What we (don't) know about permutation polytopes

Benjamin Nill
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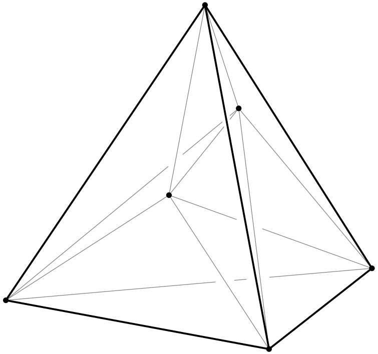
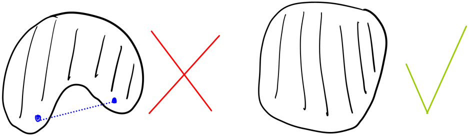


Polytopes

Convex set:

contains the connecting segment between any two points



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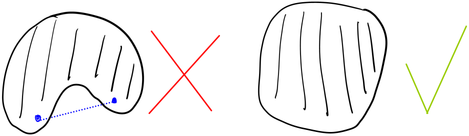

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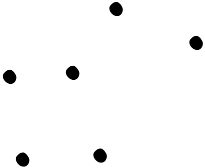

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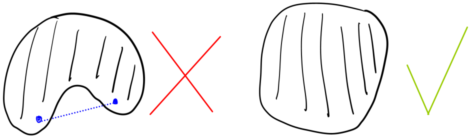

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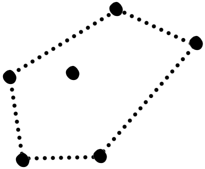
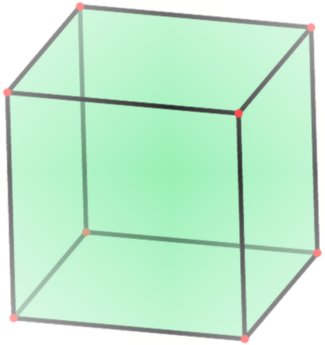


Polytopes

Polytopes: Convex hull of finite number of points



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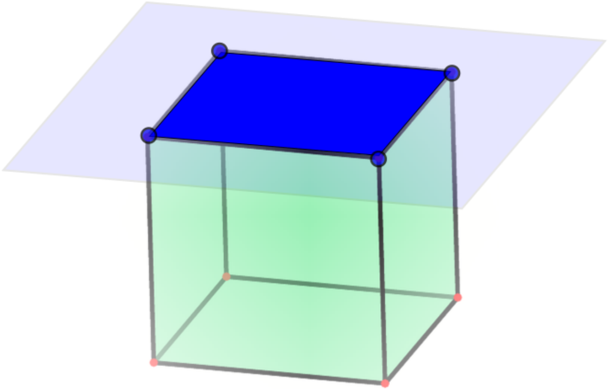
Faces: The intersection with hyperplanes with the polytope on one side

Vertices: 0-dimensional faces
Edges: 1-dimensional faces
Facets: maximal-dimensional (proper) faces



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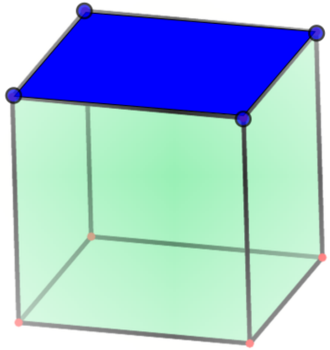

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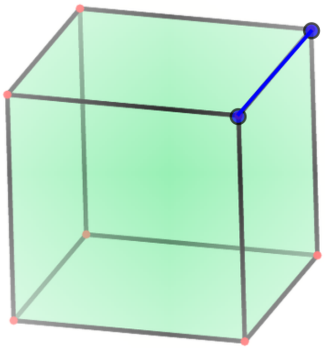

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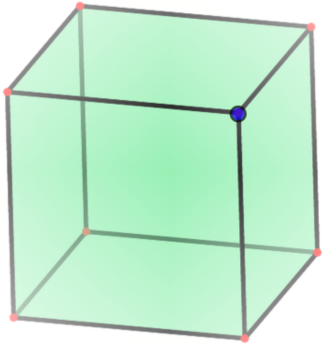
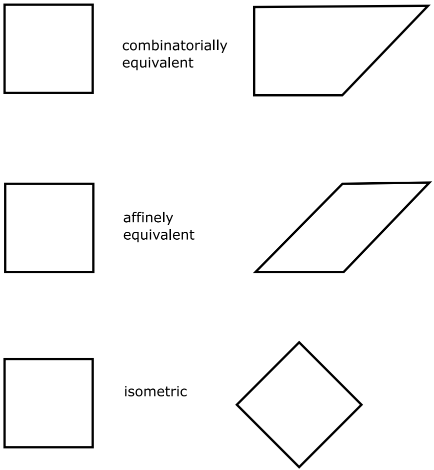


Symmetries of polytopes

Polytope ⇝ Symmetry groups



THIS TALK: Permutation polytopes

$G \leq S_n$ subgroup.

Definition

$$P(G) := \mathrm{conv}(M(g) \ : \ g \in G) \subset \mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$$

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Examples:

- $P(S_2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$
  is an interval (1-dimensional polytope) in $\mathbb{R}^4$
- $P(\langle(1\ 2\ 3\ \ \cdots\ \ d+1)\rangle)$ is d-simplex
- $P(\langle(1\ 2)\ ,\ (3\ 4)\ ,\ \ \cdots\ \ ,\ (2d-1\ \ 2d)\rangle)$ is d-cube

THIS TALK: Permutation polytopes

Two basic results:

1. G acts transitively by multiplication on vertices of P:

$$|\text{Vertices}(P(G))| = |G|.$$

2. The vertices of $P(G)$ have only 0 or 1 coordinates:

$$|G| \leq 2^{\dim(P(G))}, \text{ with equality if cube.}$$

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– *challenging representation-theoretic problems!*

Overview of talk

1. The Birkhoff polytope
2. Other special classes
3. Faces
4. Dimension
5. Equivalences

The Birkhoff polytope B_n

Definition

$B_n := P(S_n)$ is called **Birkhoff polytope**.

1. **Vertices:** all $n \times n$-permutation matrices
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   - (Canfield, McKay '09): asymptotic formula

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   - (Beck, Pixton '03): exact values known for $n \leq 10$:

     $$\mathrm{Vol}(B_{10}) = \frac{727291284016786420977508457990121862548823260052557333386607889}{828160860106766855125676318796872729344622463533089422677980721388055739956270293750883504892820848640000000}$$

The Birkhoff polytope B_n

4. **Ehrhart polynomial:**
   The function $k \mapsto |(kB_n) \cap \mathrm{Mat}_n(\mathbb{Z})|$ is a polynomial
   e.g. for B_3 : $\quad k \mapsto 1 + \frac{9}{4}k + \frac{15}{8}k^2 + \frac{3}{4}k^3 + \frac{1}{8}k^4$

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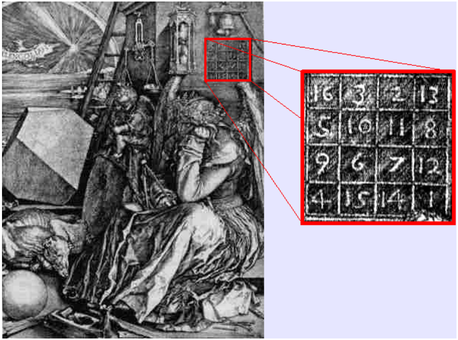

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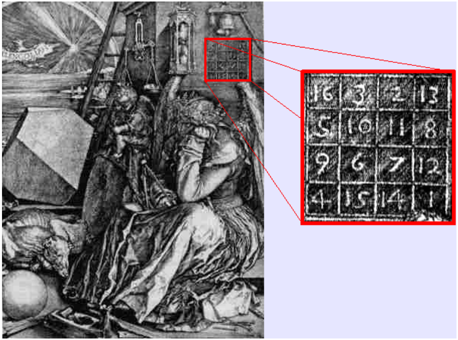

CONJECTURE 1 (De Loera et al.)

All coefficients are nonnegative.

The Birkhoff polytope B_n

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6. **Symmetry group:** Any combinatorial symmetry comes from left multiplication, right multiplication or transposition (Baumeister, Ladisch '16):

$$\mathrm{Aut}_{\mathrm{comb}}(B_n) \cong S_n \wr C_2$$

Other special classes

- $P(D_n)$ for $D_n \leq S_n$ **dihedral group** is completely understood (Baumeister, Haase, Nill, Paffenholz '14).

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- $P(D_n)$ for $D_n \leq S_n$ **dihedral group** is completely understood (Baumeister, Haase, Nill, Paffenholz '14).
- Combinatorial type and volume of $P(G)$ known if $G \leq S_n$ is **Frobenius group** (i.e. exists $H \leq G$ s.t. $\forall\, x \in G \setminus H$, $H \cap (xHx^{-1}) = \{e\}$) (Burggraf, De Loera, Omar '13).

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has dimension $ab + ac + bc - a - b - c$, abc vertices, but at least $((2^a - 2)(2^b - 2)(2^c - 2) + ab + ac + bc)/2$ many facets. (Sontag, Jaakkola '08; Baumeister, Haase, Nill, Paffenholz '12)

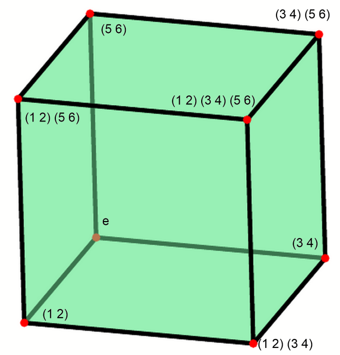
Other special classes

Computational challenge

For $(a, b, c) = (5, 6, 7)$, the permutation polytope $P(\langle z_{ab} z_{ac} z_{bc} \rangle$ has

- dimension 89,
- 210 vertices,
- but conjecturally $> 10^9$ facets.

Faces of permutation polytopes



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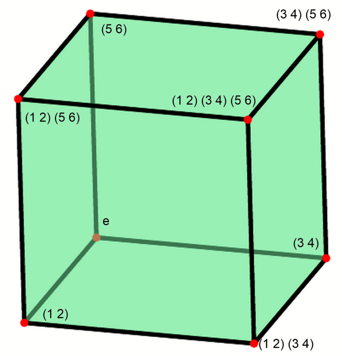

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Conjecture (Baumeister, Haase, Nill, Paffenholz '09)

Any subgroup of G whose permutation polytope is a face of $P(G)$ is a stabilizer of a partition of $\{1, \ldots, n\}$.

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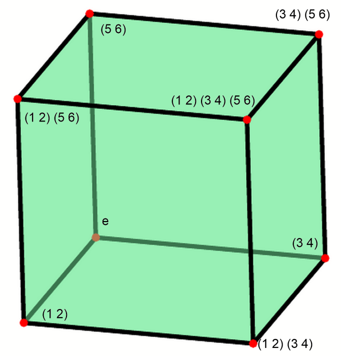

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Theorem (Haase '15)

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Faces of permutation polytopes

What about edges?

Proposition (Guralnick, Perkinson '05)

Let $g = z_1 \cdots z_r \in S_n$ be the decomposition into disjoint cycles. Then

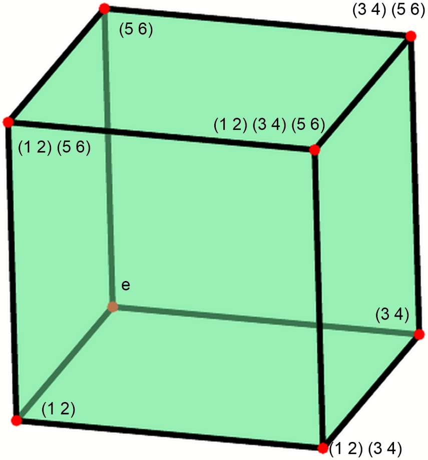
$$\left\{ \prod_{I \subseteq \{1,\dots,r\}} z_i \in G \right\}$$

are the vertices of the **smallest face** F_g of $P(G)$ that contains e and g.

$\rightsquigarrow$ e, g form an edge if and only if g is 'indecomposable'.

Faces of permutation polytopes

Example 1: $G = \langle (1\,2), (3\,4), (5\,6) \rangle$, $g = (1\,2)(3\,4)(5\,6) \in G$.
Then $F_g = P(G)$:



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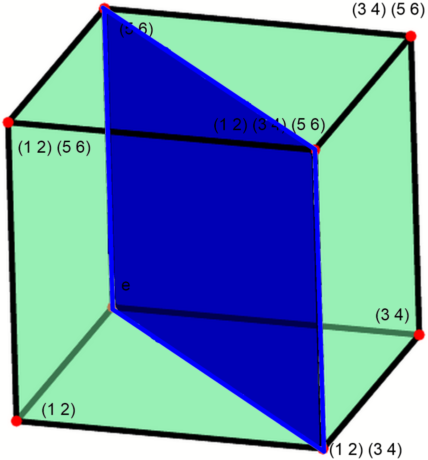

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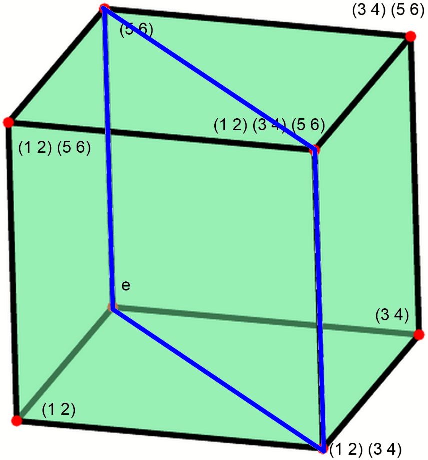

Faces of permutation polytopes

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- The smallest face containing two vertices is centrally-symmetric.
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  (For g as above, any element and its inverse is 'subelement' of g)
- $P(G)$ is a combinatorial product of two polytopes if and only if G is product of subgroups with disjoint support.
- $P(G)$ is combinatorially a crosspolytope (d-dimensional 'octahedron') if and only if d is a power of 2.

Faces of permutation polytopes

Recall: Any permutation polytope $P(G)$ of dimension d is affinely equivalent to a subpolytope of $[0, 1]^d$.

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Classification of perm. polytopes of dimension ≤ 4 (BHNP '09)

Combin. type of $P(G)$	Isom. type of G	Effective equiv. type of G
triangle	$\mathbb{Z}/3\mathbb{Z}$	$\langle(123)\rangle$
square	$(\mathbb{Z}/2\mathbb{Z})^2$	$\langle(12),(34)\rangle$
tetrahedron	$\mathbb{Z}/4\mathbb{Z}$	$\langle(1234)\rangle$
tetrahedron	$(\mathbb{Z}/2\mathbb{Z})^2$	$\langle(12)(34),(13)(24)\rangle$
triangular prism	$\mathbb{Z}/6\mathbb{Z}$	$\langle(12),(345)\rangle$
cube	$(\mathbb{Z}/2\mathbb{Z})^3$	$\langle(12),(34),(56)\rangle$
4-simplex	$\mathbb{Z}/5\mathbb{Z}$	$\langle(12345)\rangle$
B_3	S_3	$\langle(12),(123)\rangle$
prism over tetrahedron	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$\langle(1234),(56)\rangle$
prism over tetrahedron	$(\mathbb{Z}/2\mathbb{Z})^3$	$\langle(12)(34),(13)(24),(56)\rangle$
4-crosspolytope	$(\mathbb{Z}/2\mathbb{Z})^3$	$\langle(12)(34),(34)(78),(56)(78)\rangle$
product of triangles	$(\mathbb{Z}/3\mathbb{Z})^2$	$\langle(123),(456)\rangle$
prism over triang. prism	$\mathbb{Z}/6\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\langle(12),(345),(67)\rangle$
4-cube	$(\mathbb{Z}/2\mathbb{Z})^4$	$\langle(12),(34),(56),(78)\rangle$

Faces of permutation polytopes

What about classifying **faces** of permutation polytopes?

Theorem (BHNP '09)

For any d, there exists a face of a permutation polytope that is combinatorially equivalent to a crosspolytope.

Faces of permutation polytopes

Let $\mathcal{F}_d$ be the set of combinatorial types F of subpolytopes of $[0,1]^d$ such that the following condition holds:

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Theorem (BHNP '09)

For $d \leq 4$, any $F \in \mathcal{F}_d \setminus \{Q_1, Q_2\}$ is combinatorially a face of a permutation polytope.

CONJECTURE 3 (BHNP '09)

$Q_1, Q_2 \in \mathcal{F}_4$ are not combinatorially equivalent to a face of a permutation polytope.

Faces of permutation polytopes

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Bound is **sharp**: take t copies of D_4 as subgroup of S_{4t}, then combinatorial diameter is $2t = n/2$.

Dimension of representation polytopes

There is a natural generalization of a permutation polytope.

Representation polytope

Given a **real representation** $\rho : G \to \mathrm{GL}(V)$, where V is $\deg(\rho)$-dimensional real vector space.
Its **representation polytope** is defined as

$$P(G, \rho) := \mathrm{conv}(\rho(G)) \subseteq \mathrm{GL}_{\mathbb{R}}(V) \cong \mathbb{R}^{(\deg(\rho))^2}$$

Dimension of representation polytopes

Let $\mathrm{Irr}(G)$ be the set of pairwise non-isomorphic irreducible $\mathbb{C}$-representations. Any representation splits as a G-representation over $\mathbb{C}$ into irreducible components:

$$\rho \cong \sum_{\sigma \in \mathrm{Irr}(G)} c_\sigma \sigma \quad \text{for } c_\sigma \in \mathbb{Z}_{\geq 0}$$

Let $\mathrm{Irr}(\rho) = \{\sigma \in \mathrm{Irr}(G) \,:\, c_\sigma > 0\}$.

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Theorem (Guralnick, Perkinson '05)

$$\dim(P(G,\rho)) = \sum_{1_G \neq \sigma \in \mathrm{Irr}(\rho)} (\deg(\sigma))^2,$$

where 1_G is the trivial representation.

Proof uses standard representation theory.

Dimension of representation polytopes

Corollary (Guralnick, Perkinson '05)

Let ρ be permutation representation of G, and t the number of orbits of G. Then

$$\dim(P(G,\rho)) \leq (n-t)^2,$$

and equality iff at most one non-trivial irreducible component.

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Proof:

Recall: c_{1_G} equals the number of orbits t of G. Hence,

$$\sum_{1_G \neq \sigma \in \mathrm{Irr}(\rho)} \deg(\sigma) \leq \sum_{1_G \neq \sigma \in \mathrm{Irr}(\rho)} c_\sigma \deg(\sigma) = n - t.$$

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The sum

$$\sum_{1_G \neq \sigma \in \mathrm{Irr}(\rho)} (\deg(\sigma))^2 = \dim(P(G,\rho)),$$

is maximized for *one* non-trivial irreducible component. □

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and equality iff at most one non-trivial irreducible component.

Corollary (Guralnick, Perkinson '05)

Let $G \leq S_n$ transitive. Then

$$\dim(P(G)) \leq (n-1)^2,$$

and equality if and only if G is 2-transitive.

Equivalence of permutation polytopes

Corollary to dimension formula: Regular representation defines simplex.

Combin. type of $P(G)$	Isom. type of G
triangle	$\mathbb{Z}/3\mathbb{Z}$
square	$(\mathbb{Z}/2\mathbb{Z})^2$
tetrahedron	$\mathbb{Z}/4\mathbb{Z}$
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Equivalence of permutation polytopes

Observation: All these permutation groups define tetrahedron:

$\langle(1234)\rangle \leq S_4$

$\langle(1234)(5)\rangle \leq S_5$

$\langle(1234)(5678)\rangle \leq S_8$

$\langle(1234)(57)(68)\rangle \leq S_8$

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Definition/Proposition
(BHNP '09; Baumeister, Grüninger '15; Friese, Ladisch '16)

ρ_1, ρ_2 real representations of G. Then T.F.A.E.

- ρ_1, ρ_2 are **stably equivalent**
- $\mathrm{Irr}(\rho_1) \setminus \{1_G\} = \mathrm{Irr}(\rho_2) \setminus \{1_G\}$
- Exists $\alpha : P(G, \rho_1) \to P(G, \rho_2)$ affine equivalence s.t.

$$\alpha(\rho_1(g)x) = \rho_2(g)\alpha(x) \quad \text{for all } x \in P(G, \rho_1)$$

Equivalence of permutation polytopes

QUESTION 4 (BHNP '09)

Is there an implementable algorithm that solves the following problem?

Given finite group G and $S \subseteq \mathrm{Irr}(G) \setminus \{1_G\}$. Check if permutation representation with $\mathrm{Irr}(\rho) \setminus \{1_G\} = S$ exists, and if yes, find one.

This would allow to classify all permutation polytopes in small dimension d (as $|G| \leq 2^d$).

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Given permutation representation $\rho : G \to S_n$, there exists stably equivalent permutation representation $\rho' : G \to S_{n'}$ with $n' \leq 2 \dim(P(G, \rho))$.

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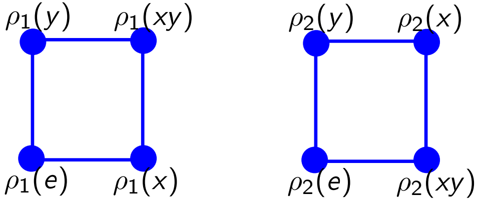
These are purely representation-theoretic challenges!

Equivalence of permutation polytopes

Stable equivalence not general enough!

Example: $G := (\mathbb{Z}_2)^2 = \{e, x, y, xy\}$ has *not stably equivalent* permutation representations ρ_1, ρ_2 with *the same* permutation polytope $P(G, \rho_1) = P(G, \rho_2)$:



Equivalence of permutation polytopes

Definition/Proposition (BHNP '09; Baumeister, Grüninger '15)

(G_1, ρ_1), (G_2, ρ_2) permutation representations. Then T.F.A.E.

- ρ_1, ρ_2 are **effectively equivalent**
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- Exists $\phi : G_1 \to G_2$ group isomorphism and $\alpha : P(G, \rho_1) \to P(G, \rho_2)$ affine equivalence s.t.

$$\alpha(\rho_1(g)x) = \rho_2(\phi(g))\alpha(x) \quad \text{for all } x \in P(G, \rho_1),\ g \in G_1$$

- Exists $\alpha : P(G_1, \rho_1) \to P(G_2, \rho_2)$ affine equivalence s.t. its restriction $\rho_1(G_1) \to \rho_2(G_2)$ is group homomorphism.

Equivalence of permutation polytopes

Example (Baumeister, Grüninger '15)

$G := (\mathbb{Z}_2)^2 \times \mathbb{Z}_4 \times \mathbb{Z}_3$ has permutation representations ρ_1, ρ_2 with

- affinely equivalent permutation polytopes, but
- **not** effectively equivalent.

Reason: The set of faces with 24 vertices that are also subgroups have different number of combinatorial types.

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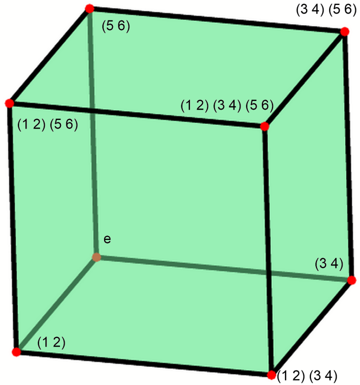
QUESTION 6 (Baumeister, Grüninger '15)

Does such an example exist if G acts **transitively**?

Equivalence of permutation polytopes

Extreme cases expected to be unique:



(BHNP '09): Unique effective equivalence class of $G \leq S_n$ if $P(G)$ is cube.

Equivalence of permutation polytopes

Let $\pi : S_n \to S_n$ standard permutation representation,
ρ permutation representation of G.

Conjecture (BHNP '09)

If $P(G, \rho)$ is affinely equivalent to $B_n = P(S_n, \pi)$,
then (G, ρ) and (S_n, π) are effectively equivalent.

Equivalence of permutation polytopes

Let $\pi : S_n \to S_n$ standard permutation representation, ρ permutation representation of G.

Theorem (Baumeister, Ladisch '16)

If $P(G, \rho)$ is affinely equivalent to $B_n = P(S_n, \pi)$, then (G, ρ) and (S_n, π) are effectively equivalent.

Proof uses symmetry group of B_n and the study of the Chermak-Delgado lattice of G.

Equivalence of permutation polytopes

(BHNP '09): conjectured that up to few exceptions ALWAYS

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Theorem (Friese, Ladisch '16)

Any elementary abelian 2-group of order $|G| \geq 2^5$ has permutation polytope $P(G, \rho)$ with $\mathrm{Aut}_{\mathsf{aff}}(P(G)) = |G|$.

Proof follows from new results on **orbit polytopes** of $G \subset \mathrm{GL}_n(\mathbb{R})$: the convex hull of the orbit Gv for $v \in \mathbb{R}^n$.

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CONJECTURE 7 (Friese, Ladisch '16)

Combinatorial and affine symmetry groups of representation polytopes are equal.