Fibers of word maps and composition factors

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Introduction

- Throughout: w denotes a reduced word in d distinct variables $X_1, \ldots, X_d$.
- For each group G, we have the word map $w_G : G^d \to G$, induced by substitution.
- Assume G is finite. What can we say about G under the assumption that w_G has a "large" fiber, say of size at least $\rho|G|^d$ for $\rho \in (0,1]$ fixed?
- Larsen and Shalev showed (see [2, Theorem 1.2]): w_S for S a large nonabelian finite simple group only has fibers of size at most $|S|^{d-\eta(w)}$, $\eta(w) > 0$.
- One can adapt parts of Larsen and Shalev's ideas to show the following (see [1, Theorem 1.1.2]):

Theorem

There are explicit functions $f_1^{(w)}, f_2^{(w)} : (0,1] \to [1,\infty)$ such that for all $\rho \in (0,1]$ and all finite groups G where w_G has a fiber of size at least $\rho|G|^d$, the following hold:

1. No finite alternating group of order larger than $f_1^{(w)}(\rho)$ is a composition factor of G.
2. No (classical) finite simple group of Lie Type of rank larger than $f_2^{(w)}(\rho)$ is a composition factor of G.

Proof sketch, part 1

- Write $w = x_1^{\epsilon_1} \cdots x_l^{\epsilon_l}$, $\epsilon_i \in \{\pm 1\}$, l the length of w, and let ι be the unique function $\{1, \ldots, l\} \to \{1, \ldots, d\}$ such that $x_i = X_{\iota(i)}$, $i = 1, \ldots, l$.
- For G a group and $\alpha_1, \ldots, \alpha_l \in \mathrm{Aut}(G)$, the *automorphic word map* $w_G^{(\alpha_1,\ldots,\alpha_l)}$ is the function $G^d \to G$, $(g_1, \ldots, g_d) \mapsto \alpha_1(g_{\iota(1)})^{\epsilon_1} \cdots \alpha_l(g_{\iota(l)})^{\epsilon_l}$.
- For finite groups G, denote by $\mathfrak{P}_w(G)$ the largest fiber size of an automorphic word map $w_G^{(\alpha_1,\ldots,\alpha_l)}$, $\alpha_1, \ldots, \alpha_l$ ranging over $\mathrm{Aut}(G)$, and set $\mathfrak{p}_w(G) := \mathfrak{P}_w(G)/|G|^d$.
- By "coset-wise counting", it is not difficult to show that $\mathfrak{P}_w(G) \leq \mathfrak{P}_w(N) \cdot \mathfrak{P}_w(G/N)$, or equivalently $\mathfrak{p}_w(G) \leq \mathfrak{p}_w(N) \cdot \mathfrak{p}_w(G/N)$, for all finite groups G and all $N \operatorname{char} G$.
- Moreover, by adaptations of proofs from [2], one can show:

Lemma

There exist $\eta'(w) > 0$ and $M(w) \in \mathbb{N}$ such that for all positive integers n and all S which are either

- a finite alternating group of order at least $M(w)$ or
- a finite simple group of Lie type of rank at least $M(w)$,

$\mathfrak{P}_w(S^n) \leq |S^n|^{d-\eta'(w)}$, or equivalently, $\mathfrak{p}_w(S^n) \leq |S^n|^{-\eta'(w)}$.

Proof sketch, part 2

- For any finite group G, if $T_1, \ldots, T_r$ are the *characteristic* composition factors of G, then $\mathfrak{p}_w(G) \leq \prod_{i=1}^r \mathfrak{p}_w(T_i) \leq \min_{i=1,\ldots,r} \mathfrak{p}_w(T_i)$.
- If S is a composition factor of G, then G has a characteristic composition factor of the form S^n, n a positive integer.
- Hence by the Lemma, if $\mathfrak{p}_w(G) \geq \rho$, then G cannot have any composition factors that are large alternating groups or simple Lie type groups of large rank.

Concluding remarks

- Open question: Do there even exist $\eta(w) > 0$ and $N(w) \in \mathbb{N}$ such that for all nonabelian finite simple groups S with $|S| \geq N(w)$ and all positive integers n, $\mathfrak{p}_w(S^n) \leq |S^n|^{-\eta(w)}$?
- This would imply that under $\mathfrak{p}_w(G) \geq \rho$, the orders of the nonabelian composition factors of G are bounded in terms of ρ and w.

References

[1] A. Bors, Fibers of automorphic word maps and an application to composition factors, submitted (2016), arXiv:1608.00131 [math.GR].

[2] M. Larsen and A. Shalev, Fibers of word maps and some applications, *J. Algebra* **354**:36–48 (2012).