Three transpositions, Graphs and Groupoids

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My congratulations to Bernd Fischer and thanks

- I appreciate the invitation to speak in honour of

Our colleague Bernd Fischer

Photographs: courtesy Ludwig Danzer
1969 Fischer theory of three transposition groups published

- In particular: wonderful constructions of the three Fischer sporadic finite simple groups

- “Three-transposition theory” caught the imagination of mathematicians world-wide and in many areas
  - In group theory, combinatorics, geometry

- My aim:
  - Trace several paths either influenced by “Three-transposition theory”
  - Or where Three-transposition groups appeared unexpectedly

- And they keep on arising ….
1969 Lecture Note, University of Warwick
1971 Inventiones paper

- Definitions:
- Group G
- family C of 3-transpositions in G:
  1) C closed under conjugation,
  2) For all x, y in C, | xy | is 1 or 2 or 3

- G called a 3-transposition group
  • if G generated a family of 3-transpositions
  • Usually refer to (G, C) as a three transposition group

- Fischer classifies all finite almost simple 3-transposition groups – beautiful concept, beautiful proof
Fischer’s classification:

- **Given** \((G, C)\) a three transposition group
- **Assume** each normal \(\{2,3\}\)-subgroup central, and \(G' = G''\)
- **Then** \(G/Z(G)\) is known explicitly: one of
  1) \(\text{Sym}(n), \text{Sp}(2n,2), \text{O}^\varepsilon(2n,2), \text{PSU}(n,2) \text{O}^\varepsilon(2n,3)\) or
  2) One of the three Fischer sporadic groups \(\text{Fi}_{22}, \text{Fi}_{23}, \text{Fi}_{24}\)
- And the class \(C\) (modulo \(Z(G)\)) was specified in each case

- This result and the underlying theory was very influential

47 MathSciNet citations, 297 cites in Google Scholar
Huge impact in Group Theory: simple group classification

- **1973 Aschbacher**: extended theory to “odd transposition groups”

Fischer groups investigated:
- **1974 Hunt**: determined conjugacy classes of $\text{Fi}_{23}$ & some character values
- **1981 Parrott**: characterised $\text{Fi}_{22}$, $\text{Fi}_{23}$, $\text{Fi}_{24}$ by their centralisers of a central involution

Inspired and underpinned studies of subgroup structure of simple groups:
- **1979 Kantor**: Subgroups of finite classical groups generated by long root elements

Even quite recently: for example
- **2006 (Chris) Parker**: 3-local characterisation of $\text{Fi}_{22}$
Geometrical and Combinatorial impact

- **1974 Buekenhout**: Fischer spaces $\Pi$
  - Partial linear space $(P, L)$ with point set $P$, line set $L$
  - Each line incident with 3 points
  - Each intersecting line pair contained in a “Subspace” $AG(2,3)$ or dual of $AG(2,2)$

- Each three transposition group $(G, C)$
  - Gives Fischer space $\Pi(G,C) = (C, L)$
  - Lines are $Sym(3)$ ‘s

- **Buekenhout**: 1-1 correspondence between connected Fischer spaces and three transposition groups with trivial centre
Geometrical and Combinatorial impact

- **1971 Fischer: diagram** $D$ of a three transposition group $(G, C)$
  - Graph with vertex set $C$
  - $\{ x, y \}$ an edge $\iff |xy| = 3$
  - [in analogy with Coxeter diagrams]

- Example $G = \text{Sym}(3)$, $C = \{ (12), (23), (13) \}$

- Paper contains diagrams like this

- So there was a combinatorial way of thinking
Geometrical and Combinatorial impact: Cuypers and (J I) Hall

- **1989 - 1997 [3 of JIH, 1 by HC, 4 joint]**: extend to infinite three transposition groups $(G, C)$ – strong use of graph theoretic methodology
- As well as the **diagram $D$**, they study
- The **commuting graph $A$** of $(G, C)$
  - Graph with vertex set $C$
  - $\{ x, y \}$ an edge $\iff |xy| = 2$
  - Commuting graph is **complement** of diagram

- Example $G = \text{Sym}(3), C = \{ (12), (23), (13) \}$
  Commuting graph is the empty graph

- Note that $G$ is a group of automorphisms of both $D$ and $A$
Geometrical and Combinatorial impact: Cuypers and (J I) Hall

- Two equivalence relations on C
  - D-relation:
    - $x \equiv_D y \iff x, y$ have same neighbour set in D
  - A-relation:
    - $x \equiv_A y \iff x, y$ have same neighbour set in A

- Both relations are G-invariant – induced G-action
  On the sets of equivalence classes

- G is irreducible if G faithful on the
  Equivalence classes for each relation

All finite three transposition groups with no nontrivial soluble normal subgroups are irreducible
Geometrical and Combinatorial impact: Cuypers and (J I) Hall

- Two equivalence relations on C
  - D-relation:
    - \( x \equiv_D y \iff x, y \) have same neighbour set in D
  - A-relation:
    - \( x \equiv_A y \iff x, y \) have same neighbour set in A

- Classification: all irreducible three transposition groups
  - Essentially same as finite case – same classical groups over possibly infinite dimensional spaces.

Ex: both relations Trivial for Sym(3)

All finite three transposition groups with no nontrivial soluble normal subgroups are irreducible
Commuting graphs and diagrams

 ASSERTION: Group $G$ and class $C$ of involutions (union of conjugacy classes; often a single class)

- Graph with vertex set $C$
- $\{ x, y \}$ an edge $\iff$ CONDITION holds

 ASSERTION: “commuting” that is $| xy | = 2$

- Motivating examples: all simply laced Weyl groups
- Bates, Bundy, Perkins, Rowley [2003 + +]
- Studied for all Coxeter groups: connectivity, diameters of components
- Many generalisations in literature
Commuting graphs and diagrams

- **Broader context:** Group $G$ and class $C$ of involutions (union of conjugacy classes; often a single class)
  - Graph with vertex set $C$
  - $\{ x, y \}$ an edge $\iff$ CONDITION holds

- **CONDITION:** $|xy| = 3$ equivalently $\langle x, y \rangle = \text{Sym}(3)$
  - Called **Sym(3) - involution graph**
  - Devillers, Giudici [2008 - several papers]
  - General theory on connectivity, automorphisms, existence of triangles

- Motivated by ....
Arose from general study of decomposing edges of a Johnson graph $J(v,k)$ “nicely” into isomorphic subgraphs [Devillers, Giudici, Li, CEP 2008]

- Exceptional example $J(12,4)$ [valency 32, 495 vertices]
  - admits $M_{12}$ decomposing into 12 copies of $\Sigma$ [valency 8, 165 vertices] admitting $M_{11}$
- Exceptional example $J(11,3)$
  - admits $M_{11}$ decomposing into 12 copies of $\Pi$ [valency 6, 55 vertices] admitting $PSL(2,11)$

- Use Witt designs to understand graphs $J(12,4)$, $\Sigma$, $\Pi$
- Or diagram geometry to understand $A_5 < PSL(2,11) < M_{11}$

Most uniform interpretation was as a set of four Involution graphs

**CONDITION:** $< x, y > = Sym(3)$ PLUS something extra
- Devillers, Giudici, Li, CEP [2010]
Commuting graphs and diagrams

- **Broader context:** Group G and class C of involutions (union of conjugacy classes; often a single class)
  - Graph with vertex set C
  - \{ x, y \} an edge \iff CONDITION holds

- **CONDITION:** \(| xy | \) lies in given set \( \pi \) of positive integers
  - \( \pi \) - Local fusion graph or Local fusion graph if \( \pi = \{ \text{all odd integers} \} \)
  - Ballantyne, Greer, Rowley [2013 - several papers]
  - For symmetric groups, sporadic simple groups: diameter at most 2

- **Theorem:** for all r, m exists G, C where local fusion graph has m components, each of diameter r
Now for something different: beginning with $M_{12}$

- Conway’s Game on $\text{PG}(3,3)$
- Start from a specified point $\infty$
- Move to a second point
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- Conway’s Game on PG(3,3)
- Start from a specified point $\infty$
- Move to a second point, say 3
- Associate move with permutation
  $$[\infty, 3] = (\infty, 3) (5, 7)$$
Now for something different: beginning with $M_{12}$

- Conway’s Game on PG(3,3)
- Start from a specified point $\infty$
- Move to a second point, say 3
- Associate move with permutation
  \[ [\infty, 3] = (\infty, 3) (5,7) \]
- Repeat: $[3,9] = (3,9) (6,12)$
- Composite move sequence
  \[ [\infty, 3, 9] = [\infty, 3] [3,9] = (\infty, 3) (5,7) (3,9) (6,12) = (\infty, 9, 3) (5,7) (6,12) \]
Now for something different: beginning with $M_{12}$

- Conway’s Game on PG(3,3)
- $L_\infty(\text{PG}(3,3)) := \text{SET of all move sequences starting with } \infty$
- "Conway’s groupoid" – subset of Sym(13) – not a group

- $\Pi_\infty(\text{PG}(3,3)) := \text{SET of all move sequences starting AND ENDING with } \infty$
- "hole stabiliser" – is a group
- Isomorphic to $M_{12}$

- Gill, Gillespie, Nixon, Semeraro: where else can we play this game?
Try a $2-(n,4,k)$ design $D$

- $n$ points, each point pair $\{a, b\}$ lies on $k$ lines [all of size 4]
- Try to define
  \[
  [a, b] = (a, b) \prod_{i=1}^{k} (c_i, d_i)
  \]
- Well defined provided the points $c_i$ and $d_i$ are pairwise distinct
- So need $D$ supersimple distinct lines have at most two common points

- $L_\infty(D) := \text{SET of all move sequences starting with distinguished point } \infty$
- $\Pi_\infty(D) := \text{SET of all move sequences starting AND ENDING with } \infty$

- Gill, Gillespie, Nixon, Semeraro: computer searches and some theory
Supersimple 2-(n,4,k) design D

- Each point pair \( \{a, b\} \) has elementary move
  \[ [a, b] = (a, b) \prod_{i=1}^{k} (c_i, d_i) \]

- For \( k=1 \) found: either Conway’s groupoid or \( \Pi_\infty(D) = \text{Alt}(n-1) \), \( L_\infty(D) = \text{Alt}(n) \)

- For \( k=2 \) found: **INTERESTING CASE** or \( \Pi_\infty(D) = \text{Sym}(n-1) \), \( L_\infty(D) = \text{Sym}(n) \)

- **INTERESTING CASE** \( n=10 \), \( \Pi_\infty(D) = \text{O}^+(4,2) \), \( L_\infty(D) = \text{Sp}(4,2) \) [a group!]

- And D satisfies:
  - Symmetric difference of two intersecting lines is also a line
  - Each 4-subset of points contains 0, 2 or 4 collinear triples
Supersimple 2-(n,4,k) designs D with

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2017 Gill, Gillespie, CEP, Semeraro:
- \( L_\infty(D) \) always a group

For \( E := \{ [a,b] \mid \text{distinct points } a, b \} \)
- \( E \) conjugacy class of \( L_\infty(D) \)
- \( (L_\infty(D), E) \) three transposition group
Supersimple 2-(n,4,k) designs D with

- Symmetric difference of intersection lines is also a line
- Each 4-subset of points contains 0, 2 or 4 collinear triples

Using the Fischer classification of three transposition groups we find

1. $\Pi_{\infty}(D) = 1$ and $L_{\infty}(D) = E(2^m)$
2. $\Pi_{\infty}(D) = O^+(2m,2)$, $L_{\infty}(D) = Sp(2m,2)$
3. $\Pi_{\infty}(D) = O^-(2m,2)$, $L_{\infty}(D) = Sp(2m,2)$
4. $\Pi_{\infty}(D) = Sp(2m,2)$, $L_{\infty}(D) = 2^{2m}.Sp(2m,2)$

D described explicitly e.g. in case 1) points and planes of $AG(m,2)$
Thank you

➢ To Professor Bernd Fischer
➢ For your beautiful mathematics
➢ Congratulations on the milestone celebrated at this conference
Thank you