Finding Reflection Factorizations
Day 2: General Metric Vector Spaces

Jon McCammond

UC Santa Barbara

September 2021
Berlin

Advertisements

I have recently been working on two different projects related to dual structures.

- (with Michael Dougherty) There is a natural copy of the dual braid complex in the complex braid arrangement complement. I talked about this recently in Ohio (slides available from the conference website) and I will also be talking about this next week in Caen.
- (with Giovanni Paolini) There is a flexible general tool for factoring isometries of quadratic spaces into reflections, and that's what I'd like to discuss today. Preprint available at `arXiv:2103.02507`.

Canonical Factorizations

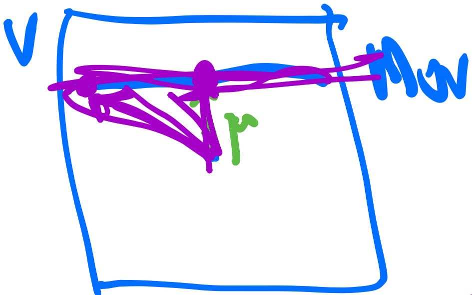
Definition (Isometry Types and Factorizations)

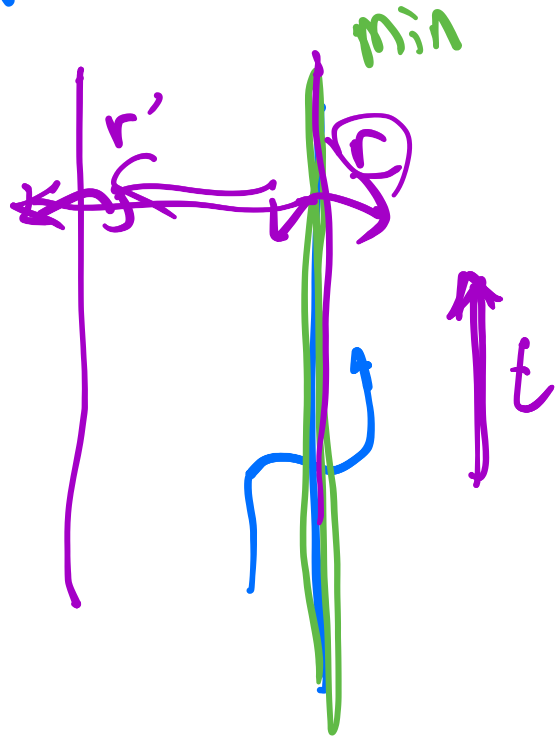
Every Euclidean isometry is either an elliptic isometry (fixing some point) or a hyperbolic isometry which has a canonical factorization as an elliptic isometry times a basic translation which commute with either other.

Remark

Using the sphere at infinity, and this factorization, we can find the basic translations below a mixing Euclidean isometry, such as a glide reflection on $\mathbf{E}^2$.

Translations below a glide reflection





$(r' \cdot r)\, r$

The trans. below this glide reflection corr. exactly to the pts in Mor.

Bipartite Coxeter Elements

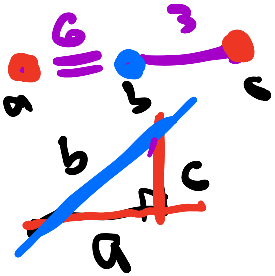
Definition

When Γ is a tree (or bipartite), there is a bipartite Coxeter elements. 2-color the vertices $S = S_r \cup S_b$. The product of elements in S_r is an involution and the product of elements in S_b is an involution. They generate a dihedral group and their product is a Coxeter element called the bipartite Coxeter element.

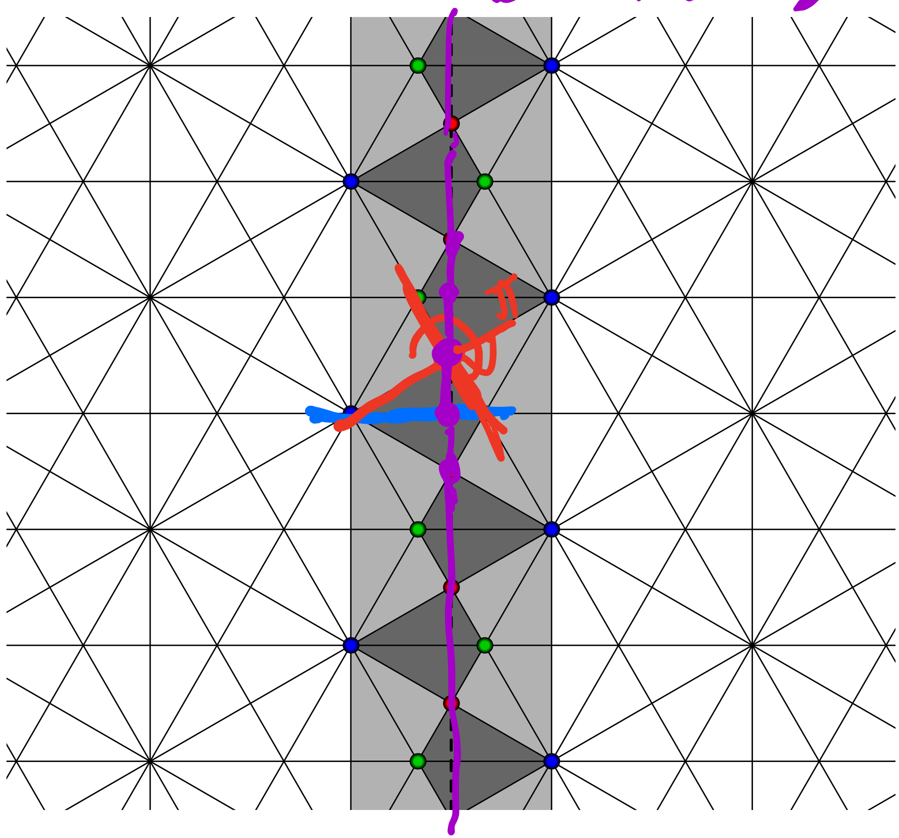
Remark

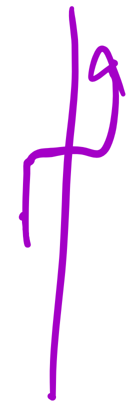
Bipartite Coxeter elements are used to find the Coxeter axis (when Euclidean) or the Coxeter plane (when spherical).

The Euclidean Coxeter Group Cox($\widetilde{G}_2$)



$D_\alpha \cong \mathbb{R} \Rightarrow \mathrm{Min}(w)$



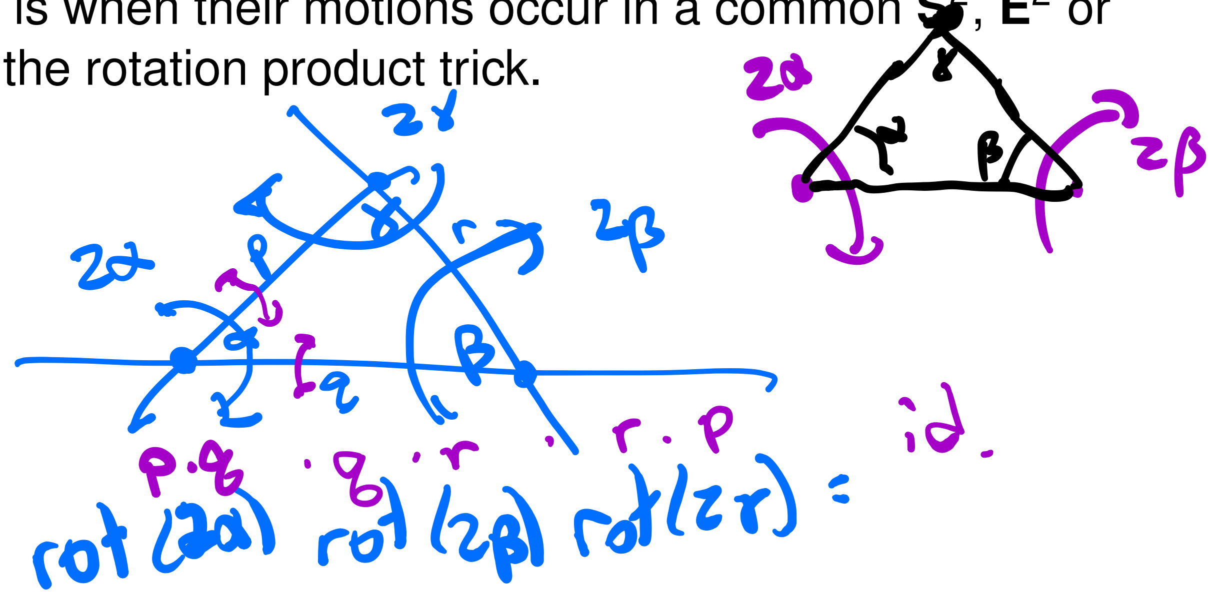


Basic Product Exercise

Question

When is the product of two basic isometries basic? And how do you find the product?

The answer is when their motions occur in a common $\mathbf{S}^2$, $\mathbf{E}^2$ or $\mathbf{H}^2$. Here is the rotation product trick.



Coarse Structure

Corollary

The product of two hyperbolic isometries of Euclidean space is never reduced.

Remark

Every factorization of $w = u \cdot v$ with $\ell(u) + \ell(v) = \ell(w)$ is either (ell,hyp), (ell,ell) or (hyp,ell). This gives the interval a coarse structure that looks like a grid.

Bowties

Proposition (Bowties)

A bounded graded poset is not a lattice if and only if it has contains a bowtie.

Remark

Big intervals of hyperbolic Euclidean isometries contain bowties, and are not lattices. These often cause the Euclidean Coxeter intervals to not be lattices.

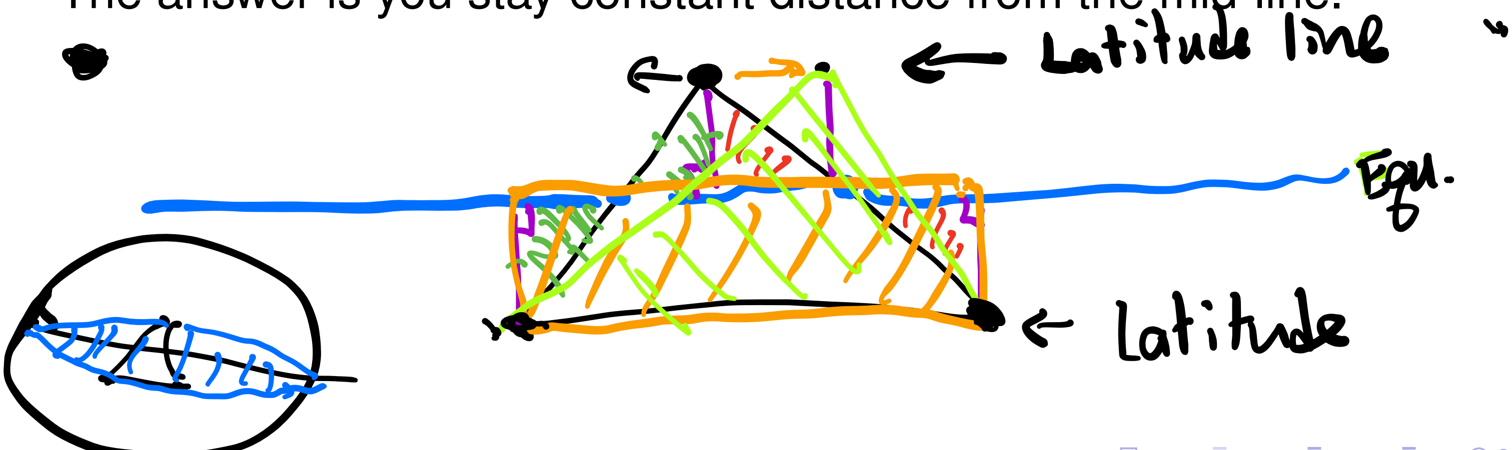
Constant Area Exercise

Before we move on to hyperbolic space and other quadratic spaces, let's do the other exercise from the first talk.

Question

What curve do you trace out when you only move one vertex of a triangle and keep the area constant?

The answer is you stay constant distance from the mid-line.



Models of Hyperbolic Space

Remark

I will assume that you're familiar with various models of hyperbolic space, such as the upper half space model, but especially the hyperboloid model with signature $(n, 1)$.

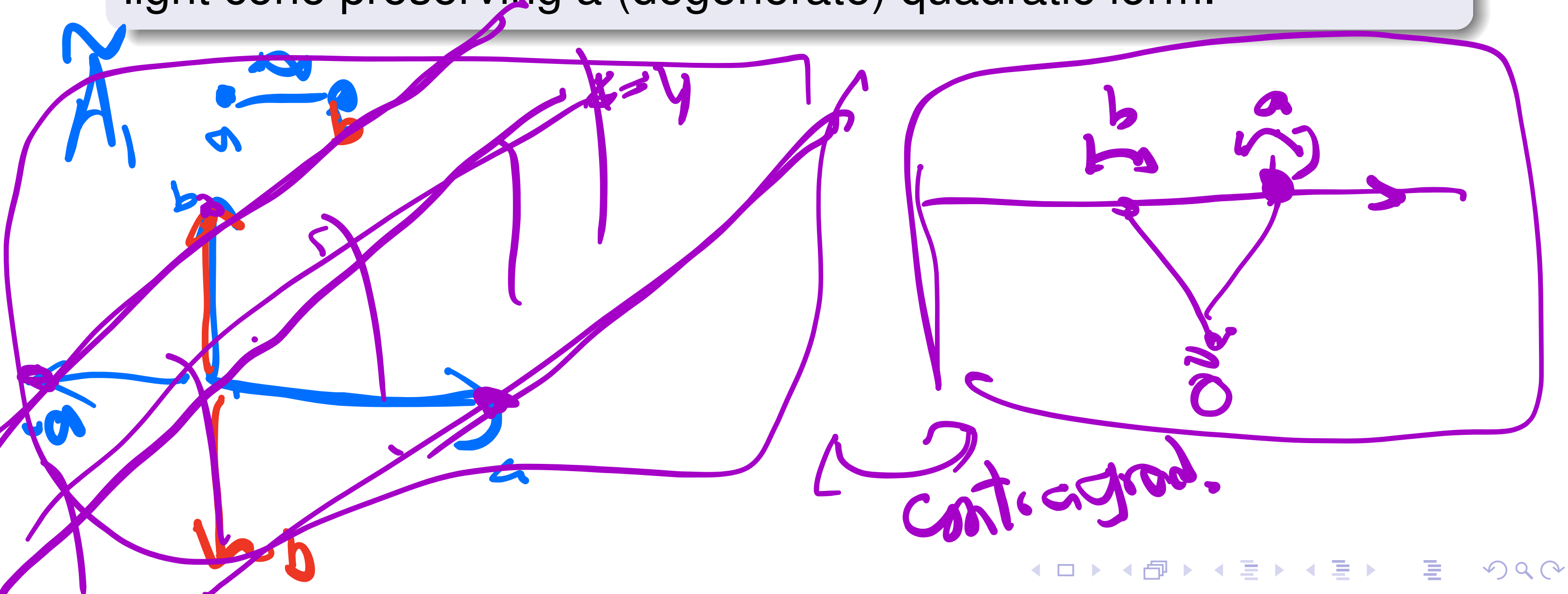
$(x,y,z)\cdot(x,y,z) = x^2+y^2-z^2$

$\mathbb{R}^{2,1}$

Euclidean Isometries in the Hyperbolic Model

Remark

To see the Euclidean isometry group inside the hyperboloid model, you need to switch to the contragradient representation. The Euclidean isometries now act on the tangent plane to the light cone preserving a (degenerate) quadratic form.



Dual vectors and linear functionals

(if time permits)

Linear Fractional Transformations

Definition

For any field $\mathbb{F}$, the group linear fractional transformations act on $\mathbb{F}P^1$ uniquely triply transitively. In particular, there is Sym_3 action which stabilizes the set $\{0, 1, \infty\}$ generated by $z \mapsto 1/z$ and $z \mapsto 1 - z$.

$z \mapsto 1/z$

$$\frac{1}{1-\frac{1}{z}} = 1 - \frac{1}{1-z}$$

$0 \leftrightarrow \infty$, 1 fixed

$z \mapsto 1-z$

$0 \leftrightarrow 1$, ∞ fixed

Acting on Mixing operators

$A \mapsto I - A \qquad A \mapsto A^{-1}$

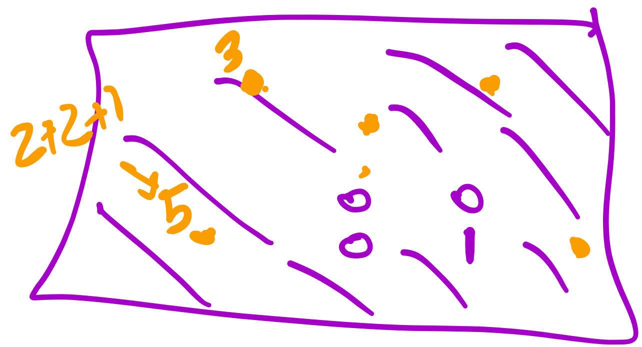
Proposition

There is a natural action of the group Sym_3 *on the set of mixing operators inside* $GL(\mathbb{C}, n)$.

Proof: use Jordan normal form and the Sym_3 action on $\mathbb{C}P^1$.

Mixing means inv. + only fixed pt is 0.

$0, 1 \in \text{Spec}(A)$



Quadratic spaces and Orthogonal groups

Let $V = \mathbb{R}^n$.

Definition (Quadratic space)

If J is a symmetric $n \times n$ matrix, define $\langle u, v \rangle = u^T J v$ to be the corresponding symmetric quadratic form. This turns $\mathbf{E}^n$ into quadratic space.

Definition

Two matrices J and K are congruent if there is an invertible matrix P such that $P^T J P = K$. Congruent matrices define the same quadratic space in different coordinates.

Sylvester's Inertia Theorem

Theorem (Sylvester)

Every real symmetric matrix J is congruent to a diagonal matrix with only $-1, 0$ and 1 on the diagonal. And the number of each is an invariant of the congruence class. This is called it signature.

Definition (Orthogonal group)

The orthogonal group of a quadratic space is
$O(V, J) = \{A \in GL(V) \mid A^T JA = J\}$

One could general orthogonal groups with respect to just these matrices, but that's not how they arise in practice.

Coxeter groups in orthogonal groups

Remark

The canonical linear representation of a Coxeter group (due to Tits) is inside the orthogonal group of the quadratic space defined by its Coxeter matrix.

In other words, we really would prefer to work in a space where J is the Coxeter matrix - not one where it is diagonal.

Remark

Finding examples of complicated matrices orthogonal with respect to a form can be quite hard, since the system of equations you need to solve ($A^T JA = J$) are quadratic.

Orthogonal mixing operators

Let A be an orthogonal mixing operator.

Proposition

If A is a mixing operator in $O(\mathbb{R}^n, J)$, then $B = (I - A)^-1$ is a mixing operator that satifisfies $BJ + JB = J$.

Here is the trivial proof. Since $A^T JA = J$ and $B = (I - A)^{-1}$, $A = (I - B^{-1})$. Substute, expand and simplify. Then multiply by $B^T \cdot$ and $\cdot B$.

$$
\begin{aligned}
(I - B^{-1})^T J (I - B^{-1}) &= J \\
J - B^{-T}J - JB^{-1} + B^{-T}JB^{-1} &= J \\
B^{-T}JB^{-1} &= B^{-T}J + JB^{-1} \\
J &= JB + BJ
\end{aligned}
$$

The important thing here is that solving for B only involves *linear* equations.

The Wall Form

Definition

Let $C = BJ$. This is the Wall Form of A. Note that $C = \frac{1}{2}J$ plus skew symmetric matrix. This map sends elements of the Lie group near $-I$ to element in the tangent plane at $-I$ (i.e. the Lie algebra. More generally, the Wall form for an arbitrary orthogonal transformation is (morally speaking) the Wall form for the transformation restricted to its move-set, on which it is mixing.

[Draw the picture of the a hexagon with $-I$]

$\langle u,v \rangle_C := u^T C v$

No longer symmetric.

asym. quad. form.

Minimal Intervals

Here is a sample theorem.

Theorem (M-Paolini)

Let $f \in O(V)$ be a minimal isometry. Then $g \mapsto \textsc{Mov}(g)$ is an order-preserving bijection between the interval $[id, f]$ and the poset of linear subspaces $U \subseteq \textsc{Mov}(f)$ that satisfy the following conditions:

(i) *$U = \{0\}$ or U is not totally singular;*

(ii) *$U^{\triangleright} = \{0\}$ or $U^{\triangleright}$ is not totally singular;*

(iii) *$\chi_f|_U$ is non-degenerate.*

In addition, the rank of $g \in [id, f]$ is equal to $\dim \textsc{Mov}(g)$.

Wall Form of a Coxeter Element

Remark

Reflection factorizations correspond to congruent forms of the Wall form matrix where the matrix is upper triangular.

Remark

The Wall form of a Coxeter element is easy to see from its Coxeter matrix. And one can perform the Hurwitz action directly on Wall forms.

Thank You