UNIVERSITY OF RIJEKA
DEPARTMENT OF MATHEMATICS

Construction of designs from the unitary group $U(3,3)$

Dean Crnković, Vedrana Mikulić Crnković and Andrea Švob

This work has been fully supported by Croatian Science Foundation under the project 1637.

Introduction

We classify transitive 3-designs and 2-designs with 28 points admitting a transitive action of the unitary group $U(3,3)$. Constructed 3-designs and the majority of 2-designs that are obtained have not been known before up to our best knowledge. Further, we construct 2-designs with 36, 56 or 63 points and strongly regular graphs on 36, 63 or 126 vertices from the simple group $U(3,3)$.

Definition 1. A t-(v,k,λ) design is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

If $\mathcal{D}$ is a t-design, then it is also an s-design, for $1 \le s \le t-1$. A 2-(v,k,λ) design is called a block design. We say that a t-(v,k,λ) design $\mathcal{D}$ is a quasi-symmetric design with intersection numbers x and y $(x<y)$ if any two blocks of $\mathcal{D}$ intersect in either x or y points.

Definition 2. A graph Γ is called a strongly regular graph with parameters (n,k,λ,μ), and it is denoted by $SRG(n,k,\lambda,\mu)$, if Γ is k-regular with n vertices and if any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours.

The construction

Using the following construction presented in [2] we obtained the results.

Theorem 3. Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $\delta_1, ..., \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G,\alpha,\delta_1,...,\delta_s) = (\Omega_2, \mathcal{B})$ is a 1-$(n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|}\sum_{i=1}^{s}|\alpha G_{\delta_i}|)$ design with $\frac{m\cdot|G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G/\bigcap_{x\in\Omega_2} G_x$ acts as an automorphism group on $(\Omega_2,\mathcal{B})$, transitively on points and blocks of the design.

If $\Delta_2 = \Omega_2$ then the set $\mathcal{B}$ consists of one block, and $\mathcal{D}(G,\alpha,\delta_1,...,\delta_s)$ is a design with parameters 1-$(n,n,1)$.

The results

For obtaining the results, we apply the method on the unitary group $U(3,3)$ which is the simple group of order 6048, and up to conjugation it has 36 subgroups.

Classification of transitive 2-designs with $v=28$ having $U(3,3)$ as an automorphism group

Here we give all block designs with 28 points on which the group $U(3,3)$ acts transitively. The designs are obtained from the group $U(3,3)$ by applying the Theorem 3. In that case, the stabilizer of a point is a subgroup of $U(3,3)$ having the biggest order *i.e.* 216 and the smallest index *i.e.* 28.

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(28, 3, 2)	1	$U(3,3):Z_2$
2-(28, 3, 8)	1	$U(3,3):Z_2$
2-(28, 3, 16)	1	$S(6,2)$
2-(28, 4, 1)	1	$U(3,3):Z_2$
2-(28, 4, 4)	1	$U(3,3):Z_2$
2-(28, 4, 32)	1	$U(3,3):Z_2$
2-(28, 4, 48)	2	$U(3,3):Z_2$
2-(28, 4, 96)	2	$U(3,3):Z_2$
2-(28, 5, 40)	1	$U(3,3):Z_2$
2-(28, 5, 80)	4	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 5, 160)	2	$U(3,3)$
	8	$U(3,3):Z_2$
	1	$S(6,2)$
2-(28, 6, 20)	1	$U(3,3):Z_2$
2-(28, 6, 30)	1	$U(3,3):Z_2$
2-(28, 6, 40)	2	$U(3,3):Z_2$
	1	$S(6,2)$
2-(28, 6, 60)	3	$U(3,3):Z_2$
2-(28, 6, 80)	1	$U(3,3)$
	1	$U(3,3):Z_2$
	1	$S(6,2)$
2-(28, 6, 120)	2	$U(3,3)$
	3	$U(3,3):Z_2$
2-(28, 6, 240)	20	$U(3,3)$
	16	$U(3,3):Z_2$
2-(28, 7, 16)	1	$S(6,2)$
2-(28, 7, 48)	1	$U(3,3):Z_2$
2-(28, 7, 56)	3	$U(3,3):Z_2$
2-(28, 7, 84)	1	$U(3,3):Z_2$
2-(28, 7, 112)	2	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 7, 168)	5	$U(3,3):Z_2$
	8	$U(3,3)$
2-(28, 7, 336)	37	$U(3,3):Z_2$
	73	$U(3,3)$
2-(28, 8, 14)	1	$U(3,3):Z_2$
2-(28, 8, 56)	3	$U(3,3):Z_2$
2-(28, 8, 112)	2	$U(3,3):Z_2$
2-(28, 8, 224)	12	$U(3,3):Z_2$
	11	$U(3,3)$
2-(28, 8, 448)	217	$U(3,3)$
	61	$U(3,3):Z_2$
	1	$S(6,2)$

Table: Block designs constructed from $U(3,3)$, $v=28$, $3 \le k \le 8$

Up to our best knowledge, the majority of the designs on 28 points have not been known before.

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(28, 9, 32)	1	$U(3,3):Z_2$
2-(28, 9, 72)	1	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 9, 96)	1	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 9, 144)	1	$U(3,3):Z_2$
2-(28, 9, 192)	5	$U(3,3):Z_2$
	4	$U(3,3)$
2-(28, 9, 288)	11	$U(3,3):Z_2$
	22	$U(3,3)$
2-(28, 9, 576)	103	$U(3,3):Z_2$
	503	$U(3,3)$
2-(28, 10, 40)	1	$S(6,2)$
2-(28, 10, 45)	1	$S(6,2)$
2-(28, 10, 60)	1	$U(3,3):Z_2$
2-(28, 10, 90)	3	$U(3,3):Z_2$
2-(28, 10, 120)	1	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 10, 180)	3	$U(3,3):Z_2$
	3	$U(3,3)$
2-(28, 10, 240)	4	$U(3,3):Z_2$
	4	$U(3,3)$
2-(28, 10, 360)	21	$U(3,3):Z_2$
	24	$U(3,3)$
2-(28, 10, 720)	136	$U(3,3):Z_2$
	996	$U(3,3)$
2-(28, 11, 110)	1	$S(6,2)$
	1	$U(3,3)$
2-(28, 11, 220)	1	$U(3,3):Z_2$
2-(28, 11, 440)	18	$U(3,3):Z_2$
	44	$U(3,3)$
2-(28, 11, 880)	1650	$U(3,3)$
	195	$U(3,3):Z_2$
	2	$S(6,2)$
2-(28, 12, 11)	1	$S(6,2)$
2-(28, 12, 44)	1	$U(3,3):Z_2$
2-(28, 12, 88)	1	$U(3,3):Z_2$
2-(28, 12, 132)	4	$U(3,3):Z_2$
2-(28, 12, 176)	1	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 12, 264)	3	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 12, 352)	8	$U(3,3):Z_2$
	6	$U(3,3)$

Table: Block designs constructed from $U(3,3)$, $v=28$, $9 \le k \le 12$

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(28, 12, 528)	24	$U(3,3):Z_2$
	46	$U(3,3)$
2-(28, 12, 1056)	218	$U(3,3):Z_2$
	2372	$U(3,3)$
	1	$S(6,2)$
2-(28, 13, 104)	1	$U(3,3):Z_2$
2-(28, 13, 208)	2	$U(3,3):Z_2$
	1	$U(3,3)$
	1	$S(6,2)$
2-(28, 13, 312)	1	$U(3,3):Z_2$
	1	$U(3,3)$
2-(28, 13, 416)	7	$U(3,3):Z_2$
	6	$U(3,3)$
	1	$S(6,2)$
2-(28, 13, 624)	19	$U(3,3):Z_2$
	59	$U(3,3)$
2-(28, 13, 1248)	260	$U(3,3):Z_2$
	2887	$U(3,3)$
2-(28, 14, 182)	1	$U(3,3)$
2-(28, 14, 208)	2	$U(3,3):Z_2$
2-(28, 14, 364)	14	$U(3,3):Z_2$
2-(28, 14, 728)	28	$U(3,3):Z_2$
	53	$U(3,3)$
2-(28, 14, 1456)	246	$U(3,3):Z_2$
	3016	$U(3,3)$

Table: Block designs constructed from $U(3,3)$, $v=28$, $12 \le k \le 14$

Transitive 2-designs with $v=36$ having $U(3,3)$ as an automorphism group

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(36, 5, 4)	2	$U(3,3)$
2-(36, 5, 12)	1	$U(3,3):Z_2$
2-(36, 5, 16)	1	$U(3,3)$
2-(36, 5, 24)	1	$U(3,3):Z_2$
	2	$U(3,3)$
2-(36, 5, 32)	1	$S(6,2)$
2-(36, 5, 48)	7	$U(3,3)$
2-(36, 5, 96)	2	$U(3,3):Z_2$
	4	$U(3,3)$
2-(36, 6, 8)	1	$S(6,2)$
2-(36, 6, 24)	1	$U(3,3):Z_2$
	2	$U(3,3)$
2-(36, 6, 36)	2	$U(3,3)$
2-(36, 6, 48)	1	$U(3,3)$
2-(36, 6, 72)	2	$U(3,3):Z_2$
	14	$U(3,3)$
2-(36, 6, 144)	6	$U(3,3):Z_2$
	25	$U(3,3)$
2-(36, 10, 72)	9	$U(3,3)$
	1	$U(3,3):Z_2$
	1	$S(6,2)$
2-(36, 10, 108)	8	$U(3,3)$
2-(36, 10, 144)	5	$U(3,3)$
	1	$U(3,3):Z_2$
2-(36, 10, 216)	14	$U(3,3):Z_2$
	227	$U(3,3)$
2-(36, 10, 432)	2	$U(3,3):Z_2$
	10	$U(3,3)$
2-(36, 11, 22)	2	$U(3,3)$
2-(36, 11, 66)	5	$U(3,3)$
2-(36, 11, 88)	2	$U(3,3):Z_2$
	14	$U(3,3)$
2-(36, 11, 132)	1	$U(3,3):Z_2$
	23	$U(3,3)$
2-(36, 11, 176)	7	$U(3,3)$
2-(36, 11, 264)	538	$U(3,3)$
2-(36, 11, 528)	39	$U(3,3)$

Table: Block designs constructed from $U(3,3)$, $v=36$, $5 \le k \le 11$

Up to our best knowledge, the majority of the listed designs have not been known before. Below, we list the rest.

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(36, 15, 6)	1	$U(3,3):Z_2$
2-(36, 15, 126)	7	$U(3,3)$
2-(36, 15, 144)	1	$U(3,3)$
	1	$U(3,3):Z_2$
2-(36, 15, 168)	2	$U(3,3):Z_2$
	20	$U(3,3)$
	1	$S(6,2)$
2-(36, 15, 252)	52	$U(3,3)$
2-(36, 15, 336)	13	$U(3,3)$
	3	$U(3,3):Z_2$
2-(36, 15, 504)	20	$U(3,3):Z_2$
	1855	$U(3,3)$
2-(36, 15, 1008)	8	$U(3,3):Z_2$
	1569	$U(3,3)$
2-(36, 16, 12)	1	$S(6,2)$
2-(36, 16, 72)	1	$U(3,3)$
2-(36, 16, 144)	5	$U(3,3)$
	5	$U(3,3):Z_2$
2-(36, 16, 192)	18	$U(3,3)$
2-(36, 16, 288)	18	$U(3,3):Z_2$
	28	$U(3,3)$
2-(36, 16, 384)	14	$U(3,3)$
	1	$U(3,3):Z_2$
2-(36, 16, 576)	1387	$U(3,3)$
	37	$U(3,3):Z_2$
2-(36, 16, 1152)	8	$U(3,3):Z_2$
	2120	$U(3,3)$

Table: Block designs constructed from $U(3,3)$, $v=36$, $15 \le k \le 16$

Following the same approach as before, we construct block designs with 56 or 63 points.

Parameters of block designs	# non-isomorphic	Full automorphism group
2-(63, 31, 15)	1	PGL
	1	$U(3,3):Z_2$
2-(63, 31, 180)	2	$U(3,3):Z_2$
	5	$U(3,3)$
2-(63, 31, 240)	77	$U(3,3)$
2-(63, 31, 360)	23	$U(3,3)$
	95	$U(3,3):Z_2$
2-(63, 31, 480)	3	$U(3,3):Z_2$
	1321	$U(3,3)$
2-(56, 11, 12)	1	$U(3,3):Z_2$
2-(56, 11, 36)	1	$U(3,3):Z_2$
	1	$U(3,3)$
2-(56, 11, 72)	1	$U(3,3)$

Table: Block designs constructed from $U(3,3)$, $v=56, 63$

New 3-designs having $U(3,3)$ as an automorphism group

If the group $U(3,3)$ acts transitively on 3-(v,k,λ) design, then by the necessary conditions for the existence of 3-designs it follows that v=28 or v=56. According to [1], a design with parameters 3-(28, 13, 528) exists but the one obtained by the method presented in [2] is not isomorphic to the one described in the literature. For the rest of 3-designs listed below, no 3-designs with the same parameter triples were known before.

Parameters of designs	# non-isomorphic	Full automorphism group
3-(28, 13, 528)	40	$U(3,3)$
3-(28, 14, 84)	1	$U(3,3)$
3-(28, 14, 168)	2	$U(3,3):Z_2$
3-(28, 14, 336)	7	$U(3,3)$
3-(28, 14, 672)	12	$U(3,3):Z_2$
	136	$U(3,3)$
3-(56, 11, 36)	1	$U(3,3)$

Table: 3-designs constructed from the group $U(3,3)$, $v=28, 56$

Quasi-symmetric designs and strongly regular graphs

Here we list transitive quasi-symmetric designs on 28 or 36 points on which the group $U(3,3)$ acts transitively. Quasi-symmetric designs with parameters 2-(28, 12, 11) and 2-(36, 16, 12) are derived or residual designs of the symmetric 2-(64, 28, 12) design with the symmetric difference property. The block design with parameters (28, 4, 1) is a Hermitian unital.

Parameters of designs	Full automorphism group
2-(28, 4, 1)	$U(3,3):Z_2$
2-(28, 12, 11)	$S(6,2)$
2-(36, 16, 12)	$S(6,2)$

Table: Quasi-symmetric designs

Parameters of graphs	Full automorphism group
(36, 14, 4, 6)	$U(3,3):Z_2$
(63, 30, 13, 15)	$U(3,3):Z_2$
(63, 30, 13, 15)	$S(6,2)$
(126, 45, 12, 18)	$(U(4,3):Z_2):Z_2$

Table: Transitive strongly regular graphs

All above constructed strongly regular graphs have been known before. Symmetric incidence matrices with all-one diagonal of the symmetric block design with parameters (36, 15, 6) and two symmetric (63, 31, 15) designs give rise to the three strongly regular graphs with 36 or 63 vertices listed in Table.

References

[1] D. L. Kreher, "t-designs", Handbook of Combinatorial Designs, 1st ed., C. J. Colbourn and J. H. Dinitz (Editors), Chapman & Hall/CRC, Boca Raton, 1996, pp. 65–84.

[2] D. Crnković, V. Mikulić, A. Švob, On some transitive combinatorial structures constructed from the unitary group $U(3,3)$, J. Statist. Plann. Inference 144 (2014), 19–40.

[3] D. Crnković, V. Mikulić Crnković, A. Švob, New 3-designs and 2-designs having $U(3,3)$ as an automorphism group, submitted