t}_s \ (s \in S) \mid \text{braid relations; } (\boldsymbol{t}_s - q)(\boldsymbol{t}_s + 1) = 0 \big\rangle.$$

$(\boldsymbol{t}_s - q)(\boldsymbol{t}_s + 1) = 0$ 'deforms' order relation $(s-1)(s+1) = 0$ in W.

Further: Degrees of constituents of $R_{T_0}^G(1_{T_0})$ are expressed by 'Schur elements' of $\mathcal{H}$ with respect to a certain symmetrising form.

Some motivation, contd

For $T \leq G$ any maximal torus (not necessarily in Borel subgroup), have Lusztig induction

$$R_T^G : \mathbb{Z}\mathrm{Irr}(T) \longrightarrow \mathbb{Z}\mathrm{Irr}(G).$$

Observation (Broué–M.–Michel (1993))

If T parametrised by regular $w \in W \Rightarrow$
$R_T^G(1_T)$ decomposes like regular character of $C_W(w)$ (a crg!).

Is there an analogue of Hecke algebra for $C_W(w)$ which explains this, a deformation of $\mathbb{C}[C_W(w)]$?

Good presentations

$W \leq \mathrm{GL}(V)$ crg.

Proposition (Coxeter, ...)

All crg have good, Coxeter-like presentations, where

- *generators are reflections,*
- *for each generator, have relation giving its order,*
- *all other relations are homogeneous, each involving at most three generators (so 'local': in dimension ≤ 3)*

These can be visualised by diagrams, à la Coxeter.

Good presentations, II

If W is truly complex, then the good presentations satisfy at least one of

- there occur reflections of order > 2, or
- there are homogeneous relations involving > 2 reflections at a time (non-symmetric)

There may be several choices of good presentation for a fixed W.

Furthermore, in general, not all parabolic subgroups can be seen from a fixed presentation.

Hecke algebras, 1st attempt

Preliminary definition (as for Iwahori–Hecke algebras):

Let $W \leq \mathrm{GL}(V)$ be a crg, with good presentation

$$W = \langle S \mid R,\ s^{|s|} = 1 \text{ for } s \in S \rangle$$

(where $S \subseteq W$ are reflections, R homogeneous relations).

The *Hecke algebra* $\mathcal{H}(W, \boldsymbol{u})$ *attached to* W and indeterminates $\boldsymbol{u} = (u_{s,j} \mid s \in S,\, 1 \leq j \leq |s|)$ is the free associative $\mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]$-algebra on generators $\{\boldsymbol{t}_s \mid s \in S\}$ and relations

- $(\boldsymbol{t}_s - u_{s,1}) \cdots (\boldsymbol{t}_s - u_{s,|s|}) = 0$ for $s \in S$,
- the homogeneous relations from R.

Problem: W may have several good presentations. Which should we take?

Example

The 3-dimensional primitive crg $G_{24} \cong \mathrm{PSL}_2(7) \times C_2$ can be generated by three reflections of order 2. It has (at least) three good presentations on three reflections:

$$\begin{aligned} G_{24} =& \langle r, s, t \mid r^2 = s^2 = t^2 = 1, \\ & \quad rsrs = srsr,\ rtr = trt,\ stst = tsts,\ srstrst = rstrstr \rangle, \\ =& \langle r, s, t \mid r^2 = s^2 = t^2 = 1, \\ & \quad rsr = srs,\ rtr = trt,\ stst = tsts,\ tsrtsrtsr = stsrtsrts \rangle, \\ =& \langle r, s, t \mid r^2 = s^2 = t^2 = 1, \\ & \quad rsr = srs,\ rtr = trt,\ stst = tsts,\ strstrstrs = trstrstrst \rangle. \end{aligned}$$

Are the corresponding Hecke algebras (as defined above) isomorphic?

The braid group

Let $V = \mathbb{C}^n$, $W \leq \mathrm{GL}(V)$ a crg.
For $s \in W$ a reflection, let $H_s := \ker_V(s-1)$ its reflecting hyperplane. Set

$$V^{\mathrm{reg}} := V \setminus \bigcup_{s \in W \text{ refl.}} H_s.$$

Theorem of Steinberg:

$$V^{\mathrm{reg}} \longrightarrow V^{\mathrm{reg}}/W$$

is an unramified covering, with Galois group W.

The *braid group* of W is the fundamental group

$$B_W := \pi_1(V^{\mathrm{reg}}/W, x_0) \qquad (\text{for some } x_0 \in V^{\mathrm{reg}}).$$

Example

For $W = \mathfrak{S}_n$ in its natural reflection representation, B_W is the Artin braid group on n strings.

Presentations of the braid group

H reflecting hyperplane $\Longrightarrow$ $C_W(H)$ generated by a reflection s_H.
s_H is *distinguished* $:\Longleftrightarrow$ its unique non-trivial eigenvalue is $\exp(2\pi i/|s_H|)$.
Set $d_H := |C_W(H)|$.

Braid reflections: Suitable lifts $\boldsymbol{s}_H \in B_W$ of distinguished $s_H \in W$ (see talks of Ivan, Jean).

Theorem (Brieskorn, Deligne (1972), Broué–M.–Rouquier (1998), Bessis (2007))

Assume W irreducible. Then B_W has a presentation on at most $\dim V + 1$ *braid reflections $\boldsymbol{s}_H$ by homogeneous positive* braid relations *in the $\boldsymbol{s}_H$. Adding the relations $\boldsymbol{s}_H^{d_H}$ yields a good presentation of W.*

Hecke algebras, II

Let $\boldsymbol{u} = (u_{s,j} \mid s \in W \text{ distinguished reflection}, 1 \leq j \leq |s|)$ be a W-invariant set of indeterminates, $A := \mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]$.

The (generic) *Hecke algebra attached to W* is the quotient

$$\mathcal{H}(W, \boldsymbol{u}) = A[B_W] / \left((\boldsymbol{s} - u_{s,1}) \ldots (\boldsymbol{s} - u_{s,|s|}) \mid \boldsymbol{s} \text{ braid-reflection} \right)$$

of the group algebra $A[B_W]$ of the braid group.

This is independent of a choice of presentation!

Examples

- For W a Coxeter group we obtain the usual generic Iwahori–Hecke algebra (with indeterminates in place of q).
- For $W = G_5$,

$$\mathcal{H}(W, \boldsymbol{u}) = \left\langle \boldsymbol{s}, \boldsymbol{t} \mid \boldsymbol{stst} = \boldsymbol{tsts}, \prod_{j=1}^{3} (\boldsymbol{s} - u_{s,j}) = \prod_{j=1}^{3} (\boldsymbol{t} - u_{t,j}) = 0 \right\rangle.$$

Hecke algebras, III

Example

For the 3-dimensional reflection group G_{24}, the three presentations of B_W on three braid reflections:

$$\begin{aligned} B_W = \langle r, s, t \mid & rsrs = srsr,\ rtr = trt, \\ & stst = tsts,\ srstrst = rstrstr\rangle, \\ = \langle r, s, t \mid & rsr = srs,\ rtr = trt, \\ & stst = tsts,\ tsrtsrtsr = stsrtsrts\rangle, \\ = \langle r, s, t \mid & rsr = srs,\ rtr = trt, \\ & stst = tsts,\ strstrstrs = trstrstrst\rangle, \end{aligned}$$

just give three presentations of the same Hecke algebra.

Hecke algebras as deformations

From the theorem on presentations of braid groups we get:

Corollary

Under the specialisation

$$u_{s,j} \mapsto \exp(2\pi ij/|s|), \qquad s \in W \text{ distinguished refl.}, \ 1 \leq j \leq |s|,$$

$\mathcal{H}(W, \boldsymbol{u})$ *becomes isomorphic to the group algebra* $\mathbb{C}[W]$ *of* W.

Long open 'Freeness Conjecture' (well-known for Coxeter groups (Tits)):

Theorem (Tits, Ariki–Koike (1993), Broué–M. (1993),..., Chavli (2018), Marin (2019), Tsuchioka(2020))

$\mathcal{H}(W, \boldsymbol{u})$ *is a free* A*-module of rank* $|W|$.

For proof, find an A-basis of $\mathcal{H}(W, \boldsymbol{u})$.

Lifting reduced expressions

Choose presentation

$$B_W = \langle \mathbf{S} \mid R \rangle$$

of the braid group, so that

$$W = \langle S \mid R,\ \text{order relations} \rangle$$

is a presentation of W, with $S \subset W$ the images of the $\boldsymbol{s} \in \mathbf{S}$.
Write $\boldsymbol{t}_s$ for the image of $\boldsymbol{s}$ in $\mathcal{H}(W, \boldsymbol{u})$.

For $w \in W$, choose reduced expression

$$w = s_1 \cdots s_r \qquad \text{with } s_i \in S$$

and let

$$\boldsymbol{w} := \boldsymbol{s}_1 \cdots \boldsymbol{s}_r \ \in B_W, \qquad \boldsymbol{t_w} := \boldsymbol{t}_{s_1} \cdots \boldsymbol{t}_{s_r} \ \in \mathcal{H}(W, \boldsymbol{u}).$$

Hope: $\{\boldsymbol{t_w} \mid w \in W\}$ is an A-basis of $\mathcal{H}(W, \boldsymbol{u})$.

Bases of $\mathcal{H}(W, \boldsymbol{u})$

For Coxeter groups,

• $\boldsymbol{w} \in B_W$ is independent of the choice of reduced expression of $w \in W$ (Lemma of Matsumoto), and

• there is a natural presentation for B_W.

Problem: for crg, in general $\boldsymbol{w}$ *depends* on

• the choice of presentation, *and*

• on the choice of reduced expression of w.

So, much more complicated arguments and computations are needed to find an A-basis of $\mathcal{H}(W, \boldsymbol{u})$....

Tits deformation theorem

Recall: have semisimple specialisation $\mathbb{C}[W]$ of $\mathcal{H}(W, \boldsymbol{u})$, by sending

$$u_{s,j} \mapsto \exp(2\pi ij/|s|).$$

Then Tits' deformation theorem shows:

Corollary (of freeness theorem)

Let W be a crg. Then over a suitable extension field K of Frac(A),

$$\mathcal{H}(W, \boldsymbol{u}) \otimes_A K \cong K[W].$$

In particular, there is a 1-1 correspondence $\mathrm{Irr}(\mathcal{H}(W, \boldsymbol{u})) \longleftrightarrow \mathrm{Irr}(W)$.

Conclusion: $\mathcal{H}(W, \boldsymbol{u})$ could be the right candidate to explain R_T^G.

Splitting fields

Which extension field K suffices?

Recall k_W = character field of W. Let $\mu(k_W)$ = roots of unity in k_W.

Theorem (M. (1998))

$\mathcal{H}(W, \boldsymbol{u})$ is split over $K_W := k_W(\boldsymbol{v})$, where $\boldsymbol{v} = (v_{s,j})$ with

$$v_{s,j}^{|\mu(k_W)|} = \exp(-2\pi ij/|s|)\, u_{s,j}.$$

Thus, over K_W, the specialisation $v_{s,j} \mapsto 1$ induces a natural bijection

$$\mathrm{Irr}(\mathcal{H}(W, \boldsymbol{u})) \longrightarrow \mathrm{Irr}(W), \qquad \chi_{\boldsymbol{v}} \mapsto \chi.$$

Example (Benson–Curtis (1972), Lusztig)

For W a Weyl group, $|\mu(k_W)| = |\mu(\mathbb{Q})| = 2$

$\Longrightarrow$ splitting field for Iwahori–Hecke algebras is obtained by extracting square roots of the indeterminates.

Symmetrizing forms

We expect Hecke algebras to carry a natural trace form:
There should exist an A-linear form

$$t : \mathcal{H}(W, \boldsymbol{u}) \longrightarrow A$$

with the following properties:

- the bilinear form $\mathcal{H} \times \mathcal{H} \to A$, $(h_1, h_2) \mapsto t(h_1 h_2)$, is symmetric and non-degenerate over A,
- t specialises to the canonical trace form on the group algebra of W,
- t restricted to a parabolic subalgebra has the same properties on that subalgebra.

Rouquier: under an additional condition, if it exists, such a t is unique.

Symmetrizing forms, contd

For Coxeter groups, such a form on $\mathcal{H}(W, \boldsymbol{u})$ is obtained by setting

$$t(\boldsymbol{t_w}) := \begin{cases} 1 & w = 1, \\ 0 & \text{else}, \end{cases}$$

for $w \in W$ (with lifted elements $\boldsymbol{t_w} \in \mathcal{H}(W, \boldsymbol{u})$ as above).

Problem: for crg, the $\boldsymbol{t_w}$ are not well-defined.

Theorem (Bremke–M. (1997), M.–Mathas (1998), Boura–Chavli–Chlouveraki–Karvounis (2020))

For almost all irreducible crg, the algebra $\mathcal{H}(W, \boldsymbol{u})$ is symmetric over A.

E.g., for $G(m, 1, n)$, t vanishes on $\boldsymbol{t_w}$ for *all* reduced expressions of all $1 \neq w \in W$.

For the proof, take above definition for *some* basis and check properties.

Schur elements

Let t denote the canonical symmetrizing form on $\mathcal{H}(W, \boldsymbol{u})$. Write

$$t = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{S_\chi} \chi_{\boldsymbol{v}},$$

with *Schur elements* $S_\chi \in K_W$.

Theorem (Geck–Iancu–M. (2000), M. (1997,2000))

The Schur elements are explicitly known for all types (assuming the existence of the symmetrizing form t).

For infinite series, determine weights of a Markov trace on $\mathcal{H}(W, \boldsymbol{u})$.

Computing Schur elements

For exceptional types, solve linear system of equations

$$t(\boldsymbol{t_w}) = \sum_{\chi} \chi_{\boldsymbol{v}}(\boldsymbol{t_w}) \frac{1}{S_\chi} = \begin{cases} 1 & w = 1 \\ 0 & \text{else} \end{cases} \qquad (w \in W).$$

How do we know $\chi_{\boldsymbol{v}}(\boldsymbol{t_w})$ on sufficiently many elements?

Construct representations explicitly.

For small dimensions ($m \leq 6$): take matrices with indeterminate entries, plug into relations, solve non-linear system.

Induction: may assume matrices known for a maximal parabolic subalgebra.

Example

For $W = G_5$, with parameters (u, v, w, x, y, z), one Schur element is

$$-\frac{(uy+vx)(vy+ux)(y-z)(uvxy+w^2z^2)(x-z)(v-w)(u-w)}{uvw^4xyz^4}.$$

In fact, the Schur elements always have total degree 0 and are of the form

$$S_\chi = m \cdot \frac{P_1}{P_2},$$

where

- m is an integer in k_W,
- P_1 is a product of cyclotomic polynomials over k_W, evaluated at monomials in the $v_{s,j}^{\pm 1}$,
- P_2 is a monomial in the $v_{s,j}^{\pm 1}$.

Decomposition of R_T^G

Recall observation: If torus $T \leq G$ parametrised by $w \in W$ regular $\Rightarrow$ $R_T^G(1_T)$ decomposes like regular character of $C_W(w)$.

Observation (M.)

The degrees of constituents of $R_T^G(1_T)$ are then given in terms of the Schur elements of a certain specialisation of $\mathcal{H}(C_W(w), \boldsymbol{u})$.

Conclusion: $\mathcal{H}(W, \boldsymbol{u})$ might definitely be the right algebra to explain R_T^G.

This conjectural explanation has so far been proved in only very few cases (Digne, Dudas, Michel, Rouquier,...)

The spetsial specialisation

We are interested in 1-parameter specialisations of $\mathcal{H}(W, \boldsymbol{u})$ through which the specialisation to $\mathbb{C}[W]$ factors.

For Iwahori–Hecke algebras, the specialisation where

$$(\boldsymbol{s} - q)(\boldsymbol{s} + 1) = 0$$

(for all distinguished $s \in W$) is particularly important.

For Hecke algebras of crg, may have reflections of order $|s| > 2$.
So consider the *spetsial* specialisation $\mathcal{H}(W, q)$ where

$$(\boldsymbol{s} - q)(\boldsymbol{s}^{|s|-1} + \boldsymbol{s}^{|s|-2} + \ldots + 1) = 0.$$

By the above, $\mathcal{H}(W, q)$ is split over $k_W(y)$, where $y^{|\mu(k_W)|} = q$.

Fake degrees

The symmetric algebra $S(V)$, the invariants $S(V)^W$, are naturally graded.

$S(V)^W_+ :=$ the invariants of degree at least 1.

$S(V)_W := S(V) \,/\, \big(S(V)^W_+\big)$ the *coinvariant algebra*.

Theorem (Chevalley (1955))

The graded W-module $S(V)_W$ affords the regular representation of W.

The *fake degree* of $\chi \in \mathrm{Irr}(W)$ is the graded multiplicity

$$R_\chi := \sum_j \langle \chi, S(V)^j_W \rangle \, z^j \;\in \mathbb{Z}[z].$$

Rationality of the reflection representation

The spetsial algebra 'knows about' W being well-generated!

For $\chi \in \mathrm{Irr}(W)$ let $D_\chi := S_1/S_\chi$, the *generic degree* of χ.

$\chi \in \mathrm{Irr}(W)$ is *special* if $R_\chi(q)$ and D_χ have same order of zero at $y = 0$.

Proposition (M.)

For an irreducible crg W the following are equivalent:

(i) *W is well-generated.*

(ii) *The reflection character of W is special.*

(iii) *The reflection representation of $\mathcal{H}(W, q)$ can be realised over $k_W(q)$.*

For example, for Coxeter groups the reflection representation of $\mathcal{H}(W, q)$ is always rational.