Small delay inertial manifolds under numerics: A numerical structural stability result

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Abstract

In this paper we formulate a numerical structural stability result for delay equations with small delay under Euler discretization. The main ingrediences of our approach are the existence and smoothness of small delay inertial manifolds, the C^1-closeness of the small delay inertial manifolds and their numerical approximation and M.-C. Li's recent result on numerical structural stability of ordinary differential equations under the Euler method.

1 Introduction

In recent years, there has been an increased interest in understanding the behavior of numerical discretizations of differential equations. One key problem is to determine how well the dynamics of the underlying equation is captured by the discretization, see e.g. [11], [14], [21], [22], [24].

It is well known that conjugacies play a fundamental role in the qualitative theory of ordinary differential equations. Indeed, when a conjugacy exists between two dynamical systems then the dynamical systems have the same orbit structure, they are qualitatively the same. We want to claim that under certain conditions the dynamics of the discretization considered as a discrete dynamical system and of the original system are the same. Thus it is natural to seek for conjugacies between the dynamical system and its numerical approximation. This yields us the concept of numerical structural stability.

Numerical structural stability results for ordinary differential equations can be found in [10], [17], [18]. Conjugacies can also be constructed near a fold bifurcation point, see [8].

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However, when one wants to prove similar results for infinite dimensional dynamical systems one has to face several difficulties. First of all, in general, structural stability fails even for the continuous problem. (A simple example showing that there is no Hartman-Grobman theorem for delay equations on a neighborhood of a hyperbolic equilibrium point can be found in [7].) On the other hand, in general (esp. for delay equations), there are no error bounds between the continuous dynamical system and its discretization on a *fixed* neighborhood of the phase space. This makes standard perturbation results unapplicable.

The usual method to overcome these difficulties is to reduce the original problem to a finite dimensional invariant manifold and apply existing results for ordinary differential equations. The aim of the present paper is to show that such a process works for delay equations with sufficiently small delay.

The outline of our proof is as follows. First we construct exponentially attractive n-dimensional invariant manifolds (small delay inertial manifolds) of class C^2. (The construction is based merely on the method of [5].) The C^2 norms of these manifolds tend to zero as the time delay goes to zero. (This allows us to obtain structural stability with respect to delay.) Secondly (via the same construction) we prove that the small delay inertial manifolds are well approximated under the Euler method. Finally we apply a recent result of [18], where a numerical structural stability result was proved for C^2 dynamical systems under the Euler discretization.

We note that for all sufficiently small *fixed* delay time the existence of small delay inertial manifolds would follow by applying the abstract result of [3]. The use of our own construction is twofold. First, we need (at least) C^2 smoothness in order to apply the main result of [18]. Secondly, we have to control the C^2 norms of the constructed manifolds as well. (It is also true, see Theorem 2 below, that the rate of the exponential attractivity can be choosen independently of the delay time.)

Recently in [2] a numerical structural stability result was proved for scalar parabolic partial differential equations under spatial discretization. Roughly speaking their method was to construct a family of inertial manifolds on which the errors tend to zero in the C^1 norm. To the contrary the dimension is fixed in our construction. It is not clear how one can prove numerical structural stability on a *fixed* inertial manifold since (at least) C^2 smoothness is crucial in deriving error estimates, but inertial manifold may loose smoothness, see [6].

Our result says that when the delay is small the delay equation is close to the "limiting ODE", i.e. when the delay time is zero. A similar result was proved in [20] under the condition that the function acting on the delayed argument is small and has small Lipschitz constant. In this paper we do not assume this smallness condition.

Small delay inertial manifolds were used in [19] to study the behavior of the attractor of the sunflower equation with small delay. Similarly, in [1] inertial manifolds were constructed for retarded semilinear parabolic partial differential equations. In order to prove the existence of the inertial manifold the delay time must satisfy some sort of smallness condition, see Thm 3.1 in [1]. The results can be applied to ordinary functional differential equations as well. They also studied the continuity properties of the inertial manifolds with respect to the delay time.

Although smoothness is very important in applications neither [19] nor [1] contain smoothness results.

We admit that our result works only for delay equations close to ODE's. It is known that even scalar delay equations may possess very reach dynamical behavior (when the delay time is sufficiently large). In recent works, see [15], [16], the complete characterization of the attractor of a scalar delay equation was presented. (Unfortunately, we cannot obtain such a complete description of the attractor of the discretization, the results in [9] shows lower semicontinuous convergence of the approximating attractors to the true one. Upper semicontinuity of the Morse decomposition of the attractor of a scalar delay equation under a "spatial-like" discretization, i.e. the delay equation was approximated by a family of ODE's with increasing dimension, was proved in [12]. The results of [9] work for the Euler method, which is a "full" discretization, i.e. the delay equation is approximated by a family of finite dimensional mappings with increasing dimension.) Although our result says something for systems in arbitrary dimension (provided the delay is small enough) for higher dimensional systems very little is known, see e.g. [4].

The rest of the paper is as follows. After a preliminary section, in Section 3 we prove the existence and smoothness of small delay inertial manifolds. In Section 4 we study the existence of asymptotic phases. We show that the small delay inertial manifolds are well approximated under the Euler method in Section 5. Finally, we show the desired numerical structural stability in Section 6.

2 Preliminaries

Consider the following delay differential equation

$$\dot{x}(t) = Ax(t) + f(x(t)) + g(x(t-\varepsilon)), \tag{1}$$

where $\varepsilon > 0$, $A \in \mathbf{R}^{n\times n}$ and $f, g \in C^2(\mathbf{R}^n, \mathbf{R}^n)$ are bounded functions with bounded derivatives. Our standard reference is [13].

Set $C_\varepsilon := C([-\varepsilon, 0], \mathbf{R}^n)$ endowed with the usual sup norm $\|\cdot\|$. Denote the C_0-semigroup on C_ε generated by the linear ordinary differential equation $\dot{x}(t) = Ax(t)$ by $\{T_\varepsilon(t)\}_{t\geq 0}$. Decompose C_ε by $\sigma(A)$ (the spectrum of A) as $C_\varepsilon = P_\varepsilon \oplus Q_\varepsilon$, where $P_\varepsilon = \pi C_\varepsilon$, $Q_\varepsilon = (id - \pi)C_\varepsilon$, and π is defined as $(\pi\phi)(\theta) := e^{A\theta}\phi(0)$, $\phi \in C_\varepsilon$, $\theta \in [-\varepsilon, 0]$. Observe that there exists a positive constant M independent of ε such that $\|\pi\| \leq M$ and $\|(id - \pi)\| \leq M$. Thus we have

(H0) for all $\varepsilon > 0$ there is a T_ε-invariant splitting $C_\varepsilon = P_\varepsilon \oplus Q_\varepsilon$ and a positive constant M independent of ε such that the norms of the associated projections $\pi : C_\varepsilon \to P_\varepsilon$, $id - \pi : C_\varepsilon \to Q_\varepsilon$ are bounded by M. Moreover, subspaces P_ε have ε-independent finite dimension.

Note that $[(id - \pi)\phi](0) = 0$ for all $\phi \in C_\varepsilon$ and thus $T_\varepsilon(t)(id - \pi)\phi = 0$ for all $t \geq \varepsilon$. We get that for all $\beta > 0$ and for all $\phi \in C_\varepsilon$

$$\|T_\varepsilon(t)(id - \pi)\phi\| \leq Me^{\beta\varepsilon}e^{-\beta t}\|\phi\|, \quad t \geq 0.$$

Set $\varepsilon_\beta = \frac{1}{\beta}$. Thus we have

(H1) for all $\beta > 0$ there exists $\varepsilon_\beta > 0$ such that for all $0 < \varepsilon \le \varepsilon_\beta$

$$\|T_\varepsilon(t)(id - \pi)\phi\| \le 3Me^{-\beta t}\|\phi\|, \quad t \ge 0.$$

Set $F_\varepsilon(\phi) := f(\phi(0)) + g(\phi(-\varepsilon))$, $\phi \in C_\varepsilon$. Then

(H2) $F_\varepsilon \in C^2(C_\varepsilon, \mathbf{R}^n)$ is bounded with bounded derivatives. Moreover, the bounds are independent of ε, i.e. $\|F_\varepsilon\|_{C^2} \le K$ for all $\varepsilon > 0$.

Finally,

(H3) there exists an $\omega > 0$ independent of ε such that for all $\varepsilon > 0$

$$\|T_\varepsilon(t)\pi\phi\| \le Me^{\omega|t|}\|\phi\|, \quad t \in \mathbf{R}.$$

These properties will be frequently used in later sections. Finally, the Banach space of bounded linear, resp. bilinear operators between Banach spaces E_1, E_2 (endowed with the induced operator norm) will be denoted by $L(E_1, E_2)$, resp. $L^2(E_1, E_2)$.

3 Existence and smoothness of small delay inertial manifolds

The main result of this section is the following theorem.

Theorem 1 *For all $\varepsilon > 0$ small enough there exists a $\Phi_\varepsilon \in C^2(P_\varepsilon, Q_\varepsilon)$ such that* $\mathrm{graph}(\Phi_\varepsilon) = \{\phi + \Phi_\varepsilon(\phi) \;:\; \phi \in P_\varepsilon\}$ *is an exponentially attractive invariant manifold for (1). Moreover, $\|\Phi_\varepsilon\|_{C^2(B,Q_\varepsilon)} \to 0$ as $\varepsilon \to 0+$ for all closed bounded sets $B \subset P_\varepsilon$.*

Before we turn to the proof of Theorem 1, which is the content of the following subsections, we make some remarks.

Remark 1. Our proof works for equations of the form

$$\dot{x}(t) = L_\varepsilon x_t + F_\varepsilon(x_t)$$

possessing properties (H0)-(H3), where T_ε is the C_0 semigroup generated by $\dot{x}(t) = L_\varepsilon x_t$.

Remark 2. If we assume instead of (H2) that $F_\varepsilon \in C^k$ with ε-independent C^k norm then the manifold will be C^k as well. In this case a larger spectral gap is required and an induction step is inserted.

Remark 3. For the largest value of ε^* for which the theorem is still true can be estimated as $\varepsilon^* \le (72NMK)^{-1}$, see the proof below.

3.1 Existence

Choose a natural number $N \in \mathbf{N}$ such that $\omega < NMK$ holds. Set $\beta > 72NMK$, $\alpha := 8NMK$, $\delta_0 := NMK$, $\omega_1 := \alpha - \omega > 0$, $\omega_2 := \beta - \alpha > 0$ and $\varepsilon \in [0, \varepsilon_\beta]$.

The fundamental matrix solution X of

$$\dot{x}(t) = Ax(t) \tag{2}$$

on C_ε is defined to be the (unique) matrix solution of (2) with initial value X_0 at zero, where X_0 is the $n \times n$ matrix function on $[-\varepsilon, 0]$ defined by $X_0(\theta) = 0$ for $-\varepsilon \le \theta < 0$ and $X_0(0) = I$. Let $X_0^{P_\varepsilon} = e^{A\cdot}$ and $X_0^{Q_\varepsilon} = X_0 - X_0^{P_\varepsilon}$. Then we have the following exponential estimates

$$\|e^{\alpha t} T_\varepsilon(t)\pi\phi\| \le Me^{\omega_1 t}\|\phi\| \quad t \le 0,$$

$$\|e^{\alpha t} T_\varepsilon(t)(id - \pi)\phi\| \le 3Me^{-\omega_2 t}\|\phi\| \ \ t \ge 0,$$

$$\|e^{\alpha t} T_\varepsilon(t)X_0^{P_\varepsilon}\| \le Me^{\omega_1 t} \quad t \le 0,$$

and

$$\|e^{\alpha t} T_\varepsilon(t)X_0^{Q_\varepsilon}\| \le 3Me^{-\omega_2 t} \quad t \ge 0.$$

Define the Banach space

$$S_\eta := \{Y \ : \ Y : \mathbf{R}^- \to C_\varepsilon \text{ is continuous and } \sup_{t\in\mathbf{R}^-} e^{\eta t}\|Y(t)\| < \infty\}$$

with norm

$$|Y|_\eta := \sup_{t\in\mathbf{R}^-} e^{\eta t}\|Y(t)\|.$$

Fix an arbitrary $\phi \in P_\varepsilon$ and for $Y \in S_\eta$ we define

$$\mathcal{T}_\varepsilon(Y)(t) := T_\varepsilon(t)\phi + \int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}F_\varepsilon(Y(s))ds + \int_{-\infty}^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}F_\varepsilon(Y(s))ds, \quad t \le 0.$$

Lemma 1 *For all $\phi \in P_\varepsilon$ and $\delta \in [0, \delta_0]$ operator $\mathcal{T}_\varepsilon$ maps $S_{\alpha-\delta}$ into itself and is a uniform 1/3-contraction.*

Proof of Lemma 1. Let $\phi \in P_\varepsilon$ and $Y \in S_{\alpha-\delta}$ be given. Then

$$\|e^{(\alpha-\delta)t}\mathcal{T}_\varepsilon(Y)(t)\| \le Me^{(\omega_1-\delta)t}\|\phi\| + MK\int_0^t e^{(\omega_1-\delta)(t-s)}e^{(\alpha-\delta)s}ds$$

$$+3MK\int_{-\infty}^t e^{-(\omega_2+\delta)(t-s)}e^{(\alpha-\delta)s}ds$$

which shows that $|\mathcal{T}_\varepsilon(Y)|_{\alpha-\delta} < \infty$.

Let $Y_1, Y_2 \in S_{\alpha-\delta}$ be given. Then

$$\|e^{(\alpha-\delta)t}(\mathcal{T}_\varepsilon(Y_1) - \mathcal{T}_\varepsilon(Y_2))(t)\| \le MK \int_0^t e^{(\omega_1-\delta)(t-s)} e^{(\alpha-\delta)s} \|Y_1(s) - Y_2(s)\| ds$$

$$+3MK \int_{-\infty}^t e^{-(\omega_2+\delta)(t-s)} e^{(\alpha-\delta)s} \|Y_1(s) - Y_2(s)\| ds$$

$$\le MK(\frac{1}{\omega_1 - \delta} + \frac{3}{\omega_2 + \delta})|Y_1 - Y_2|_{\alpha-\delta}$$

which implies that

$$|\mathcal{T}_\varepsilon(Y_1) - \mathcal{T}_\varepsilon(Y_2)|_{\alpha-\delta} \le MK(\frac{1}{\omega_1 - \delta} + \frac{3}{\omega_2 + \delta})|Y_1 - Y_2|_{\alpha-\delta}.$$

Note that $\omega_1 - \delta \ge 6NMK$ and $\omega_2 + \delta \ge 18NMK$ whenever $\delta \in [0, \delta_0]$, and the desired contraction property follows. QED

Denote the fixed point of $\mathcal{T}_\varepsilon$ by $Y_\varepsilon^\delta(\phi)$. Since $S_{\alpha-\delta} \subset S_\alpha$ we have by uniqueness that the $Y_\varepsilon^\delta(\phi) = Y_\varepsilon^0(\phi)$. Set $Y_\varepsilon(\phi) = Y_\varepsilon^0(\phi)$. Now we can define a mapping $\Phi_\varepsilon : P_\varepsilon \to Q_\varepsilon$ by $\Phi_\varepsilon(\phi) := [(id - \pi)Y_\varepsilon(\phi)](0)$. This mapping defines our small delay inertial manifold. In what follows we prove that $\Phi_\varepsilon \in C^2(P_\varepsilon, Q_\varepsilon)$. In order to do so we prove that $Y_\varepsilon \in C^2(P_\varepsilon, S_{2\alpha})$.

Choose an arbitrary bounded closed ball $B \subset P_\varepsilon$. Redefine operator $\mathcal{T}_\varepsilon$ on the space $C(B, S_{\alpha-\delta})$ by setting for $Y \in C(B, S_{\alpha-\delta})$, $\phi \in B$

$$(\mathcal{T}_\varepsilon(Y))(\phi, t) := T_\varepsilon(t)\phi + \int_0^t T_\varepsilon(t-s) X_0^{P_\varepsilon} F_\varepsilon(Y(\phi, t)) ds + \int_{-\infty}^t T_\varepsilon(t-s) X_0^{Q_\varepsilon} F_\varepsilon(Y(\phi, t)) ds, \quad t \le 0.$$

The proof of the following lemma goes along the same line as the proof of Lemma 1 and thus it is omitted. (The extra continuity of $\mathcal{T}_\varepsilon(Y)$ can be seen from the definition.)

Lemma 2 *For all $\delta \in [0, \delta_0]$ operator $\mathcal{T}_\varepsilon$ maps $C(B, S_{\alpha-\delta})$ into itself and is a uniform $1/3$ contraction.*

Moreover, the fixed points are independent of δ, and equal to $Y_\varepsilon|_B$. Thus we obtain that our manifold is continuous. Moreover,

$$\sup_{\phi \in B} \|\Phi_\varepsilon(\phi)\| = \sup_{\phi \in B} \| \int_{-\infty}^0 T_\varepsilon(-s) X_0^{Q_\varepsilon} F_\varepsilon(Y_\varepsilon(\phi, s)) ds\| \le 3MK \int_{-\infty}^0 e^{\beta s} ds = 3MK\varepsilon_\beta \to 0.$$

3.2 Smoothness

Choose an arbitrary sequence $\delta_0 > \delta_1 > \delta_2 > \cdots > 0$. With the help of suitably chosen sequences we prove that the C^1 property is preserved under a certain loss of exponential weights when we apply $\mathcal{T}_\varepsilon$. Similar result holds for the second derivative, see Lemma 4 below. When we apply these results in the proof of Lemma 7 we have to be able to control the exponential weights. This motivates the introduction of some constant $0 < \Delta < \delta_0$ after the proof of Lemma 4.

Lemma 3 *If $Y \in C^1(P_\varepsilon, S_{\alpha-\delta_i})$ then $\mathcal{T}_\varepsilon(Y) \in C^1(P_\varepsilon, S_{\alpha-\delta_{i+1}})$, for $i = 0, 1, \ldots$.*

Proof of Lemma 3. Let $Y \in C^1(P_\varepsilon, S_{\alpha-\delta_i})$ be fixed. Then $DY \in C(P_\varepsilon, L(P_\varepsilon, S_{\alpha-\delta_i}))$, where DY denotes the Fréchet derivative of Y. Differentiate formally $\mathcal{T}_\varepsilon(Y)$ to obtain

$$(D\mathcal{T}_\varepsilon(DY) \cdot [\psi])(t, \phi) := T_\varepsilon(t)\psi + \int_0^t T_\varepsilon(t-s) X_0^{P_\varepsilon} DF_\varepsilon(Y(\phi, s)) \cdot [DY(\phi) \cdot [\psi]](s) ds$$

$$+ \int_{-\infty}^t T_\varepsilon(t-s) X_0^{Q_\varepsilon} DF_\varepsilon(Y(\phi, s)) \cdot [DY(\phi) \cdot [\psi]](s) ds.$$

We claim that $D(\mathcal{T}_\varepsilon(Y)) = D\mathcal{T}_\varepsilon(DY)$.

Let $\phi_1, \phi_2 \in P_\varepsilon$ be given. Define

$$I := \|e^{(\alpha-\delta_{i+1})t}(\mathcal{T}_\varepsilon(Y)(\phi_1, t) - \mathcal{T}_\varepsilon(Y)(\phi_2, t) - D\mathcal{T}_\varepsilon(DY) \cdot [\phi_1 - \phi_2](\phi_2, t))\|.$$

Since $D\mathcal{T}_\varepsilon(DY)$ is linear and continuous in $\psi \in P_\varepsilon$ it suffices to show that

$$\sup_{t \in \mathbf{R}^-} I = o(\|\phi_1 - \phi_2\|)$$

as $\|\phi_1 - \phi_2\| \to 0$. To this end let $\eta > 0$ be given and write $I \leq I_1 + I_2$ where

$$I_1 = \|e^{(\alpha-\delta_{i+1})t}(\int_0^t T_\varepsilon(t-s) X_0^{P_\varepsilon}(F_\varepsilon(Y(\phi_1, s)) - F_\varepsilon(Y(\phi_2, s))$$

$$- DF_\varepsilon(Y(\phi_2, s)) \cdot [DY(\phi_2) \cdot [\phi_1 - \phi_2]](s)) ds)\|$$

and

$$I_2 = \|e^{(\alpha-\delta_{i+1})t}(\int_{-\infty}^t T_\varepsilon(t-s) X_0^{Q_\varepsilon}(F_\varepsilon(Y(\phi_1, s)) - F_\varepsilon(Y(\phi_2, s))$$

$$- DF_\varepsilon(Y(\phi_2, s)) \cdot [DY(\phi_2) \cdot [\phi_1 - \phi_2]](s)) ds)\|.$$

We prove that $I_1 = o(\|\phi_1 - \phi_2\|)$ as $\|\phi_1 - \phi_2\| \to 0$. Choose $T < 0$ so that

$$\frac{2MK\|Y\|_{C^1}}{\omega_1 - \delta_{i+1}} e^{(\delta_i - \delta_{i+1})T} < \eta/2.$$

There are two cases.

Case $t \geq T$.

Write

$$I_1 = \|e^{(\alpha-\delta_{i+1})t} \int_0^t T_\varepsilon(t-s) X_0^{P_\varepsilon}(\int_0^1 DF_\varepsilon(uY(\phi_1, s) + (1-u)Y(\phi_2, s))$$

$$\cdot [Y(\phi_1, s), Y(\phi_2, s)] du - DF_\varepsilon(Y(\phi_2, s)) \cdot [DY(\phi_2) \cdot [\phi_1 - \phi_2](s)) ds\|$$

$$\leq \|e^{(\alpha-\delta_{i+1})t}\int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(\int_0^1 (DF_\varepsilon(uY(\phi_1,s)+(1-u)Y(\phi_2,s))$$
$$-DF_\varepsilon(Y(\phi_2,s)))\cdot[DY(\phi_2)\cdot[\phi_1-\phi_2]](s)du)ds\|$$
$$+\|e^{(\alpha-\delta_{i+1})t}\int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(\int_0^1 DF_\varepsilon(uY(\phi_1,s)+(1-u)Y(\phi_2,s))$$
$$\cdot[Y(\phi_1,s)-Y(\phi_2,s)-DY(\phi_2)\cdot[\phi_1-\phi_2](s)]du)ds\|.$$

Now we choose $\kappa>0$ such that if $\|\phi_1-\phi_2\|<\kappa$ then

$$\|DF_\varepsilon(uY(\phi_1,s)+(1-u)Y(\phi_2,s))-DF_\varepsilon(Y(\phi_2,s))\|\leq\frac{\eta(\omega_1-\delta_{i+1})}{2M\|Y\|_{C^1}}$$

for all $u\in[0,1]$ and $s\in[T,0]$, and

$$\sup_{s\in\mathbf{R}^-}\{e^{(\alpha-\delta_i)s}\|Y(\phi_1,s)-Y(\phi_2,s)-DY(\phi_2)\cdot[\phi_1-\phi_2](s)\|\}\leq\frac{\eta(\omega_1-\delta_{i+1})}{2MK}\|\phi_1-\phi_2\|$$

hold. It is easy to see that in this case

$$I_1\leq\eta\|\phi_1-\phi_2\|.$$

Case $t<T$.

Write $I_1=I_1^1+I_1^2$ where

$$I_1^1=\|e^{(\alpha-\delta_2)t}\int_T^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(F_\varepsilon(Y(\phi_1,s))-F_\varepsilon(Y(\phi_2,s))$$
$$-DF_\varepsilon(Y(\phi_2,s))\cdot[DY(\phi_2)\cdot[\phi_1-\phi_2]](s))ds)\|$$

and

$$I_1^2=\|e^{(\alpha-\delta_2)t}\int_0^T T_\varepsilon(t-s)X_0^{P_\varepsilon}(F_\varepsilon(Y(\phi_1,s))-F_\varepsilon(Y(\phi_2,s))$$
$$-DF_\varepsilon(Y(\phi_2,s))\cdot[DY(\phi_2)\cdot[\phi_1-\phi_2]](s))ds)\|.$$

We have

$$I_1^1\leq\frac{2MK\|Y\|_{C^1}}{\omega_1-\delta_2}e^{(\delta_1-\delta_2)T}\|\phi_1-\phi_2\|<\eta/2\|\phi_1-\phi_2\|.$$

A similar argument as in Case $t\geq T$ shows that

$$I_1^2\leq\eta/2\|\phi_1-\phi_2\|$$

and thus in both cases $I_1\leq\eta\|\phi_1-\phi_2\|$ whenever $\|\phi_1-\phi_2\|$ is small enough.

The proof of $I_2=o(\|\phi_1-\phi_2\|)$ is similar and is omitted. QED

Lemma 4 *If* $Y \in C^1(P_\varepsilon, S_{\alpha-\delta_i}) \cap C^2(P_\varepsilon, S_{2\alpha-\delta_i})$ *then* $\mathcal{T}_\varepsilon(Y) \in C^1(P_\varepsilon, S_{\alpha-\delta_{i+1}}) \cap C^2(P_\varepsilon, S_{2\alpha-\delta_{i+1}})$ *for* $i = 0, 1, \ldots$.

Proof of Lemma 4. Let $Y \in C^1(P_\varepsilon, S_{\alpha-\delta_i}) \cap C^2(P_\varepsilon, S_{2\alpha-\delta_i})$ be fixed. Then $DY \in C(P_\varepsilon, L(P_\varepsilon, S_{\alpha-\delta_i}))$ and $D^2Y \in C(P_\varepsilon, L^2(P_\varepsilon, S_{2\alpha-\delta_i}))$, where D^2Y is the Fréchet derivative of DY. The previous lemma shows that $\mathcal{T}_\varepsilon(Y) \in C^1(P_\varepsilon, S_{\alpha-\delta_{i+1}})$ and the derivative $D(\mathcal{T}_\varepsilon(Y)) = D\mathcal{T}_\varepsilon(DY)$ where

$$(D\mathcal{T}_\varepsilon(DY) \cdot [\psi])(t, \phi) = T_\varepsilon(t)\psi + \int_0^t T_\varepsilon(t-s) X_0^{P_\varepsilon} DF_\varepsilon(Y(\phi, s)) \cdot [DY(\phi) \cdot [\psi]](s) ds$$

$$+ \int_{-\infty}^t T_\varepsilon(t-s) X_0^{Q_\varepsilon} DF_\varepsilon(Y(\phi, s)) \cdot [DY(\phi) \cdot [\psi]](s) ds.$$

Differentiate $D\mathcal{T}_\varepsilon(DY)$ to obtain

$$(D^2\mathcal{T}_\varepsilon(D^2Y) \cdot [\psi_1, \psi_2])(t, \phi)$$

$$:= \int_0^t T_\varepsilon(t-s) X_0^{P_\varepsilon} (D^2F_\varepsilon(Y(\phi, s)) \cdot [DY(\phi) \cdot [\psi_1](s), DY(\phi) \cdot [\psi_2](s)]$$

$$+ DF_\varepsilon(Y(\phi, s)) \cdot [D^2Y(\phi) \cdot [\psi_1, \psi_2](s)]) ds$$

$$+ \int_{-\infty}^t T_\varepsilon(t-s) X_0^{Q_\varepsilon} (D^2F_\varepsilon(Y(\phi, s)) \cdot [DY(\phi) \cdot [\psi_1](s), DY(\phi) \cdot [\psi_2](s)]$$

$$+ DF_\varepsilon(Y(\phi, s)) \cdot [D^2Y(\phi) \cdot [\psi_1, \psi_2](s)]) ds.$$

We claim that $D^2(\mathcal{T}_\varepsilon(Y)) = D^2\mathcal{T}_\varepsilon(D^2Y)$. Let $\phi_1, \phi_2 \in P_\varepsilon$ be given. Define

$$I := \| e^{(2\alpha-\delta_{i+1})t} ((D\mathcal{T}_\varepsilon(DY) \cdot [\psi])(\phi_1, t) - (D\mathcal{T}_\varepsilon(DY) \cdot [\psi])(\phi_2, t)$$

$$- (D^2\mathcal{T}_\varepsilon(D^2Y) \cdot [\phi_1 - \phi_2, \psi])(\phi_2, t)) \|.$$

Since $D^2\mathcal{T}_\varepsilon(D^2Y)$ is bilinear and continuous in $\psi_1, \psi_2 \in P_\varepsilon$ it suffices to show that

$$\sup_{\psi \in P_\varepsilon, \|\psi\| \le 1} \sup_{t \in \mathbf{R}^-} I = o(\|\phi_1 - \phi_2\|)$$

as $\|\phi_1 - \phi_2\| \to 0$. To this end let $\eta > 0$ be given and write $I = I_1 + I_2$ where

$$I_1 = \| e^{(2\alpha-\delta_{i+1})t} (\int_0^t T_\varepsilon(t-s) X_0^{P_\varepsilon} (DF_\varepsilon(Y(\phi_1, s)) \cdot [DY(\phi_1) \cdot [\psi](s)]$$

$$- DF_\varepsilon(Y(\phi_2, s)) \cdot [DY(\phi_2) \cdot [\psi](s)]$$

$$- D^2F_\varepsilon(Y(\phi_2, s)) \cdot [DY(\phi_2) \cdot [\phi_1 - \phi_2](s), DY(\phi_2) \cdot [\psi](s)]$$

$$-DF_\varepsilon(Y(\phi_2,s))\cdot[D^2Y(\phi_2)\cdot[\phi_1-\phi_2,\psi](s)]))ds\|$$

and

$$I_2=\|e^{(2\alpha-\delta_{i+1})t}(\int_{-\infty}^{t}T_\varepsilon(t-s)X_0^{Q_\varepsilon}(DF_\varepsilon(Y(\phi_1,s))\cdot[DY(\phi_1)\cdot[\psi](s)]$$
$$-DF_\varepsilon(Y(\phi_2,s))\cdot[DY(\phi_2)\cdot[\psi](s)]$$
$$-D^2F_\varepsilon(Y(\phi_2,s))\cdot[DY(\phi_2)\cdot[\phi_1-\phi_2](s),DY(\phi_2)\cdot[\psi](s)]$$
$$-DF_\varepsilon(Y(\phi_2,s))\cdot[D^2Y(\phi_2)\cdot[\phi_1-\phi_2,\psi](s)]))ds\|.$$

We prove that $I_1=o(\|\phi_1-\phi_2\|)$ as $\|\phi_1-\phi_2\|\to 0$. Choose $T<0$ so that

$$\frac{2MK(\|Y\|_{C^1}+\|Y\|_{C^2})}{\omega_1+\alpha-\delta_i}e^{(\delta_i-\delta_{i+1})T}<\eta/2.$$

There are two cases.

Case $t\geq T$.

Write

$$I_1\leq\|e^{(2\alpha-\delta_{i+1})t}\int_0^t T_\varepsilon X_0^{P_\varepsilon}DF_\varepsilon(Y(\phi_2,s))$$
$$\cdot[DY(\phi_1)\cdot[\psi](s)-DY(\phi_2)\cdot[\psi](s)-D^2Y(\phi_2)\cdot[\phi_1-\phi_2,\psi](s)]ds\|$$
$$+\|e^{(2\alpha-\delta_{i+1})t}\int_0^t T_\varepsilon X_0^{P_\varepsilon}(\int_0^1 D^2F_\varepsilon(uY(\phi_1,s)+(1-u)Y(\phi_2,s))$$
$$\cdot[DY(\phi_1)\cdot[\psi](s)-DY(\phi_2)\cdot[\psi](s),Y(\phi_1,s)-Y(\phi_2,s)-DY(\phi_2)\cdot[\phi_1-\phi_2](s)]du)ds\|$$
$$+\|e^{(2\alpha-\delta_{i+1})t}\int_0^t T_\varepsilon X_0^{P_\varepsilon}(\int_0^1 D^2F_\varepsilon(uY(\phi_1,s)+(1-u)Y(\phi_2,s))-D^2F_\varepsilon(Y(\phi_2,s))$$
$$\cdot[DY(\phi_2)\cdot[\psi](s),DY(\phi_2)\cdot[\phi_1-\phi_2](s)]du)ds.$$

Now choose $\kappa>0$ such that if $\|\phi_1-\phi_2\|<\kappa$ then

$$\sup_{s\in\mathbf{R}^-}\|e^{(\alpha-\delta_i)s}(DY(\phi_1)-DY(\phi_2))\cdot[\psi](s)\|\leq\|\psi\|,$$

$$\|D^2F_\varepsilon(uY(\phi_1,s)+(1-u)Y(\phi_2,s))-D^2F_\varepsilon(Y(\phi_2,s))\|<\frac{\eta(\omega_1+\alpha-\delta_i)}{3M\|Y\|_{C^1}^2}$$

for all $u\in[0,1]$ and $s\in[T,0]$,

$$\sup_{s\in\mathbf{R}^-}\|e^{(\alpha-\delta_i)s}(Y(\phi_1,s)-Y(\phi_2,s)-DY(\phi_2)\cdot[\phi_1-\phi_2](s))\|$$

$$<\frac{\eta(\omega_1+\alpha-\delta_i)}{3MK}\|\phi_1-\phi_2\|$$

and

$$\sup_{s\in\mathbf{R}^-} \|e^{(2\alpha-\delta_i)s}(DY(\phi_1)\cdot[\psi](s) - DY(\phi_2)\cdot[\psi](s) - D^2Y(\phi_2)\cdot[\phi_1-\phi_2,\psi](s))\|$$

$$< \frac{\eta(\omega_1+\alpha-\delta_i)}{3MK}\|\psi\|\cdot\|\phi_1-\phi_2\|$$

hold. Then we have that

$$I_1 \leq \eta\|\psi\|\cdot\|\phi_1-\phi_2\|.$$

Case $t < T$.

Write $I_1 = I_1^1 + I_1^2$ where

$$\begin{aligned} I_1^1 = \|e^{(2\alpha-\delta_{i+1})t}(\int_T^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(DF_\varepsilon(Y(\phi_1,s))\cdot[DY(\phi_1)\cdot[\psi](s)] \\ -DF_\varepsilon(Y(\phi_2,s))\cdot[DY(\phi_2)\cdot[\psi](s)] \\ -D^2F_\varepsilon(Y(\phi_2,s))\cdot[DY(\phi_2)\cdot[\phi_1-\phi_2](s), DY(\phi_2)\cdot[\psi](s)] \\ -DF_\varepsilon(Y(\phi_2,s))\cdot[D^2Y(\phi_2)\cdot[\phi_1-\phi_2,\psi](s)]))ds\| \end{aligned}$$

and

$$\begin{aligned} I_1^2 = \|e^{(2\alpha-\delta_{i+1})t}(\int_0^T T_\varepsilon(t-s)X_0^{P_\varepsilon}(DF_\varepsilon(Y(\phi_1,s))\cdot[DY(\phi_1)\cdot[\psi](s)] \\ -DF_\varepsilon(Y(\phi_2,s))\cdot[DY(\phi_2)\cdot[\psi](s)] \\ -D^2F_\varepsilon(Y(\phi_2,s))\cdot[DY(\phi_2)\cdot[\phi_1-\phi_2](s), DY(\phi_2)\cdot[\psi](s)] \\ -DF_\varepsilon(Y(\phi_2,s))\cdot[D^2Y(\phi_2)\cdot[\phi_1-\phi_2,\psi](s)]))ds\|. \end{aligned}$$

We have

$$I_1^1 \leq \frac{2MK(\|Y\|_{C^1}+\|Y\|_{C^2})}{\omega_1+\alpha-\delta_i}e^{(\delta_i-\delta_{i+1})T}\|\psi\|\cdot\|\phi_1-\phi_2\| < \eta/2\|\psi\|\cdot\|\phi_1-\phi_2\|.$$

A similar argument as in Case $t \geq T$ shows that

$$I_1^2 \leq \eta/2\|\psi\|\cdot\|\phi_1-\phi_2\|$$

and thus $\sup_{t\in\mathbf{R}^-} I_1 \leq \eta\|\psi\|\cdot\|\phi_1-\phi_2\|$ whenever $\|\phi_1-\phi_2\|$ is small enough. The proof for I_2 is similar and is omitted. QED

Let $B \subset P_\varepsilon$ be a closed bounded ball and fix $0 < \Delta < \delta_0$. For $Y \in C(B, S_{\alpha-\Delta})$ define the operator $D\mathcal{T}_{\varepsilon,Y}$ on $C(B, L(P_\varepsilon, S_{\alpha-\delta}))$, $\delta \in [0,\Delta]$ by setting

$$\begin{aligned} (D\mathcal{T}_{\varepsilon,Y}(\mathcal{Y})\cdot[\psi])(t,\phi) := T_\varepsilon(t)\psi + \int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}DF_\varepsilon(Y(\phi,s))\cdot[\mathcal{Y}(\phi)\cdot[\psi](s)]ds \\ + \int_{-\infty}^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}DF_\varepsilon(Y(\phi,s))\cdot[\mathcal{Y}(\phi)\cdot[\psi](s)]ds. \end{aligned}$$

It is easy to see (e.g. Lemma 2) that $D\mathcal{T}_{\varepsilon,Y}$ maps $C(B, L(P_\varepsilon, S_{\alpha-\delta}))$ into $C(B, L(P_\varepsilon, S_{\alpha-\delta}))$ for all $\delta \in [0,\Delta]$ and is a uniform 1/3-contraction. Denote the fixed points by $\mathcal{Y}_Y^\delta$.

Lemma 5 *The fixed points* $\mathcal{Y}_Y^\delta$ *are independent of* δ *and their norms are uniformly bounded, i.e. there exists a constant* Ω_1 *independent of* Y, δ *and* ε *such that*

$$\|\mathcal{Y}_Y^\delta\|_{C(B,L(P_\varepsilon,S_{\alpha-\delta}))} \le \Omega_1.$$

Proof of Lemma 5. Since $C(B,L(P_\varepsilon,S_{\alpha-\delta})) \subset C(B,L(P_\varepsilon,S_\alpha))$ by uniqueness we have that the fixed points are independent of δ.

With norm $\|\cdot\| = \|\cdot\|_{C(B,L(P_\varepsilon,S_{\alpha-\delta}))}$ we have that

$$\|\mathcal{Y}_Y^\delta\| = \|D\mathcal{T}_{\varepsilon,Y}(\mathcal{Y}_Y^\delta)\| \le \sup_{\phi\in B}\ \sup_{\psi\in P_\varepsilon,\|\psi\|\le 1}\ \sup_{t\in\mathbf{R}^-} I,$$

where

$$\begin{aligned} I = {} & \|e^{(\alpha-\delta)t}T_\varepsilon(t)\psi\| \\ & +\|e^{(\alpha-\delta)t}\int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}DF_\varepsilon(Y(\phi,s))\cdot[\mathcal{Y}_Y^\delta(\phi)\cdot[\psi](s)]ds\| \\ & +\|e^{(\alpha-\delta)t}\int_{-\infty}^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}DF_\varepsilon(Y(\phi,s))\cdot[\mathcal{Y}_Y^\delta(\phi)\cdot[\psi](s)]ds\|. \end{aligned}$$

By a simple calculation

$$I \le M\|\psi\| + \frac{MK}{\omega_1-\delta}\|\mathcal{Y}_Y^\delta\|\cdot\|\psi\| + \frac{3MK}{\omega_2+\delta}\|\mathcal{Y}_Y^\delta\|\cdot\|\psi\| \le M\|\psi\| + (1/3)\|\mathcal{Y}_Y^\delta\|\cdot\|\psi\|.$$

Hence

$$\|\mathcal{Y}_Y^\delta\| \le M + (1/3)\|\mathcal{Y}_Y^\delta\|$$

and the Lemma is proved by setting $\Omega_1 = (3/2)M$. QED

Set $\mathcal{Y}_Y = \mathcal{Y}_Y^0$.

Similarly, for $Y \in C^1(B,S_{\alpha-\Delta/2})$ define the operator $D^2\mathcal{T}_{\varepsilon,Y}$ on $C(B,L^2(P_\varepsilon,\ S_{2\alpha-\delta}))$, $\delta \in [0,\Delta]$ by setting

$$\begin{aligned} & (D^2\mathcal{T}_{\varepsilon,Y}(\Psi)\cdot[\psi_1,\psi_2])(t,\phi) := \\ & \int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(D^2F_\varepsilon(Y(\phi,s))\cdot[DY(\phi)\cdot[\psi_1](s),DY(\phi)\cdot[\psi_2](s)] \\ & \qquad +DF_\varepsilon(Y(\phi,s))\cdot[\Psi(\phi)\cdot[\psi_1,\psi_2](s)])ds \\ & +\int_{-\infty}^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}(D^2F_\varepsilon(Y(\phi,s))\cdot[DY(\phi)\cdot[\psi_1](s),DY(\phi)\cdot[\psi_2](s)] \\ & \qquad +DF_\varepsilon(Y(\phi,s))\cdot[\Psi(\phi)\cdot[\psi_1,\psi_2](s)])ds. \end{aligned}$$

It is easy to see (e.g. Lemma 2) that $D^2\mathcal{T}_{\varepsilon,Y}$ maps $C(B,L^2(P_\varepsilon,S_{2\alpha-\delta}))$ into $C(B,L^2(P_\varepsilon,S_{2\alpha-\delta}))$ for all $\delta\in[0,\Delta]$ and is a uniform 1/3-contraction. Denote the fixed points by Ψ_Y^δ.

Lemma 6 *The fixed points are independent of δ and there is constant Ω_2 independent of δ and ε such that*

$$\|\Psi_Y^\delta\|_{C(B,L^2(P_\varepsilon,S_{2\alpha-\delta}))} \leq \Omega_2 \|Y\|^2_{C^1(B,S_{\alpha-\Delta/2})}.$$

Proof of Lemma 6. Since $C(B,L^2(P_\varepsilon,S_{2\alpha-\delta})) \subset C(B,L^2(P_\varepsilon,S_{2\alpha}))$ by uniqueness we have that the fixed points are independent of δ.

With norm $\|\cdot\| = \|\cdot\|_{C(B,L^2(P_\varepsilon,S_{2\alpha-\delta}))}$ we have that

$$\|\Psi_Y^\delta\| = \|D^2\mathcal{T}_{\varepsilon,Y}(\Psi_Y^\delta)\| \leq \sup_{\phi\in B} \sup_{\psi_1,\psi_2\in P_\varepsilon,\|\psi_1\|,\|\psi_2\|\leq 1} \sup_{t\in\mathbf{R}^-} I,$$

where

$$I = e^{(2\alpha-\delta)t}\|\int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(D^2F_\varepsilon(Y(\phi,s))\cdot[DY(\phi)\cdot[\psi_1](s), DY(\phi)\cdot[\psi_2](s)]$$

$$+DF_\varepsilon(Y(\phi,s))\cdot[\Psi_Y^\delta(\phi)\cdot[\psi_1,\psi_2](s)])ds\|$$

$$+e^{(2\alpha-\delta)t}\|\int_{-\infty}^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}(D^2F_\varepsilon(Y(\phi,s))\cdot[DY(\phi)\cdot[\psi_1](s), DY(\phi)\cdot[\psi_2](s)]$$

$$+DF_\varepsilon(Y(\phi,s))\cdot[\Psi_Y^\delta(\phi)\cdot[\psi_1,\psi_2](s)])ds\|.$$

By a simple calculation

$$I \leq (\frac{MK}{\omega_1+\alpha-\delta} + \frac{3MK}{\omega_2-\alpha+\delta})\|Y\|^2_{C^1(B,S_{\alpha-\Delta/2})}\|\psi_1\|\cdot\|\psi_2\|$$

$$+(\frac{MK}{\omega_1+\alpha-\delta} + \frac{3MK}{\omega_2-\alpha+\delta})\|\Psi_Y^\delta\|\cdot\|\psi_1\|\cdot\|\psi_2\| \leq (1/3)(\|Y\|^2_{C^1(B,S_{\alpha-\Delta/2})} + \|\Psi_Y^\delta\|)\|\psi_1\|\cdot\|\psi_2\|.$$

Hence

$$\|\Psi_Y^\delta\| \leq (1/3)(\|Y\|^2_{C^1(B,S_{\alpha-\Delta/2})} + \|\Psi_Y^\delta\|)$$

and the lemma is proved by setting $\Omega_2 = 1/2$. QED

Set $\Psi_Y = \Psi_Y^0$. Now we are in a position to prove the C^2 smoothness of the small delay inertial manifold.

Lemma 7 *The small delay inertial manifold is C^2 smooth, i.e. $Y_\varepsilon \in C^1(B,S_\alpha)\cap C^2(B,S_{2\alpha})$. Moreover, $DY_\varepsilon = \mathcal{Y}_{Y_\varepsilon}$ and $D^2Y_\varepsilon = \Psi_{Y_\varepsilon}$.*

Proof of Lemma 7. Set $Y^0 \equiv 0$ and $Y^{n+1} := \mathcal{T}_\varepsilon(Y^n)$. Fix a sequence $\delta_0 > \delta_1 > \delta_2 > \cdots > \Delta > 0$. By Lemma 2, $Y^n \to Y_\varepsilon$ in $C(B,S_{\alpha-\Delta})$. Moreover, by Lemma 3, $Y^n \in C^1(B,S_{\alpha-\delta_n})$ and $DY^{n+1} = D\mathcal{T}_{\varepsilon,Y^n}(DY^n)$. Thus $\{Y^n\}_{n\geq 0} \subset C^1(B,S_{\alpha-\Delta})$. In what follows we show that $\{Y^n\}_{n\geq 0}$

is a Cauchy sequence in $C^1(B, S_{\alpha-\Delta/2})$. Clearly, it is enough to prove that $\{DY^n\}_{n\geq 0}$ is Cauchy in $C(B, L(P_\varepsilon, S_{\alpha-\Delta/2}))$. In the estimates below $\|\cdot\|$ stands for the norm of $C(B, L(P_\varepsilon, S_{\alpha-\Delta/2}))$:

$$\|DY^n - \mathcal{Y}_{Y^n}\| \leq 1/3\|DY^{n-1} - \mathcal{Y}_{Y^{n-1}}\| + \|\mathcal{Y}_{Y^n} - \mathcal{Y}_{Y^{n-1}}\|.$$

With $L \in \mathbf{N}$ we set $e_L := \sup_{n,m\geq L} \|\mathcal{Y}_{Y^n} - \mathcal{Y}_{Y^m}\|$. By an inductive application of the above estimate we have

$$\|DY^n - \mathcal{Y}_{Y^n}\| \leq (1/3)^{n-L}\|DY^L - \mathcal{Y}_{Y^L}\| + 3/2e_L$$

and thus for $m \geq n \geq L$

$$\|DY^m - DY^n\| \leq 2(1/3)^{n-L}\|DY^L - \mathcal{Y}_{Y^L}\| + 3e_L.$$

It remains to prove that $e_L \to 0$ as $L \to \infty$. Since

$$\|\mathcal{Y}_{Y^n} - \mathcal{Y}_{Y^m}\| \leq 1/3\|\mathcal{Y}_{Y^n} - \mathcal{Y}_{Y^m}\| + \|D\mathcal{T}_{\varepsilon,Y^n}(\mathcal{Y}_{Y^n}) - D\mathcal{T}_{\varepsilon,Y^m}(\mathcal{Y}_{Y^n})\|$$

it is enough to prove that

$$\sup_{n,m\geq L} \|D\mathcal{T}_{\varepsilon,Y^n}(\mathcal{Y}_{Y^n}) - D\mathcal{T}_{\varepsilon,Y^m}(\mathcal{Y}_{Y^n})\| \to 0$$

as $L \to \infty$. By a simple calculation we have

$$\|D\mathcal{T}_{\varepsilon,Y^n}(\mathcal{Y}_{Y^n}) - D\mathcal{T}_{\varepsilon,Y^m}(\mathcal{Y}_{Y^n})\| \leq \sup_{\phi\in B} \sup_{\psi\in P_\varepsilon, \|\psi\|\leq 1} \sup_{t\in\mathbf{R}^-} (I_1 + I_2),$$

where

$$I_1 = \|e^{(\alpha-\Delta/2)t} \int_0^t T_\varepsilon(t-s) X_0^{P_\varepsilon}(DF_\varepsilon(Y^n(\phi,s)) - DF_\varepsilon(Y^m(\phi,s))) \cdot [\mathcal{Y}_{Y^n}(\phi) \cdot [\psi](s)]ds\|$$

and

$$I_2 = \|e^{(\alpha-\Delta/2)t} \int_{-\infty}^t T_\varepsilon(t-s) X_0^{Q_\varepsilon}(DF_\varepsilon(Y^n(\phi,s)) - DF_\varepsilon(Y^m(\phi,s))) \cdot [\mathcal{Y}_{Y^n}(\phi) \cdot [\psi](s)]ds\|.$$

Consider I_1. Let $\eta > 0$ be given. Choose $T < 0$ so that

$$\frac{2KM\Omega_1}{\omega_1 - \Delta/2} e^{(\Delta/2)T} < \eta/2.$$

There are two cases.
Case $t \geq T$.
Choose L so large such that

$$\|DF_\varepsilon(Y^n(\phi,s)) - DF_\varepsilon(Y^m(\phi,s))\| \leq \frac{\eta(\omega_1 - \Delta/2)}{2M\Omega_1}$$

holds for $n, m \geq L$, $\phi \in B$ and $s \in [T, 0]$. Then

$$I_1 \leq \eta/2\|\psi\|.$$

Case $t < T$.

Write $I_1 = I_1^1 + I_1^2$ where

$$I_1^1 = \|e^{(\alpha-\Delta/2)t} \int_T^t T_\varepsilon(t-s) X_0^{P_\varepsilon}(DF_\varepsilon(Y^n(\phi,s)) - DF_\varepsilon(Y^m(\phi,s))) \cdot [\mathcal{Y}_{Y^n}(\phi) \cdot [\psi](s)] ds\|$$

and

$$I_1^2 = \|e^{(\alpha-\Delta/2)t} \int_0^T T_\varepsilon(t-s) X_0^{P_\varepsilon}(DF_\varepsilon(Y^n(\phi,s)) - DF_\varepsilon(Y^m(\phi,s))) \cdot [\mathcal{Y}_{Y^n}(\phi) \cdot [\psi](s)] ds\|.$$

It is easy to see that

$$I_1^1 \leq \eta/2\|\psi\|$$

while I_1^2 can be handled as I_1 in Case $t \geq T$, so we have that $I_1 \leq \eta\|\psi\|$.

The proof for $I_2 < \eta\|\psi\|$ is similar and is omitted. Hence $e_L \to 0$ as $L \to \infty$ and $\{Y^n\}_{n\geq 0} \subset C^1(B, S_{\alpha-\Delta/2})$ is a Cauchy sequence and $DY_\varepsilon = \mathcal{Y}_{Y_\varepsilon}$.

Let us turn to the C^2 property. By Lemma 4, $Y^n \in C^1(B, S_{\alpha-\delta_n}) \cap C^2(B, S_{2\alpha-\delta_n})$ and $D^2Y^{n+1} = D^2\mathcal{T}_{\varepsilon,Y^n}(DY^n)$. We prove that $\{Y^n\}_{n\geq 0}$ is a Cauchy sequence in $C^2(B, S_{2\alpha})$. Clearly, it is enough to prove that $\{D^2Y^n\}_{n\geq 0}$ is Cauchy in $C(B, L^2(P_\varepsilon, S_{2\alpha}))$. Note that $\{Y^n\}_{n\geq 0}$ is bounded in $C^1(B, S_{\alpha-\Delta/2})$, i.e. there exist a constant Ω_3 such that

$$\|Y^n\|_{C^1(B,S_{\alpha-\Delta/2})} \leq \Omega_3 \text{ for } n = 0, 1, 2, \ldots.$$

By Lemma 6 there exists a constant Ω_4 independent of $\delta \in [0, \Delta]$, $n \geq 0$ and ε such that

$$\|\Psi_{Y^n}\|_{C(B,L^2(P_\varepsilon,S_{2\alpha-\delta}))} \leq \Omega_4.$$

With norm $\|\cdot\| = \|\cdot\|_{C(B,L^2(P_\varepsilon,S_{2\alpha}))}$ we have that

$$\|D^2Y^n - \Psi_{Y^n}\| \leq 1/3\|D^2Y^{n-1} - \Psi_{Y^{n-1}}\| + \|\Psi_{Y^n} - \Psi_{Y^{n-1}}\|.$$

For $L \in \mathbf{N}$ we set $e_L := \sup_{n,m\geq L} \|\Psi_{Y^n} - \Psi_{Y^m}\|$. By an inductive application of the estimate above we have that

$$\|D^2Y^n - \Psi_{Y^n}\| \leq (1/3)^{n-L}\|D^2Y^L - \Psi_{Y^L}\| + 3/2e_L$$

and thus for $m \geq n \geq L$

$$\|D^2Y^m - D^2Y^n\| \leq 2(1/3)^{n-L}\|D^2Y^L - \Psi_{Y^L}\| + 3e_L.$$

It remains to prove that $e_L \to 0$ as $L \to \infty$. Since

$$\|\Psi_{Y^n} - \Psi_{Y^m}\| \le 1/3\|\Psi_{Y^n} - \Psi_{Y^m}\| + \|D^2\mathcal{T}_{\varepsilon,Y^n}(\Psi_{Y^n}) - D^2\mathcal{T}_{\varepsilon,Y^m}(\Psi_{Y^n})\|$$

it is enough to prove that

$$\sup_{n,m\ge L} \|D^2\mathcal{T}_{\varepsilon,Y^n}(\Psi_{Y^n}) - D^2\mathcal{T}_{\varepsilon,Y^m}(\Psi_{Y^n})\| \to 0$$

as $L \to \infty$. Write

$$\|D^2\mathcal{T}_{\varepsilon,Y^n}(\Psi_{Y^n}) - D^2\mathcal{T}_{\varepsilon,Y^m}(\Psi_{Y^n})\| \le \sup_{\phi\in B} \sup_{\psi_1,\psi_2\in P_\varepsilon,\|\psi_1\|,\|\psi_2\|\le 1} \sup_{t\in\mathbf{R}^-} (I_1 + I_2 + I_3 + I_4 + I_5 + I_6),$$

where

$$I_1 = \|e^{2\alpha t} \int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(D^2F_\varepsilon(Y^n(\phi,s))$$
$$\cdot[(DY^n(\phi) - DY^m(\phi))\cdot[\psi_1](s), (DY^n(\phi) - DY^m(\phi))\cdot[\psi_2](s)])ds\|$$

$$I_2 = \|e^{2\alpha t} \int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(D^2F_\varepsilon(Y^n(\phi,s)) - D^2F_\varepsilon(Y^m(\phi,s)))$$
$$\cdot[DY^m(\phi)\cdot[\psi_1](s), DY^m(\phi)\cdot[\psi_2](s)]ds\|$$

$$I_3 = \|e^{2\alpha t} \int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(DF_\varepsilon(Y^n(\phi,s)) - DF_\varepsilon(Y^m(\phi,s)))\cdot[\Psi_{Y^n}(\phi)\cdot[\psi_1,\psi_2](s)]ds\|$$

$$I_4 = \|e^{2\alpha t} \int_{-\infty}^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}(D^2F_\varepsilon(Y^n(\phi,s))$$
$$\cdot[(DY^n(\phi) - DY^m(\phi))\cdot[\psi_1](s), (DY^n(\phi) - DY^m(\phi))\cdot[\psi_2](s)])ds\|$$

$$I_5 = \|e^{2\alpha t} \int_{-\infty}^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}(D^2F_\varepsilon(Y^n(\phi,s)) - D^2F_\varepsilon(Y^m(\phi,s)))$$
$$\cdot[DY^m(\phi)\cdot[\psi_1](s), DY^m(\phi)\cdot[\psi_2](s)]ds\|$$

and

$$I_6 = \|e^{2\alpha t} \int_{-\infty}^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}(DF_\varepsilon(Y^n(\phi,s)) - DF_\varepsilon(Y^m(\phi,s)))\cdot[\Psi_{Y^n}(\phi)\cdot[\psi_1,\psi_2](s)]ds\|.$$

Let $\eta > 0$ be given. Consider first I_1. Since $\{Y^n\}_{n\ge 0}$ is Cauchy in $C^1(B, S_{\alpha-\Delta/2})$ we can choose L so large such that for $n, m \ge L$

$$\|DY^n - DY^m\|_{C(B,L(P_\varepsilon,S_\alpha))} \le \frac{\eta(\omega_1+\alpha)}{3MK}$$

holds and thus

$$I_1 \le \eta/3\|\psi_1\|\cdot\|\psi_2\|.$$

Let us estimate I_2. Choose $T < 0$ so that

$$\frac{2MK\Omega_3^2}{\omega_1 + \alpha} e^{\Delta T} \leq \eta/3.$$

There are two cases.
Case $t \geq T$. Choose L so large such that

$$\|D^2 F_\varepsilon(Y^n(\phi, s)) - D^2 F_\varepsilon(Y^m(\phi, s))\| \leq \frac{\eta(\omega_1 + \alpha)}{3MK\Omega_3^2}$$

holds for all $n, m \geq L$, $\phi \in B$ and $s \in [T, 0]$. Then

$$I_2 \leq \eta/3\|\psi_1\| \cdot \|\psi_2\|.$$

Case $t < T$.
Write $I_2 = I_2^1 + I_2^2$ where

$$I_2^1 = \|e^{2\alpha t} \int_T^t T_\varepsilon(t-s) X_0^{P_\varepsilon}(D^2 F_\varepsilon(Y^n(\phi, s)) - D^2 F_\varepsilon(Y^m(\phi, s)))$$
$$\cdot [DY^m(\phi) \cdot [\psi_1](s), DY^m(\phi) \cdot [\psi_2](s)] ds\|$$

and

$$I_2^2 = \|e^{2\alpha t} \int_0^T T_\varepsilon(t-s) X_0^{P_\varepsilon}(D^2 F_\varepsilon(Y^n(\phi, s)) - D^2 F_\varepsilon(Y^m(\phi, s)))$$
$$\cdot [DY^m(\phi) \cdot [\psi_1](s), DY^m(\phi) \cdot [\psi_2](s)] ds\|.$$

We have

$$I_2^1 \leq \eta/3\|\psi_1\| \cdot \|\psi_2\|$$

and a similar estimate for I_2^2 goes along the same line as in Case $t \geq T$.
Now we turn to I_3. Recall that

$$\|\Psi_{Y^n}\|_{C(B, L^2(P_\varepsilon, S_{2\alpha - \Delta}))} \leq \Omega_4.$$

Choose $T < 0$ so that

$$\frac{2MK\Omega_4}{\omega_1 + \alpha} e^{\Delta T} \leq \eta/3.$$

There are two cases.
Case $t \geq T$.
Choose L so large such that

$$\|DF_\varepsilon(Y^n(\phi, s)) - DF_\varepsilon(Y^m(\phi, s))\| \leq \frac{\eta(\omega_1 + \alpha)}{3MK\Omega_4}$$

holds for $n, m \geq L$, $\phi \in B$ and $s \in [T, 0]$. Then

$$I_3 \leq \eta/3 \|\psi_1\| \cdot \|\psi_2\|.$$

Case $t < T$.

Write $I_3 = I_3^1 + I_3^2$ where

$$I_3^1 = \|e^{2\alpha t} \int_T^t T_\varepsilon(t-s) X_0^{P_\varepsilon} (DF_\varepsilon(Y^n(\phi, s)) - DF_\varepsilon(Y^m(\phi, s))) \cdot [\Psi_{Y^n}(\phi) \cdot [\psi_1, \psi_2](s)] ds\|$$

and

$$I_3^2 = \|e^{2\alpha t} \int_0^T T_\varepsilon(t-s) X_0^{P_\varepsilon} (DF_\varepsilon(Y^n(\phi, s)) - DF_\varepsilon(Y^m(\phi, s))) \cdot [\Psi_{Y^n}(\phi) \cdot [\psi_1, \psi_2](s)] ds\|.$$

We have

$$I_3^1 \leq \eta/3 \|\psi_1\| \cdot \|\psi_2\|$$

and $I_3^2 \leq \eta/3 \|\psi_1\| \cdot \|\psi_2\|$ as in Case $t \geq T$.

The proof of $I_{4,5,6} \leq \eta \|\psi_1\| \cdot \|\psi_2\|$ is similar and is omitted. Hence $e_L \to 0$ as $L \to \infty$ and $\{Y^n\}_{n \geq 0}$ is a Cauchy sequence in $C^2(B, S_{2\alpha})$ and $D^2 Y_\varepsilon = \Psi_{Y_\varepsilon}$ and the lemma is proved. QED

Note that by Lemmata 6 and 7, there exists a constant Ω_5 independent of ε such that

$$\|Y_\varepsilon\|_{C^2(B, S_{2\alpha})} \leq \Omega_5.$$

We have

$$\|D\Phi_\varepsilon\|_{C(B, L(P_\varepsilon, Q_\varepsilon))} = \sup_{\phi \in B} \sup_{\psi \in P_\varepsilon, \|\psi\| \leq 1} \| \int_{-\infty}^0 T_\varepsilon(-s) X_0^{Q_\varepsilon} DF_\varepsilon(Y_\varepsilon(\phi, s)) \cdot [DY_\varepsilon(\phi) \cdot [\psi](s)] ds\|$$

$$\leq \frac{3MK\Omega_5}{\omega_2 - \alpha} \to 0$$

as $\beta \to \infty$ (or $\varepsilon_\beta \to 0$). Similarly,

$$\|D^2\Phi_\varepsilon\|_{C(B, L^2(P_\varepsilon, Q_\varepsilon))} = \sup_{\phi \in B} \sup_{\psi_1, \psi_2 \in P_\varepsilon, \|\psi_1\|, \|\psi_2\| \leq 1} \| \int_{-\infty}^0 T_\varepsilon(-s) X_0^{Q_\varepsilon} (D^2 F_\varepsilon(Y_\varepsilon(\phi, s))$$

$$\cdot [DY_\varepsilon(\phi) \cdot [\psi_1](s), DY_\varepsilon(\phi) \cdot [\psi_2](s)] + DF_\varepsilon(Y_\varepsilon(\phi, s)) \cdot [D^2 Y_\varepsilon(\phi) \cdot [\psi_1, \psi_2](s)]) ds\|$$

$$\leq \frac{3MK(\Omega_5^2 + \Omega_5)}{\omega_2 - \alpha} \to 0$$

as $\beta \to \infty$. QED

We end this section with estimating the C^2 norm of the small delay inertial manifolds. For $\psi, \psi_1, \psi_2 \in P_\varepsilon$, $\|\psi\|, \|\psi_1\|, \|\psi_2\| \le 1$, $\phi \in B$ we have that

$$\|D\Phi_\varepsilon(\phi)\cdot[\psi]\| \le 3MK\int_{-\infty}^0 e^{\beta s}\|DY_\varepsilon(\phi)\cdot[\psi](s)\|ds \le \frac{3MK\Omega_1}{\beta-\alpha+\delta} \to 0$$

as $\beta \to \infty$ and

$$\|D^2\Phi_\varepsilon(\phi)\cdot[\psi_1,\psi_2]\| \le 3MK\int_{-\infty}^0 e^{\beta s}(\|DY_\varepsilon(\phi)\cdot[\psi_1](s)\|\cdot\|DY_\varepsilon(\phi)\cdot[\psi_2](s)\|$$

$$+\|D^2Y_\varepsilon(\phi)\cdot[\psi_1,\psi_2](s)\|)ds \le \frac{3MK(\Omega_1^2+\Omega_4)}{\beta-2\alpha+\delta} \to 0$$

as $\beta \to \infty$. It remains to prove the invariance and the exponential attractivity of the small delay inertial manifolds.

3.3 Invariance

Denote the solution of (1) starting from $\phi = x_0 \in C_\varepsilon$ by x_t, $t \ge 0$. Observe that $Y_\varepsilon(\pi x_0, s)$, $s \le 0$ is a backward (in time) solution of (1) on the small delay inertial manifold starting from $\pi x_0 + \Phi_\varepsilon(\pi x_0)$. Thus

$$\Phi_\varepsilon(\pi x_t) = \int_{-\infty}^0 T_\varepsilon(-s)X_0^{Q_\varepsilon}F_\varepsilon(\pi x_{s+t}+\Phi_\varepsilon(\pi x_{s+t}))ds$$

$$= \int_{-\infty}^t T_\varepsilon(t-u)X_0^{Q_\varepsilon}F_\varepsilon(\pi x_u+\Phi_\varepsilon(\pi x_u))du$$

$$= T_\varepsilon(\int_{-\infty}^0 T_\varepsilon(-s)X_0^{Q_\varepsilon}F_\varepsilon(\pi x_s+\Phi_\varepsilon(\pi x_s))ds) + \int_0^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}F_\varepsilon(\pi x_s+\Phi_\varepsilon(\pi x_s))ds$$

$$= T_\varepsilon(t)\Phi_\varepsilon(\pi x_0) + \int_0^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}F_\varepsilon(\pi x_s+\Phi_\varepsilon(\pi x_s))ds.$$

Now let $x_0 \in \mathrm{graph}\Phi_\varepsilon$, i.e. $(id-\pi)x_0 = \Phi_\varepsilon(\pi x_0)$. By the variation of constants formula we have that

$$(id-\pi)x_t = T_\varepsilon(t)\Phi_\varepsilon(\pi x_0) + \int_0^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}F_\varepsilon(x_s)ds.$$

Hence

$$(id-\pi)x_t - \Phi_\varepsilon(\pi x_t) = \int_0^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}(F_\varepsilon(x_s)-F_\varepsilon(\pi x_s+\Phi_\varepsilon(\pi x_s)))ds$$

and

$$\|(id-\pi)x_t - \Phi_\varepsilon(\pi x_t)\| \le 3MK\int_0^t e^{-\beta(t-s)}\|(id-\pi)x_s - \Phi_\varepsilon(\pi x_s)\|ds$$

from which it follows that $\|(id-\pi)x_t - \Phi_\varepsilon(\pi x_t)\| \equiv 0$.

3.4 Exponential attractivity

Let x_t be an arbitrary solution of (1). Define

$$v(t) := (id - \pi)x_t - \Phi_\varepsilon(\pi x_t).$$

By a simple calculation

$$v(t) = T_\varepsilon(t)v(0) + \int_0^t T_\varepsilon(t-s)X_0^{Q_\varepsilon}(F_\varepsilon(x_s) - F_\varepsilon(\pi x_s - \Phi_\varepsilon(\pi x_s)))ds,$$

hence

$$\|v(t)\| \leq 3Me^{-\beta t}\|v(0)\| + 3MK\int_0^t e^{-\beta(t-s)}\|v(s)\|ds$$

from which it follows (by using the Gronwall inequality) that

$$\|v(t)\| \leq 3Me^{-\mu t}\|v(0)\|$$

where $\mu = \beta - 3MK > 0$. This proves exponential attractivity and the proof of the theorem is now complete. QED

4 Existence of asymptotic phases

In this section we prove that our small delay inertial manifolds have asymptotic phases. Namely, we have the following

Theorem 2 *For all solution x_t of (1) there exists a solution $\overline{x_t}$ of (1) on the small delay inertial manifold such that*

$$\|x_t - \overline{x_t}\| \leq const \cdot e^{-\mu t},$$

where $\mu = \beta - 3MK$.

Proof. Let x_t, $t \geq 0$ be an arbitrary solution of (1). Let $\overline{x_t}$ be the unknown solution of (1) on the small delay inertial manifold, i.e.

$$\overline{x_t} = T_\varepsilon(t)\overline{x_0} + \int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}F_\varepsilon(\pi\overline{x_s} + \Phi_\varepsilon(\pi\overline{x_s}))ds.$$

Recall that $v(t) = (id - \pi)x_t + \Phi_\varepsilon(\pi x_t)$ and

$$\|v(t)\| \leq 3Me^{-\mu t}\|v(0)\|$$

by the exponential attractivity. Set $w(t) = \overline{x_t} - \pi x_t$. A simple calculation yields that

$$w(t) = T_\varepsilon(t)w(0) + \int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(F_\varepsilon(\pi x_s + w(s) + \Phi_\varepsilon(\pi x_s + w(s)))$$

$$-F_\varepsilon(\pi x_s + v(s) + \Phi_\varepsilon(\pi x_s)))ds.$$

Define the Banach space

$$S_\mu^+ := \{w : \mathbf{R}^+ \to P_\varepsilon \ : \ w \text{ is continuous and } \sup_{t\in\mathbf{R}^+} \|w(t)\|e^{\mu t} < \infty\}$$

with norm

$$|w|_\mu := \sup_{t\in\mathbf{R}^+} \|w(t)\|e^{\mu t}.$$

Define operator $\mathcal{F}$ on S_μ^+ by setting

$$\mathcal{F}(w)(t) := \int_t^\infty T_\varepsilon(t-s)X_0^{P_\varepsilon}(F_\varepsilon(\pi x_s + w(s) + \Phi_\varepsilon(\pi x_s + w(s)))$$

$$-F_\varepsilon(\pi x_s + v(s) + \Phi_\varepsilon(\pi x_s)))ds.$$

It is easy to see that

$$|\mathcal{F}(w)|_\mu \leq \frac{MK((1+\|\Phi_\varepsilon\|_{C^1})|w|_\mu + 3M\|v(0)\|)}{\mu-\omega} < \infty$$

and

$$|\mathcal{F}(w_1) - \mathcal{F}(w_2)|_\mu \leq \frac{MK(1+\|\Phi_\varepsilon\|_{C^1})}{\mu-\omega}|w_1-w_2|_\mu \leq 1/2|w_1-w_2|_\mu.$$

Denote the fixed point of $\mathcal{F}$ by w^*. Then

$$w^*(t) = \int_t^\infty T_\varepsilon(t-s)X_0^{P_\varepsilon}(F_\varepsilon(\pi x_s + w^*(s) + \Phi_\varepsilon(\pi x_s + w^*(s))) - F_\varepsilon(\pi x_s + v(s) + \Phi_\varepsilon(\pi x_s)))ds$$

$$= T_\varepsilon(t)(\int_0^\infty T_\varepsilon(-s)X_0^{P_\varepsilon}(F_\varepsilon(\pi x_s + w^*(s) + \Phi_\varepsilon(\pi x_s + w^*(s))) - F_\varepsilon(\pi x_s + v(s) + \Phi_\varepsilon(\pi x_s)))ds)$$

$$+\int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(F_\varepsilon(\pi x_s + w^*(s) + \Phi_\varepsilon(\pi x_s + w^*(s))) - F_\varepsilon(\pi x_s + v(s) + \Phi_\varepsilon(\pi x_s)))ds$$

$$= T_\varepsilon(t)w^*(0) + \int_0^t T_\varepsilon(t-s)X_0^{P_\varepsilon}(F_\varepsilon(\pi x_s + w^*(s) + \Phi_\varepsilon(\pi x_s + w^*(s)))$$

$$-F_\varepsilon(\pi x_s + v(s) + \Phi_\varepsilon(\pi x_s)))ds.$$

Hence $\overline{x_t} := \pi x_t + w^*(t)$ is a solution on the small delay inertial manifold such that

$$\|\pi\overline{x_t} - \pi x_t\| \leq |w^*|_\mu e^{-\mu t}.$$

Finally, the result follows by observing that

$$\|\overline{x_t} - x_t\| \leq \|\pi\overline{x_t} - \pi x_t\| + \|v(t)\| + \|\Phi_\varepsilon(\pi x_t) - \Phi_\varepsilon(\pi\overline{x_t})\|.$$

QED

5 Discretization of small delay inertial manifolds

For $N \in \mathbf{N}$ set $h = \varepsilon/N$ and consider the Euler-discretization of (1) with set-size h, i.e. we consider the map on $\mathbf{R}^n$ defined by

$$y_{k+1} = (I + hA)y_k + h(f(y_k) + g(y_{k-N})), \tag{3}$$

where y_k is the approximating value of the exact solution $x(kh)$. An initial value $\phi \in C_\varepsilon$ of (1) gives rise to an initial value $\phi_h := (y_0, \ldots, y_{-N}) \in \mathbf{R}^{n\times(N+1)}$ of (3) by setting $y_i := \phi(ih)$, $i = 0, \ldots, -N$.

Identify the space $\mathbf{R}^{n\times(N+1)}$ (endowed with the usual max norm) with a subspace of C_ε consisting of piecewise linear continuous funtions defined on the mesh-points $\{ih \ : \ i = 0, \ldots, -N\}$. Denote this subspace by C^h_ε. Define the projection $\pi^h_\varepsilon : C_\varepsilon \to C^h_\varepsilon$ by $\pi^h_\varepsilon \phi =$ piecewise linear continuous extension from the values on the mesh points. The map (3) gives rise to a map on C^h_ε

$$x_{k+1} = T_{h,\varepsilon} x_k + hE_0 F_{h,\varepsilon}(x_k), \tag{4}$$

where

$$T_{h,\varepsilon} = \begin{bmatrix} I + hA & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ 0 & \ldots & 0 & 0 \\ 0 & \cdots & I & 0 \end{bmatrix},$$

$E_0 = \pi^h_\varepsilon X_0$ and $F_{h,\varepsilon} = f(x_k(0)) + g(x_k(-\varepsilon))$.

From now on we assume that $\varepsilon > 0$ is so small such that $(I + \varepsilon A)$ is invertible. Let us decompose the space C^h_ε by $\sigma(I + hA)$ as $C^h_\varepsilon = P^h_\varepsilon \oplus Q^h_\varepsilon$, where $P^h_\varepsilon = \pi_h C^h_\varepsilon$, $Q^h_\varepsilon = (id - \pi_h)C^h_\varepsilon$ and the projection π_h is defined by setting $\pi_h \phi_h(ih) := (I + hA)^i \phi_h(0)$ for $i = 0, \ldots, -N$. Note that the above splitting is $T_{h,\varepsilon}$-invariant for all ε and h.

The proof of the properties below is straightforward and thus is omitted.

$(\mathrm{H0})_h$ For all ε and h we have that $\dim P^h_\varepsilon = n$, there exists a constant M independent of ε and h such that $\|\pi^h_\varepsilon\|, \|\pi_h\|, \|id - \pi_h\| \le M$,

$(\mathrm{H1})_h$ for all $\beta > 0$ there exists $\varepsilon_\beta > 0$ such that for all $\varepsilon \in (0, \varepsilon_\beta]$ and for all h

$$\|T^k_{h,\varepsilon}(id - \pi_h)x\| \le 3Me^{-\beta kh}\|x\|, \quad k \ge 0,$$

$(\mathrm{H2})_h$ for all h the function $F_{h,\varepsilon} \in C^2(C^h_\varepsilon, \mathbf{R}^n)$ is bounded with bounded derivatives. Moreover, the bounds are independent of ε and h, i.e. $\|F_{h,\varepsilon}\|_{C^2} \le K$ for all ε and h,

$(\mathrm{H3})_h$ there exists an $\omega > 0$ independent of ε and h such that for all ε and h

$$\|T^k_{h,\varepsilon}\pi_h x\| \le Me^{\omega|kh|}\|x\|, \quad k \in \mathbf{Z}.$$

The following theorems can be proved exactly the same way as Theorem 1. Details are left to the reader.

Theorem 3 *For all ε small enough and for all h there exists a $\Phi_\varepsilon^h \in C^2(P_\varepsilon^h, Q_\varepsilon^h)$ such that graph$(\Phi_\varepsilon^h) = \{x + \Phi_\varepsilon^h(x) \ : \ x \in P_\varepsilon^h\}$ is an exponentially attractive invariant manifold for (4) with an asymptotic phase.*

Note that there exists an approximating small delay inertial manifold for all step-sizes. This is due to the fact that ε is small, and thus the ODE part alone is well approximated by the Euler method.

In what follows we study the behavior of Φ_ε^h with fixed ε as $h \to 0+$.

Let $\{f_1, \dots, f_n\}$ be the usual orthonormal basis in $\mathbf{R}^n$. Then $\{\phi_1, \dots, \phi_n\}$ and $\{\phi_1^h, \dots, \phi_n^h\}$ are bases of P_ε and P_ε^h, respectively, where $\phi_i(\theta) = e^{A\theta} f_i$ for $\theta \in [-\varepsilon, 0]$, $i = 1, \dots, n$ and $\phi_i^h(jh) = (I + hA)^j f_i$ for $j = 0, \dots, -N$, $i = 1, \dots, n$. For sake of simplicity we write $\phi_i = e^{A\cdot} f_i$ and $\phi_i^h = (I + hA)^{\cdot} f_i$. Let us define the linear bijection $\mathcal{P}_h : P_\varepsilon \to P_\varepsilon^h$ by setting $\mathcal{P}_h \phi := \sum_{i=1}^n \alpha_i \phi_i^h$ whenever $\phi = \sum_{i=1}^n \alpha_i \phi_i$.

Lemma 8 (i) *There exist a constant M_1 independent of ε and h such that $\|\mathcal{P}_h\| \le M_1$ for all ε and h,*

(ii) *there exists a continuous function $l : \mathbf{R}^+ \to \mathbf{R}^+$ such that $l(0) = 0$ and*

$$\|\mathcal{P}_h T_\varepsilon(-h) - T_{h,\varepsilon}^{-1} \mathcal{P}_h\|_{L(P_\varepsilon, P_\varepsilon^h)} \le l(h)h,$$

(iii) *for all $T > 0$*

$$\sup_{kh \in [0,T]} \|\pi_h^\varepsilon T_\varepsilon(kh) X_0^{Q_\varepsilon} - T_{h,\varepsilon}^k (id - \pi_h) E_0\| \to 0$$

as $h \to 0+$,

(iv)

$$\|\mathcal{P}_h - \pi_h^\varepsilon|_{P_\varepsilon}\|_{L(P_\varepsilon, C_\varepsilon^h)} \to 0$$

as $h \to 0+$,

(v)

$$F_\varepsilon(\phi) = F_{h,\varepsilon}(\pi_\varepsilon^h \phi),$$
$$DF_\varepsilon(\phi) \cdot [\psi] = DF_{h,\varepsilon}(\pi_\varepsilon^h \phi) \cdot [\pi_\varepsilon^h \psi],$$

(vi) *for all h*

$$\pi_h E_0 = \mathcal{P}_h X_0^{P_\varepsilon}.$$

Proof of Lemma 8. Let $\phi \in P_\varepsilon$, $\|\phi\| \le 1$. By definition $1 \ge \|\phi\| = \sup_{\theta\in[-\varepsilon,0]} \|e^{A\theta} \sum_{i=1}^n \alpha_i f_i\| \ge \|\sum_{i=1}^n \alpha_i f_i\|$ which shows that $\sum_{i=1}^n |\alpha_i| \le n$. Thus $\|\mathcal{P}_h\phi\| = \|\sum_{i=1}^n \alpha_i \phi_i^h\| \le n \max_{i=1,\dots,n} \|\phi_i^h\|$. Observe that $(I+hA)^\cdot \to e^{A\cdot}$ uniformly on the interval $[-\varepsilon, 0]$ and $e^{A\cdot}$ is uniformly bounded in $\varepsilon \in [0, \varepsilon_\beta]$. Thus there exists a constant K_1 independent of h, ε such that $\max_{i=1,\dots,n} \|\phi_i^h\| \le K_1$. This proves (i).

Let us prove (ii). Let, as before, $\phi \in P_\varepsilon$, $\|\phi\| \le 1$. Then $\phi = e^{A\cdot} \sum_{i=1}^n \alpha_i f_i$ and $T_\varepsilon(-h)\phi = e^{A\cdot} e^{-Ah} \sum_{i=1}^n \alpha_i f_i$. On the other hand $T_{h,\varepsilon}^{-1} \mathcal{P}_h \phi = (I+hA)^\cdot (I+hA)^{-1} \sum_{i=1}^n \alpha_i f_i$. Combining these formulas we get that

$$\|\mathcal{P}_h T_\varepsilon(-h)\phi - T_{h,\varepsilon}^{-1}\mathcal{P}_h\phi\| \le n \max_{j=0,-1,\dots,-N} \|(I+hA)^j\| \cdot \|e^{-Ah} - (I+hA)^{-1}\|.$$

Since $\|(I+hA)^j\|$ is uniformly bounded with some constant K_2 (independent of h, ε) and $\|e^{-Ah} - (I+hA)^{-1}\| \le K_3 h^2$ with some constant K_3 (independent of h, ε) the desired result follows with $l(h) = nK_2K_3h$.

Recall that $X_0^{Q_\varepsilon}(\theta) = -e^{A\theta}$, if $-\varepsilon \le \theta < 0$ and $X_0^{Q_\varepsilon}(0) = 0$. Then

$$T_\varepsilon(kh) X_0^{Q_\varepsilon}(jh) = \begin{cases} 0 & \text{if } -kh \le jh \le 0 \\ -e^{A(jh+kh)} & \text{if } jh < -kh \end{cases}$$

for $j = 0, -1, \dots, -N$ which shows that $T_\varepsilon(kh)X_0^{Q_\varepsilon} = 0$ if $kh \ge \varepsilon$. Similarily, $(id - \pi_h)E_0(jh) = -(I+hA)^j$, if $j = -1, \dots, -N$ and $(id - \pi_h)E_0(0) = 0$. Then

$$T_{h,\varepsilon}^k (id - \pi_h) E_0(j) = \begin{cases} 0 & \text{if } -k \le j \le 0 \\ -(I+hA)^{j+k} & \text{if } j < -k \end{cases}$$

for $j = 0, -1, \dots, -N$ which shows that $T_{h,\varepsilon}^k (id-\pi_h)E_0 = 0$ if $k \ge N$. Without loss of generality we assume that $T \le \varepsilon$. Then

$$\sup_{kh\in[0,T]} \|\pi_h^\varepsilon T_\varepsilon(kh) X_0^{Q_\varepsilon} - T_{h,\varepsilon}^k (id-\pi_h)E_0\| = \sup_{kh\in[0,T]} \sup_{j=0,-1,\dots,-N} \|-e^{A(jh+kh)} + (I+hA)^{j+k}\| \le K_4 h$$

with some constant K_4 (independent of h, ε) which proves (iii).

Let $\phi \in P_\varepsilon$, $\|\phi\| \le 1$. Then

$$\|\mathcal{P}_h\phi - \pi_h^\varepsilon \phi\| \le n \sup_{j=0,-1,\dots,-N} \|(I+hA)^j - e^{Ajh}\| \le nK_5 h$$

with some suitable constant K_5 (independent of h, ε) and (iv) follows.

Finally, it is straightforward to check (v)-(vi) and the lemma is proved. QED

Now we are in a position to formulate the main result of this section.

Theorem 4 *Let ε be fixed such that (1) has a small delay inertial manifold Φ_ε and (4) has an approximating small delay inertial manifold Φ_ε^h. Then for all $B \subset P_\varepsilon$ closed ball we have that*

$$\|\pi_h^\varepsilon \Phi_\varepsilon - \Phi_\varepsilon^h \circ \mathcal{P}_h\|_{C^1(B, C_\varepsilon^h)} \to 0$$

as $h \to 0+$.

Proof. Without loss of generality we assume that constants $M, \omega, \alpha, \beta, K, N, \delta_0, \Delta$ have the same value as in Section 2.

Define the Banach space

$$S_\eta^h := \{Y_h \ : \ Y_h : \mathbf{Z}^- \to C_\varepsilon^h \text{ is continuous and } \sup_{k \in \mathbf{Z}^-} e^{\eta k h} \|Y_h(k)\| < \infty\}$$

with norm

$$|Y_h|_\eta = \sup_{k \in \mathbf{Z}^-} e^{\eta k h} \|Y_h(k)\|.$$

Let $B \subset P_\varepsilon^h$ be an arbitrary bounded closed ball. We define operator $\mathcal{T}_\varepsilon^h$ on $C(B, S_\eta^h)$ by setting

$$(\mathcal{T}_\varepsilon^h(Y_h))(\phi_h, k) = T_{h,\varepsilon}^k \phi_h - \sum_{i=-1}^{k} T_{h,\varepsilon}^{k-1-i} \pi_h h E_0 F_{h,\varepsilon}(Y_h(\phi_h, i))$$

$$+ \sum_{i=-\infty}^{k-1} T_{h,\varepsilon}^{k-1-i} (id - \pi_h) h E_0 F_{h,\varepsilon}(Y_h(\phi_h, i)), \quad \phi_h \in B, \ k \in \mathbf{Z}^-.$$

Then for all $\delta \in [0, \delta_0]$, $\mathcal{T}_\varepsilon^h : C(B, S_{\alpha-\delta}^h) \to C(B, S_{\alpha-\delta}^h)$ is a uniform 1/3 contraction.

Then the approximating small delay inertial manifold is defined as

$$\Phi_\varepsilon^h = (id - \pi_h) Y_\varepsilon^h(0),$$

where Y_ε^h is the fixed point of $\mathcal{T}_\varepsilon^h$.

We define the operator $D\mathcal{T}_{\varepsilon, Y_\varepsilon^h}^h$ on $C(B, L(P_\varepsilon^h, S_{\alpha-\delta}^h))$, $\delta \in [0, \Delta]$ by setting

$$(D\mathcal{T}_{\varepsilon, Y_\varepsilon^h}^h(\mathcal{Y}_h)) \cdot [\psi_h])(\phi_h, k) := T_{h,\varepsilon}^k \psi_h$$

$$+ \sum_{i=-1}^{k} T_{h,\varepsilon}^{k-1-i} \pi_h h E_0 DF_{h,\varepsilon}(Y_\varepsilon^h(\phi_h, i)) \cdot [\mathcal{Y}_h(\phi_h) \cdot [\psi_h](i)]$$

$$+ \sum_{i=-\infty}^{k-1} T_{h,\varepsilon}^{k-1-i} \pi_h h E_0 DF_{h,\varepsilon}(Y_\varepsilon^h(\phi_h, i)) \cdot [\mathcal{Y}_h(\phi_h) \cdot [\psi_h](i)].$$

Then $D\mathcal{T}_{\varepsilon, Y_\varepsilon^h}^h$ maps $C(B, L(P_\varepsilon^h, S_{\alpha-\delta}^h))$ into $C(B, L(P_\varepsilon^h, S_{\alpha-\delta}^h))$ for all $\delta \in [0, \Delta]$ and is a uniform 1/3 contraction. Its fixed point is DY_ε^h, the derivative of Y_ε^h.

Define the bounded linear operator $\Pi_h : C(B, S_\mu) \to C(\mathcal{P}_h(B), S^h_\mu)$ by setting for $Y \in C(B, S_\mu)$

$$(\Pi_h Y)(\mathcal{P}_h\phi)(k) := \pi^\varepsilon_h Y(\phi)(kh).$$

Lemma 9 *For all* $Y \in C(B, S_{\alpha-\delta})$, $\delta \in (0, \Delta]$

$$\|\Pi_h \mathcal{T}_\varepsilon Y - \mathcal{T}^h_\varepsilon \Pi_h Y\|_{C(\mathcal{P}_h(B), S^h_\alpha)} \to 0$$

as $h \to 0+$.

Proof of Lemma 9. Let $Y \in C(B, S_{\alpha-\delta})$ be fixed. We have that

$$\|\Pi_h \mathcal{T}_\varepsilon Y - \mathcal{T}^h_\varepsilon \Pi_h Y\|_{C(\mathcal{P}_h(B), S^h_\alpha)} \le \sup_{\phi\in B} \sup_{k\in \mathbf{Z}^-} (I_1 + I_2 + I_3),$$

where

$$I_1 = e^{\alpha kh} \|\pi^h_\varepsilon T_\varepsilon(kh)\phi - T^k_{h,\varepsilon}\mathcal{P}_h\phi\|,$$

$$I_2 = e^{\alpha kh}\|\pi^h_\varepsilon \int_0^{kh} T_\varepsilon(kh-s)X_0^{P_\varepsilon}F_\varepsilon(Y(\phi,s))ds - \sum_{i=-1}^{k} T^{k-1-i}_{h,\varepsilon}\pi_h h E_0 F_{h,\varepsilon}(\pi^h_\varepsilon Y(\phi, ih))\|$$

and

$$I_3 = e^{\alpha kh}\|\pi^h_\varepsilon \int_{-\infty}^{kh} T_\varepsilon(kh-s)X_0^{Q_\varepsilon}F_\varepsilon(Y(\phi,s))ds - \sum_{i=-\infty}^{k-1} T^{k-1-i}_{h,\varepsilon}(id-\pi_h) h E_0 F_{h,\varepsilon}(\pi^h_\varepsilon Y(\phi, ih))\|.$$

We estimate each component separately. First we estimate I_1 as

$$I_1 \le e^{\alpha kh}(\|\mathcal{P}_h T_\varepsilon(kh)\phi - \pi^h_\varepsilon T_\varepsilon(kh)\phi\| + \|T^k_{h,\varepsilon}\mathcal{P}_h\phi - \mathcal{P}_h T_\varepsilon(kh)\phi\|)$$

$$\le Me^{\omega_1 kh}\|\mathcal{P}_h - \pi^h_\varepsilon|_{P_\varepsilon}\| \cdot \|\phi\| + e^{\alpha kh}\sum_{j=0}^{|k|-1} \|T^{k+j}_{h,\varepsilon}\mathcal{P}_h T_\varepsilon(-jh)\phi - T^{k+j+1}_{h,\varepsilon}\mathcal{P}_h T_\varepsilon(-(j+1)h)\phi\|$$

$$\le (M\|\mathcal{P}_h - \pi^h_\varepsilon|_{P_\varepsilon}\| + e^{\alpha kh}\sum_{j=0}^{|k|-1} Me^{-\omega(k+j+1)h}\|T^{-1}_{h,\varepsilon}\mathcal{P}_h - \mathcal{P}_h T_\varepsilon(-h)\|_{L(P_\varepsilon, P^h_\varepsilon)} Me^{\omega jh})\|\phi\|$$

$$\le (M\|\mathcal{P}_h - \pi^h_\varepsilon|_{P_\varepsilon}\| + M^2|k|he^{\omega_1 kh}l(h))\|\phi\| \le (M\|\mathcal{P}_h - \pi^h_\varepsilon|_{P_\varepsilon}\| + M^2\frac{l(h)}{\omega_1})\|\phi\|$$

which shows that $\sup_{\phi\in B}\sup_{k\in\mathbf{Z}^-} I_1 \to 0$ as $h \to 0+$.

Now we estimate I_2 as

$$I_2 \le e^{\alpha kh}\|\sum_{i=-1}^{k} T^{k-1-i}_{h,\varepsilon}(\pi_h E_0 - \mathcal{P}_h X_0^{P_\varepsilon}) h F_{h,\varepsilon}(\pi^h_\varepsilon Y(\phi, ih))\|$$

$$+e^{\alpha kh}\|\sum_{i=-1}^{k}(T_{h,\varepsilon}^{k-1-i}\mathcal{P}_h X_0^{P_\varepsilon}-\mathcal{P}_h T_\varepsilon((k-1-i)h)X_0^{P_\varepsilon})hF_\varepsilon(Y(\phi,ih))\|$$

$$+\|\mathcal{P}_h-\pi_\varepsilon^h|_{P_\varepsilon}\|e^{\alpha kh}\|\sum_{i=-1}^{k}T_\varepsilon((k-1-i)h)X_0^{P_\varepsilon}hF_\varepsilon(Y(\phi,ih))\|$$

$$+e^{\alpha kh}\|\pi_\varepsilon^h\|\cdot\|\sum_{i=-1}^{k}T_\varepsilon((k-1-i)h)X_0^{P_\varepsilon}hF_\varepsilon(Y(\phi,ih))-\int_0^{kh}T_\varepsilon(kh-s)X_0^{P_\varepsilon}F_\varepsilon(Y(\phi,s))ds\|$$

$$=I_2^1+I_2^2+I_2^3+I_2^4.$$

Further, we have that

$$I_2^1=0,$$

$$I_2^2\le\sum_{i=-1}^{k}e^{\alpha(i+1)h}\frac{M^2l(h)}{\omega_1}Kh\le\frac{M^2Kl(h)h}{\omega_1(1-e^{-\alpha h})},$$

(here we used the estimates obtained for I_1 previously)

$$I_2^3\le\|\mathcal{P}_h-\pi_\varepsilon^h|_{P_\varepsilon}\|e^{\alpha kh}\sum_{i=-1}^{k}MKhe^{-\omega(k-1-i)h}\le\|\mathcal{P}_h-\pi_\varepsilon^h|_{P_\varepsilon}\|MKh\sum_{i=-1}^{k}e^{\omega_1(k-1-i)h}$$

$$\le\frac{\|\mathcal{P}_h-\pi_\varepsilon^h|_{P_\varepsilon}\|MKh}{1-e^{-\omega_1 h}}.$$

It is clear form these estimates that $\sup_{\phi\in B}\sup_{k\in\mathbf{Z}^-}(I_2^1+I_2^2+I_2^3)\to 0$ as $h\to 0+$. It remains to check this property for I_2^4. To this end et $\eta>0$ be given. We choose $T<0$ such that

$$\frac{MK}{\omega_1}e^{\alpha T}\le\eta/2.$$

There are two cases.
Case $kh\ge T$.

Since the integral is now on a compact interval it follows simply that $I_2^4\le\eta/2$ for all h small enough.
Case $kh<T$.

Write

$$I_2^4\le+M_1e^{\alpha kh}\|\sum_{i=-1}^{[T/h]}T_\varepsilon((k-1-i)h)X_0^{P_\varepsilon}hF_\varepsilon(Y(\phi,ih))-\int_0^{[T/h]h}T_\varepsilon(kh-s)X_0^{P_\varepsilon}F_\varepsilon(Y(\phi,s))ds\|$$

$$+M_1e^{\alpha kh}\|\sum_{i=[T/h]-1}^{k}T_\varepsilon((k-1-i)h)X_0^{P_\varepsilon}hF_\varepsilon(Y(\phi,ih))-\int_{[T/h]h}^{kh}T_\varepsilon(kh-s)X_0^{P_\varepsilon}F_\varepsilon(Y(\phi,s))ds\|$$

$$= I_2^{4,1} + I_2^{4,2}.$$

As in Case $kh \geq T$ the desired result follows for $I_2^{4,1}$ while

$$I_2^{4,2} \leq \sum_{i=[L/h]-1}^{k} Me^{\omega_1(k-1-i)h} Khe^{\alpha(i+1)h} + \int_{[T/h]h}^{kh} Me^{\omega_1(kh-s)} Ke^{\alpha s} ds$$

$$\leq \Big(\frac{MKh}{1-e^{-\omega_1 h}} + \frac{MK}{\omega_1}\Big)e^{\alpha T}.$$

Let us turn to estimating I_3 as

$$I_3 \leq e^{\alpha kh} \|\Big(\sum_{i=-\infty}^{k-1} T_{h,\varepsilon}^{k-1-i}(id-\pi_h)E_0 - \sum_{i=-\infty}^{k-1} \pi_\varepsilon^h T_\varepsilon((k-1-i)h)X_0^{Q_\varepsilon}\Big) hF_{h,\varepsilon}(\pi_\varepsilon^h Y(\phi, ih))\|$$

$$+ e^{\alpha kh} \| \sum_{i=-\infty}^{k-1} \pi_\varepsilon^h T_\varepsilon((k-1-i)h)X_0^{Q_\varepsilon} hF_\varepsilon(Y(\phi, ih)) - \int_{-\infty}^{kh} \pi_\varepsilon^h T_\varepsilon(kh-s)X_0^{Q_\varepsilon} F_\varepsilon(Y(\phi, s))ds\|$$

$$= I_3^1 + I_3^2.$$

First we note that I_3^2 can be handled as I_2^4 and thus we omit the details. It remains to prove that I_3^1 tends to 0 as h tends to 0. To this end let $\eta > 0$ be given. We choose a $T < 0$ such that

$$\frac{6M(1+M_1)K}{\omega_2} e^{\alpha T} \leq \eta/2.$$

There are two cases.

Case $kh < T$.

In this case

$$I_3^1 \leq 3M \sum_{i=-\infty}^{k-1} e^{-\omega_2(k-1-i)h} Khe^{\alpha(i+1)h} + 3MM_1 \sum_{i=-\infty}^{k-1} e^{-\omega_2(k-1-i)h} Khe^{\alpha(i+1)h}$$

$$\leq \Big(\frac{3MKh}{1-e^{-\omega_2 h}} + \frac{3MM_1Kh}{1-e^{-\omega_2 h}}\Big)e^{\alpha T} < \eta/2$$

for all h sufficiently small.

Case $kh \geq T$. In this case

$$I_3^1 \leq |T| \sup_{jh\in[0,|T|]} \|T_{h,\varepsilon}^j(id-\pi_h)E_0 - \pi_\varepsilon^h T_\varepsilon(jh)X_0^{Q_\varepsilon}\| e^{\alpha kh} K + \sum_{i=-\infty}^{[T/h]} 3M(1+M_1)e^{-\omega_2(k-1-i)h} Khe^{\alpha(i+1)h}$$

$$\leq |T| \sup_{jh\in[0,|T|]} \|T_{h,\varepsilon}^j(id-\pi_h)E_0 - \pi_\varepsilon^h T_\varepsilon(jh)X_0^{Q_\varepsilon}\| K + \frac{3M(1+M_1)Kh}{1-e^{-\omega_2 h}} e^{\alpha T} < \eta$$

for all h small enough. The proof of the Lemma is now complete. QED

With norm $\|\cdot\| = \|\cdot\|_{C(\mathcal{P}_h(B),S^h_\alpha)}$ we have that

$$\|\Pi_h Y_\varepsilon - Y^h_\varepsilon\| \leq \|\Pi_h \mathcal{T}_\varepsilon(Y_\varepsilon) - \mathcal{T}^h_\varepsilon(\Pi_h Y_\varepsilon)\| + (1/3)\|\Pi_h Y_\varepsilon - Y^h_\varepsilon\|$$

which shows that $\|\Pi_h Y_\varepsilon - Y^h_\varepsilon\| \to 0$ as $h \to 0+$.

Since the derivatives of Y_ε and Y^h_ε have a similar integral representation, the same type of argument can be used to finish the proof of the Theorem. We only sketch the main ideas, the rest is left to the reader.

Redefine the bounded linear operator $\Pi_h : C(B, L(P_\varepsilon, S_{\alpha-\delta})) \to C(\mathcal{P}_h(B), L(P^h_\varepsilon, S^h_{\alpha-\delta}))$ by setting

$$(\Pi_h \mathcal{Y})(\mathcal{P}_h\phi) \cdot [\mathcal{P}_h\psi](k) := \pi^h_\varepsilon \mathcal{Y}(\phi) \cdot [\psi](kh).$$

Lemma 10 *For all* $\mathcal{Y} \in C(B, L(P_\varepsilon, S_{\alpha-\delta}))$, $\delta \in (0, \Delta/2]$

$$\|D\mathcal{T}^h_{\varepsilon,Y^h_\varepsilon}(\Pi_h\mathcal{Y}) - \Pi_h D\mathcal{T}_{\varepsilon,Y_\varepsilon}(\mathcal{Y})\|_{C(\mathcal{P}_h(B),L(P^h_\varepsilon,S^h_\alpha))} \to 0$$

as $h \to 0+$.

Proof of Lemma 10. Let $\mathcal{Y} \in C(B, L(P_\varepsilon, S_{\alpha-\delta}))$ be fixed. We have that

$$\|D\mathcal{T}^h_{\varepsilon,Y^h_\varepsilon}(\Pi_h\mathcal{Y}) - \Pi_h D\mathcal{T}_{\varepsilon,Y_\varepsilon}(\mathcal{Y})\|_{C(\mathcal{P}_h(B),L(P^h_\varepsilon,S^h_\alpha))} \leq \sup_{\psi\in P_\varepsilon,\|\psi\|\leq 1} \sup_{\phi\in B} \sup_{k\in\mathbf{Z}^-} (I_1 + I_2 + I_3),$$

where

$$I_1 = e^{\alpha kh}\|\pi^h_\varepsilon T_\varepsilon(kh)\psi - T^k_{h,\varepsilon}\mathcal{P}_h\psi\|,$$

$$I_2 = e^{\alpha kh}\|\pi^h_\varepsilon \int_0^{kh} T_\varepsilon(kh-s)X_0^{P_\varepsilon} DF_\varepsilon(Y_\varepsilon(\phi,s)) \cdot [\mathcal{Y}(\phi)\cdot[\psi](s)]ds$$
$$- \sum_{i=-1}^{k} T^{k-1-i}_{h,\varepsilon}\pi_h h E_0 DF_{h,\varepsilon}(Y^h_\varepsilon(\mathcal{P}_h\phi,i)) \cdot [\pi^h_\varepsilon\mathcal{Y}(\phi)\cdot[\psi](ih)]\|$$

and

$$I_3 = e^{\alpha kh}\|\pi^h_\varepsilon \int_{-\infty}^{kh} T_\varepsilon(kh-s)X_0^{Q_\varepsilon} DF_\varepsilon(Y_\varepsilon(\phi,s)) \cdot [\mathcal{Y}(\phi)\cdot[\psi](s)]ds$$
$$- \sum_{i=-\infty}^{k-1} T^{k-1-i}_{h,\varepsilon}(id-\pi_h) h E_0 DF_{h,\varepsilon}(Y^h_\varepsilon(\mathcal{P}_h\phi,i)) \cdot [\pi^h_\varepsilon\mathcal{Y}(\phi)\cdot[\psi](ih)]\|.$$

Notice that I_1 was already treated in the previous lemma.

For I_2 we write

$$I_2 \leq e^{\alpha kh}\|\sum_{i=-1}^{k} T^{k-1-i}_{h,\varepsilon}\pi_h E_0 h(DF_{h,\varepsilon}(Y^h_\varepsilon(\mathcal{P}_h\phi,i)) - DF_{h,\varepsilon}(Y_\varepsilon(\phi,ih))) \cdot [\pi^h_\varepsilon\mathcal{Y}(\phi)\cdot[\psi](ih)]\|$$

$$+e^{\alpha kh}\| \sum_{i=-1}^{k} T_{h,\varepsilon}^{k-1-i}\pi_h E_0 h DF_{h,\varepsilon}(Y_\varepsilon(\phi, ih)) \cdot [\pi_\varepsilon^h \mathcal{Y}(\phi) \cdot [\psi](ih)]$$

$$-\pi_\varepsilon^h \int_0^{kh} T_\varepsilon(kh-s) X_0^{P_\varepsilon} DF_\varepsilon(Y_\varepsilon(\phi, s)) \cdot [\mathcal{Y}(\phi) \cdot [\psi](s)] ds\|$$

$$= I_2^1 + I_2^2.$$

The second term can be estimated as I_2 in the previous lemma. Let us estimate the first term. Let $\eta > 0$ be given. We choose $T < 0$ so that

$$\frac{MKh\|\mathcal{Y}\|}{1-e^{-\omega_2 h}} e^{\delta T} \leq \eta/2.$$

There are two cases.
Case $kh \geq T$.

If h is sufficiently small then

$$\|DF_{h,\varepsilon}(Y_\varepsilon^h(\mathcal{P}_h\phi, i)) - DF_{h,\varepsilon}(Y_\varepsilon(\phi, ih))\| \leq \frac{(1-e^{-\omega_2 h})\eta}{2MKh\|\mathcal{Y}\|}$$

holds for all $\phi \in B$, $ih \in [T, 0]$. Hence

$$I_2^1 \leq \eta/2$$

Case $kh < T$.

Write

$$I_2^1 \leq I_2^{1,1} + I_2^{1,2},$$

where

$$I_2^{1,1} = e^{\alpha kh}\| \sum_{i=[T/h]-1}^{k} T_{h,\varepsilon}^{k-1-i}\pi_h E_0 h (DF_{h,\varepsilon}(Y_\varepsilon^h(\mathcal{P}_h\phi, i)) - DF_{h,\varepsilon}(Y_\varepsilon(\phi, ih))) \cdot [\pi_\varepsilon^h \mathcal{Y}(\phi) \cdot [\psi](ih)]\|$$

and

$$I_2^{1,2} = e^{\alpha kh}\| \sum_{i=-1}^{[T/h]} T_{h,\varepsilon}^{k-1-i}\pi_h E_0 h (DF_{h,\varepsilon}(Y_\varepsilon^h(\mathcal{P}_h\phi, i)) - DF_{h,\varepsilon}(Y_\varepsilon(\phi, ih))) \cdot [\pi_\varepsilon^h \mathcal{Y}(\phi) \cdot [\psi](ih)]\|.$$

The second term can be estimated as I_2^1 in Case $kh \geq T$ while

$$I_2^{1,1} \leq \eta/2.$$

By putting an extra term in the estimate of I_3 as well we have

$$I_3 \leq e^{\alpha kh}\| \sum_{i=-\infty}^{k} T_{h,\varepsilon}^{k-1-i}(id - \pi_h) E_0 h (DF_{h,\varepsilon}(Y_\varepsilon^h(\mathcal{P}_h\phi, i)) - DF_{h,\varepsilon}(Y_\varepsilon(\phi, ih))) \cdot [\pi_\varepsilon^h \mathcal{Y}(\phi) \cdot [\psi](ih)]\|$$

$$+e^{\alpha kh}\| \sum_{i=-\infty}^{k} T_{h,\varepsilon}^{k-1-i}(id-\pi_h)E_0 hDF_{h,\varepsilon}(Y_\varepsilon(\phi,ih))\cdot[\pi_\varepsilon^h \mathcal{Y}(\phi)\cdot[\psi](ih)]$$

$$-\pi_\varepsilon^h \int_{-\infty}^{kh} T_\varepsilon(kh-s)X_0^{Q_\varepsilon} DF_\varepsilon(Y_\varepsilon(\phi,s))\cdot[\mathcal{Y}(\phi)\cdot[\psi](s)]ds\|$$

and the proof can be finished in a similar way. QED

We end this section with two remarks concerning the rate of the convergence of approximating small delay inertial manifolds and the convergence of higher order derivatives.

Remark 4. In the proof of (iii) we observed that $T(t)X_0^{Q_\varepsilon}=0$, resp. $T_{h,\varepsilon}^k(id-\pi_h)E_0=0$ if $t\geq\varepsilon$, resp. $k\geq N$. Moreover, the proof of Lemma 8 shows that $l(h)$ and the convergences in (iii) and (iv) are bounded by $O(h)$. Thus our manifolds have the form

$$\Phi_\varepsilon(\phi)=\int_{-\varepsilon}^{0} T_\varepsilon(-s)X_0^{Q_\varepsilon}F_\varepsilon(Y_\varepsilon(\phi,s))ds$$

and

$$\Phi_\varepsilon^h(\mathcal{P}_h\phi)=\sum_{i=-N-1}^{-1} T_{h,\varepsilon}^{-1-i}(id-\pi_h)E_0F_{h,\varepsilon}(Y_\varepsilon^h(\mathcal{P}_h\phi,i))$$

and the speed of the convergence can be estimated by $O(h)$ if the speed of the convergence of $Y_\varepsilon^h(\mathcal{P}_h,\cdot)$ to $Y_\varepsilon(\phi,\cdot)$ can be estimated by $O(h)$ on the compact interval $[-\varepsilon,0]$. It is easy to see that in this case every term of the estimates in the proof of Lemma 9 are bounded by $O(h)$. Similar convergence result holds true for the derivatives as well.

Remark 5. Since the second derivatives of Y_ε and Y_ε^h have similar integral/sum representations the same type of arguments used in Lemmata 9 and 10 can be repeated. Moreover, if f,g are of class C^k with bounded derivatives then (via the same procedure) the convergence of the higher order derivatives can be proved as well. Since our application requires only C^2 smoothness and C^1 closeness we omit the (largely technical) details.

6 A numerical structural stability result

The solution flow of equation (1) on the small delay inertial manifold is given by the ordinary differenatial equation

$$\dot{y}(t)=Ay(t)+f(y(t))+H(y(t)), \tag{5}$$

where $y(t)\in\mathbf{R}^n$ and $H(y)=g(y+\Phi_\varepsilon(e^{A\cdot}y)(-\varepsilon))$. The solution flow of (5) is denoted by $\varphi_{1,\varepsilon}^t$.

On the other hand, the Euler method on the approximating small delay inertial form takes the form

$$y_{k+1}=(I+hA)y_k+h(f(y_k)+H_h(y_k))=:E_h(y_k), \tag{6}$$

where $y_k \in \mathbf{R}^n$ and $H_h(y) = g(y + \Phi_\varepsilon^h((I + hA)^\cdot y)(-N))$. Observe that (6) is the Euler discretization of

$$\dot{y}(t) = Ay(t) + f(y(t)) + H_h(y(t)). \tag{7}$$

The solution flow of (7) is denoted by $\varphi_{2,h}^t$.

Finally, denote the solution flow of the "limiting ODE", i.e.

$$\dot{y}(t) = Ay(t) + f(y(t)) + g(y(t))$$

by φ^t.

Assume that φ^t flows into a bounded closed ball B along the boundary. Assume further that the chain recurrent set of φ^t is hyperbolic and φ^t satisfies the strong transversality condition. (For definitions we refer to [23]).

It is known that there is an $\eta > 0$ such that if $\|H - g\|_{C^1(B)} < \eta$ then there exist a homeomorphism G_1 on B and a continuous function $\tau_1 : B \to \mathbf{R}$ such that for all $y \in B$

$$G_1 \circ \varphi^{\tau_1(y)}(y) = \varphi_{1,\varepsilon}^t \circ G_1(y).$$

Since $\|H - g\|_{C^1(B)} \to 0$ as $\varepsilon \to 0$ by Theorem 1, we can choose an $\varepsilon > 0$ such that $\|H - g\|_{C^1(B)} < \eta/2$. (Structural stability with respect to delay.)

Moreover, for all h small enough we find (by Theorem 3) that $\|H - H_h\|_{C^1(B)} < \eta/2$ and thus a similar result holds for $\varphi_{2,h}^t$, i.e. there exist a homeomorphism G_2 on B and a continuous function $\tau_2 : B \to \mathbf{R}$ such that for all $y \in B$

$$G_2 \circ \varphi^{\tau_2(y)}(y) = \varphi_{2,h}^t \circ G_2(y).$$

Finally, by using that H_h is C^2 we can apply the main result of [18].

Corollary 1 *For all h small enough there is a homeomorphism G_h on B and a continuous function $\tau_h : B \to \mathbf{R}$ such that for all $y \in B$*

$$G_h \circ \varphi_{2,h}^{\tau_h(y)}(y) = E_h^{1/h} \circ G_h(y)$$

and $G_h(y) \to y$.

Combining these result we obtain conjugacies between the Euler method and the "limiting ODE" as well as between the Euler method and the solutions of (1) on the small delay inertial manifold.

We note that similar results are valid when we assume that φ^t is Morse-Smale gradient-like, however in this case there is no reparameterization needed, see [18].

Finally we mention that beyond numerical structural stability results one may apply results concerning the persistence of invariant sets under discretization studied for finite-dimensional systems directly to delay equations with small delay.

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