ORBITAL AND WEAK SHADOWING IN DYNAMICAL SYSTEMS

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Abstract. We study weak and orbital shadowing properties of dynamical systems related to the following approach: we look for exact trajectories lying in small neighborhoods of approximate ones (or containing approximate ones in their small neigborhoods) or for exact trajectories such that the Hausdorff distances between their closures and closures of approximate trajectories are small.

These properties are characterized for linear diffeomorphisms. We also study some C^1-open sets of diffeomorphisms defined in terms of these properties. It is shown that the C^1-interior of the set of diffeomorphisms having the orbital shadowing property coincides with the set of structurally stable diffeomorphisms.

1.**Introduction.**

One of the main applications of the shadowing theory is rigorization of results of numerical study of dynamical systems.

Consider a dynamical system generated by a homeomorphism φ of a metric space (M, dist). For a point $x \in M$, we denote by $O(x, \varphi)$ its trajectory in the system φ, i.e., the set

$$O(x, \varphi) = \{\varphi^k(x) : k \in \mathbf{Z}\}.$$

[1]The research of the first author was supported in part by the Russian Foundation for Basic Research, grant 99-01-00683, and by DFG research group "Spectral analysis, asymptotic distributions, and stochastic dynamics," University of Bielefeld, Germany.

We say that a sequence $\xi = \{x_k \in M : k \in \mathbf{Z}\}$ is a d-pseudotrajectory of φ if the inequalities

$$\mathrm{dist}(\varphi(x_k), x_{k+1}) < d, \quad k \in \mathbf{Z}, \tag{1}$$

hold. A d-pseudotrajectory is a natural model of computer output in a process of numerical investigation of the system φ. In this case, the value d in (1) measures errors of the method, round-off errors, etc.

The usual shadowing property of φ [1] is formulated as follows: given $\epsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\xi = \{x_k\}$ we can find a point $p \in M$ with the property

$$\mathrm{dist}(\varphi^k(p), x_k) < \epsilon, \quad k \in \mathbf{Z}.$$

Of course, if φ has the shadowing property formulated above, then results of its numerical study (with a proper accuracy) reflect its qualitative structure.

Usually, one studies the geometric pattern of the set of trajectories of a system under investigation. In this case, the main objects of interest are just the sets $\{\varphi^k(x)\}$ while the indices k of individual points $\varphi^k(x)$ are irrelevant.

Developing the latter approach, let us give the following definitions. Let $N(a, A)$ be the a-neighborhood of a set $A \subset M$ and let $\mathrm{dist}_H(.,.)$ be the Hausdorff distance on the set of compact subsets of M.

First let us define two weak shadowing properties.

We say that φ has the first weak shadowing property (1WSP) if given $\epsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory ξ of φ we can find a point $p \in M$ with the property

$$\xi \subset N(\epsilon, O(p, \varphi)). \tag{2}$$

This property was introduced in [2] and called there the weak shadowing property.

We say that φ has the second weak shadowing property (2WSP) if given $\epsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory ξ of φ we can find a point $q \in M$ with the property

$$O(q, \varphi) \subset N(\epsilon, \xi). \tag{3}$$

Finally, we say that φ has the orbital shadowing property (OSP) if given $\epsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory ξ we can find a point $r \in M$ with the property

$$\operatorname{dist}_H \left(\overline{O(r,\varphi)},\, \overline{\xi} \right) < \epsilon \tag{4}$$

(as usual, $\overline{A}$ is the closure of a set A).

It is easy to see that φ has the OSP if and only if it has the 1WSP and 2WSP simultaneously and, in addition, for a d-pseudotrajectory ξ, the points p and q in (2) and (3) can be chosen to be the same.

In this paper, we study mostly the 2WSP and OSP and their relations to the classical stability properties, such as structural stability and Ω-stability.

In Sec.2, we give conditions under which a linear diffeomorphism has the 2WSP. In Sec.3, we study some C^1-open sets of diffeomorphisms defined in terms of the 2WSP. In Sec.4, we show that the C^1-interior of the set of diffeomorphisms having the OSP coincides with the set of structurally stable diffeomorphisms.

2. **2WSP and linear diffeomorphisms.**

Fix a nonsingular matrix A and consider the corresponding linear diffeomorphism

$$\varphi(x) = Ax$$

of the space $\mathbf{C}^n$.

As usual, we say that a matrix A is hyperbolic if its eigenvalues λ_j satisfy the inequalities $|\lambda_j| \neq 1$.

In the following Lemmas $1-3$, we assume that $\varphi(x) = Ax$ has the 2WSP. Obviously, φ has the 2WSP if and only if the diffeomorphism $\varphi'(x) = Jx$, where J is a Jordan form of A, has this property. Thus, we assume below that the matrix A coincides with its Jordan form.

Lemma 1. *If λ is an eigenvalue of the matrix A such that $|\lambda| = 1$, then any Jordan block of A corresponding to λ is one-dimensional.*

Proof. To get a contradiction, assume that A has a Jordan block J of dimension $k \times k$ with $k > 1$ corresponding to an eigenvalue λ with $|\lambda| = 1$. We may assume that the first two rows of A are

$$(\lambda, 0, \dots, 0)$$

and

$$(1, \lambda, 0, \dots, 0).$$

Let $d > 0$ correspond to $\epsilon = 1$ in the definition of the 2WSP. Construct a sequence $\xi = \{x_k \in \mathbf{C}^n : k \in \mathbf{Z}\}$ as follows. Denote by $x^{(i)}$ the ith component of a vector $x \in \mathbf{C}^n$. Set $x_0^{(1)} = 1$ and $x_0^{(i)} = 0$ for $i = 2, \dots, n$. For $k \geq 0$, we define $x_{k+1}^{(1)}$ by the equalities

$$x_{k+1}^{(1)} = \lambda x_k^{(1)} \left(1 + \frac{d}{2|x_k^{(1)}|}\right). \tag{5}$$

For $k \leq 0$, we define $x_{k-1}^{(1)}$ by the equalities

$$x_{k-1}^{(1)} = \lambda^{-1} x_k^{(1)} \left(1 + \frac{d}{2|x_k^{(1)}|}\right). \tag{6}$$

For $i = 2, \dots, n$, we set

$$x_{k+1}^{(i)} = (Ax_k)^{(i)}, \ k \geq 0, \ \text{and} \ x_{k-1}^{(i)} = \left(A^{-1}x_k\right)^{(i)}, \ k \leq 0.$$

Since

$$|Ax_k - x_{k+1}| = \left|(Ax_k - x_{k+1})^{(1)}\right| = \left|\frac{\lambda d x_k^{(1)}}{2|x_k^{(1)}|}\right| = \frac{d}{2},$$

the sequence ξ is a d-pseudotrajectory of φ.

By the choice of d, there exists a point $x \in \mathbf{C}^n$ with the following property: for any $m \in \mathbf{Z}$ we can find an index $k(m)$ such that

$$\left|A^m x - x_{k(m)}\right| < 1. \tag{7}$$

Note that

$$(A^m x)^{(1)} = \lambda^m x^{(1)}, \ m \in \mathbf{Z}, \ \text{and} \ (A^m x)^{(2)} = m\lambda^{m-1} x^{(1)} + \lambda^m x^{(2)}, \ m > 0,$$

hence inequalities (7) imply that

$$\left|\lambda^m x^{(1)} - x_{k(m)}^{(1)}\right| < 1, \ m \in \mathbf{Z}, \tag{8}$$

and

$$\left|m\lambda^m x^{(1)} + \lambda^m x^{(2)} - x_{k(m)}^{(2)}\right| < 1, \ m > 0. \tag{9}$$

It is easy to see that

$$\left|x_{k+1}^{(1)}\right| = \left|x_k^{(1)}\right| + \frac{d}{2},\ k \geq 0, \text{ and } \left|x_{k-1}^{(1)}\right| = \left|x_k^{(1)}\right| + \frac{d}{2},\ k \leq 0, \tag{10}$$

hence

$$\left|x_k^{(1)}\right| \to \infty \text{ as } |k| \to \infty. \tag{11}$$

Since the value $|\lambda^m x^{(1)}|$ does not depend on m, it follows from relations (8) and (11) that the set $\{k(m) : m \in \mathbf{Z}\}$ is bounded. Hence, the set $\{x_{k(m)}^{(2)}\}$ is also bounded, and we deduce from inequalities (9) that $x^{(1)} = 0$. Since

$$\left|x_k^{(1)}\right| \geq 1$$

(see inequalities (10)), we get a contradiction with inequalities (8). Our lemma is proved.

Lemma 2. *If λ is an eigenvalue of the matrix A such that $|\lambda| = 1$, then there exists a natural number m such that $\lambda^m = 1$.*

Proof. To get a contradiction, assume that A has an eigenvalue $\lambda = \cos a + i \sin a$ such that the ratio a/π is irrational. Assume that according to the standard basis $v_1, \dots, v_n$ of the space $\mathbf{C}^n$ consisting of unit vectors, the first row of the matrix A is

$$(\lambda, 0, \dots, 0).$$

Then v_1 is an eigenvector corresponding to λ.

Let $d > 0$ correspond to $\epsilon = 1$ in the definition of the 2WSP. Let

$$M = 2\left[\frac{4}{d} + 1\right]$$

(as usual, [.] is the integer part). Fix a number R_0 such that

$$2R_0 > M. \tag{12}$$

Construct a sequence $\xi = \{x_k \in \mathbf{C}^n : k \in \mathbf{Z}\}$ as follows. Set $x_0^{(1)} = R_0 + 1$ and $x_k^{(i)} = 0$ for $i = 2, \dots, n$ and all $k \in \mathbf{Z}$. For $k \geq 0$, we define $x_{k+1}^{(1)}$ by equalities (5), for $k \leq 0$, we define $x_{k-1}^{(1)}$ by equalities (6). The same reasons as in Lemma 1 show that the sequence ξ is a d-pseudotrajectory of φ and that relations (10) hold.

By the choice of d, there exists a point $x \in \mathbf{C}^n$ with the following property: for any $m \in \mathbf{Z}$ we can find an index $k(m)$ such that inequality (7) is valid.

Let L be the subspace of $\mathbf{C}^n$ spanned by the vector v_1. Take a point $x \in \mathbf{C}^n$ and set $R = |x^{(1)}|$. Our assumption on λ implies that the points $(A^k x)^{(1)}$ (the projections of the points $A^k x$ to L) belong to the circle

$$S_R = \{y \in L : |y| = R\}$$

and form a dense subset of this circle. It follows from our choice of $x_0^{(1)}$ and from inequalities (10) that if $|x^{(1)}| \le R_0$, then

$$\left| x_k^{(1)} - (A^m x)^{(1)} \right| \ge 1$$

for any k and m, hence such a point x does not have the above-formulated property.

Denote by $N'(R)$ the 1-neighborhood of the circle S_R in L. It follows from inequalities (10) that the number of points x_k of our pseudotrajectory ξ such that $x_k^{(1)} \in N'(R)$ does not exceed M. For a point $y \in L$, let $N'(y)$ be the 1-neighborhood of y in L. It is easy to see that if $R > 1$ and $|y| > 1$, then the length of the arc $S_R \cap N'(y)$ is less than π.

It follows that the total measure m_R of the set of points of S_R covered by the union of the sets $N'(x_k^{(1)})$ is less than $M\pi$. By the choice of R_0 (see (12)), the inequalities

$$m_R < M\pi < 2\pi R_0$$

hold, hence if $R = |x^{(1)}| \ge R_0$, then there exists an open subset of the circle S_R not covered by the union of the sets $N'(x_k^{(1)})$. As was mentioned above, the points $(A^k x)^{(1)}$ form a dense subset of S_R, hence there exists a point $(A^k x)^{(1)}$ not belonging to the union of the sets $N'(x_k^{(1)})$. This means that there exists $m \in \mathbf{Z}$ such that inequality (7) is valid for no indices $k(m)$. The lemma is proved.

Lemma 3. *Assume that the matrix A has the form $A = \mathrm{diag}\,(\Lambda, B)$, where Λ is a diagonal $l \times l$ matrix,*

$$\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_l),$$

such that $|\lambda_i| = 1, i = 1, \dots, l$, and B is a hyperbolic matrix. If $\mu_1, \dots, \mu_{n-l}$ are the eigenvalues of B, then either

$$|\mu_j| < 1, \; j = 1, \dots, n - l, \tag{13}$$

or

$$|\mu_j| > 1,\ j = 1, \dots, n-l. \tag{14}$$

Proof. To get a contradiction, assume that A has the form

$$A = \mathrm{diag}(\Lambda, B, C), \tag{15}$$

where the matrices B and C are hyperbolic, the eigenvalues of B satisfy analogs of inequalities (13), and the eigenvalues of C satisfy analogs of inequalities (14). Standard arguments (see [1], for example) show that it is possible to choose coordinates so that the inequalities

$$||B|| < 1 \text{ and } \left|\left|C^{-1}\right|\right| < 1 \tag{16}$$

hold, where $||.||$ is the operator norm.

Let $d > 0$ correspond to $\epsilon = 1$ in the definition of the 2WSP. Let B be an $l_1 \times l_1$ matrix and let C be an $l_2 \times l_2$ matrix. Represent

$$\mathbf{C}^n = \mathbf{C}^l \times \mathbf{C}^{l_1} \times \mathbf{C}^{l_2}$$

according to the form (15) of the matrix A and let us write $x \in \mathbf{C}^n$ as follows: $x = (v, v^{(1)}, v^{(2)})$, where

$$v \in \mathbf{C}^l, \quad v^{(1)} \in \mathbf{C}^{l_1}, \text{ and } v^{(2)} \in \mathbf{C}^{l_2}.$$

Fix two vectors

$$v' \in \mathbf{C}^{l_1} \text{ and } v'' \in \mathbf{C}^{l_2}$$

with $|v'| = |v''| = 1$ and consider a sequence $\xi = \{x_k \in \mathbf{C}^n : k \in \mathbf{Z}\}$ constructed as follows:

$$x_k = \left(v_k, B^k v', C^k v''\right),$$

where $v_k^{(i)}$, the ith component of v_k, is defined by

$$v_k^{(i)} = \frac{kd}{2}\lambda_i^k.$$

It is easy to see that ξ is a d-pseudotrajectory of φ.

By the choice of d, there exists a point $x \in \mathbf{C}^n$ with the following property: for any $m \in \mathbf{Z}$ we can find an index $k(m)$ such that inequality (7) is valid.

If $x = (y, y', y'')$ and $y^{(i)}$ is the ith component of y, then the inequalities

$$\left| y^{(i)} - \frac{k(m)d}{2} \lambda_i^{k(m)} \right| < 1, \; i = 1, \ldots, l, \tag{17}$$

$$\left| B^m y' - B^{k(m)} v' \right| < 1, \tag{18}$$

and

$$\left| C^m y'' - C^{k(m)} v'' \right| < 1 \tag{19}$$

hold for all $m \in \mathbf{Z}$.

It follows from inequalities (17) that the sequence $\{k(m)\}$ is bounded. If $y' \neq 0$, then inequalities (16) and (18) imply that $k(m) \to -\infty$ as $m \to -\infty$. Similarly, if $y'' \neq 0$, then inequalities (16) and (19) imply that $k(m) \to \infty$ as $m \to \infty$. Thus, it remains to consider the case $y' = 0, y'' = 0$. In this case, we get a contradictory pair of inequalities

$$\left| B^{k(m)} v' \right| < 1 \text{ and } \left| C^{k(m)} v'' \right| < 1$$

(recall that $|v'| = |v''| = 1$). The obtained contradiction proves our lemma.

Theorem 1. *A diffeomorphism $\varphi(x) = A'x$ of $\mathbf{C}^n$ has the 2WSP if and only if the matrix A' has a Jordan form A satisfying one of the following conditions:*

(1) A is a hyperbolic matrix;

(2) there exists a natural number m such that either

(2.1) $A^m = E$, where E is the identity matrix,

or

(2.2) $A^m = \operatorname{diag}(E, B)$, where the eigenvalues of B satisfy either condition (13) *or condition* (14).

Proof. 1. Necessity. Assume that a diffeomorphism $\varphi(x) = A'x$ has the 2WSP.

If the matrix A' is not hyperbolic, then either $A' = \Lambda$, where every eigenvalue λ of Λ satisfies the condition $|\lambda| = 1$, or A' has a Jordan form A such that $A = \operatorname{diag}(\Lambda, C)$, where Λ is as above, while the matrix C is hyperbolic. In both cases, it follows from Lemmas 1 – 3 that there exists a natural number m such that A^m satisfies either condition (2.1) or condition (2.2).

2. Sufficiency. It was noted above that if A is a Jordan form of A', then the diffeomorphisms $\varphi(x) = A'x$ and $\varphi'(x) = Ax$ have or do not have the 2WSP simultaneously. In addition, it is easy to see that if, for some natural m, the diffeomorphism $\psi(x) = A^m x$ has the 2WSP, then the diffeomorphism $\varphi(x) = Ax$ has the same property. Thus, it is enough to consider a matrix A satisfying either condition (1) of our theorem or an analog of conditions (2.1) or (2.2) (with A^m replaced by A).

It is well known (see [1, Theorem 3.2.1], for example) that if A is a hyperbolic matrix, then the diffeomorphism $\varphi(x) = Ax$ has the usual shadowing property, hence it has the 2WSP.

It is easy to see that if $A = E$ (i.e., if φ=id), then φ has the 2WSP.

Thus, it remains to consider a diffeomorphism $\varphi(x) = Ax$, where the matrix A satisfies an analog of condition (2.2). Represent

$$\mathbf{C}^n = \mathbf{C}^l \times \mathbf{C}^{l_1}$$

according to the representation

$$A = \operatorname{diag}(E, B)$$

and let $x = (v, v')$, where $v \in \mathbf{C}^l$ and $v' \in \mathbf{C}^{l_1}$. For definiteness, assume that the eigenvalues of the matrix B satisfy an analog of condition (13).

Fix a positive ϵ. Since the diffeomorphism $\psi(v') = Bv'$ has the usual shadowing property, there exists $d > 0$ such that if $\{v'_k\}$ is a d-pseudotrajectory of ψ, then there is a point v' such that

$$|\psi^k(v') - v'_k| < \epsilon/2, \; k \in \mathbf{Z}. \tag{20}$$

If $\xi = \{v_k, v'_k\}$ is a d-pseudotrajectory of φ, then obviously $\{v'_k\}$ is a d-pseudotrajectory of ψ. Find a point v' for which relations (20) hold. Since

$$\left|\psi^k(v')\right| = \left|B^k v'\right| \to 0 \text{ as } k \to \infty,$$

there exists an index κ such that $|v'_\kappa| < \epsilon$. For the fixed point $p = (v_\kappa, 0)$ of the diffeomorphism φ, the inclusion

$$O(p, \varphi) = \{p\} \subset N(\epsilon, \xi)$$

holds. Thus, φ has the 2WSP. Our theorem is proved.

Remark. Note that, for a diffeomorphism $\varphi(x) = Ax$ of $\mathbf{C}^n$, the following statements are equivalent:

(1) φ has the usual shadowing property;

(2) φ has the orbital shadowing property;

(3) φ has the first weak shadowing property;

(4) the matrix A is hyperbolic.

The equivalence of statements (1) and (4) was established by Morimoto [3] (later his proof was refined by Kakubari [4], see also [1, Theorem 3.2.1]), the equivalence of statements (2), (3) and (4) follows from the proof of Theorem 3.2.1 of [1].

3. **Open sets related to the 2WSP.**

Let M be a closed smooth n-dimensional manifold with a Riemannian metric dist. As before, we denote by $N(a, A)$ the a-neighborhood of a set $A \subset M$.

Let us introduce two classes of diffeomorphisms related to the 2WSP.

Let us say that a diffeomorphism φ has the uniform 2WSP if there exists a neighborhood W of φ with respect to the C^1 topology having the following property: given $\epsilon > 0$ we can find $d > 0$ such that if $\xi = \{x_k\}$ is a d-pseudotrajectory of a diffeomorphism $\psi \in W$, then there is a point $x \in M$ such that

$$O(x, \psi) \subset N(\epsilon, \xi).$$

Denote by I_U the set of diffeomorphims of M having the uniform 2WSP.

Let us say that a diffeomorphism φ has the Lipschitz 2WSP if there exist two positive numbers d_0 and $\mathcal{L}$ such that if $\xi = \{x_k\}$ is a d-pseudotrajectory of φ with $d \le d_0$, then there is a point $x \in M$ such that

$$O(x, \psi) \subset N(\mathcal{L}d, \xi).$$

Denote by I_L the C^1-interior of the set of diffeomorphims of M having the Lipschitz 2WSP.

Theorem 2. *If a diffeomorphism φ of M belongs to $I_U \cup I_L$, then φ satisfies Axiom A and the no-cycle condition.*

Proof. Denote by $\mathcal{F}$ the set of diffeomorphisms φ having the following property: there exists a C^1-neighborhood W of φ such that any periodic point of any diffeomorphism $\psi \in W$ is hyperbolic. It was shown by Hayashi and Aoki [5,6] that any diffeomorphism $\varphi \in \mathcal{F}$ satisfies Axiom A and the no-cycle condition.

We claim that $I_U \cup I_L \subset \mathcal{F}$. To get a contradiction, assume that there exists a diffeomorphism $\varphi \in (I_U \cup I_L) \setminus \mathcal{F}$. Note that the set I_U is C^1-open, hence the set $I_U \cup I_L$ is C^1-open. Take a C^1-neighborhood W of φ lying in $I_U \cup I_L$ and find a diffeomorphism $\varphi' \in W$ having a nonhyperbolic periodic point p. Let m be the period of p.

Passing from φ' to its C^1-small perturbation φ'', we may assume that the derivative $D(\varphi'')^m(p)$ has an eigenvalue equal to 1.

First let us assume that $m = 1$. In this case, it is easy to see that we can find a diffeomorphism $\psi \in W$ having the following properties:

- there is a neighborhood U of p with local coordinates $y = (y_1, \dots, y_n)$ such that p is the origin;

- there is a number $a > 0$ such that, for $y \in U$ with $|y| < a$,

$$\psi(y) = (y_1, By'),$$

where $y' = (y_2, \dots, y_n)$, and B is a hyperbolic matrix (here and below, $|.|$ is the Euclidean norm with respect to the corresponding local coordinates).

Standard reasons show that we can find a positive number b_0 with the following property: for any $b \in (0, b_0)$ there is a diffeomorphism $\psi_b \in W$ such that

$$\psi_b(y) = (y_1 + by_1^2, By') \text{ for } |y| < a \tag{21}$$

with respect to the coordinates y introduced above.

Let us first consider the case where the initial diffeomorphism $\varphi \in I_U$ (and hence we can choose our neighborhood W belonging to I_U). In this case, we may assume that W has the property described by the definition of the uniform 2WSP. Fix $\epsilon = a/2$. Assume that there exists the corresponding d (suitable for any diffeomorphism in W). Take $b \in (0, b_0)$ such that $ba^2 < 4d$. Consider the sequence $\xi = \{x_k\}$, where

$$x_k = \left(y_1^k, \dots, y_n^k\right) \text{ with } y_1^k = \frac{a}{2} \text{ and } y_j^k = 0,\ j = 2, \dots, n.$$

It follows from (21) that

$$|\psi_b(x_k) - x_{k+1}| = \frac{ba^2}{4}. \tag{22}$$

By the choice of b, ξ is a d-pseudotrajectory of ψ_b. Formula (21) implies that the only complete trajectory of ψ_b in the neighborhood $N(a,0)$ (with respect to the coordinates y) is the fixed point $y=0$, hence the $a/2$-neighborhood of ξ contains no complete trajectories of ψ_b. The obtained contradiction completes the consideration of the case $\varphi \in I_U$.

Now we consider the case where the initial diffeomorphism $\varphi \in I_L$ (and hence we can choose our neighborhood W belonging to I_L). Fix a number $b \in (0, b_0)$ and take d_0 and $\mathcal{L}$ such that ψ_b has the Lipschitz 2WSP with these constants. Take a positive number $c_0 < 1/2$ such that $2ba^2c_0^2 < d_0$. Then, for any $c \in (0, c_0)$, the sequence $\xi = \{x_k\}$, where

$$x_k = \left(y_1^k, \dots, y_n^k\right) \text{ with } y_1^k = ac \text{ and } y_j^k = 0, j = 2, \dots, n,$$

is a $2ba^2c^2$-pseudotrajectory of ψ_b. By our assumption, there exists a point x such that

$$O(x, \psi) \subset N(2\mathcal{L}ba^2c^2, \xi).$$

It was noted that the fixed point $y=0$ is the only complete trajectory of ψ_b in the neighborhood $N(a,0)$, hence the inequality

$$ca \le 2\mathcal{L}ba^2c^2$$

holds for any $c \in (0, c_0)$. Since the latter inequality cannot hold for all small c, we again get a contradiction.

To complete the proof, let us consider the case $m > 1$. We treat in detail only the case $\varphi' \in I_U$.

Standard reasons show that we can find a diffeomorphism $\psi \in W$ having the following properties:

(1) p is a periodic point of ψ of period m (we denote $p_i = \psi^i(p), i = 0, \dots, m$);

(2) if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the derivative $D\psi^m(p)$, then $\lambda_1 = 1$ and $|\lambda_j| \neq 1$ for $j = 2, \dots, n$;

(3) we can introduce local coordinates $y = (y_1, \dots, y_n)$ in disjoint neighborhoods U_i of the points p_i (our notation of coordinates is the same for all i but this will lead to no confusion) so that p_i is the origin in U_i, and

(3.1) if L_0 is the subspace of the tangent space T_pM (identified with $\mathbf{R}^n$) corresponding to the eigenvalue λ_1 of $D\psi^m(p)$, then the spaces $L_i = D\psi^i(p)L$, $i = 0, \dots, m-1$, coincide with the subspaces

$$\{y : y_2 = \dots = y_n = 0\}$$

in the coordinates of the corresponding neighborhoods;

(3.2) there is a number $a > 0$ such that the mapping $f^{(i)}$, the restriction of ψ to the set $U_i' = U_i \cap \{|y| < a\}$, maps U_i' into U_{i+1} (of course, $U_m = U_0$) and is given by the formula

$$f^{(i)}(y) = (g_i y_1, B_i y'),$$

where $y' = (y_2, \dots, y_n)$.

Since ψ is a diffeomorphism, $g_i \neq 0, i = 0, \dots, m-1$. It follows from (2), (3.1), and (3.2) that $g_0 g_1 \cdots g_{m-1} = 1$ and that the matrix $B = B_{m-1} \cdots B_0$ is hyperbolic (its eigenvalues are $\lambda_2, \dots, \lambda_n$). Let

$$G_1 = \min_{0 \le i \le m-1} |g_i g_{i-1} \cdots g_0|, \quad G_2 = \max_{0 \le i \le m-1} |g_i g_{i-1} \cdots g_0|, \text{ and } G = \frac{G_1}{G_2}.$$

There exists a positive number b_0 with the following property: for any $b \in (0, b_0)$ there is a diffeomorphism $\psi_b \in W$ coinciding with ψ outside U_{m-1} and such that $f_b(y)$, the restriction of ψ_b to the set U_{m-1}', is given by the formula

$$f_b(y) = (y_1 + b y_1^2, B_{m-1} y'). \tag{23}$$

Assume that W has the property described in the definition of the uniform 2WSP, fix $\epsilon = aG/2$, and find the corresponding d. Take $b \in (0, b_0)$ such that $ba^2 < 4d$. Consider the sequence $\xi = \{x_k\}$ constructed as follows. Represent a number $k \in \mathbf{Z}$ in the form $k = mj + i$, where $j, i \in \mathbf{Z}$ and $0 \le i \le m-1$. Let x_k be the point of the neighborhood U_i with coordinates

$$y_1 = g_{i-1} \cdots g_0 \frac{a}{2G_2}, \quad y_t = 0, \ t = 2, \dots, n.$$

Note that, for any point x_k, its first coordinate y_1 satisfies the inequalities

$$\epsilon = G_1 \frac{a}{2G_2} \le |y_1| \le G_2 \frac{a}{2G_2} = \frac{a}{2}. \tag{24}$$

If $i \neq m-1$, then $\psi_b(x_k) = x_{k+1}$. If $i = m-1$, then

$$|\psi_b(x_k) - x_{k+1}| \le b \left(g_{m-2} \cdots g_0\right)^2 \frac{a^2}{4G_2^2} \le b \frac{a^2}{4},$$

hence ξ is a d-pseudotrajectory of ψ_b.

Formula (23) implies that the trajectory of the periodic point p is the only complete trajectory of ψ_b belonging to the union of the sets U_i', hence it follows from inequalities (24) that the set $N(\epsilon, \xi)$ contains no complete trajectories of ψ_b. The obtained contradiction completes the proof of Theorem 2.

Denote by I_{UL} the set of diffeomorphisms φ having the following property: there exists a C^1-neighborhood W of φ and two positive numbers d_0 and $\mathcal{L}$ such that any diffeomorphism $\psi \in W$ has the Lipschitz 2WSP with constants d_0 and $\mathcal{L}$. Obviously, I_{UL} is an open subset of $I_U \cap I_L$. By Theorem 2, any diffeomorphism $\varphi \in I_{UL}$ satisfies Axiom A and the no-cycle condition.

Denote by SS the set of structurally stable diffeomorphisms.

Theorem 3. *There exist diffeomorphisms belonging to $I_{UL} \setminus SS$.*

To prove this theorem, we need the following statement (a corollary of [1, Theorem 1.2.5]).

Lemma 4. *Let p be a hyperbolic fixed point of a diffemorphism φ. There exist neighborhoods $W(p)$ of φ in the C^1 topology and $U(p)$ of the point p in M and positive numbers $d(p)$ and $\mathcal{L}(p)$ having the following property. If $\{x_k\}$ is a d-pseudotrajectory of a diffeomorphism $\psi \in W(p)$ with $d \le d(p)$ and there exists a number l such that the inclusions $x_k \in U(p)$ hold for $k \ge l$, then there is a point x satisfying the inequalities*

$$\operatorname{dist}\left(\psi^k(x), x_{k+l}\right) \le \mathcal{L}(p)d, \ k \ge 0.$$

Proof of Theorem 3. Let φ be an Ω-stable diffeomorphism such that its nonwandering set $\Omega(\varphi)$ consists of fixed points. We claim that $\varphi \in I_{UL}$.

Since the set $\Omega(\varphi)$ is hyperbolic [7], it consists of a finite number of fixed points, let

$$\Omega(\varphi) = \{p_1, \dots, p_N\}.$$

Apply Lemma 4 and find the corresponding neighborhoods $W(p_i)$, $U(p_i)$, and numbers $d(p_i)$, $\mathcal{L}(p_i)$, $i = 1, \dots, N$. Let

$$W_1 = \bigcap_{1 \le i \le N} W(p_i), \ d_1 = \min_{1 \le i \le N} d(p_i), \ \text{and } \mathcal{L}_1 = \max_{1 \le i \le N} \mathcal{L}(p_i).$$

It is easy to see that we can find the neighborhoods $U(p_i)$ and W_1 (decreasing them, if necessary) so that there exists a number $\epsilon > 0$ such that, for any

i, any diffeomorphism $\psi \in W_1$ has a unique fixed point $p_i' \in U(p_i)$ and, in addition,

$$\Omega(\psi) \cap N(\epsilon, U(p_i)) = \{p_i'\}. \tag{25}$$

Since the diffeomorphism φ is Ω-stable, there exists a continuous Lyapunov function

$$V : M \to [1, N]$$

such that

$$V(\varphi(x)) \leq V(x), \text{ and } V(\varphi(x)) = V(x) \text{ if and only if } x \in \Omega(\varphi)$$

(see [8]). Set

$$M' = M \setminus \bigcup_{1 \leq i \leq N} U(p_i).$$

The set M' is compact, hence there exists a positive number a such that

$$V(\varphi(x)) - V(x) \leq -3a \text{ for } x \in M'. \tag{26}$$

The continuous function V is uniformly continuous on M, hence there exists a positive number δ such that $|V(x) - V(y)| < a$ for $x, y \in M$ with $\text{dist}(x, y) < \delta$. Find a C^1-neighborhood W' of φ such that $\text{dist}(\varphi(x), \psi(x)) < \delta$ for $x \in M$ and $\psi \in W'$.

Set

$$d_0 = \min\left(d_1, \frac{\epsilon}{\mathcal{L}_1 + 1}, \delta\right).$$

Let us show that $W = W_1 \cap W'$, d_0, and $\mathcal{L} = \mathcal{L}_1 + 1$ have the property described in the definition of the set I_{UL}. Fix a diffeomorphism $\psi \in W$ and let $\xi = \{x_k\}$ be a d-pseudotrajectory of ψ such that $d \leq d_0$.

The choice of W and d_0 implies the inequalities

$$V(x_{k+1}) - V(x_k) = V(x_{k+1}) - V(\psi(x_k)) + V(\psi(x_k)) - V(\varphi(x_k)) +$$
$$+ V(\varphi(x_k)) - V(x_k) < -a. \tag{27}$$

Since the function V is bounded, it follows from inequalities (27) that the set $\{k : x_k \in M'\}$ is finite, hence there exists an index l and a neighborhood $U(p_j)$ such that the inclusions $x_k \in U(p_j)$ hold for $k \geq l$.

By Lemma 4, there exists a point x such that

$$\text{dist}\left(\psi^k(x), x_{k+l}\right) \leq \mathcal{L}_1 d, \ k \geq 0. \tag{28}$$

Since $\mathcal{L}_1 d < \epsilon$, the points $\psi^k(x)$, $k \geq 0$, belong to $N(\epsilon, U(p_j))$, and it follows from relation (25) that

$$\psi^k(x) \to p'_j \text{ as } k \to \infty.$$

Hence, there exists an index $m > 0$ such that

$$\operatorname{dist}\left(\psi^m(x), p'_j\right) < d. \tag{29}$$

It follows from inequalities (28) and (29) that

$$O(p'_j, \psi) = \{p'_j\} \in N(\mathcal{L}d, x_{l+m}).$$

Hence, $\varphi \in I_{UL}$. To complete the proof, it remains to note that there exist structurally unstable Ω-stable diffeomorphisms with nonwandering set $\Omega(\varphi)$ consisting of fixed points.

Let us describe some differences between the 1WSP and 2WSP.

It was mentioned in the proof of Theorem 1 that the mapping $\varphi(x) = x$ has the 2WSP (for any phase space). It is easy to see that, for any manifold M with $\dim M > 1$, the identity mapping does not have the 1WSP.

We do not have an example of a system having the 2WSP and not having the 1WSP but it is possible to find such examples for "uniform" variants of these properties. Let us define the uniform 1WSP similarly to the uniform 2WSP (see the beginning of this section).

It was noted in [9] that Mañé constructed in [10] a C^1-open set O of diffeomorphisms of the three-dimensional torus T^3 with the following properties:

– any diffeomorphism $\varphi \in O$ has a dense trajectory (it is easy to see that in this case any $\varphi \in O$ has the uniform 1WSP);

– any diffeomorphism $\varphi \in O$ is not Anosov (hence it is not Ω-stable).

It follows from Theorem 3 that any $\varphi \in O$ has the uniform 1WSP and does not have the uniform 2WSP.

Let us also mention a diffeomorphism φ of the two-dimensional torus T^2 studied in [11] (see also [1, Section 2.3]). The nonwandering set $\Omega(\varphi)$ consists of 4 hyperbolic fixed points,

$$\Omega(\varphi) = \{p_1, p_2, p_3, p_4\},$$

where p_1 is a sink, p_4 is a source, and p_2, p_3 are saddles such that

$$W^s(p_2) \cup \{p_3\} = W^u(p_3) \cup \{p_2\}$$

(i.e., φ has the so-called saddle connection). It is assumed that the eigenvalues of $D\varphi(p_2)$ are $-\mu, \nu$ with $\mu > 1, 0 < \nu < 1$, and the eigenvalues of $D\varphi(p_3)$ are $-\lambda, \kappa$ with $\kappa > 1, 0 < \lambda < 1$ (in addition, it is assumed that φ satisfies some local linearity conditions).

It follows from the results of [12] that φ does not have the usual shadowing property. Plamenevskaya showed that φ has the WSP if and only if the value $\log(\mu)/\log(\nu)$ is irrational. Since φ satisfies the no-cycle condition, φ is Ω-stable, and it follows from the proof of Theorem 3 that $\varphi \in I_{UL}$. Thus, the usual shadowing property, the weak shadowing property, and the second weak shadowing property are related to quite different characteristics of φ.

In this connection, let us also note the following statement proved in [9]: the C^1-interior of the set of diffeomorphisms of a closed surface having the WSP consists of structurally stable diffeomorphisms.

4. C^1-interior of the set of diffeomorphisms having the OSP.

In this section, we prove the following result.

Theorem 4. *If φ is in the C^1-interior of the set of diffeomorphisms having the OSP, then φ is structurally stable.*

Proof. Denote by I_O the C^1-interior of the set of diffeomorphisms φ having the OSP.

First we claim that $I_O \subset \mathcal{F}$. To get a contradiction, assume that there exists $\varphi \in I_O \setminus \mathcal{F}$. Take a neighborhood W of φ lying in I_O and find a diffeomorphism $\varphi' \in W$ having a nonhyperbolic periodic point q of period m. We assume that $D(\varphi')^m(q)$ has an eigenvalue equal to 1 (see the proof of Theorem 3). To simplify presentation, we assume that $m = 1$ (the case of a periodic point of minimal period $m > 1$ is considered similarly) and write φ instead of φ'. Applying a C^1-small perturbation (so that the perturbed diffeomorphism denoted again by φ belongs to W), we may assume that φ has the following property. With respect to some local coordinates $(x_1, \dots, x_n)$, q is the origin and there exists a number $a > 0$ such that φ maps a point $x = (x_1, y)$ with $|x| < 4a$, where $y = (x_2, \dots, x_n)$, to the point (x_1, By), where B is a hyperbolic matrix.

In this case, the segment

$$\Delta = \{(x_1, 0, \dots, 0) : 0 < |x_1| < 4a\}$$

consists of fixed points of φ

It was assumed that φ is in the C^1-interior of the set of diffeomorphisms having the OSP. Take $\epsilon = a$ and find the corresponding $d < \epsilon$ from the definition of the OSP. There exists a natural number m such that the sequence $\xi = \{x_k : k \in \mathbf{Z}\}$, where

$$x_k = \begin{cases} 0 & \text{for} \;\; k < 0, \\ (2ak/m, 0, \dots, 0) & \text{for} \;\; 0 \le k \le m, \\ (2a, 0, \dots, 0) & \text{for} \;\; k > m, \end{cases}$$

is a d-pseudotrajectory of φ. Let $x \in N(\epsilon, x_0)$ be a point such that

$$\mathrm{dist}_H \left(\overline{\xi}, \overline{O(x, \varphi)} \right) < \epsilon.$$

Since the matrix B is hyperbolic, for any point (x_1, y) with $y \neq 0$, its trajectory leaves the set $\{|x| < 4a\}$, hence the inclusion

$$O(x, \varphi) \subset N(2a, \xi)$$

implies that $x = (b, 0, \cdots, 0)$.

The inclusion

$$\xi \subset N(\epsilon, \overline{O(z, \varphi)})$$

implies that $|b| < a$ and $|b - 2a| < a$. The obtained contradiciton proves that $I_O \subset \mathcal{F}$, hence any diffeomorphism $\varphi \in I_O$ satisfies Axiom A and the no-cycle condition (and hence φ is Ω-stable).

It is known [6] that to establish the structural stability of a diffeomorphism $\varphi \in I_O$ it remains to prove the following statement: if p and q are periodic points of φ, then their stable manifold $W^s(p)$ and unstable manifold $W^u(q)$ are transverse.

To get a contradiction, let us assume that there is a diffeomorphism $\varphi \in I_O$ having periodic points p and q and a point r of nontransverse intersection of $W^s(p)$ and $W^u(q)$. Note that in this case the point r is wandering (stable and unstable manifolds of points of Ω are always transverse for an Ω-stable diffeomorphism) and p and q are in different basic sets.

To simplify presentation, we assume that p and q are fixed points.

First let us show that, for any C^1-neighborhood W of φ, it is possible to find a diffeomorphism $\psi \in W$ with the following properties:

– p and q are fixed points of ψ;

– ψ is linear in a neighborhood of p;

– r is a point of nontransverse intersection of the stable manifold $W^s(p, \psi)$ of p with respect to ψ and the unstable manifold $W^u(q, \psi)$ of q with respect to ψ.

Indeed, let us introduce local coordinates x near p so that p is the origin. Then, for any $\delta > 0$, we may use the standard (bump function) procedure to linearize φ in $N(\delta, 0)$ so that there exists a diffeomorphism φ_δ such that

– $\varphi_\delta(x) = \varphi(x)$ for $x \notin N(4\delta, 0)$

and

– $\varphi_\delta(x) = D\varphi(p)x$ for $x \in N(\delta, 0)$.

Standard estimates show that φ_δ converges to φ as $\delta \to 0$ with respect to the C^1-topology. Then, for any fixed $a > 0$, the local stable manifold $W^s_a(p, \varphi_\delta)$ of size a converges to $W^s_a(p, \varphi)$ as $\delta \to 0$ with respect to the C^1-topology.

Fix a small neighborhood U of p not containing the point r and find $a > 0$ such that $W^s_a(p, \varphi)$ and $W^s_a(p, \varphi_\delta)$ for small δ are in U. Fix a natural number m such that $\varphi^m(r) \in W^s_a(p, \varphi)$.

Since the point r is wandering, there exists a small neighborhood V of r such that

$$\varphi^k(V) \cap V = \emptyset$$

for $k \neq 0$.

It follows from our considerations above (since m is fixed) that

$$\varphi_\delta^{-k}(W^s_a(p, \varphi_\delta)) \to \varphi^{-k}(W^s_a(p, \varphi))$$

for $k = 0, \dots, m$ as $\delta \to 0$ with respect to the C^1-topology.

Hence, it is possible to construct diffeomorphisms ψ_δ such that

– $\psi_\delta(x) = \varphi(x)$ for $x \notin V$;

– ψ_δ maps the intersection $\varphi^{-m}(W^s_a(p, \varphi)) \cap V$ to $\varphi_\delta^{-m+1}(W^s_a(p, \varphi_\delta)) \cap \varphi(V)$;

– $\psi_\delta \to \varphi$ as $\delta \to 0$ with respect to the C^1-topology.

It follows that we can find a diffeomorphism ψ with the desired properties in any C^1-neighborhood of φ. In what follows, we denote ψ again by φ (and we do the same after C^1-small modifications of φ described below).

In the following proof, we have to consider various types of behavior of φ near the fixed point p. Let us select the case to which we pay the main attention. We consider this case in detail (and describe the necessary modifications for the remaining cases).

Let Σ be the spectrum of $D\varphi(p)$. Denote $\dim W^s(p) = s$ and $\dim W^u(p) = u$ $(s + u = \dim M)$.

Case 1. The relation

$$\Sigma \cap \{\lambda \in \mathbf{C} : |\lambda| > 1\} \subset \mathbf{R} \tag{30}$$

and the inequality $u \geq 2$ hold.

By (30), in local coordinates $(y; z)$ in a neighborhood U of p, the matrix $D\varphi(p)$ has the block-diagonal form $D\varphi(p) = \mathrm{diag}(A, B)$, where $||A|| < 1$, $||(B)^{-1}|| < 1$, and the eigenvalues of B are real. Applying an arbitrarily C^1-small perturbation of φ, we may assume that

$$B = \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_u \end{pmatrix},$$

where $1 < |\lambda_1| < \cdots < |\lambda_u|$.

Note that we use the notation $(y; z)$, where the coordinate y is s-dimensional and the coordinate z is u-dimensional.

Let $E^s = \{z = 0\}$ and $E^u = \{y = 0\}$. Obviously, the component of the intersection $W^s(p) \cap U$ containing p belongs to E^s and the component of the intersection $W^u(p) \cap U$ containing p belongs to E^u.

Take $a > 0$ such that if $|(y; z)| < 4a$, then $(y; z) \in U$. We may assume that that our point r of nontransverse intersection satisfies the inequality $|r| < a$ with respect to our local coordinates. Let $r' = \varphi^{-1}(r)$, let $L = T_r W^u(q)$, and let L_1 be the affine space $L_1 = L + r$.

We consider a C^1-small perturbation of φ (again denoted φ) with the following property:

– there exists a small open (with respect to the inner topology of $W^u(q)$) disk $\mathcal{C} \subset W^u(q)$ such that $r' \in \mathcal{C}$ and

$$\varphi(\mathcal{C}) \subset L_1. \tag{31}$$

The nontransversality of $W^u(q)$ and $W^s(p)$ at r means that

$$T_r M \neq L + E^s.$$

In other words, if L' is the projection of L to E^u parallel to E^s, then the nontransversality means that $\dim L' < u$.

The latter inequality allows us to modify φ (for the last time) and assume that the following statement holds:

$$L' \cap \{z_2 = \ldots = z_u = 0\} = \{0\} \tag{32}$$

(here $z_1, \ldots, z_u$ are the coordinates in E^u).

It is easy to see that there exists a constant $c > 0$ such that

$$\frac{|z_1|}{|z_2| + \ldots + |z_u|} \leq c \tag{33}$$

if $z = (z_1, \ldots, z_u) \in L' \setminus \{0\}$.

For $\epsilon > 0$, let $C^u(\epsilon, r)$ be the connected component of $W^u(q) \cap N(\epsilon, r)$ containing r.

Then $C^u(\epsilon, r) \subset \varphi(\mathcal{C}) \subset L_1$ if ϵ is small enough.

Let Ω_1 and Ω_2 be the basic sets of φ containing q and p, respectively. Let E_1 be the line $\{z_2 = \ldots = z_u = 0\}$ in E^u through p.

Case 1 is divided into two subcases.

Case 1.1. $(E_1 \cap U) \setminus \Omega \neq \emptyset$.

Since the line E_1 is φ-invariant in U, there are wandering points of φ in any neighborhood of p. Let us assume for definiteness that

$$\zeta = (0; a, 0, \ldots, 0) \notin \Omega.$$

In this case, there exists a basic set Ω_3 such that $\zeta \in W^s(\Omega_3)$ (we denote by $W^s(\Omega_i)$ and $W^u(\Omega_i)$ the stable and unstable manifolds of a basic set Ω_i). Note that Ω_3 is different from Ω_1 and Ω_2.

Assume that the point r has coordinates $r = (r_y; 0)$.

For a natural number m, put $\zeta_m = (r_y; a{\lambda_1}^{-m}, 0, \ldots, 0)$. Then

$$\varphi^k(\zeta_m) = (A^k r_y; a{\lambda_1}^{k-m}, 0, \ldots, 0)$$

for $0 \leq k \leq m-1$. Consider a sequence $\xi_m = \{x_k : k \in \mathbf{Z}\}$ defined as follows:

$$x_k = \begin{cases} \varphi^k(r) & \text{for} \quad k < 0, \\ \varphi^k(\zeta_m) & \text{for} \quad 0 \leq k \leq m-1, \\ \varphi^{k-m}(\zeta) & \text{for} \quad k \geq m. \end{cases}$$

It follows from our assumptions that for any $d > 0$ there exists $m_0 > 0$ such that if $m \geq m_0$, then ξ_m is a d-pseudotrajectory of φ.

Lemma 5. *There exists $\epsilon_0 > 0$ such that if*

$$\mathrm{dist}_H\left(\overline{\xi_m}, \overline{O(x,\varphi)}\right) < \epsilon \tag{34}$$

for some $\epsilon \in (0, \epsilon_0)$ and x, then

$$O(x,\varphi) \cap N(\epsilon, r) \subset C^u(\epsilon, r).$$

To prove Lemma 5, we need some preliminaries. The following statement is well known.

Lemma 6. *If Ω_i is a basic set of an Ω-stable diffeomorphism φ, then for any $\alpha > 0$ there exists $\beta > 0$ such that if*

$$x \in W^u(\Omega_i) \cap N(\beta, \Omega_i),$$

then

$$O^-(x,\varphi) \subset N(\alpha, \Omega_i).$$

Here and below,

$$O^-(x,\varphi) \text{ and } O^+(x,\varphi)$$

are the negative semi-trajectory and the positive semi-trajectory of x, respectively.

Proof (of Lemma 5.) Consider a Lyapunov function V applied in the proof of Theorem 3. Since V is constant on basic sets, we denote by V_i the value of V on a basic set Ω_i. Since

$$r \in W^u(\Omega_1) \cap W^s(\Omega_2) \text{ and } \zeta \in W^u(\Omega_2) \cap W^s(\Omega_3),$$

we can find a positive number α such that

$$V_1 - V_2, V_2 - V_3 > 2\alpha.$$

Decreasing the number a chosen above, we may assume that

$$V_2 - \alpha < V(x) < V_2 + \alpha \tag{35}$$

for $x \in N(4a, p)$.

Consider a d-pseudotrajectory ξ_m constructed above and denote $v_k = V(x_k)$. Since V decreases along trajectories, we have the inequalities

$$v_k < v_m \text{ for } k > m$$

and

$$v_k > v_0 \text{ for } k < 0.$$

In addition, $v_m < V_2 < v_0$ since $x_0 \in W^s(p)$ and $x_m \in W^u(p)$. Hence, if a point x' is close to $x_0 = r$, then $V(x') > V_2 > V_3$, and it follows that

$$x' \in W^u(\Omega_i), \tag{36}$$

where $i \neq 2, 3$.

Obviously (under a proper choice of a) there exists $\delta > 0$ such that if $i \notin \{1, 2, 3\}$, then

$$\operatorname{dist}(x_k, \Omega_i) \geq \delta$$

for all k. Hence, if $i \neq 1$ in (36), then inequality (34) cannot hold for small $\epsilon > 0$.

Thus, it is enough to consider the case of a point

$$x' \in O(x, \varphi) \cap W^s(\Omega_1) \cap N(\epsilon, r).$$

Since $V_1 \geq V_2 + 2\alpha$ and $v_k \leq V_2 + \alpha$ (see (35)), there exists a positive constant β such that

$$x_k \notin N(2\beta, \Omega_1) \text{ for } k \geq 0. \tag{37}$$

The Stable Manifold Theorem implies that there exists $\epsilon_1 > 0$ such that for any $l_1 < 0$ we can find $E = E(\epsilon_1, l_1)$ with the following property: for any $\epsilon \in (0, E(\epsilon_1, l_1))$ and for any

$$x' \in N(\epsilon, r) \setminus C^u(\epsilon, r) \tag{38}$$

there exists $l < l_1$ such that

$$\varphi^l(x') \notin N(2\epsilon_1, q). \tag{39}$$

Find $l_2 < 0$ such that

$$\varphi^k(r) \in N(\epsilon_1, \Omega_1) \text{ for } k < l_2 \tag{40}$$

and denote

$$2\gamma = \min_{l_2 \le k \le 0} \mathrm{dist}(\varphi^k(r), \Omega).$$

Obviuosly, $\gamma > 0$. It follows from Lemma 6 that there exist $\epsilon_2 > 0$ and $l_3 < 0$ such that

$$\varphi^l(x') \in N(\min(\beta, \gamma), \Omega_1) \tag{41}$$

for any $x' \in N(\epsilon_2, r) \cap W^u(\Omega_1)$ and any $l \le l_3$.

Set

$$\epsilon_3 = \min(\epsilon_1, \beta, \gamma) \text{ and } l_4 = \min(l_2, l_3).$$

Take $0 < \epsilon_0 < \min(E(\epsilon_1, l_4), \epsilon_2, \epsilon_3)$. We claim that this ϵ_0 has the desired property.

Fix $\epsilon \in (0, \epsilon_0)$, take a point x' such that (38) holds, and find $l < l_4$ such that (41) is satisfied. Let $x'' = \varphi^l(x')$.

Relations (37) and (41) imply that

$$\mathrm{dist}(x'', x_k) \ge \beta \text{ for } k \ge 0.$$

Relation (41) and the definition of γ imply that

$$\mathrm{dist}(x'', x_k) \ge \gamma \text{ for } l_2 \le k \le 0.$$

Finally, relations (39) and (40) imply that

$$\mathrm{dist}(x'', x_k) \ge \epsilon_1 \text{ for } k \le l_2.$$

Hence,

$$\mathrm{dist}_H\left(\overline{O(x, \varphi)}, \overline{\xi_m}\right) \ge \epsilon_3 > \epsilon.$$

Lemma 5 is proved.

The inclusions

$$\zeta \in W^s(\Omega_3) \text{ and } r \in W^s(\Omega_1)$$

imply that there exists a number $\gamma > 0$ such that

$$\mathrm{dist}(x, \chi) \ge \gamma$$

for any point

$$x \in O^+(\zeta, \varphi) \cup O^-(r, \varphi)$$

and any point $\chi = (0; z_1, \ldots, z_u)$ with $|z_i| \le \gamma$.

It follows from the construction of ξ_m that there exists $\epsilon > 0$ such that the set

$$I = \left\{ (y; z) : |y| < \epsilon; |z_1| < \epsilon; \frac{\gamma}{|\lambda_u|} \leq |z_2| + \ldots + |z_u| \leq \gamma \right\}$$

has the following property:

$$\operatorname{dist}(x_k, I) > \epsilon, \quad k \in \mathbf{Z}. \tag{42}$$

It was assumed that φ has the OSP, hence there exists m_0 such that for any $m \geq m_0$ there is a point $\eta_m = (r_m; \chi_m)$ such that

$$\operatorname{dist}_H \left(\overline{\xi_m}, \overline{O(\eta_m, \varphi)} \right) < \epsilon. \tag{43}$$

Of course, we may assume that $\eta_m \in N(\epsilon, r)$. Applying Lemma 5 (and decreasing ϵ if necessary), we see that $\eta_m \in C^u(\epsilon, r)$ for $m \geq m_0$, hence

$$\eta_m = (r_m; z_{1,m}, \ldots, z_{u,m}) \in L_1$$

for large m. By (33), $\eta'_m := |z_{2,m}| + \ldots + |z_{u,m}| \neq 0$. Since the OSP implies that $\eta'_m \to 0$ as $m \to 0$, there exist numbers $k(m)$ such that $\varphi^k(\eta_m) \in \{|x| \leq 4a\}$ for $0 \leq k \leq k(m)$,

$$\frac{\gamma}{|\lambda_u|} \leq |\lambda_2|^{k(m)} |z_{2,m}| + \ldots + |\lambda_u|^{k(m)} |z_{u,m}| \leq \gamma,$$

and $k(m) \to \infty$ as $m \to \infty$.

Since

$$\varphi^{k(m)}(\zeta_m) = \left(A^{k(m)} r_m; \lambda_1^{k(m)} z_{1,m}, \ldots, \lambda_u^{k(m)} z_{u,m} \right)$$

and

$$\frac{|z_{1,m}||\lambda_1|^{k(m)}}{|z_{2,m}||\lambda_2|^{k(m)} + \ldots + |z_{u,m}||\lambda_u|^{k(m)}} \leq$$

$$\leq \frac{|z_{1,m}||\lambda_1|^{k(m)}}{\eta'_m |\lambda_2|^{k(m)}} \leq c \left| \frac{\lambda_1}{\lambda_2} \right|^{k(m)} \to 0$$

as $m \to \infty$, the points $\varphi^{k(m)}(\eta_m) \in I$ for large m. We get the desired contradiction with inequality (43).

Now let us consider the following case.

Case 1.2. $E_1 \cap U \subset \Omega$. Note that in this case the basic set Ω_2 is nontrivial, i.e., it contains an infinite set of different periodic points.

Consider the cone

$$K = \{(y; z) : |z_2| + \ldots + |z_u| < |z_1|/4\}.$$

Obviously, there exists an open (with respect to the interior topology of $W^u(p)$) disk $D \subset W^u(p)$ centered at the point $\zeta = (0; a, \ldots, 0)$ such that $D \subset K \cap E^u$.

By our assumption, the point ζ is nonwandering, hence ζ is a point of transverse intersection of $W^u(p)$ with the stable manifold $W^s(p_0)$ of some point $p_0 \in \Omega$. Periodic points are dense in Ω, hence there is a sequence of periodic points p_k converging to p_0. Since the stable manifolds of p_k tend (in the C^1-topology) to $W^s(p_0)$ (on compact subsets with respect to the inner topology), there exists a periodic point $p' \neq p$ and a point

$$\zeta = (0; z'_1, \ldots, z'_u) \in W^s(p') \cap D.$$

We fix these points p' and ζ and do not change them in the following below process of constructing pseudotrajectories ξ_m. Note that $z'_1 \neq 0$.

Note that since $p \neq p'$, there exists $b > 0$ such that

$$\mathrm{dist}(\varphi^k(\zeta), p) \geq b \text{ for } k \geq 0.$$

Fix a natural number m, denote

$$\zeta_m = \left(r_y; z'_1 \lambda_1^{-m}, \ldots, z'_u \lambda_u^{-m}\right),$$

and define a sequence ξ_m by the same formulas as in case 1.1.

The same reasons as in case 1.1 prove an analog of Lemma 5 for ξ_m.

Since

$$\frac{|\lambda_2^{-k} z'_2| + \ldots + |\lambda_u^{-k} z'_u|}{\left|\lambda_1^{-k} z'_1\right|} \leq \frac{|z'_2| + \ldots + |z'_u|}{|z'_1|}$$

for $0 \leq k \leq m$, the inclusions $\varphi^k(\zeta_m) \in K$ hold for $k \geq 0$, Hence, we can find $\epsilon > 0$ such that the set I defined by the same formula as in case 1.1 has property (42). We complete the proof in case 1.2 similarly to the previuos case.

Case 2. The equality dim$W^u(p) = 1$ holds. In this case, the matrix $D\varphi(p)$ has exactly one eigenvalue λ such that $|\lambda| > 1$; of course, λ is real.

In the notation of case 1, the nontransversality of $W^u(q)$ and $W^s(p)$ at r means that

$$L \subset E^s. \tag{44}$$

Consider the point $\zeta = (0; a)$ (as previously, we assume that $\{|x| < 4a\} \subset U$).

First let us assume that $\zeta \notin \Omega$. For a natural m, we take $\zeta_m = (0; a\lambda^{-m})$ and define a pseudotrajectory $\xi_m = \{x_m\}$ similarly to case 1.1. An analog of Lemma 5 shows that if $\epsilon > 0$ is small enough and, for a point x, inequality (34) holds, then

$$O(x, \varphi) \cap N(\epsilon, p) \subset C^u(\epsilon, r) \subset W^s(p),$$

hence any shadowing trajectory belongs to $W^u(q) \cap W^s(p)$. Obviously, this leads to a contradiction.

The case $\zeta \in \Omega$ is treated similarly.

Case 3. The matrix $D\varphi(p)$ has complex eigenvalues λ such that $|\lambda| > 1$. As previously, let Σ be the spectrum of $D\varphi(p)$. Replacing φ by a C^1-small perturbation, we may assume that any circle $C_\mu = \{|\lambda| = \mu\} \subset \mathbf{C}$ of radius $\mu > 1$ such that $C_\mu \cap \Sigma \neq \emptyset$ contains either one simple real eigenvalue of $D\varphi(p)$ or two simple complex conjugated eigenvalues.

Let $\{\mu_1, \dots, \mu_v\}$ be the set of all radii of the mentioned circles C_μ numbered so that $1 < \mu_1 < \dots < \mu_v$. If the circle C_{μ_1} contains a real eigenvalue, then the proof does not differ from the proof given in case 1.

Consider the case where the circle C_{μ_1} contains two eigenvalues

$$\lambda_{1,2} = \mu_1(\cos\theta \pm i \sin\theta). \tag{45}$$

Let E^u_1 be the two-dimensional subspace of E^u with real coordinate $z' = (z_1, z_2)$ corresponding to the pair (45). Denote by E^u_2 the subspace with coordinate $z'' = (z_3, \dots, z_u)$ corresponding to the remaining eigenvalues λ with $|\lambda| > \mu_1$.

We assume that the matrix $D\varphi(p)$ represented in the block-diagonal form $D\varphi(p) =$diag(A, B_1, B_2) corresponding to the decomposition $x = (y; z', z'')$ has the following property: a scalar block corresponds to a real eigenvalue and a (2×2)-block corresponds to a complex conjugated pair. Obviously, in this case the inequality $|z''| < \beta|z'|$ with $\beta > 0$ implies the inequalities $|B_2^{-k}z''| < \beta|B_1^{-k}z'|$ for $k \geq 0$.

Let Π be the projection to $E^u = E_1^u \oplus E_2^u$ parallel to E^s. Denote by Π_1 and Π_2 the projections to E_1^u parallel to $E_2^u \oplus E^s$ and to E_2^u parallel to $E_1^u \oplus E^s$, respectively.

Let

$$L' = \Pi_1 T_r W^u(q).$$

Since it was assumed that $W^s(p)$ and $W^u(q)$ are nontransverse at r, we may perturb φ so that $\dim L' < \dim E_1^u = 2$.

Case 3.1. $\dim L' = 1$. Replacing φ by a C^1-small perturbation, we may assume that, for the number θ in (45), the ratio θ/π is rational. Find a natural number l such that

$$(\cos\theta \pm i\sin\theta)^l = 1. \tag{46}$$

Geometrically, condition (46) means the following: there exist l lines $E_1 = L', E_2, \dots, E_l$ in the 2-dimensional space E_1^u such that the linear mapping $x \mapsto D\varphi(p)x$ takes E_i to E_{i+1} (where $E_{l+1} = E_1$).

Take a point $\zeta = (0; z_1^*, z_2^*, 0, \dots, 0)$ such that $|\zeta| = a$ and ζ does not belong to the union $E = E_1 \cup \ldots \cup E_l$. Let E' be the line in E_1^u containing the origin and the point ζ. There exist l lines $E_1' = E', E_2', \dots, E_l'$ such that the mapping $x \mapsto Df(p)x$ takes E_i' to E_{i+1}' (where $E_{l+1}' = E_1'$). Let $E^* = E_1' \cup \ldots \cup E_l'$.

Let $\zeta \notin \Omega$. Fix a natural number m, let $\zeta_m = (r_y; B_1^{-m}(z_1^*, z_2^*), 0, \dots, 0)$, and define a pseudotrajectory ξ_m similarly to case 1.1. Find $\gamma \in (0, a/2)$ such that

$$\mathrm{dist}(\varphi^k(\zeta), p) \geq 3\gamma \text{ for } k \geq 0$$

and

$$\mathrm{dist}(\varphi^k(r), p) \geq 3\gamma \text{ for } k \leq 0.$$

For a line E_i, consider the set

$$I(E_1) = \left\{ x \in E_i : \frac{\gamma}{\mu_v} \leq \mathrm{dist}(x, p) \leq \gamma \right\}.$$

Let $I_1 = I(E_1) \cup \ldots \cup I(E_l)$.

Denote by I_2 the set

$$I_2 = \left\{(0; 0, 0, z'') : \frac{\gamma}{\mu_v} \leq |z''| \leq \gamma\right\}.$$

Find a positive number β such that the cone

$$K = \{(y; z', z'') : |z''| < \beta|z'|\}$$

has the following property:

$$\Pi_2 K_a \subset \left\{(0; 0, 0, z'') : |z''| < \frac{\gamma}{2\mu_v}\right\},$$

where

$$K_a = K \cap \{|x| \leq a\}.$$

There exists a number $\epsilon \in (0, \gamma)$ having the following properties:
– if $\Pi_1 x \in I_1$, then $\mathrm{dist}(x, E^*) > \epsilon$;
– if $|\Pi_1 x| \leq \gamma$ and $\Pi_2 x \in I_2$, then $\mathrm{dist}(x, K) > \epsilon$.

It was assumed that φ has the OSP, hence, for m large enough, there exist points $\eta_m \in N(\epsilon, r)$ satisfying inequality (43).

It follows from our assumption concerning the structure of $D\varphi(p)$ that

$$\varphi^k(\zeta_m) \in K \text{ for } 0 \leq k \leq m. \tag{47}$$

An analog of Lemma 5 shows that if $\epsilon > 0$ is small enough, then $\Pi_1\eta_m \in L'$. Hence, while the inequality $|\varphi^k(\eta_m)| < 4a$ holds, we have the inclusions $\Pi_1\varphi^k(\eta_m) \in E$.

It is easy to see that, for m large, there exist numbers $k(m) \to \infty$ as $m \to \infty$ such that

$$|\Pi\varphi^{k(m)}(\eta_m)| \leq \gamma \text{ for } 0 \leq k \leq k(m)$$

and there exist indices $j \in \{1, 2\}$ such that

$$\frac{\gamma}{\mu_v} \leq |\Pi_j \eta'_m| \leq \gamma, \tag{48}$$

where $\eta'_m = \varphi^{k(m)}(\eta_m)$.

If $j = 1$ in (48), then it follows from the inequality $\text{dist}(\Pi_1\eta'_m, E^*) > \epsilon$ that

$$\text{dist}(\eta'_m, x_k) > \epsilon \text{ for } 0 \leq k \leq m$$

(recall that $\Pi_1 x_k \in E^*$ for $0 \leq k \leq m$). Since $|\eta'_m| < 2\gamma$, it follows from the choice of γ and ϵ that $\text{dist}(\eta'_m, x_k) > \epsilon$ for all k, and we get a contradiction with inequality (43).

If $j = 2$ in (48), then the inclusion $\Pi_2\eta'_m \in I_2$ and the inequality $|\eta'_m| < 2\gamma < a$ imply that $\text{dist}(\eta'_m, K) > \epsilon$. Combining the latter inequality with inclusions (47), we again get the desired contradiction with inequality (43).

If $\zeta \in \Omega$, we apply a similar construction parallel to the arguments of case 2.2.

Case 3.2. $\dim L' = 0$. In this case, our arguments are the same as in case 2.

The theorem is proved.

Remark. An analog of Theorem 4 for the case of the usual shadowing property was proved in [13] involving quite different ideas.

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