Fischer's Monsters

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Abstract

This is an expanded version of a talk given in honour of Bernd Fischer, at a meeting to celebrate his 80th birthday. It consists of a brief survey of some things we know about the Monster, and some things we do not yet know, including the following topics. Introduction to the Fischer groups and Fischer's Monsters. The 6-transposition property and its possible relationship to E_8. The character table and Moonshine. The Ronan–Smith diagram of the 2-local geometry. Constructions of the 196883 dimensional representation by Griess, Conway, computer (existence and uniqueness). The current state of play for maximal subgroups.

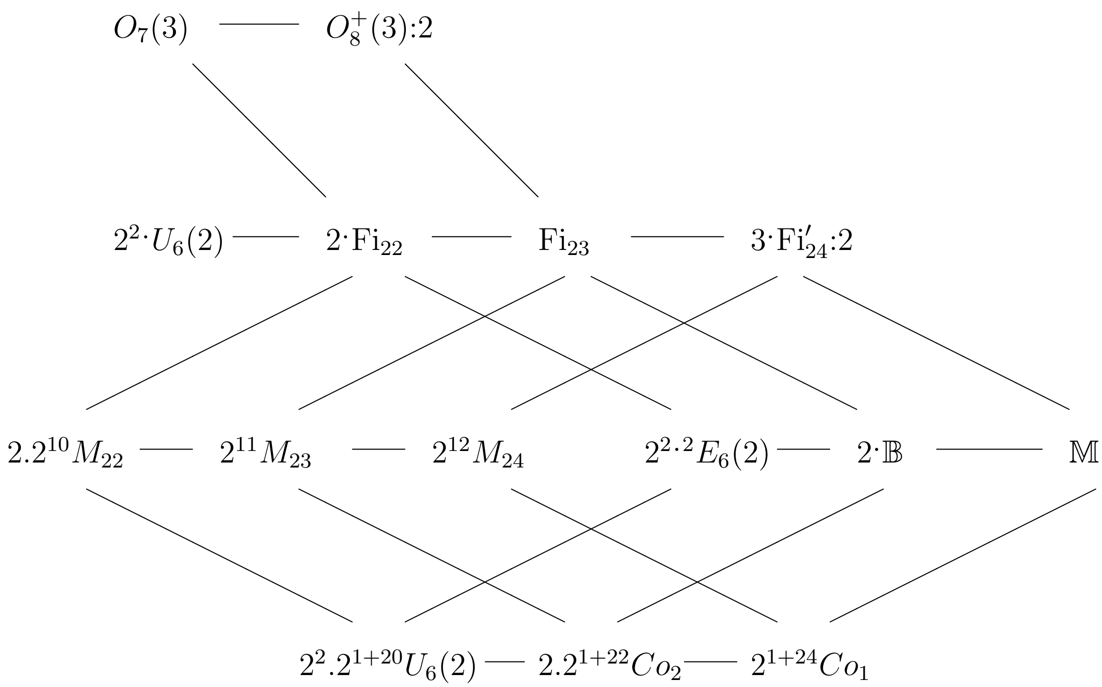
1 Introduction

I first heard about the Fischer groups in a lecture course given by Conway in Cambridge in the academic year 1978–9. This course was mainly devoted to the Mathieu groups and Conway groups, but also included an introduction to the Fischer 3-transposition groups and the Monsters.

The story begins with 3-transposition groups, groups like the symmetric groups that are generated by a conjugacy class of involutions, with the property that the product of every pair of elements in the class has order 1, 2 or 3. Fischer's classification of such groups in the late 1960s/early 1970s produced three new examples, which he called $M(22)$, $M(23)$, $M(24)$. The number denotes the maximum number of mutually commuting 3-transpositions. It is almost but not quite true that $M(22) < M(23) < M(24)$ (see the diagram).

Fischer then showed that $M(22) < {}^2E_6(2)$, the latter being a 4-transposition group. Moreover, the double cover of $M(22)$ embeds in a double cover of

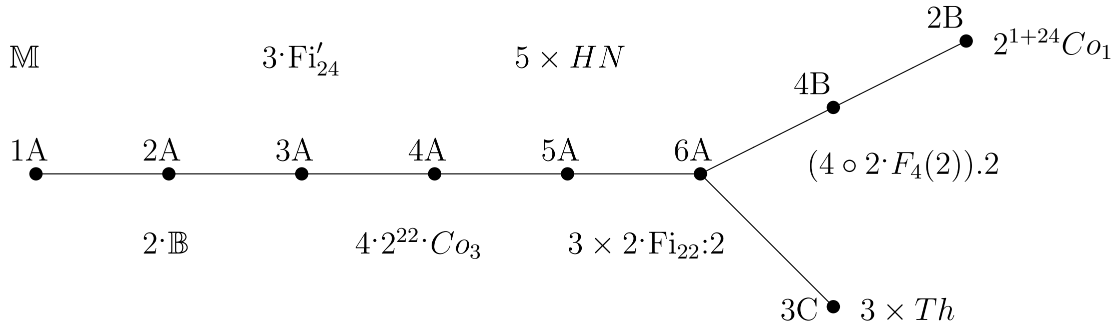
${}^2E_6(2)$. The latter is an involution centralizer in another sporadic simple group, Fischer's Baby Monster group. But the outer automorphism of order 3 of ${}^2E_6(2)$ gives another double cover, making $2^{2\cdot 2}E_6(2)$, which is contained in a double cover of the Baby Monster. Hence there is the possibility of a new sporadic simple group with involution centralizer $2{\cdot}\mathbb{B}$. The Monster was born. And the rest, as they say, is history.



2 6-transpositions

The Baby Monster is a 4-transposition group, and the Monster is a 6-transposition group. There seems to be something very special about the 6-transposition property, both in general, and specifically in the Monster. For a start, the product of two 6-transpositions in the Monster lies in one of 9 conjugacy classes:

- counting up to 6: $1A, 2A, 3A, 4A, 5A, 6A$;
- counting up in 2s: $2B, 4B$;
- and in 3s: $3C$.



The mysterious fact that this is the extended E_8 Dynkin diagram, with the orders of the elements matching up nicely with the coefficients of the simple roots in the highest root, is still not explained. Perhaps it is just a coincidence. Or perhaps it is evidence of a deep connection between the largest exceptional Lie group, and the largest sporadic simple group?

3 Moonshine

In any case, this picture enabled Norton to analyse the permutation representation of the Monster on its nearly 10^{20} transpositions, and prove that there is a representation of degree 196883. It is relatively easy to see that this is the smallest possible degree of a faithful representation. With this character in place, Fischer, Livingstone and Thorne computed the entire character table, allegedly in smoke-filled rooms in Birmingham.

To John McKay, the most striking thing about this character table was the occurrence of the number 196883, which reminded him of the number 196884, which is the first coefficient in the Fourier expansion of the j function, from the theory of modular forms. The story has often been told of the difficulty he had in persuading anyone that this apparent coincidence was worth investigating. The fact that

$$196883 + 1 = 196884$$

seemed hardly a deep result! Eventually, however, it led to a whole series of "Moonshine" conjectures by Conway and Norton, and a proof of many of them by Borcherds, who won a Fields Medal for this work.

4 Nets

(This section was omitted from the talk itself, due to lack of time.)

Let us return to the idea of 6-transpositions. Given a pair of 6-transpositions, (a, b) say, there are two obvious ways to "braid" them, to get new pairs (b^a, a) and (b, a^b), with the same product as the original: $b^a a = abaa = ab$ and $ba^b = bbab = ab$. Repeating one of these operations we get

$$(a, b) \to (aba, a) \to (ababa, aba) \to (abababa, ababa) \to (ababababa, abababa) \to$$

If, for example ab has order 4, then

$$(ababababa, abababa) = ((ab)^4 a, (ab)^4 b) = (a, b)$$

so we have a cycle of length 4. Similarly for other orders up to 6.

Now suppose we have a triple (a, b, c) of 6-transpositions. Then we can braid a and b as above, keeping the product equal to abc; and similarly we can braid b and c. We can also braid a and c, by mapping (a, b, c) to (c^a, b^{ca}, a), for example. In each case we get a cycle of length at most 6. We can even fit together the three cycles at a vertex (a, b, c), provided we quotient by a suitable equivalence relation.

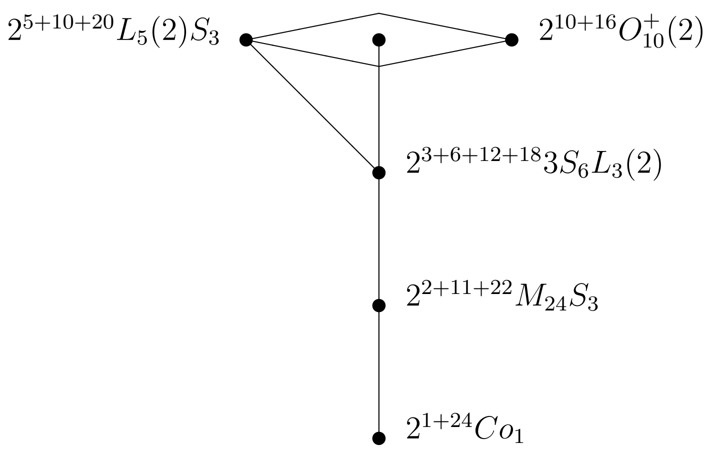
What we find is that first braiding a and b we get (b^a, a, c), and then braiding a and c we get (b^a, c^a, a). This we regard as equivalent to (a, b, c): the equivalence relation is generated by conjugation in the group, together with cyclic rotation of the triple: $(a, b, c) \to (b, c, a)$. Thus braiding a and b in one direction gives (b, a^b, c), and braiding b and c in the other gives (a, c^b, b). These two triples are equivalent, by rotation followed by conjugation by b. Hence the (a, b) cycle fits together with the (b, c) cycle along this edge.

A similar calculation allows us to fit the (a, c) cycle to the (a, b) cycle and to the (b, c) cycle, so that three faces fit together at each vertex. Doing the same thing at every vertex, we eventually end up with a closed polyhedron. There are two cases: either all faces are hexagons, and the polyhedron is topolgically a torus; or some faces have fewer than six sides, and the polyhedron is topologically a sphere. Moreover, we can calculate the total number of hexagons, pentagons, squares, etc. Hence, Euler's formula tells us exactly how many "nets" of genus 0 there are. This number is 13575.

It is not known how many nets of genus 1 there are.

5 Geometry

It was apparent from quite early on that the subgroup structure of the Monster mimics that of the groups of Lie type, but in many different characteristics at once. The most obvious case is $p = 2$, in which we get the following Ronan–Smith diagram, analogous to the Dynkin diagram. Note first the isomorphism $Sp_4(2) \cong S_6$, so that S_6 has Dynkin diagram of type B_2. But already in M_{24} we have a non-split extension $3^{\cdot}S_6$, which we notate with a B_2 diagram with an extra node (representing S_3) trapped inside.



The subgroups described by the nodes of the diagram behave very much like maximal parabolic subgroups in a group of Lie type in characteristic 2. But the Monster has similar collections of "parabolic" subgroups also for $p = 3$, 5, 7 and even 13.

6 Construction

The 2-local geometry was used by Griess to construct the group, is the sense of proving its existence. He built the 196883-dimensional representation from the lowest two nodes of the diagram. In fact counting the numbers of irreducible constituents for the three groups involved in the diagram (the alternating sum of the centralizer dimensions is $1 - 3 - 5 + 7 = 0$) shows that there is essentially only one such group: this is an important part, due to Thompson, of showing that the Monster is unique.

$$
\begin{tikzcd}[column sep=small]
\begin{pmatrix} S^2(24)-1 \\ + \\ \text{monomial} \\ + \\ 24\otimes 2^{12} \end{pmatrix} \quad \begin{matrix} 299 \\ + \\ 98280 \\ + \\ 98304 \end{matrix} & 2^{1+24}Co_1 \arrow[dr, dash] & & 2^{2+11+22}(M_{24}\times S_3) \arrow[dl, dash] & \begin{matrix} 1+22 \\ + \\ 828 \\ + \\ 48576 \\ + \\ 147456 \end{matrix} \\
\begin{matrix} 1+22+276 \\ + \\ 552+48576+49152 \\ + \\ 98304 \end{matrix} & & 2^{1+1+11+11+11}(M_{24}\times 2) & &
\end{tikzcd}
$$

Later, Beth Holmes and I repeated this construction on a computer, working over the field of order 3 in order to obtain a representation of the Monster that we could actually compute in. Richard Parker tells me that he now has a Meataxe that will multiply dense matrices of this size in 25 minutes, but in fact we work without writing down any actual matrices. The first computer construction, however, was done over the field of order 2, with 3-local subgroups instead. (This picture was not drawn in the lecture.)

$$
\begin{tikzcd}[column sep=small]
\begin{pmatrix} 12\otimes 12^{*}-1-1 \\ + \\ \text{semimonomial} \\ + \\ 12.66.12\otimes 3^6+* \end{pmatrix} \quad \begin{matrix} 142 \\ + \\ 65520 \\ + \\ 131220 \end{matrix} & 3^{1+12}2Suz.2 \arrow[dr, dash] & & 3^{2+5+10}(M_{11}\times 2S_4) \arrow[dl, dash] & \begin{matrix} 10 \\ + \\ 528 \\ + \\ 3564+17820 \\ + \\ 46656 \\ + \\ 128304 \end{matrix} \\
\begin{matrix} 10+132 \\ + \\ 396+3564+17820+11664+32076 \\ + \\ 2\times 17496+96228 \end{matrix} & & 3^{1+1+5+5+5}(M_{11}\times 2S_3) & &
\end{tikzcd}
$$

7 The Griess–Norton algebra

(This section was omitted from the lecture.)

Norton showed there is a commutative non-associative algebra structure on the representation of degree 196883, and worked out many of its properties. Griess's construction of the Monster made this algebra more explicit. He also adjoined an identity element to his algebra. Restricting the character of degree 196883 to the double cover of the Baby Monster, it breaks up as 1 + 4371 + 96255 + 96256. Hence the 6-transpositions correspond to 1-dimensional subspaces of the algebra. These 1-spaces are spanned by idempotents.

Much current research is being done, for example by Sasha Ivanov and his students, investigating properties of this algebra, and in particular, properties of these idempotents.

8 Maximal subgroups

My particular interest in the Monster has always been its subgroup structure. By the time I started work on this, a great deal was already known about the subgroups of the Monster, beginning of course with the work of Bernd Fischer himself. Many interesting maximal subgroups had been constructed, and Norton had done a lot of work on restricting the possibilities for any further maximal subgroups. In particular, he had completely classified the maximal p-local subgroups for $p \geq 5$; the subgroups isomorphic to A_5; all the non-local subgroups which contain an A_5 with $5A$-elements; and all $(2, 3, 7A)$-generated subgroups.

I classified the 3-local subgroups, but could not do the 2-local case; Meierfrankenfeld and Shpectorov did that. Norton and I worked together on more of the non-local cases, and reduced the problem to about forty cases.

Then my student Beth Holmes used the computer constructions to deal with most of these cases (over 20 in her PhD, at least 10 more as a post-doc). The most interesting part of her work is the discovery of a fair number of previously unknown maximal subgroups, specifically

$$L_2(71), L_2(59), L_2(29){:}2, L_2(19){:}2$$

Later I discovered a subgroup $L_2(41)$, which had previously been missed because we thought (wrongly) that we had proved it could not be there.

By 2005, it was claimed that there were only three cases outstanding, namely $L_2(13)$, $U_3(8)$, and $Sz(8)$. However, a number of other cases were never written up for publication, making it desirable at least to repeat the calculations. In the past three years or so, I have worked on some of these cases, and summarise the results:

- there is a (unique) $L_2(41)$: computational, 2013.
- there is no $L_2(27)$: computational, 2014, repeating unpublished work.
- there is no $13B$ type $L_2(13)$: computational, 2015.
- there is no $Sz(8)$: theoretical, 2016, relying on 2-local classification.
- the $U_3(8)$ is unique: theoretical, 2017? (submitted)
- the $U_3(4)$ is unique: computational, 2017? (in preparation)
- there is no $7B$ type $L_2(8)$: computational, 2017? (preprint undergoing checking)

The remaining questions are now just the following:

- $L_2(13)$ of $13A$ type;
- $L_2(16)$ of $5B$ type (probably already answered by Holmes, but not written up).

9 Conclusion

After more than 40 years, there is still a lot we do not know about the Monster. It has provided an enormously fruitful field for research, and shows every sign of continuing to do so for a long time to come.