Complex Braid Groups

Ivan Marin, Université d'Amiens (UPJV)

Part 1 : Presentations
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Notations for complex reflection groups

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$\mathcal{R}^*$ is in 1-1 correspondence with $\mathcal{A}$,

$$s \mapsto \mathrm{Ker}(s-1), \quad H \mapsto s_H$$

Classification of irreducible CRG's

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- its non-diagonal reflections belong to $G(r, r, n) < W$ and have the form

$$\mathrm{Id}_u \oplus \begin{pmatrix} 0 & \zeta_e^{-k} \\ \zeta_e^k & 0 \end{pmatrix} \oplus \mathrm{Id}_{n-2-u}$$

In addition to these, there are 34 exceptional groups $G_4, \dots, G_{37}$, half of them in rank 2.

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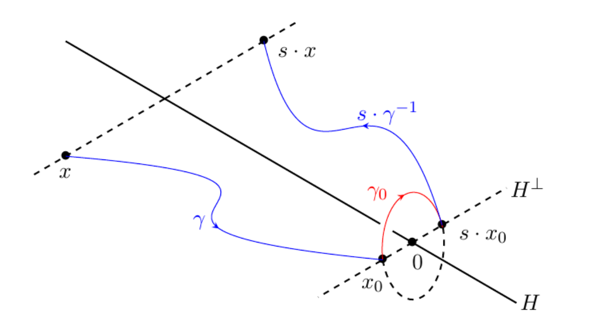

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Definition

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The following is easy to prove

Proposition

For every braided reflection σ, we have $\ell(\sigma) = 1$.

Braided reflections and presentations of B

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Proposition

The kernel of $B \twoheadrightarrow W$ is (normally) generated by the $\sigma^{m(\sigma)}$, for σ running among the collection of all braided reflections.

As a consequence, any presentation of B with generators braided reflections will provide a presentation of W, as soon as the set of generators contains representatives for every conjugacy class of reflections.

Lemma

Two braided reflections are conjugates inside B if and only if their images are conjugates inside W.

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Its image inside $B = \pi_1(X/W)$ is denoted z_B.

Lemma

$z_B \in Z(B)$ and $z_B^{|Z(W)|} = z_P$.

1 Definitions

2 Braids

3 Braid groups of $G(de, e, n)$

4 A few words about exceptional groups

Braid groups of surfaces

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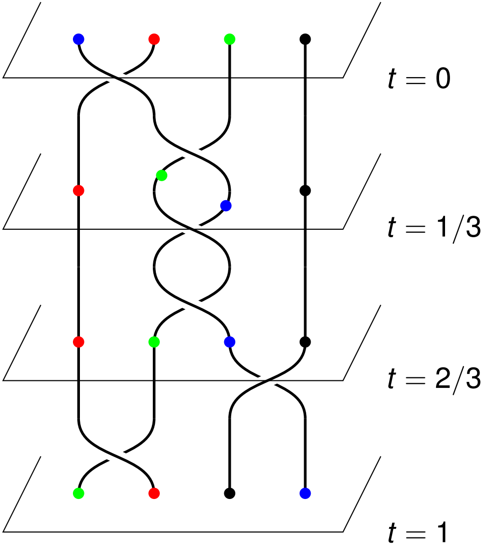
$$\{\underline{z} = (z_1, \dots, z_n) \in \Sigma^n \mid i \neq j \Rightarrow z_i \neq z_j\}$$

by the action of $\mathfrak{S}_n$ by permutation of the coordinates.

The usual braid group : $\mathcal{B}_n = \mathcal{B}(\Sigma), \Sigma = \mathbb{C}$



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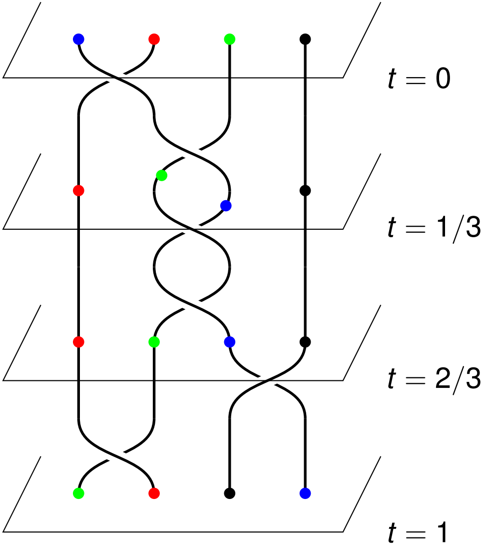

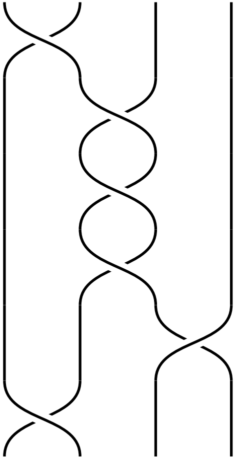

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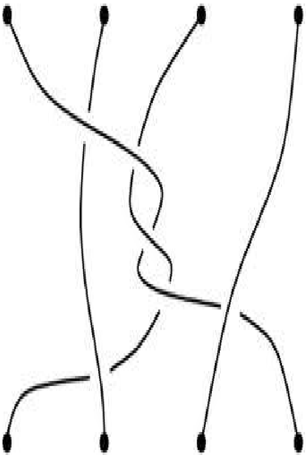

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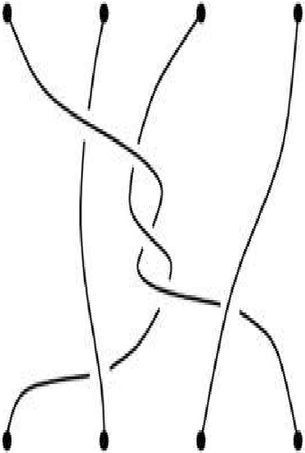

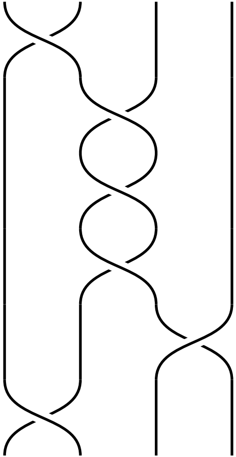

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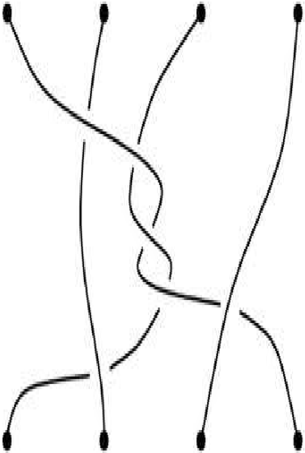

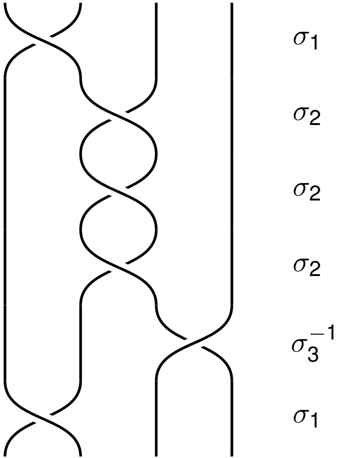

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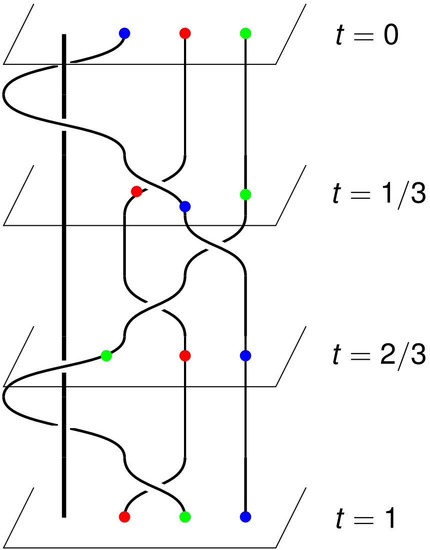

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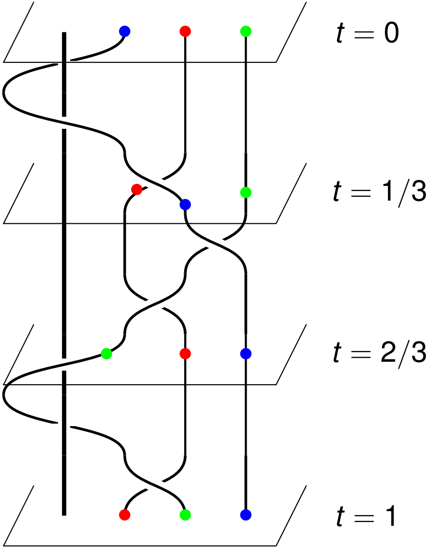

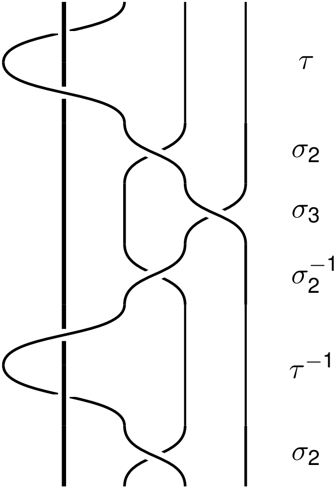

$\mathcal{B}_n^*$ and $\mathcal{B}_n$

From the projection map $\pi : \mathcal{B}_{n+1} \twoheadrightarrow \mathfrak{S}_{n+1}$, and taking for $\Sigma = \mathbb{C} \setminus \{1\}$, we get that

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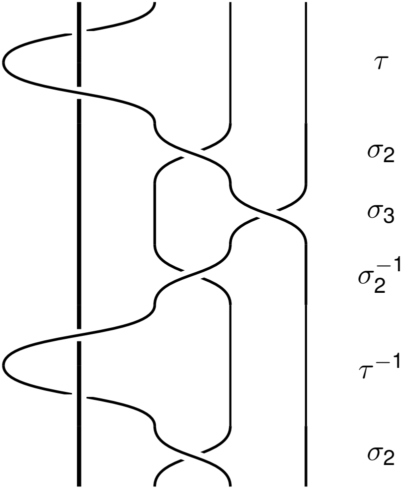

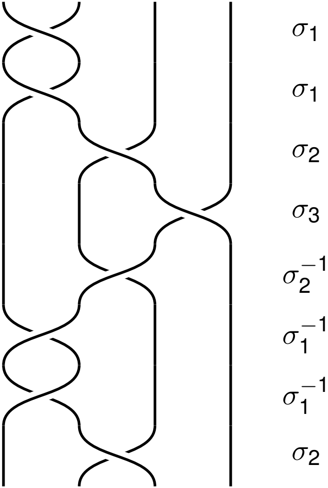

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It can be illustrated as follows.

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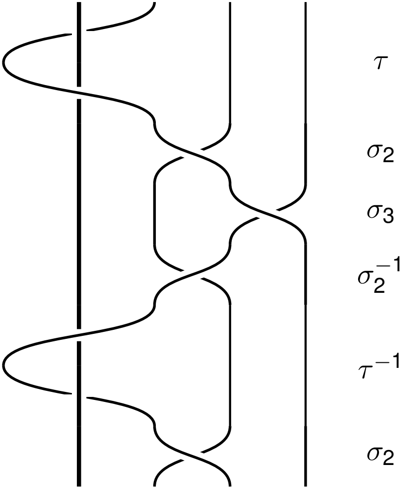

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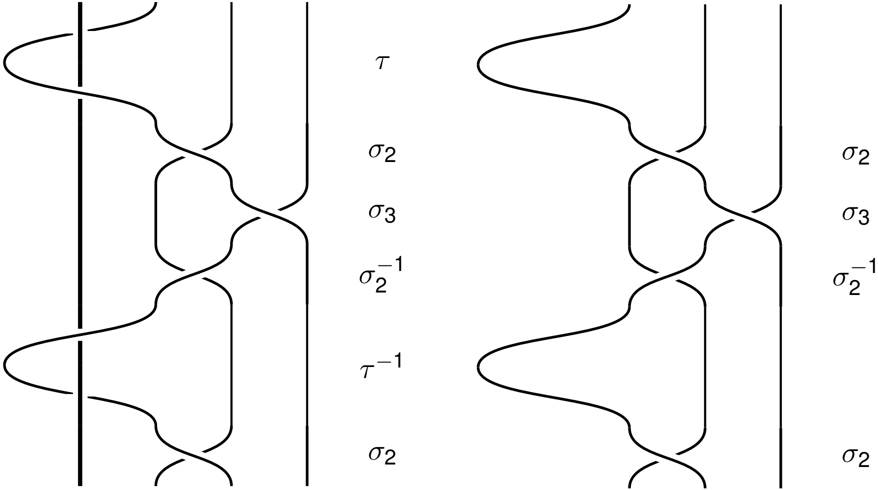

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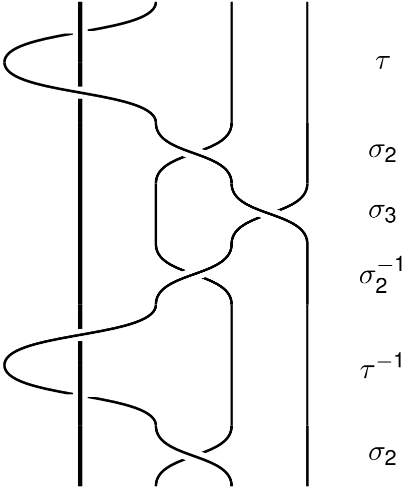

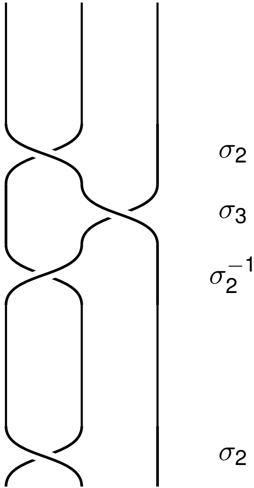

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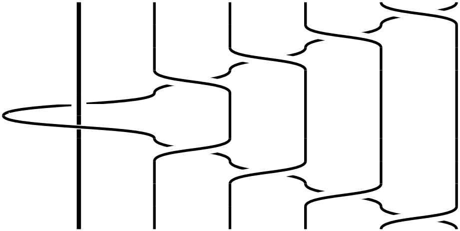

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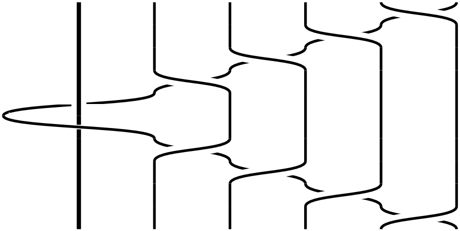

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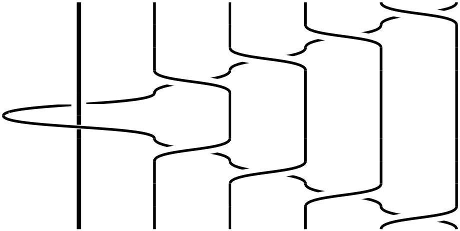

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$$\mathcal{B}_n^* \simeq \mathcal{B}_n \ltimes \mathcal{F}_n$$

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$$\tilde{\mathcal{B}}^*_{n-1} = \left\langle \tau, \sigma_2, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2 \\ \sigma_i\tau = \tau\sigma_i, i > 2 \\ \sigma_2\tau\sigma_2\tau = \tau\sigma_2\tau\sigma_2 \end{array} \right\rangle \tag{1}$$

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It is then sufficient to check that $\mathcal{B}_2 = \langle \sigma_1 \rangle \simeq \mathbb{Z} \simeq \tilde{\mathcal{B}}_2$ to prove by induction that the presentations are correct.

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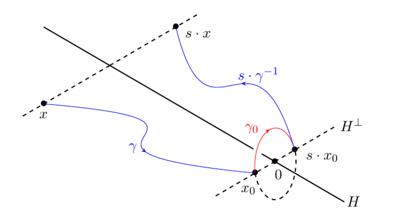

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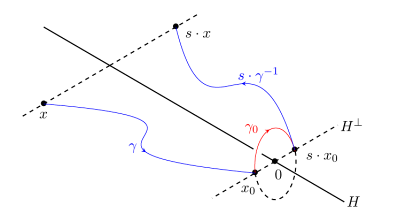

$$B = \langle \sigma \mid \sigma \in B \text{ braided reflection } \rangle$$

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So we already have presentations for them.

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This includes

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- the conjugacy problem
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as we shall see in Part 2.

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Moreover, the Reidemeister-Schreier method provides a presentation for this group.

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We start from the known presentation for $\tilde{\mathcal{B}}_n^*$ (with some shift of indices).

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$$\left\langle \begin{array}{l} \sigma_{1,0}, \ldots, \sigma_{1,e-1} \\ \boldsymbol{\tau}, \sigma_2, \ldots, \sigma_{n-1} \end{array} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2 \\ \sigma_i\boldsymbol{\tau} = \boldsymbol{\tau}\sigma_i, i \geq 2 \\ \sigma_{1,k}\sigma_2\sigma_{1,k} = \sigma_2\sigma_{1,k}\sigma_2 \\ \sigma_{1,k}\sigma_j = \sigma_j\sigma_{1,k}, j \geq 3 \\ \sigma_{1,k}\sigma_{1,k+1} = \sigma_{1,k+1}\sigma_{1,k+2}, 0 \leq k \leq e-3 \\ \sigma_{1,e-2}\sigma_{1,e-1} = \sigma_{1,e-1}.\boldsymbol{\tau}\sigma_{1,0}\boldsymbol{\tau}^{-1} \\ \boldsymbol{\tau}^{-1}\sigma_{1,e-1}\boldsymbol{\tau}\sigma_{1,0} = \sigma_{1,0}\sigma_{1,1} \end{array} \right\rangle \tag{3}$$

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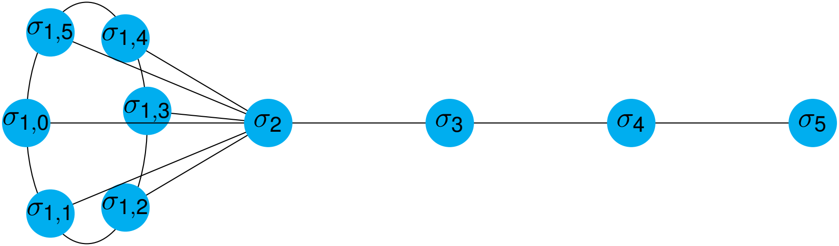
This yields the following presentation for $\mathcal{B}_n(e)$.

$$\left\langle \begin{array}{l} \sigma_{1,k}, k \in \mathbb{Z}/e\mathbb{Z} \\ \sigma_2, \dots, \sigma_{n-1} \end{array} \middle| \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2 \\ \sigma_{1,k} \sigma_2 \sigma_{1,k} = \sigma_2 \sigma_{1,k} \sigma_2 \\ \sigma_{1,k} \sigma_j = \sigma_j \sigma_{1,k}, j \geq 3 \\ \sigma_{1,k} \sigma_{1,k+1} = \sigma_{1,k+1} \sigma_{1,k+2}, k \in \mathbb{Z}/e\mathbb{Z} \end{array} \right\rangle \qquad (4)$$

Standard diagrams for complex braid groups

For the groups $W = G(e, e, n)$, $e \geq 1$, $B = \mathcal{B}_n(e)$:



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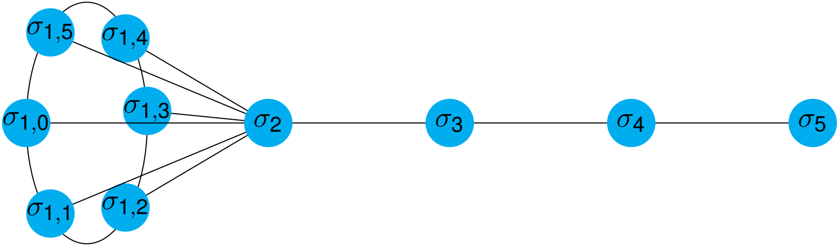

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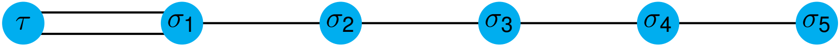

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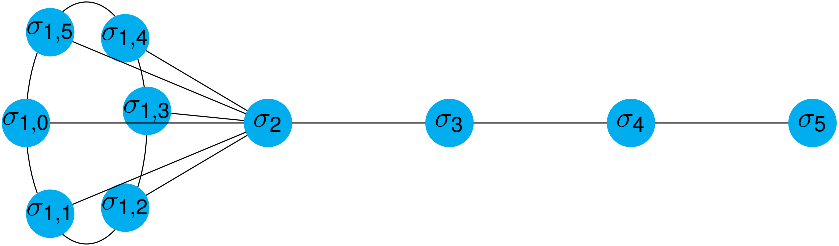

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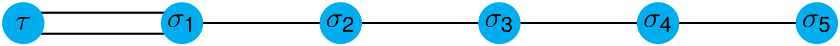

For the groups $W = G(de, e, n)$, $d > 1$, $B = \mathcal{B}_n^*(e)$ is a nice subgroup of $\mathcal{B}_n^*$.

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The discriminantal viewpoint

General theorems tell us that

$$\mathbb{C}[x_1, \ldots, x_n]^W \simeq \mathbb{C}[f_1, \ldots, f_n]$$

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$$(X - z_1)(X - z_2)\ldots(X - z_n) = X^n - f_1 X^{n-1} + \cdots + (-1)^n f_n$$

expressed as a polynomial in the f_i's

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However, most of the time these groups are more easily dealt using the fact that

$$P = \pi_1(\mathbb{C}^2 \setminus \bigcup \mathcal{A}) \simeq \pi_1\left(\mathbb{C}^\times\right) \times \pi_1\left(\mathbb{C} \setminus \{|\mathcal{A}| - 1 \text{ points}\}\right) \simeq \mathbb{Z} \times F_{|\mathcal{A}|-1}$$

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$$\begin{aligned} G_{25} &= \mathcal{B}_4/\sigma_i^3 \\ G_{26} &= \mathcal{B}_3^*/\langle \tau^2, \sigma_i^3 \rangle \\ G_{32} &= \mathcal{B}_5/\sigma_i^3 \end{aligned}$$

Case 2.b : other groups of rank ≥ 3

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