Complex Braid Groups

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Part 2 : Standard monoids
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Notations for complex reflection groups

Let $W < \mathrm{GL}(V)$ be a complex reflection group, $n = \dim V$

$$W = \langle \mathcal{R} \rangle \quad \mathcal{R} = \{s \in W; \dim \mathrm{Ker}(s-1) = n-1\}$$

The collection of its *reflecting hyperplanes* is the *hyperplane arrangement*

$$\mathcal{A} = \{\mathrm{Ker}(s-1), s \in \mathcal{R}\}$$

For $H \in \mathcal{A}$, $W_H = \{w \in W; w_{|H} = \mathrm{Id}_H\}$ is cyclic, isomorphic to its image under $\det : W_H \to \mathbb{C}^\times$.

The generator of W_H mapped to $\exp(2\pi\mathrm{i}/|W_H|)$ is a reflection s_H called the *distinguished reflection* associated to H. The collection of all distinguished reflections is denoted $\mathcal{R}^*$.

$\mathcal{R}^*$ is in 1-1 correspondence with $\mathcal{A}$,

$$s \mapsto \mathrm{Ker}(s-1), \quad H \mapsto s_H$$

Classification of irreducible CRG's

The main series is made of the groups $W = G(de, e, n)$ of

- $n \times n$ *monomial* matrices
- with nonzero entries inside μ_r, $r = de$
- whose product belongs to μ_d.

Of course $G(r, r, n) < G(de, e, n) < G(r, 1, n)$.

- W contains diagonal reflections, of the form $\mathrm{diag}(1, \dots, 1, \zeta, 1, \dots)$ if and only if $d > 1$.
- its non-diagonal reflections belong to $G(r, r, n) < W$ and have the form

$$\mathrm{Id}_u \oplus \begin{pmatrix} 0 & \zeta_e^{-k} \\ \zeta_e^k & 0 \end{pmatrix} \oplus \mathrm{Id}_{n-2-u}$$

In addition to these, there are 34 exceptional groups $G_4, \dots, G_{37}$, half of them in rank 2.

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Their braid groups are

- the braid group $\mathcal{B}_n$ for $G(1, 1, n)$
- the punctured braid group $\mathcal{B}_n^*$ for $G(r, 1, n) = G(d, 1, n)$ when $d > 1$
- a finite index normal subgroup $\mathcal{B}_n^*(e)$ of $\mathcal{B}_n^*$ when $d > 1$ and $e > 1$
- a quotient $\mathcal{B}_n(e)$ of $\mathcal{B}_n^*(e)$ for $G(e, e, n) = G(r, r, n)$.

Preliminaries : Garside monoids

- A monoid is called *cancellative* if, for all $a, b, c \in M$, $ac = bc$ implies $a = b$ and $ca = cb$ implies $a = b$
- An element $a \in M$ left-divides $c \in M$ if $\exists b \in M \ \ ab = c$. Then, c is a right-multiple of a, and one writes $a \prec c$. Similarly, a right-divides $c \in M$ if $\exists b \in M \ \ ba = c$ and c is then a left-multiple of a, and one writes $c \succ a$.
- Two elements a, b admit a right lowest common multiple (lcm) if they admit a right common multiple $c = \mathrm{lcm}_R(a, b)$ such that, $\forall m \in M \ a \prec m, b \prec m \Rightarrow c \prec m$. They admit a left lcm if they admit a left common multiple $c = \mathrm{lcm}_L(a, b)$ such that $\forall m \in M \ m \succ a, m \succ b \Rightarrow m \succ c$.
- Two elements a, b admit a left greatest common divisor (gcd) if they admit a left common divisor $c = \mathrm{gcd}_L(a, b)$ such that, $\forall m \in M \ m \prec a, m \prec b \Rightarrow m \prec c$. They admit a right gcd if they admit a right common divisor $c = \mathrm{gcd}_R(a, b)$ such that $\forall m \in M \ a \succ m, b \succ m \Rightarrow c \succ m$.

If $M^{\times} = 1$ and M is cancellable, these lcm's and gcd's are uniquely defined.
An element $a \in M$ is called *reducible* if there exists $b, c \in M$ with $b, c \notin M^{\times}$ such that $a = bc$. It is called *irreducible* if it is not invertible and not reducible.

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Definition

An *homogeneous monoid* is a monoid M together with a *length function*, that is a monoid morphism $\ell : M \to \mathbb{N}$, such that M is generated by the elements of length > 0.

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Under these conditions, every element of M is a product of irreducible elements. They are called the *atoms* of the monoid M.

Definition

The *group of fractions* of M is by definition a group $\mathrm{Frac}(M)$ together with a morphism of monoids $M \to \mathrm{Frac}(M)$ such that every $M \to G$ for G a group factors

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & \mathrm{Frac}(M) \arrow[d, dotted] \\ & G \end{tikzcd}$$

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For S a set of generators and R a collection of relations, if M is presented as $\langle S \mid R\rangle^+$, then $\langle S \mid R\rangle$ is a presentation of $\mathrm{Frac}(M)$.

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Definition

An element of a monoid M is said to be *balanced* if the sets of its left and right divisors are the same.

Definition

An homogeneous monoid M is said to have the Garside property, or to be a Garside monoid, if it is cancellable, and if it has the following properties

- any two elements of M admit *gcd*'s and *lcm*'s on the right and on the left
- M admits a balanced element Δ whose set of divisors is finite and generates M.

The chosen element Δ is called a *Garside element* for M.

Preferred Garside element

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For $m \in M$, we set

$$\mathrm{Div}_L(m) = \{a \in M \mid a \prec m\} \ \ \mathrm{Div}_R(m) = \{a \in M \mid m \succ a\}$$

and, if m is balanced $\mathrm{Div}(m) = \mathrm{Div}_L(m) = \mathrm{Div}_R(m)$.

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How to deal with $\mathcal{B}_n^*(e)$?

Finite index normal subgroups of Garside groups

Reminder

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Since F is finite, from a finite set of generators of $C_G(x)$ one gets a finite set of generators of $\mathrm{Ker}(\Phi_{C_G(x)})$ by Schreier's Lemma.

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So : is there $b \in H$ with $bc^{-1} \in C_G(x) = \langle g_1, \dots, g_r \rangle$?

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Let $x, y \in H$.If x, y are not conjugates in G we are done. Otherwise, let $c \in G$ with $y = x^c$.Then $y = x^b, b \in H \Leftrightarrow bc^{-1} \in C_G(x)$. So : is there $b \in H$ with $bc^{-1} \in C_G(x) = \langle g_1, \dots, g_r \rangle$? Actually equivalent to checking whether $\Phi(c) \in \langle \Phi(g_1), \dots, \Phi(g_r) \rangle < F$.

Preliminaries : Garside interval monoids

Let W be a group generated by a set S, and $\ell_S : S \to \mathbb{N} = \mathbb{Z}_{\geq 0}$ the length with respect to S.

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- $a \prec b$ means $\ell_S(b) = \ell_S(a) + \ell_S(a^{-1} \cdot b)$,
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For $c \in W$,

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and c is balanced if $\mathrm{Div}_L(c) = \mathrm{Div}_R(c)$.

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By definition, the interval monoid attached to (W, S, c) with c balanced is defined by generators and relations

$$M(c) = \langle \mathrm{Div}(c) \mid z = xy \ \text{ if } z = x \cdot y \text{ and } \ell_S(z) = \ell_S(x) + \ell_S(y) \rangle^+$$

Preliminaries : Garside interval monoids

Theorem

If $(\mathrm{Div}(c), \prec)$ *and* $(\mathrm{Div}(c), \succ)$ *are lattices, then* $M(c)$ *is Garside with Garside element* c*.*

Moreover, the poset structures on $\mathrm{Div}(c)$ are the same inside W and inside $M(c)$.

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and both of them are Garside, with a preferred Garside element.

1 Definitions

2 Standard monoid for $G(e, e, n)$

3 Standard monoid for $G(d, 1, n)$

4 A word on exceptional groups

Standard monoid for $G(e, e, n)$

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The presentation for B^+ :

$$\left\langle \begin{array}{l} \sigma_{1,k}, k \in \mathbb{Z}/e\mathbb{Z} \\ \sigma_2, \ldots, \sigma_{n-1} \end{array} \middle| \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_{1,k} \sigma_2 \sigma_{1,k} = \sigma_2 \sigma_{1,k} \sigma_2 \\ \sigma_{1,k} \sigma_j = \sigma_j \sigma_{1,k}, j \geq 3 \\ \sigma_{1,k} \sigma_{1,k+1} = \sigma_{1,k+1} \sigma_{1,k+2}, k \in \mathbb{Z}/e\mathbb{Z} \end{array} \right\rangle^+$$

Standard monoid for $G(e, e, n)$

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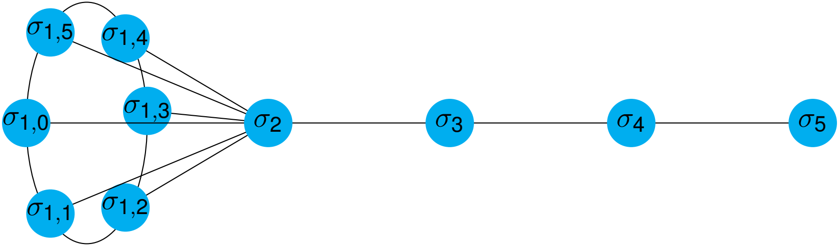

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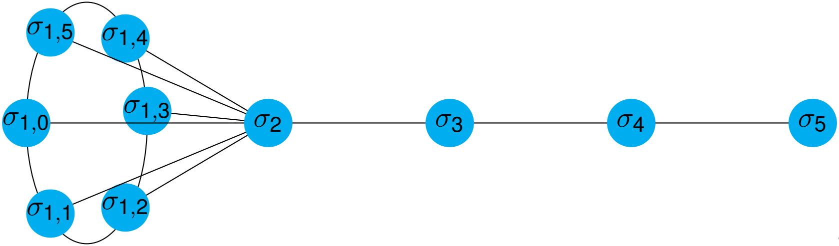

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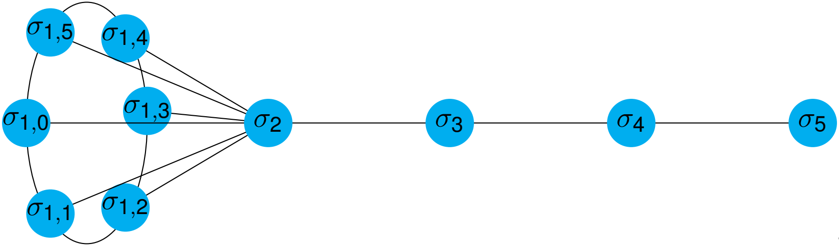

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For $e \geq 1$ and $W = G(e, e, n)$, B^+ is an interval Garside monoid for the group B, with respect to the set $S = \{s_i, i \geq 2\} \cup \{s_{1,k}, k \in \mathbb{Z}/e\mathbb{Z}\}$, identified with its set of atoms.

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while for $e = 1$ we have for preferred Garside element

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For this we need to introduce the following remarkable elements, for $1 \leq i \leq j$

- $[j \cdots i] = (j, j-1, \dots, i) = s_{j-1} \dots s_{i+1} s_i$
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and we set

$$\begin{array}{rcl} [\mathbf{j} \cdots \mathbf{i}] & = & \sigma_{j-1} \dots \sigma_{i+1} \sigma_i \in \mathcal{M} \\ [\mathbf{i} \cdots \mathbf{j}] & = & \sigma_i \sigma_{i+1} \dots \sigma_{j-1} \in \mathcal{M} \end{array}$$

where $\mathcal{M}$ is the free monoid on the atoms of B^+.

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The output $R(w)$ of the algorithm is then a word of the form $R_2(w) R_3(w) \ldots R_n(w)$ where $R_i(w)$ is computed inside $G(e, e, i)$.

Neaime's algorithm for $G(e, e, n)$: example.

$$w = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \zeta_7^3 & 0 & 0 \\ 0 & 0 & \zeta_7^4 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in G(7,7,4) \rightsquigarrow R_4(w) = [\mathbf{4} \cdots \mathbf{1}] = \sigma_3\sigma_2\sigma_1$$

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Then the matrix obtained inside $G(7,7,2)$ is equal to s_1 hence $R_2(w) = \sigma_1$ and $R(w) = R_2(w)R_3(w)R_4(w) = \sigma_1\sigma_2\sigma_{1,4}\sigma_1\sigma_3\sigma_2\sigma_1$.

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We have $\mathrm{Div}(J) = \mathfrak{S}_n < G(e, e, n)$.

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Lemma ($r = 1$)

- *If* $\hat{w}$ *is diagonal, then* $\ell(s_{1,k} w) = \ell(w) - 1$ *if and only if the bottom content of* $\hat{w}$ *is not* 1 *;*
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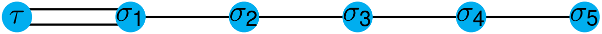
We consider the monoid B^+ with presentation

$$\left\langle \tau, \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2 \\ \sigma_i\tau = \tau\sigma_i, i > 1 \\ \sigma_1\tau\sigma_1\tau = \tau\sigma_1\tau\sigma_1 \end{array} \right\rangle \tag{1}$$



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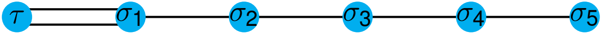

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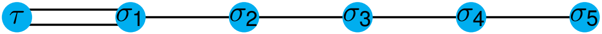

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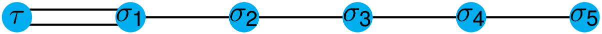

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We denote

$$\begin{aligned} t &= \mathrm{diag}(\zeta_d^{-1}, 1, \ldots, 1)\\ s_i &= (i, i+1) \in \mathfrak{S}_n < W\end{aligned}$$

the images of τ and σ_i in W.

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- Start by $w \in W$. If $n = 1$, $w = \zeta_d^{-k}$ for some k with $0 \leq k < d$; then output $\tau^k \in \mathcal{M}$.
- Otherwise, consider the last row. Its content is ζ_d^{-k} with $0 \leq k < d$, and belongs to some column of index $j \leq n$.
- If $k = 0$, then consider $w' = w[j \cdots n]$; otherwise, consider $w' = w[j \cdots 1]t^{-k}[1 \cdots n]$.
- Then $w' \in G(d, 1, n-1)$ and we apply the same algorithm recursively to it, getting some $R(w') \in \mathcal{M}$.
- Finally, return $R(w) = R(w')[\mathbf{n} \cdots \mathbf{j}]$ in the first case ($k = 0$), and $R(w) = R(w')[\mathbf{n} \cdots \mathbf{1}]\tau^k[\mathbf{1} \cdots \mathbf{j}]$ in the second one.

Neaime's algorithm for $\mathcal{B}_n^*$

Again : $\mathcal{M}$ is the free monoid on the generators, $[i \cdots j]$,$[\mathbf{i} \cdots \mathbf{j}]$ as before.

- Start by $w \in W$. If $n = 1$, $w = \zeta_d^{-k}$ for some k with $0 \leq k < d$; then output $\tau^k \in \mathcal{M}$.
- Otherwise, consider the last row. Its content is ζ_d^{-k} with $0 \leq k < d$, and belongs to some column of index $j \leq n$.
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The output $R(w)$ of the algorithm is then a word of the form

$$R_1(w)R_2(w)R_3(w)\dots R_n(w)$$

where $R_i(w)$ is computed inside $G(d, 1, i)$.

Neaime's algorithm for $\mathcal{B}_n^*$

Example

For

$$w = \begin{pmatrix} 0 & 0 & \zeta_7^{-2} \\ \zeta_7^{-1} & 0 & 0 \\ 0 & \zeta_7^2 & 0 \end{pmatrix} \in G(7,1,3)$$

one gets

$$\begin{aligned} R_3(w) &= [\mathbf{3}\cdots\mathbf{1}]\tau^5[\mathbf{1}\cdots\mathbf{2}] &= \sigma_2\sigma_1\tau^5\sigma_1 \\ R_2(w) &= \tau\sigma_1 \\ R_1(w) &= \tau^2 \end{aligned}$$

hence

$$R(w) = R_1(w)R_2(w)R_3(w) = \tau^2.\tau\sigma_1.\sigma_1\tau^5\sigma_1\sigma_2$$

Neaime's algorithm for $\mathcal{B}_n^*$

$d > 1$

Proposition

For every $w \in G(d, 1, n)$, we have $\ell_S(w) = \ell(R(w))$.

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The set $\mathrm{Div}(c)$ is made of the monomial matrices whose nonzero entries are either 1 or ζ_d^{-1}.

Connection with the 'real' theory

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CRG	Coxeter type	standard monoid
$G(2, 1, n)$	B_n/C_n	classic
$G(2, 2, n)$	D_n	classic
$G(1, 1, n)$	A_{n-1}	classic
$G(e, e, n)$	$I_2(e)$	dual

Elementary monoids

Definition

The monoid $M(r,s)$ is presented by generators $u_1, u_2, \dots, u_r$ and relations

$$\underbrace{u_1 u_2 u_3 \dots}_{s} = \underbrace{u_2 u_3 u_4 \dots}_{s} = \cdots = \underbrace{u_r u_1 u_2 \dots}_{s}$$

where $\underbrace{u_1 u_2 u_3 \dots}_{s}$ represents the unique subword of length s starting with u_1 of the infinite word $(u_1 u_2 \dots u_r)(u_1 u_2 \dots u_r)\dots$.

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In the cases we are interested in, we shall check that these are actually Garside interval monoids.

Elementary monoids

- $W = G(e, e, 2)$ with $S = \{s, t\}$,

$$s = (1, 2) \quad t = \begin{pmatrix} 0 & \zeta_e \\ \zeta_e^{-1} & 0 \end{pmatrix}$$

and $c = sts \cdots = tst \dots$.
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  and $c = sts \cdots = tst \dots$.
  Then $M(2,e)$ is an interval monoid for B.
- $W = G_7, G_{11}, G_{19}$ and $G(4,2,2)$: $M(3,3)$.
- $W = G_{12}$: $M(3,4)$
- $W = G_{22}$: $M(3,5)$

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- Obviously, some of the exceptional groups are 'real', so one can use Coxeter's theory and get an interval monoid for them
- Some of them are *not*, but share the same space X/W as a 'real' group W_0

Let then B^+ the corresponding Artin monoid with set of atoms S. It is an interval monoid w.r.t. W_0, with preferred Garside element Δ_0.

Theorem

- *We have $B = \mathrm{Frac}(B^+)$, the atoms of B^+ being mapped to braided reflections.*
- *Setting $|Z(W)| = m$, we have $B^+ = M_S(c)$ where $c = \zeta_m^{-1}\mathrm{Id}$.*

Use of the 'real theory'

W	$\lvert ZW \rvert$	$\lvert \mathrm{Div}(c) \rvert$	c	W_0	$\lvert ZW_0 \rvert$
G_4	2	19	Δ_0^2	$I_2(3)$	1
G_8	4	19	Δ_0^2	$I_2(3)$	1
G_{16}	10	19	Δ_0^2	$I_2(3)$	1
G_5	6	8	Δ_0	$I_2(4)$	2
G_{10}	12	8	Δ_0	$I_2(4)$	2
G_{18}	30	8	Δ_0	$I_2(4)$	2
G_{20}	6	51	Δ_0^2	$I_2(5)$	1
G_6	4	12	Δ_0	$I_2(6)$	2
G_9	8	12	Δ_0	$I_2(6)$	2
G_{17}	20	12	Δ_0	$I_2(6)$	2
G_{21}	12	20	Δ_0	$I_2(10)$	2
G_{25}	3	211	Δ_0^2	A_3	1
G_{26}	6	48	Δ_0	B_3	2
G_{32}	6	3651	Δ_0^2	A_4	1

Bessis monoids

For the groups G_{24}, G_{27}, G_{29}, G_{33} and G_{34}, none of this works, but Bessis constructed suitable monoids using the fact that they are 'well-generated'[1].

The element c is a Springer regular element, the length is computed from all the reflections, but not all of them are atoms.

It is then a nontrivial task to get from this a 'short' presentation with geometric meaning.

For the last group G_{31}, one needs a Garside *category* to deal with a corresponding braid *groupoid*, and no Garside monoid is known for this case.

1. applied to the 'real' case, this yields the 'dual braid monoid'

For $W = G_{24}$, one gets that $M(c)$ is presented by generators b_i, $i = 1, \ldots, 14$ such that $b_i = s_i$ for $i \leq 3$, and circular relations depicted as follows

$$\begin{tikzcd} b_1 \arrow[r] & b_2 \arrow[dl] & b_3 \arrow[d] \\ b_4 \arrow[u] & b_1 \arrow[ur] & b_5 \arrow[l] \end{tikzcd} \quad \begin{tikzcd} b_2 \arrow[r] & b_{11} \arrow[dl] & b_7 \arrow[d] \\ b_{10} \arrow[u] & b_5 \arrow[ur] & b_8 \arrow[l] \end{tikzcd} \quad \begin{tikzcd} b_6 \arrow[r] & b_8 \arrow[dl] & b_{14} \arrow[d] \\ b_9 \arrow[u] & b_6 \arrow[ur] & b_{12} \arrow[l] \end{tikzcd} \quad \begin{tikzcd} b_{11} \arrow[r] & b_{12} \arrow[dl] & b_9 \arrow[r] & b_{14} \arrow[dl] \\ b_{13} \arrow[u] & b_5 \arrow[ur] & b_{11} \arrow[l] & \end{tikzcd}$$

$$\begin{tikzcd} b_6 \arrow[r] & b_{13} \arrow[d] \\ b_1 \arrow[u] & b_{10} \arrow[l] \end{tikzcd} \quad \begin{tikzcd} b_7 \arrow[r] & b_9 \arrow[d] \\ b_1 \arrow[u] & b_{12} \arrow[l] \end{tikzcd} \quad \begin{tikzcd} b_2 \arrow[r] & b_3 \arrow[d] \\ b_6 \arrow[u] & b_7 \arrow[l] \end{tikzcd} \quad \begin{tikzcd} b_2 \arrow[r] & b_8 \arrow[d] \\ b_{13} \arrow[u] & b_{14} \arrow[l] \end{tikzcd} \quad \begin{tikzcd} b_3 \arrow[r] & b_8 \arrow[d] \\ b_4 \arrow[u] & b_{11} \arrow[l] \end{tikzcd} \quad \begin{tikzcd} b_4 \arrow[r] & b_5 \arrow[d] \\ b_{10} \arrow[u] & b_{12} \arrow[l] \end{tikzcd}$$

representing the relations

$$\begin{array}{lclclcl} b_1 b_2 & = & b_2 b_4 & = & b_4 b_1 & & \\ b_6 b_{13} & = & b_{13} b_{10} & = & b_{10} b_1 & = & b_1 b_6 \\ \ldots & & & & & & \end{array}$$