Connecting Orbits with Bifurcating End Points

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Abstract

In this paper we consider the bifurcation of transversal heteroclinic orbits in discrete time dynamical systems. We assume that a non-hyperbolic transversal heteroclinic orbit exists at some critical parameter value. This situation appears, for example, when one end point undergoes a fold or flip bifurcation. In these two cases the bifurcation analysis of the orbit is performed in detail. In particular, we prove, using implicit function techniques that the orbit can be continued beyond the bifurcation point. Finally, we show numerical computations for the fold and for the flip bifurcation.

Keywords: Discrete time dynamical systems, bifurcation analysis, continuation, non-hyperbolic transversal heteroclinic orbits, numerical approximation.

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1 Introduction

Codimension 2 bifurcations of connecting orbits in continuous time dynamical systems are well understood but in the discrete case the theory is much less developed, cf. [19], [7], [8], [2], [14].

In this paper we consider discrete time dynamical systems of the form

$$x_{n+1} = f(x_n, \lambda), \quad n \in \mathbb{Z}, \tag{1}$$

where $f : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$ is sufficiently smooth and a diffeomorphism w.r.t. the x-variable. It is our aim to analyze the bifurcation of transversal heteroclinic orbits, when one end point loses its hyperbolicity at a critical parameter, while transversality remains valid for all λ. Here transversality is meant in the sense of **A5**, see below.

Assume that two branches of fixed points $\xi_\pm(\lambda)$ exist, where $\xi_-(\lambda)$ is hyperbolic for all λ and $\xi_+(\bar\lambda)$ possesses one center eigenvalue. Further assume that a transversal heteroclinic orbit $\left(\bar x_n(\bar\lambda)\right)_{n\in\mathbb{Z}}$ exists, connecting the fixed points $\xi_-(\bar\lambda)$ and $\xi_+(\bar\lambda)$, i.e. $(\bar x_n(\bar\lambda))_{n\in\mathbb{Z}}$ is a solution of (1) satisfying $\lim_{n\to\pm\infty} \bar x_n(\bar\lambda) = \xi_\pm(\bar\lambda)$.

At the critical parameter $\bar\lambda$, such a non-hyperbolic heteroclinic orbit is displayed in Figure 1. Note that every point of this orbit is lying in the intersection of the unstable manifold $W^u\big(\xi_-\big)$ and the center-stable manifold $W^{sc}\big(\xi_+\big)$.

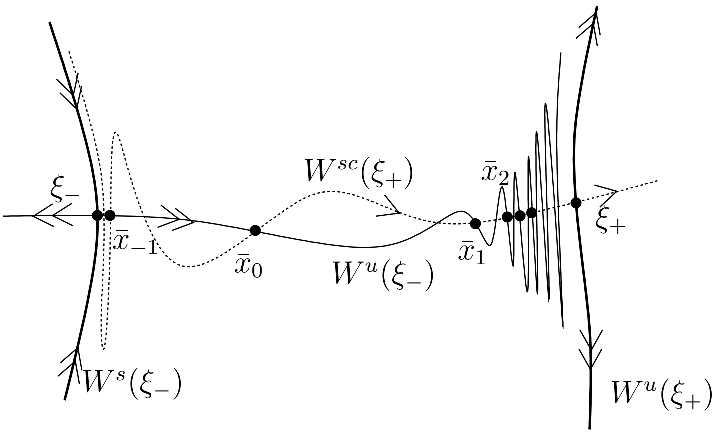

Figure 1: *Schematic picture of a non-hyperbolic transversal heteroclinic orbit as intersection of the unstable manifold of some hyperbolic fixed point ξ_- and the center-stable manifold of another non-hyperbolic fixed point ξ_+.*

By solving the boundary value problem

$$\begin{aligned} x_{n+1} &= f(x_n, \bar\lambda), \quad n = n_-, \ldots, n_+ - 1, && (2)\\ b(x_{n_-}, x_{n_+}) &= 0, && (3) \end{aligned}$$

with some appropriately chosen boundary operator $b \in C^1(\mathbb{R}^{2k}, \mathbb{R}^k)$, one can obtain a finite approximation of the transversal heteroclinic orbit.

It turns out that this approach works for both hyperbolic (cf. [13], [5], [9]) and non-hyperbolic orbits (cf. [10], [3]). In contrast to the hyperbolic case, for non-hyperbolic orbits, a boundary operator of sufficiently high order is required.

Moreover, uniqueness of the solution $x_J := (x_n)_{n=n_-,\dots,n_+}$ of (2), (3) can only be assured within some ball

$$\|\bar{x}_{|J} - x_J\|_\infty \leq \begin{cases} \delta, & \text{if both end points are hyperbolic,} \\ \frac{\delta}{n_+}, & \text{if one end point is non-hyperbolic,} \end{cases}$$

where $J = [n_-, n_+] \cap \mathbb{Z}$, $\bar{x}_{|J}$ is the restriction of the exact orbit to the finite interval J, and δ is a sufficiently small constant.

This non-hyperbolic situation arises, for example, when one fixed point undergoes a fold or a flip bifurcation, while the second fixed point stays hyperbolic. Since the bifurcation of fixed points is well understood, see [14], [20], we are interested in the bifurcation analysis of the corresponding saddle to fold and saddle to flip orbits in a neighborhood of the critical parameter $\bar{\lambda}$.

To understand these bifurcations, we prove in Section 2 that a transversal point of intersection of the unstable manifold of $\xi_-(\bar{\lambda})$ and the center-stable manifold of $\xi_+(\bar{\lambda})$ can be continued into a neighborhood $U(\bar{\lambda})$. The proof uses implicit function techniques.

In Section 3 we discuss the fate of a heteroclinic orbit. For the fold bifurcation the orbit can only exist on one side of the critical parameter, whereas in the flip case the orbit bifurcates to an orbit, converging towards a period two orbit in positive time. Furthermore, we have to cope with the fact, that the center-stable manifold is generally not unique. Therefore we must make an effort to assure that the continued points of intersection of the corresponding manifolds belong to heteroclinic connecting orbits.

For two examples, we show in Section 4 the numerical continuation of these orbits through the bifurcation.

A survey of numerical methods and examples for non-degenerate connecting orbits is given in [4].

2 Continuation of transversal heteroclinic orbits

We begin this section by presenting our basic assumptions and notations to guarantee the existence of a transversal non-hyperbolic heteroclinic orbit.

2.1 Assumptions

Consider a discrete time dynamical system

$$x_{n+1} = f(x_n, \lambda), \quad n \in \mathbb{Z}. \tag{4}$$

A1 Let $f \in C^\infty(\mathbb{R}^k \times \mathbb{R}, \mathbb{R}^k)$ and let $f(\cdot, \lambda)$ be a diffeomorphism for all $\lambda \in \mathbb{R}$.

A2 At the parameter $\bar\lambda$ the map $f(\cdot, \bar\lambda)$ possesses two fixed points $\xi_+(\bar\lambda)$ and $\xi_-(\bar\lambda)$.

Definition 1 A **heteroclinic orbit** $\bar x_{\mathbb{Z}}(\lambda) := \big(\bar x_n(\lambda)\big)_{n\in\mathbb{Z}}$ is a solution of the difference equation (4) satisfying $\lim_{n\to\pm\infty} \bar x_n(\lambda) = \xi_\pm(\lambda)$. For any $n \in \mathbb{Z}$ the point $\bar x_n(\lambda)$ is called a **heteroclinic point**.

With $k_{\pm\kappa}$, $\kappa \in \{s, c, u, sc\}$ we denote the dimension of the stable, center, unstable and center-stable subspace of $f'\big(\xi_\pm(\bar\lambda), \bar\lambda\big)$, respectively. The corresponding subspaces and manifolds are denoted by $X^\kappa\big(\xi_\pm(\bar\lambda)\big)$ and $W^\kappa\big(\xi_\pm(\bar\lambda)\big)$, $\kappa \in \{s, c, u, sc\}$.

A3 Let $k_{-c} = 0$, $k_{+c} = 1$ and $k_{-u} + k_{+sc} = k$.

In **A3** we assume that the fixed point $\xi_+(\bar\lambda)$ possesses a one-dimensional center manifold and that $\xi_-(\bar\lambda)$ is hyperbolic. Thus an orbit, converging towards $\xi_+(\bar\lambda)$ via the center-stable manifold, generically has a component in the slow center direction. This technical assumption is stated in **A4**.

A4 Let $\bar x_{\mathbb{Z}}(\bar\lambda)$ be a heteroclinic orbit such that $\bar x_0(\bar\lambda) \in W^{sc}\big(\xi_+(\bar\lambda)\big) \setminus W^s\big(\xi_+(\bar\lambda)\big)$.

Finally, we assume that the unstable manifold of $\xi_-(\bar\lambda)$ and the center-stable manifold of $\xi_+(\bar\lambda)$ intersect transversally.

A5 The invariant manifolds $W^u\big(\xi_-(\bar\lambda)\big)$ and $W^{sc}\big(\xi_+(\bar\lambda)\big)$ have transversal intersections at $\bar x_{\mathbb{Z}}(\bar\lambda)$, i.e. $T_{\bar x_n(\bar\lambda)} W^u\big(\xi_-(\bar\lambda)\big) \cap T_{\bar x_n(\bar\lambda)} W^{sc}\big(\xi_+(\bar\lambda)\big) = \{0\}$ for all $n \in \mathbb{Z}$.

Here $T_x W$ denotes the tangent space of the manifold W at the point x.

Using implicit function techniques, we prove in Section 2.2 that the transversal intersections of $W^{sc}(\xi_+(\bar\lambda))$ and $W^u(\xi_-(\bar\lambda))$ at the critical parameter $\bar\lambda$ can be continued into a neighborhood $U(\bar\lambda)$. Since every point of the heteroclinic orbit $\bar x_{\mathbb{Z}}(\bar\lambda)$ lies in the intersection of these manifolds, the whole orbit can also be continued into $U(\bar\lambda)$. But the continuation of $\bar x_{\mathbb{Z}}(\bar\lambda)$ is not necessarily a heteroclinic orbit, since existence of two fixed points $\xi_\pm(\lambda)$ is not guaranteed for all $\lambda \in U(\bar\lambda)$. In case of the flip bifurcation, for example, the orbit converges towards a two-periodic orbit for $\lambda > \bar\lambda$, whereas in the fold case, there is no fixed point the orbit can converge to for $\lambda > \bar\lambda$. We analyze this in detail in Section 3.

2.2 Continuation of transversal intersections

For the forthcoming analysis, it is helpful to consider the extended system (cf. [14])

$$z_{n+1} := \begin{pmatrix} x_{n+1} \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} f(x_n, \lambda_n) \\ \lambda_n \end{pmatrix} =: \tilde f(z_n), \quad n \in \mathbb{Z}. \tag{5}$$

Two fixed points of (5) are given by $\zeta_\pm = \big(\xi_\pm(\bar\lambda), \bar\lambda\big)$. The center-stable manifold of ζ_+ possesses locally the graph representation (cf. Appendix (A.1), Theorem 5)

$$\tilde W^{sc}_{\mathrm{loc}}(\zeta_+) = \left\{ \tilde h^+(\tilde\eta) : \tilde\eta \in Z^+_{sc} \cap \tilde U(0) \right\}, \tag{6}$$

where $\tilde U(0)$ is a sufficiently small neighborhood of 0. Here the center-stable and the unstable subspace of ζ_+ are denoted by $Z^+_{sc} = X^{sc}\big(\xi_+(\bar\lambda)\big) \times \mathbb{R}$ and $Z^+_u = X^u\big(\xi_+(\bar\lambda)\big) \times \{0\}$, respectively. The map $\tilde h^+$ is of the form

$$\tilde h^+(\tilde\eta) = \zeta_+ + \tilde\eta + \tilde\phi^+_{sc}(\tilde\eta), \quad \tilde\eta \in Z^+_{sc},$$

where $\tilde\phi^+_{sc} : Z^+_{sc} \to Z^+_u$ is smooth and

$$\tilde\phi^+_{sc}(0) = 0, \quad D\tilde\phi^+_{sc}(0) = 0.$$

In a sufficiently small neighborhood $U_1(0)$, $\tilde h^+$ has the form

$$\tilde h^+(\tilde\eta) = \begin{pmatrix} \xi_+(\bar\lambda) \\ \bar\lambda \end{pmatrix} + \begin{pmatrix} \eta \\ \mu \end{pmatrix} + \begin{pmatrix} \tilde\phi^+_{sc_1}(\eta,\mu) \\ 0 \end{pmatrix}, \quad \tilde\eta = \begin{pmatrix} \eta \\ \mu \end{pmatrix}, \ \eta \in X^{sc}\big(\xi_+(\bar\lambda)\big), \ \mu \in U_1(0).$$

Here $\tilde\phi^+_{sc_1}$ denotes the first block component of $\tilde\phi^+_{sc}$. Thus we obtain the representation

$$\tilde h^+(\tilde\eta) = \begin{pmatrix} \hat h^+(\eta,\lambda) \\ \lambda \end{pmatrix}, \quad \eta \in X^{sc}\big(\xi_+(\bar\lambda)\big), \ \lambda \in U_2(\bar\lambda),$$

where $\hat h^+(\eta,\lambda) := \xi_+(\bar\lambda) + \eta + \tilde\phi^+_{sc_1}(\eta, \bar\lambda - \lambda)$ and $U_2(\bar\lambda)$ is a sufficiently small neighborhood of $\bar\lambda$.

Note that also the fixed point ξ_- has a one-dimensional center component in the extended system (5). With a similar calculation for the center-unstable manifold of ζ_- we get:

$$\begin{aligned} \tilde W^{sc}_{\mathrm{loc}}(\zeta_+) &= \left\{ \begin{pmatrix} \hat h^+(\eta,\lambda) \\ \lambda \end{pmatrix} : \eta \in X^{sc}\big(\xi_+(\bar\lambda)\big) \cap V_1(0), \lambda \in V_2(\bar\lambda) \right\}, \\ \tilde W^{uc}_{\mathrm{loc}}(\zeta_-) &= \left\{ \begin{pmatrix} \hat h^-(\gamma,\lambda) \\ \lambda \end{pmatrix} : \gamma \in X^{uc}\big(\xi_-(\bar\lambda)\big) \cap V_3(0), \lambda \in V_4(\bar\lambda) \right\}, \end{aligned} \tag{7}$$

where $V_1(0), V_2(\bar\lambda), V_3(0), V_4(\bar\lambda)$ are sufficiently small neighborhoods. Therefore these manifolds are foliated over the parameter λ. Let $\Pi_\lambda := \big\{(x,\mu) : x \in \mathbb{R}^k, \mu = \lambda\big\}$, then

$$\tilde W^{sc}_\lambda(\zeta_+) := \tilde W^{sc}_{\mathrm{loc}}(\zeta_+) \cap \Pi_\lambda = \left\{ \begin{pmatrix} \hat h^+(\eta,\lambda) \\ \lambda \end{pmatrix} : \eta \in X^{sc}\big(\xi_+(\bar\lambda)\big) \cap V_1(0) \right\}$$

defines a family of local invariant manifolds for $\lambda \in V_2(\bar\lambda)$, see Figure 2.

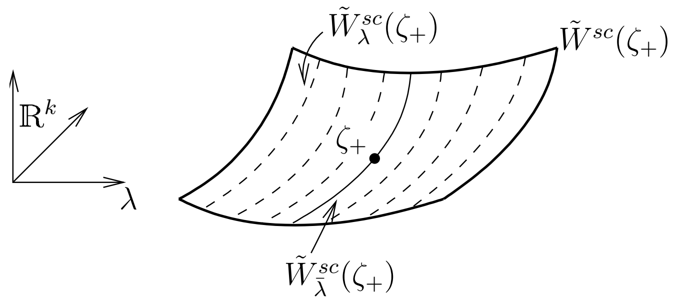

Figure 2: *Foliated center-stable manifold of the fixed point ζ_+ in the extended system (5).*

Note that the local graph representation of the center-stable manifold of $\xi_+(\bar\lambda)$ in the original system (4) can also be described in terms of the function $\hat{h}^+$:

$$W^{sc}_{\mathrm{loc}}\big(\xi_+(\bar\lambda)\big) = \left\{\hat{h}^+(\eta,\bar\lambda) : \eta \in X^{sc}\big(\xi_+(\bar\lambda)\big) \cap V_1(0)\right\}. \tag{8}$$

Similarly

$$W^{uc}_{\mathrm{loc}}\big(\xi_-(\bar\lambda)\big) = \left\{\hat{h}^-(\gamma,\bar\lambda) : \gamma \in X^{uc}\big(\xi_-(\bar\lambda)\big) \cap V_3(0)\right\}. \tag{9}$$

For the extended system (5), the dimension of the center-stable and of the center-unstable manifold increase by one:

$$\dim \tilde{W}^{sc}(\zeta_+) = k_{+sc} + 1, \quad \dim \tilde{W}^{uc}(\zeta_-) = k_{-u} + 1.$$

Since the space has the dimension $k+1$ and $\dim \tilde{W}^{sc}(\zeta_+) + \dim \tilde{W}^{uc}(\zeta_-) = k+2$, these manifolds generically have a one-dimensional transversal intersection (cf. assumption **A5**). This is stated in the following theorem, cf. [10, Satz 5.1].

Theorem 2 *Assume **A1** to **A5**. Then there exists a neighborhood $U(\bar\lambda)$ such that the two manifolds $\tilde{W}^{sc}_{\lambda}\big(\xi_+(\bar\lambda),\bar\lambda\big)$ and $\tilde{W}^{uc}_{\lambda}\big(\xi_-(\bar\lambda),\bar\lambda\big)$ intersect transversally at a smooth curve $\sigma : U(\bar\lambda) \to \mathbb{R}^{k+1}$ such that $\sigma(\bar\lambda) = \big(\bar{x}_N(\bar\lambda),\bar\lambda\big)$ for some $N \in \mathbb{Z}$.*

Proof: Due to our assumptions the existence of a transversal heteroclinic orbit $\bar{x}_{\mathbb{Z}}(\bar\lambda)$, connecting the fixed points $\xi_\pm(\bar\lambda)$, is guaranteed. By Appendix A.1, Theorem 6,

$$\big(\bar{x}_n(\bar\lambda),\bar\lambda\big) \in \tilde{W}^{sc}_{\mathrm{loc}}(\zeta_+) \quad \text{for all } n \geq N.$$

Using the local graph representation of $\tilde{W}^{sc}_{\mathrm{loc}}(\zeta_+)$, $\bar{x}_N(\bar\lambda) = \hat{h}^+(\bar\eta,\bar\lambda)$ holds for some $\bar\eta \in X^{sc}\big(\xi_+(\bar\lambda)\big)$. Since this orbit converges towards the fixed point $\xi_-(\bar\lambda)$ as $n \to -\infty$, an index $L \in \mathbb{N}$ exists, such that

$$\big(\bar{x}_N(\bar\lambda),\bar\lambda\big) \in \tilde{f}^L\left(\tilde{W}^{uc}_{\mathrm{loc}}(\zeta_-)\right).$$

For this reason $\bar{x}_N(\bar{\lambda}) = f^L\left(\hat{h}^-(\bar{\gamma},\bar{\lambda}),\bar{\lambda}\right)$, with a suitably chosen $\bar{\gamma} \in X^{uc}\big(\xi_-(\bar{\lambda})\big)$. Here $f^{\pm L}(x,\lambda)$ is recursively defined for $L \geq 2$, $L \in \mathbb{N}$ by

$$f^{\pm L}(x,\lambda) = f^{\pm 1}\left(f^{\pm L \mp 1}(x,\lambda),\lambda\right).$$

In the next step of proof, we show that a transversal point of intersection of $W^{sc}_{\text{loc}}\big(\xi_+(\bar{\lambda})\big)$ and $W^u\big(\xi_-(\bar{\lambda})\big)$ persists for $\lambda \in U(\bar{\lambda})$.

Consider the operator

$$\Omega : \begin{array}{rcl} X^{sc}\big(\xi_+(\bar{\lambda})\big) \times X^u\big(\xi_-(\bar{\lambda})\big) \times \mathbb{R} & \to & \mathbb{R}^k \\ (\eta,\gamma,\lambda) & \mapsto & \hat{h}^+(\eta,\lambda) - f^L\left(\hat{h}^-(\gamma,\lambda),\lambda\right). \end{array}$$

Obviously $\Omega(\bar{\eta},\bar{\gamma},\bar{\lambda}) = 0$ and

$$\begin{aligned} D_{\eta,\gamma}\Omega(\bar{\eta},\bar{\gamma},\bar{\lambda})\begin{pmatrix}\eta\\ \gamma\end{pmatrix} &= D_\eta\hat{h}^+(\bar{\eta},\bar{\lambda})\eta - D_\gamma\left[f^L\left(\hat{h}^-(\bar{\gamma},\bar{\lambda}),\bar{\lambda}\right)\right]\gamma \\ &= D_\eta\hat{h}^+(\bar{\eta},\bar{\lambda})\eta - \left(D_\gamma f^L\Big(f^{-L}\big(\bar{x}_N(\bar{\lambda}),\bar{\lambda}\big),\bar{\lambda}\Big)D_\gamma\hat{h}^-(\bar{\gamma},\bar{\lambda})\right)\gamma, \end{aligned}$$

where $\eta \in X^{sc}\big(\xi_+(\bar{\lambda})\big)$, $\gamma \in X^u\big(\xi_-(\bar{\lambda})\big)$. To find an analytic representation of the tangent-space of $W^{sc}_{\text{loc}}\big(\xi_+(\bar{\lambda})\big)$, we use the local graph representation (8). It holds

$$T_{\bar{x}_N(\bar{\lambda})}W^{sc}_{\text{loc}}\big(\xi_+(\bar{\lambda})\big) = \mathcal{R}\big(D_\eta\hat{h}^+(\bar{\eta},\bar{\lambda})\big).$$

To derive a similar result for $T_{\bar{x}_N(\bar{\lambda})}W^u\big(\xi_-(\bar{\lambda})\big)$ an additional transformation is needed, such that the local graph representation (9) can be applied:

$$\begin{aligned} T_{\bar{x}_N(\bar{\lambda})}W^u\big(\xi_-(\bar{\lambda})\big) &= D_\gamma f^L\Big(f^{-L}\big(\bar{x}_N(\bar{\lambda}),\bar{\lambda}\big),\bar{\lambda}\Big)T_{f^{-L}\left(\bar{x}_N(\bar{\lambda}),\bar{\lambda}\right)}W^u_{\text{loc}}\big(\xi_-(\bar{\lambda})\big) \\ &= D_\gamma f^L\Big(f^{-L}\big(\bar{x}_N(\bar{\lambda}),\bar{\lambda}\big),\bar{\lambda}\Big)\mathcal{R}\big(D_\gamma\hat{h}^-(\bar{\gamma},\bar{\lambda})\big). \end{aligned}$$

From the transversality assumption **A5** it follows that $D_{\eta,\gamma}\Omega(\bar{\eta},\bar{\gamma},\bar{\lambda})$ is non-singular. The implicit function theorem can be applied and guarantees the existence of two smooth curves $\eta(\lambda)$, $\gamma(\lambda)$, such that $\Omega\big(\eta(\lambda),\gamma(\lambda),\lambda\big) = 0$ for $\lambda \in U(\bar{\lambda})$. Furthermore, the matrix $D_{\eta,\gamma}\Omega\big(\eta(\lambda),\gamma(\lambda),\lambda\big)$ is non-singular for sufficiently small $|\lambda - \bar{\lambda}|$. Therefore, the two manifolds $\tilde{W}^{sc}_\lambda\left(\xi_+(\bar{\lambda}),\bar{\lambda}\right)$ and $\tilde{W}^{uc}_\lambda\left(\xi_-(\bar{\lambda}),\bar{\lambda}\right)$ intersect transversally for $|\lambda - \bar{\lambda}|$ sufficiently small.

In other words, the transversal point of intersection $\left(\bar{x}_N(\bar{\lambda}),\bar{\lambda}\right)$ can be continued to

$$\big(\bar{x}_N(\lambda),\lambda\big) = \sigma(\lambda) := \tilde{h}^+\big(\eta(\lambda),\lambda\big)$$

for $\lambda \in U(\bar{\lambda})$. ∎

3 Bifurcations of saddle-fold and saddle-flip orbits

From Theorem 2 we know that the transversal intersection of the unstable manifold of $\xi_-(\bar\lambda)$ and the center-stable manifold of $\xi_+(\bar\lambda)$ can be continued in a neighborhood of $\bar\lambda$. Since the center-stable manifold is in general not unique, we have to determine whether the continued points belong to a heteroclinic connecting orbit. Especially one has to assure the existence of two branches of fixed points $\xi_\pm(\lambda)$ for $\lambda \in U(\bar\lambda)$. In case of the fold bifurcation, for example, the fixed point exists only on one side of $\bar\lambda$.

To simplify the notation, we call a heteroclinic orbit, whose end point $\xi_+(\lambda)$ undergoes a fold or flip bifurcation at $\lambda = \bar\lambda$, saddle-fold or saddle-flip orbits, respectively.

3.1 Continuation through a saddle-fold orbit

Without loss of generality let $\bar\lambda = 0$ and $\xi_+(0) = 0$. Assume **A1** to **A5** and let $\xi_+(\lambda)$ undergo a fold bifurcation at the parameter value $\lambda = 0$. Note that the normal form of the fold bifurcation is given by (cf. [14])

$$x \mapsto g(x,\lambda) = \lambda + x \pm x^2. \tag{10}$$

From a symmetry argument $(x,\lambda) \mapsto (-x,-\lambda)$, one sees that it is sufficient to consider the "+" case of (10). For $-1 < \lambda \leq 0$ this map possesses two branches of fixed points $\mu^s(\lambda) = -\sqrt{-\lambda}$ and $\mu^u(\lambda) = \sqrt{-\lambda}$.

By $\xi_+^s(\lambda)$ and $\xi_+^u(\lambda)$, we denote the corresponding branches of $\xi_+(\lambda)$ in the original system (4), see Figure 3 for an illustration.

The following theorem (cf. [10, Satz 5.2]) describes the bifurcation of a saddle-fold orbit.

Theorem 3 *With the assumptions given in this section, there exists a neighborhood $U(0)$, such that the transversal heteroclinic orbit $x_{\mathbb{Z}}(0)$ can be continued for $\lambda < 0$ to $x_{\mathbb{Z}}(\lambda)$, where the map $\lambda \mapsto x_n(\lambda)$ is smooth for any $n \in \mathbb{Z}$. Furthermore, we get for all $\lambda \in U(0)$*

$$\lim_{n\to\infty} x_n(\lambda) = \xi_+^s(\lambda), \qquad \lim_{n\to-\infty} x_n(\lambda) = \xi_-(\lambda).$$

Proof: It is sufficient to verify the convergence of $x_n(\lambda)$ towards the fixed point $\xi_+^s(\lambda)$ for $\lambda < 0$, $\lambda \in U(0)$ as $n \to \infty$. The corresponding result for the negative half-orbit can be obtained in a similar way.

According to Appendix A.2, Theorem 8, we can transform the extended system

$$\begin{pmatrix} x_{n+1} \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} f(x_n,\lambda_n) \\ \lambda_n \end{pmatrix} \tag{11}$$

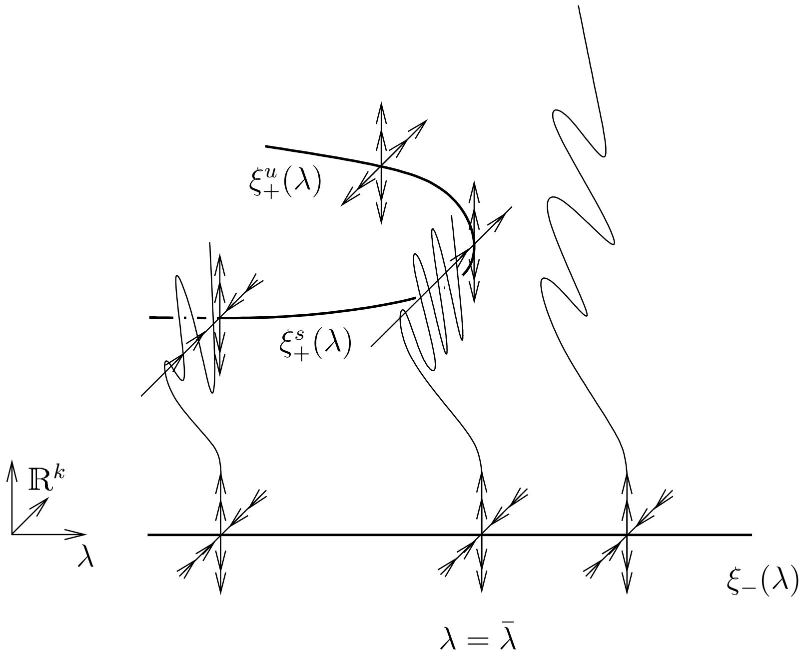

Figure 3: *Illustration of the manifolds, when one fixed point undergoes a fold bifurcation.*

for $(x, \lambda) \in U_1\big(0,0\big)$ by a homeomorphism Ψ into

$$\Psi \begin{pmatrix} x_{n+1} \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} s_{n+1} \\ u_{n+1} \\ w_{n+1} \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} As_n \\ Bu_n \\ g(w_n, \lambda_n) \\ \lambda_n \end{pmatrix}, \tag{12}$$

where the matrix $A := f_x\big(0,0\big)_{|X^s(0)}$ is stable, $B := f_x\big(0,0\big)_{|X^u(0)}$ is unstable and g is the normal form of the fold bifurcation.

By assumption **A4** the original system (4) possesses a heteroclinic orbit $x_{\mathbb{Z}}(0)$ at the parameter $\lambda = 0$, where $x_0(0) \in W^{sc}(0) \setminus W^s(0)$. For a sufficiently large $M \in \mathbb{N}$ we find $\big(x_m(0), 0\big) \in U_1\big(0,0\big)$ for all $m \geq M$. Thus

$$\Psi\big(x_m(0), 0\big) = \big(s_m(0), u_m(0), w_m(0), 0\big)$$

converges towards the fixed point 0 in the transformed system (12), as $m \to \infty$. Therefore the inequality $w_m(0) < 0$ holds, since the "+"-case of the normal form is considered.

In the extended system (11), $\big(s_m(0), u_m(0), w_m(0), 0\big)$, $m \geq M$ lies in any center-stable manifold of the fixed point $\zeta_+ = (0,0)$, cf. Appendix A.1, Theorem 6. By Theorem 2, any transversal point of intersection $(x_m(0), 0)$ of the manifolds $\tilde{W}_0^{sc}(\zeta_+)$ and $\tilde{W}_0^{uc}(\zeta_-)$ can be continued to a smooth curve $(x_m(\lambda), \lambda)$ for sufficiently small λ.

Next we prove for sufficiently small $\lambda < 0$ the convergence of $x_m(\lambda)$ towards the fixed point $\xi_+^s(\lambda)$ as $m \to \infty$.

The map

$$\lambda \mapsto \big(s_m(\lambda), u_m(\lambda), w_m(\lambda), \lambda\big)$$

is continuous, because Ψ is a homeomorphism and $x_m(\cdot)$ is continuous according to Theorem 2. Choose an m such that $-\frac{1}{2} < w_m(0) < 0$ holds. Then there exists a neighborhood $U_2(0)$ with

$$-\tfrac{1}{2} < w_m(\lambda) < -\sqrt{-\lambda} \quad \text{for } \lambda < 0,\ \lambda \in U_2(0).$$

Since the "+"-case of the normal form (10) is considered, $w_m(\lambda)$ converges towards $-\sqrt{-\lambda}$ as $m \to \infty$. Furthermore, a neighborhood $U_3(0) \subset U_2(0)$ and an $M \in \mathbb{N}$ exist, such that

$$\big(s_m(\lambda), u_m(\lambda), w_m(\lambda), \lambda\big) \in W^{sc}_{\mathrm{loc}}(0,0) \quad \text{for all } m \geq M,\ \lambda \in U_3(0),$$

see Appendix A.1, Theorem 6. We apply Appendix A.2, Lemma 9 by setting

$$h(w_m, \lambda) = \begin{pmatrix} g(w_m, \lambda) \\ \lambda \end{pmatrix}$$

and obtain $u_m(\lambda) = 0$. Since $s_m(\lambda)$ converges exponentially fast towards 0, applying the inverse transformation completes the proof. ■

Note that for $\lambda > 0$ a transversal intersection of the corresponding manifolds in the extended system (5) exists according to Theorem 2, but there is no fixed point $\xi_+(\lambda)$ the orbit can converge to, see Figure 3.

3.2 Continuation through a saddle-flip orbit

In this section we assume without loss of generality $\bar{\lambda} = 0$ and $\xi_+(0) = 0$. Assume **A1** to **A5** and let $\xi_+(\lambda)$ undergo a flip bifurcation at $\lambda = 0$. The normal form of the flip bifurcation is given by (cf. [14]) $w \mapsto (1+\lambda)w \pm w^3$. Due to assumption **A4**, the "+" case of the normal form is excluded. Otherwise an orbit, possessing a center component, cannot converge towards the fixed point $\xi_+(0)$ at $\lambda = 0$. Therefore, we consider the normal form:

$$g(w, \lambda) := (1+\lambda)w - w^3. \tag{13}$$

By $\eta_{1,2}(\lambda)$ we denote the branch of period-two orbits, bifurcating from the fixed point 0 at the parameter value $\lambda = 0$.

For the forthcoming analysis, it is suitable to consider the squared map

$$h(x, \lambda) := f\big(f(x, \lambda), \lambda\big).$$

At the critical parameter $\lambda = 0$ the fixed point $\xi_+(\lambda)$ of $h(\cdot, \lambda)$ undergoes a pitchfork bifurcation. Let

$$\xi_{1,2}(\lambda) := \begin{cases} \xi_+(\lambda), & \lambda \le 0, \\ \eta_{1,2}(\lambda), & \lambda > 0. \end{cases}$$

In the following theorem (cf. [10, Satz 5.8]), we analyze the bifurcation of an $h(\cdot, \lambda)$-orbit.

Theorem 4 *Let the assumptions described above, be fulfilled. Denote by $x_{\mathbb{Z}}(0)$ a solution of*

$$x_{n+1}(0) = h\big(x_n(0), 0\big), \quad n \in \mathbb{Z} \tag{14}$$

connecting the fixed points $\xi_\pm(0)$. Then there exists a neighborhood $U(0)$, such that a continuation $x_{\mathbb{Z}}(\lambda)$ of $x_{\mathbb{Z}}(0)$ exists for $\lambda \in U(0)$. For any $n \in \mathbb{Z}$ the map $\lambda \mapsto x_n(\lambda)$, $\lambda \in U(0)$ is smooth, $\lim\limits_{n\to-\infty} x_n(\lambda) = \xi_-(\lambda)$ holds for all $\lambda \in U(0)$ and

$$\textit{either } \Big[\lim_{n\to\infty} x_n(\lambda) = \xi_1(\lambda) \quad \forall \lambda \in U(0)\Big] \textit{ or } \Big[\lim_{n\to\infty} x_n(\lambda) = \xi_2(\lambda) \quad \forall \lambda \in U(0)\Big]$$

hold.

Proof: As in the proof of Theorem 3, it is sufficient to show the convergence of the positive half-orbit.

According to Appendix A.2, Theorem 8, a homeomorphism Ψ exists, transforming the extended system

$$\begin{pmatrix} x_{n+1} \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} h(x_n, \lambda_n) \\ \lambda_n \end{pmatrix}$$

for $(x, \lambda) \in U_1\big(\xi_+(0), 0\big)$ into

$$\Psi \begin{pmatrix} x_{n+1} \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} s_{n+1} \\ u_{n+1} \\ w_{n+1} \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} As_n \\ Bu_n \\ g(w_n, \lambda_n) \\ \lambda_n \end{pmatrix}, \tag{15}$$

where the matrix $A := h_x\big(\xi_+(0), 0\big)_{|X^s(0)}$ is stable and $B := h_x\big(\xi_+(0), 0\big)_{|X^u(0)}$ is unstable. A point $\big(x(0), 0\big) \in U_1\big(0, 0\big)$ lying in the transversal intersection of the corresponding manifolds, can be continued due to Theorem 2 to a smooth curve $\big(x(\lambda), \lambda\big)$ for $\lambda \in U_2(0)$. In the transformed system this curve has the form $\big(s(\lambda), u(\lambda), w(\lambda), \lambda\big)$, where the map

$$\lambda \mapsto \big(s(\lambda), u(\lambda), w(\lambda), \lambda\big), \quad \lambda \in U_2(0)$$

is continuous. Note that the center-part $w(\lambda)$ cannot vanish according to assumption **A4**. Furthermore, a neighborhood $U_3(0)$ exists such that $\operatorname{sign}\big(w(\lambda)\big) = \operatorname{sign}\big(w(0)\big)$ for all $\lambda \in U_3(0)$.

We consider the following two branches of fixed points $(\tilde{\xi}_j(\lambda), \lambda)$, $j \in \{1,2\}$ of the transformed system (15), defined by

$$\left(\tilde{\xi}_j(\lambda), \lambda\right) := \Psi\left(\xi_j(\lambda), \lambda\right) = \begin{cases} (0,0,0,\lambda) & \text{for } \lambda \le 0, \\ \left(0,0,(-1)^j\sqrt{\lambda},\lambda\right) & \text{for } \lambda > 0. \end{cases}$$

Due to our construction the point $\big(s(\lambda), u(\lambda), w(\lambda), \lambda\big)$ lies in the local center-stable manifold of the fixed point $(0,0,0,0)$ of (15) and the $s(\lambda)$-part decreases exponentially fast under iteration with this map. To study the behavior of $w(\lambda)$ we consider the scalar equation (13) and get

$$\begin{aligned} w_n(0) < 0 \;&\Rightarrow\; \lim_{n\to\infty} w_n(\lambda) = [\tilde{\xi}_1(\lambda)]_w \text{ for all } \lambda \in U_3(0), \\ w_n(0) > 0 \;&\Rightarrow\; \lim_{n\to\infty} w_n(\lambda) = [\tilde{\xi}_2(\lambda)]_w \text{ for all } \lambda \in U_3(0). \end{aligned}$$

From Appendix A.2, Lemma 9, $u(\lambda) = 0$ follows for all $\lambda \in U_3(0)$. Finally, a transformation back to the original system (14) completes the proof. ■

This result can also be used, to analyze the bifurcation of the corresponding $f(\cdot,\lambda)$-orbits. Due to Theorem 4 we can continue an $h(\cdot,0)$-orbit $x^1_{\mathbb{Z}}(0)$ to $x^1_{\mathbb{Z}}(\lambda)$ for $\lambda \in U(0)$. By defining $x^2_{\mathbb{Z}}(\lambda) := \big(f(x^1_n(\lambda),\lambda)\big)_{n\in\mathbb{Z}}$, we obtain a second branch of $h(\cdot,\lambda)$-orbits. Note that both orbits converge for $\lambda \le 0$ to the same fixed point $\xi_+(\lambda)$ from opposite sides. For $\lambda > 0$ one orbit converges to the fixed point $\eta_1(\lambda)$, while the other orbit converges to $\eta_2(\lambda)$.

Next we construct the corresponding $f(\cdot,\lambda)$-orbit $y_{\mathbb{Z}}(\lambda)$ by

$$\begin{aligned} y_{2n}(\lambda) &= x^1_n(\lambda), \; n \in \mathbb{Z}, \\ y_{2n+1}(\lambda) &= x^2_n(\lambda), \; n \in \mathbb{Z}. \end{aligned}$$

The orbit $y_{\mathbb{Z}}(0)$ converges towards the fixed point $\xi_+(0)$ in an alternating way, since $D_x f\big(\xi_+(0),0\big)$ has an eigenvalue -1, see Figure 4. For $\lambda > 0$, $y_{\mathbb{Z}}(\lambda)$ tends to the

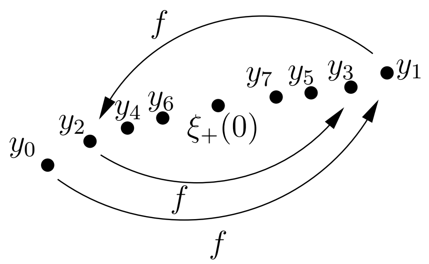

Figure 4: *An $f(\cdot,0)$-orbit in a neighborhood of the fixed point $\xi_+(0)$.*

two-periodic orbit $\big(\eta_1(\lambda), \eta_2(\lambda)\big)$ as $n \to \infty$. The intersection of the corresponding manifolds is illustrated in Figure 5.

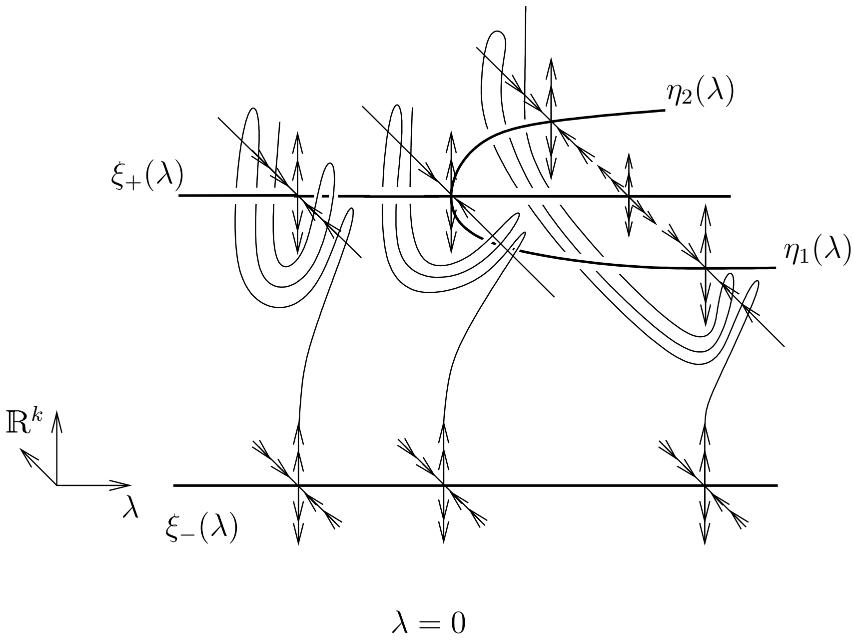

Figure 5: *Illustration of the manifolds, when the fixed point* $\xi_+(\lambda)$ *undergoes a flip bifurcation at* $\lambda = 0$.

In the space of bounded sequences

$$S_{\mathbb{Z}} = \left\{ x_{\mathbb{Z}} \in (\mathbb{R}^k)^{\mathbb{Z}} : \sup_{n \in \mathbb{Z}} \|x_n\| \leq \infty \right\},$$

the two $h(\cdot, \lambda)$-orbits $x^1_{\mathbb{Z}}(\lambda)$ and $x^2_{\mathbb{Z}}(\lambda)$ are for any $\lambda \in U(0)$ far away from each other. Note that the map $\lambda \mapsto x^{1,2}_n(\lambda)$ is for any fixed $n \in \mathbb{Z}$ differentiable in $S_{\mathbb{Z}}$ w.r.t. $\|\cdot\|_\infty$ at $\lambda = 0$ (see Theorem 2) but the branch of fixed points $\xi_+(\lambda)$ is not differentiable at $\lambda = 0$. Therefore $x^{1,2}_{\mathbb{Z}}(\lambda)$ cannot be differentiable in $S_{\mathbb{Z}}$ at $\lambda = 0$, see Figure 6.

Finally we remark that in [11] a model function is introduced that allows us to study the flip bifurcation explicitly. In particular, one can analyze precisely the way in which the exponential rate of convergence towards the fixed point turns into a polynomial one as $\lambda \to 0$.

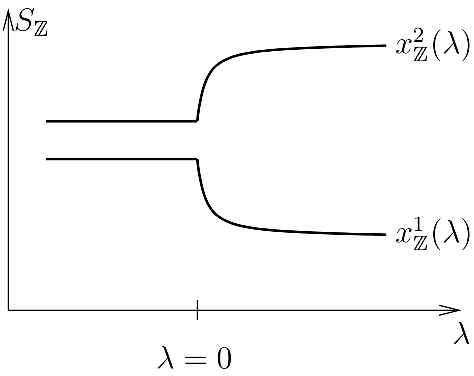

Figure 6: *Two heteroclinic orbits, plotted schematically in $S_{\mathbb{Z}}$ over the parameter λ.*

4 Examples

In this section we consider two examples, having a saddle-fold and a saddle-flip orbit at the parameter value $\lambda = \bar{\lambda}$, respectively.

For approximating a (non-hyperbolic) heteroclinic orbit, we solve the boundary value problem (2), (3) using projection boundary conditions, see [10], [3]. In a second step, we perform the numerical continuation of this finite orbit-segment due to a predictor-corrector method, described in [1, Chapter 10].

4.1 Numerical continuation through a saddle-fold orbit

Consider the map

$$f: \begin{array}{rcl} \mathbb{R}^2 \times \mathbb{R} & \to & \mathbb{R}^2 \\ (x,\lambda) & \mapsto & \begin{pmatrix} 2x_2 \\ x_1 - \lambda - 2x_2 + 4x_2^2 - 8x_2^4 \end{pmatrix}. \end{array}$$

Figure 7 shows the bifurcation diagram of fixed points, obtained with CONTENT, cf. [15].

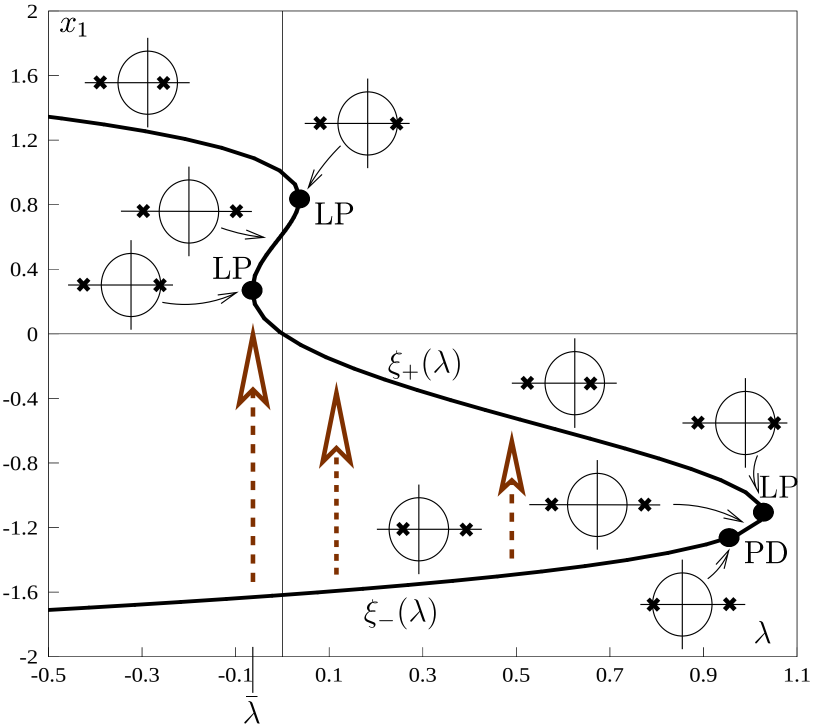

Figure 7: *Bifurcation diagram of fixed points of f, projected onto the (λ, x_1)-plane. The big arrows symbolize the existence of heteroclinic orbits.*

We select two branches of fixed points and denote them by $\xi_\pm(\lambda)$, see Figure 7. At the critical parameter $\lambda = \bar\lambda \approx -0.06475733358165706$ the assumptions **A1**–**A5** are fulfilled, thus a non-hyperbolic heteroclinic $f(\cdot, \bar\lambda)$-orbit can be approximated by a finite orbit segment (cf. [10, Theorem 4.3], [3]). Furthermore, Theorem 3 allows us to continue this orbit segment w.r.t. the parameter λ for $\lambda > \bar\lambda$.

The continued orbits are plotted in Figure 8 and a neighborhood of the fixed point $\xi_+(\lambda)$, projected onto the (λ, x_1)-space is displayed in Figure 9. Note that the length of the orbit is fixed during the continuation. Figure 9 indicates that the exponential rate of convergence for $\lambda > \bar\lambda$, turns into a polynomial one at the bifurcation parameter $\lambda = \bar\lambda$.

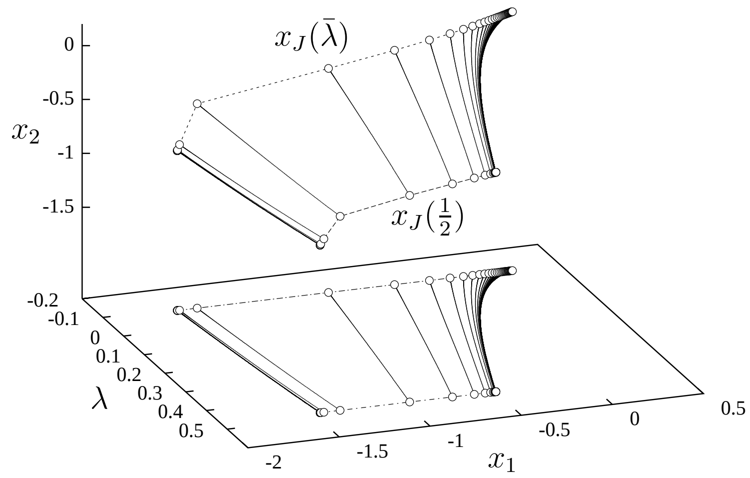

Figure 8: *Continuation of an $f(\cdot, \frac{1}{2})$-orbit-segment of length $n_- = 10$, $n_+ = 30$ over the parameter λ. The first and the last orbit, calculated during the continuation are also plotted and the points of these orbits are connected with dotted lines. Furthermore, the projection onto the (x_1, λ)-plane is displayed.*

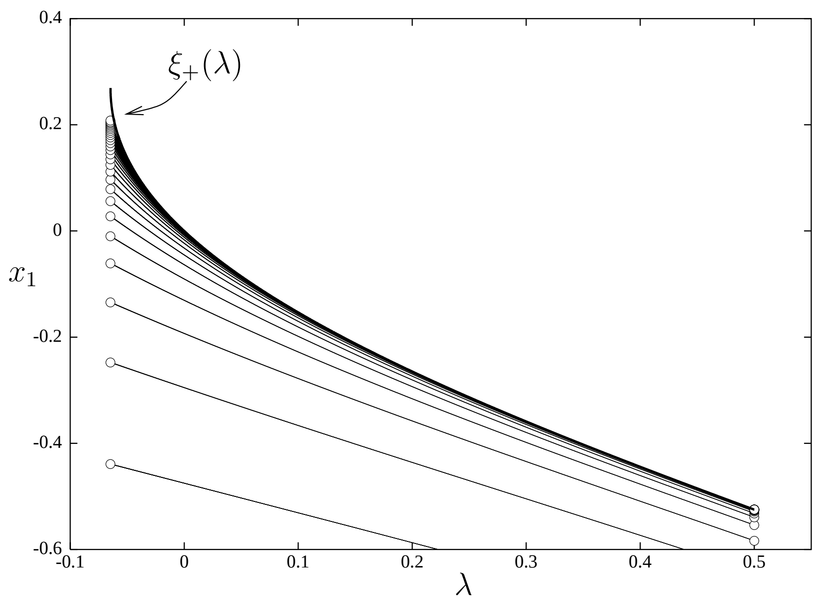

Figure 9: *A neighborhood of the fixed point $\xi_+(\lambda)$, projected onto the (λ, x_1)-plane. The branch of fixed points $\xi_+(\lambda)$ is also displayed.*

4.2 Numerical continuation through a saddle-flip orbit

Consider the map

$$f: \begin{array}{rcl} \mathbb{R}^2 \times \mathbb{R} & \to & \mathbb{R}^2 \\ (x, \lambda) & \mapsto & \begin{pmatrix} \left(\frac{1}{2} - \lambda\right) x_1 + x_1^3 + \frac{2}{5} x_1^4 + x_2 \\ \frac{3}{2} x_1 \end{pmatrix}. \end{array}$$

At $\lambda = 0$ the fixed point $\xi_+(0) = 0$ undergoes a flip bifurcation and the assumptions **A1**–**A5** are satisfied. Figure 10 shows the corresponding bifurcation diagram of fixed points, obtained with Content, cf. [15].

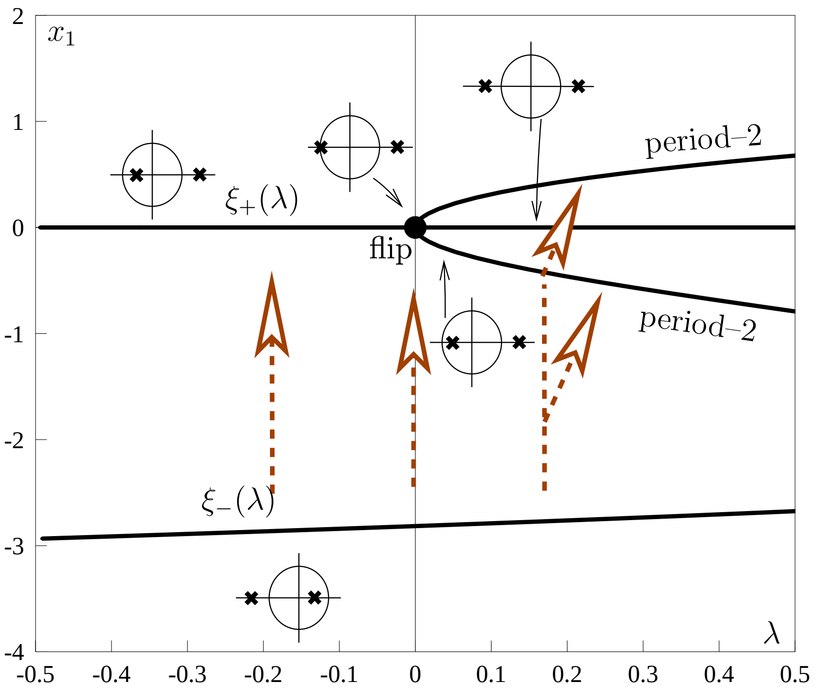

Figure 10: *Bifurcation diagram of fixed points of f, projected onto the (λ, x_1)-plane. The big arrows mark the fixed points and two-periodic orbits, connected during the numerical continuation.*

Since the orbit converges towards a two-periodic orbit for $\lambda > 0$, we perform the continuation for the squared map $f^2(\cdot, \lambda) := f\big(f(\cdot, \lambda), \lambda\big)$ and calculate a family of orbit segments $x_J(\lambda)$, $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$ numerically. For the original system, we then get the corresponding $f(\cdot, \lambda)$-orbit of double length by

$$\Big(x_{n_-}(\lambda), f\big(x_{n_-}(\lambda), \lambda\big), \ldots, x_{n_+}(\lambda), f\big(x_{n_+}(\lambda), \lambda\big)\Big).$$

Figure 11 contains the continuation picture of a saddle-flip orbit. Additionally, a projection onto the (λ, x_1)-plane is plotted in Figure 12. In this picture, the fixed points and two-periodic orbits are also displayed.

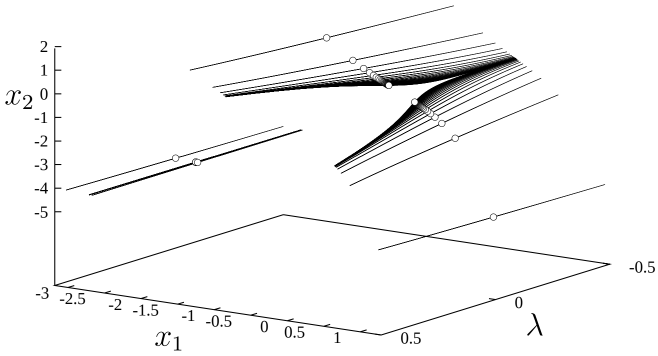

Figure 11: *Continuation of a saddle-flip orbit over the parameter λ. The small circles denote the saddle-flip orbit at $\lambda = 0$.*

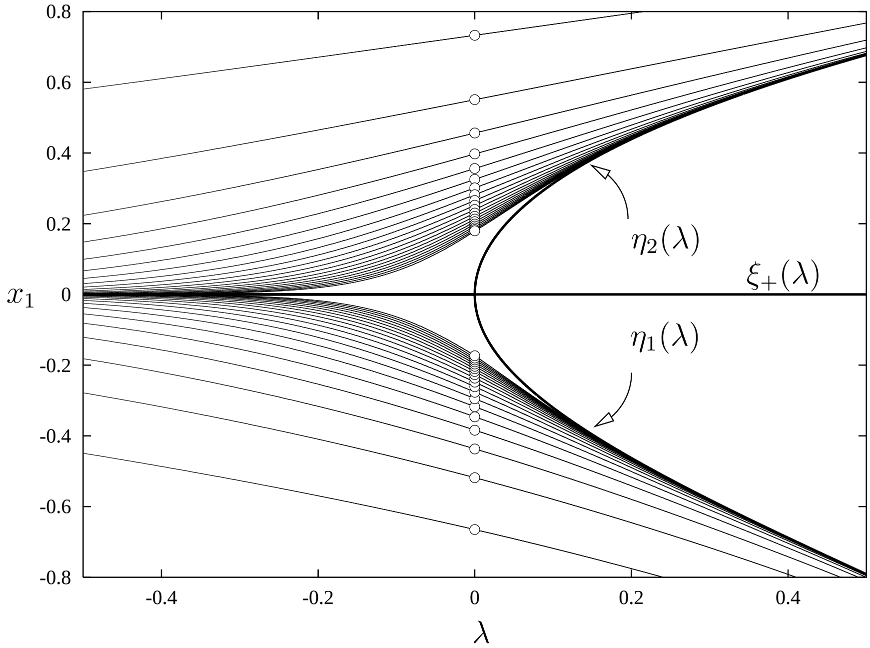

Figure 12: *A neighborhood of the flip bifurcation at $\lambda = 0$, projected onto the (λ, x_1)-plane. Additionally, the fixed points and two-periodic orbits are displayed.*

A Appendix

A.1 Locally invariant manifolds

Some basic properties of the center manifold are summarized in this appendix.

Consider the system

$$\begin{array}{rcl} x & \mapsto & Cx + f(x,y), \\ y & \mapsto & Hy + g(x,y), \end{array} \quad (x,y) \in \mathbb{R}^c \times \mathbb{R}^{su}, \tag{16}$$

where the matrix $C \in \mathbb{R}^{c,c}$ possesses only center eigenvalues and the matrix $H \in \mathbb{R}^{su,su}$ is hyperbolic. The functions f and g are C^r-smooth, $r \geq 2$, defined in a neighborhood of 0 and

$$\begin{array}{rclcrcl} f(0,0) & = & 0, & \qquad & Df(0,0) & = & 0, \\ g(0,0) & = & 0, & \qquad & Dg(0,0) & = & 0. \end{array}$$

Obviously $(x,y) = (0,0)$ is a fixed point of (16). The following theorem (cf. [6, Chapter 2.8], [20, Chapter 2.1D], [14, Chapter 5.1.2]) guarantees the existence of a center manifold.

Theorem 5 *There exists a locally invariant manifold of (16) possessing the graph representation*

$$W^c_{\text{loc}}(0) = \big\{(x,y) \in \mathbb{R}^c \times \mathbb{R}^{su} : y = h(x), |x| \leq \delta,\ h(0) = 0, Dh(0) = 0\big\}, \tag{17}$$

where $h : \mathbb{R}^c \to \mathbb{R}^{su}$ is C^r-smooth and $\delta > 0$ is chosen sufficiently small.

A locally invariant manifold having the representation (17) is called a *center manifold*. In general this manifold is not unique. For a counter example for maps that can be studied explicitly, see [11]. The dynamics, reduced to this manifold are described by the c-dimensional system

$$w \mapsto Cw + f\big(w, h(w)\big), \quad w \in \mathbb{R}^c.$$

The following theorem (cf. [16, Theorem III.7]) provides a criterion, to check if a given $x \in \mathbb{R}^k$ lies in W^s_{loc}, W^u_{loc}, W^{sc}_{loc}, W^{uc}_{loc} or W^c_{loc}.

Theorem 6 *Let U be a neighborhood of* 0 *and $f : U \to \mathbb{R}^k$ a C^r-diffeomorphism ($r \geq 2$) having the fixed point* 0.

Then there exists for each of the manifolds W^s_{loc}, W^u_{loc}, W^{sc}_{loc}, W^{su}_{loc} and W^u_{loc} a ball V around 0*, such that the following assertions hold:*

- $W^s_{\text{loc}} = \big\{x \in V :\ f^n(x) \in V$ *for all* $n \geq 0$ *and* $f^n(x) \to 0$ *exponentially fast as* $n \to \infty\big\}$.
- $f\,(W^{sc}_{\text{loc}}) \cap V \subset W^{sc}_{\text{loc}}$. *If* $f^n(x) \in V$ *for all* $n \geq 0$, *then* $x \in W^{sc}_{\text{loc}}$.

- $f\left(W_{\text{loc}}^{c}\right) \cap V \subset W_{\text{loc}}^{c}$. *If* $f^n(x) \in V$ *for all* $n \in \mathbb{Z}$, *then* $x \in W_{\text{loc}}^{c}$.
- $f\left(W_{\text{loc}}^{uc}\right) \cap V \subset W_{\text{loc}}^{uc}$. *If* $f^n(x) \in V$ *for all* $n \leq 0$, *then* $x \in W_{\text{loc}}^{uc}$.
- $W_{\text{loc}}^{u} = \big\{x \in V : \; f^n(x) \in V$ *for all* $n \leq 0$ *and* $f^n(x) \to 0$ *exponentially fast as* $n \to -\infty\big\}$.

A.2 Equivalence of dynamical systems

Definition 7 Two discrete time dynamical systems $x_{n+1} = F(x_n)$ and $y_{n+1} = G(y_n)$, $n \in \mathbb{Z}$ are called **topologically equivalent**, if a homeomorphism Υ exists such that

$$F = \Upsilon^{-1} \circ G \circ \Upsilon. \tag{18}$$

Consider the dynamical system

$$x_{n+1} = f(x_n, \lambda), \quad x_n \in \mathbb{R}^k, \ \lambda \in \mathbb{R}, \ n \in \mathbb{Z} \tag{19}$$

and the corresponding extended system

$$\begin{pmatrix} x_{n+1} \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} f(x_n, \lambda_n) \\ \lambda_n \end{pmatrix}, \quad n \in \mathbb{Z}.$$

Assume $\xi = 0$ is a fixed point at the parameter $\lambda = 0$. Denote by $W_{\lambda}^{c}(0)$ the foliated local center manifold and let

$$w \mapsto g(w, \lambda) \tag{20}$$

be the restriction of (19) to $W_{\lambda}^{c}(0)$, $A := f_x(0,0)_{|X^s}$, $B := f_x(0,0)_{|X^u}$.

The next theorem provides a (topological) normal form (cf. [17], [18], [12]).

Theorem 8 (Šošitaĭšvili, 1972) *The systems (19) and*

$$\begin{pmatrix} s_{n+1} \\ u_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} As_n \\ Bu_n \\ g(w_n, \lambda) \end{pmatrix}, \quad s_n \in \mathbb{R}^{k_s}, \ u_n \in \mathbb{R}^{k_u}, \ w_n \in \mathbb{R}^{k_c}, \ n \in \mathbb{Z} \tag{21}$$

are locally topologically equivalent in a neighborhood of $(0,0)$. *Furthermore, the map (20) can be replaced by any topologically equivalent map.*

Finally we show that points on the local center-stable manifold cannot have an unstable component, assuming the center component stays in a small neighborhood of 0 under iteration with f, cf. [10, Lemma A.5].

Consider the system

$$\begin{pmatrix} s \\ u \\ w \end{pmatrix} \mapsto f \begin{pmatrix} s \\ u \\ w \end{pmatrix} = \begin{pmatrix} As \\ Bu \\ h(w) \end{pmatrix},$$

where the matrix A is stable, and B is unstable. The map h satisfies $h(0) = 0$ and $Dh(0)$ possesses only center eigenvalues.

The local graph representation of the center-stable manifold of the fixed point 0 is

$$W^{sc}_{\mathrm{loc}}(0) = \left\{ \begin{pmatrix} s \\ 0 \\ w \end{pmatrix} + \phi^{sc} \begin{pmatrix} s \\ 0 \\ w \end{pmatrix} : \begin{pmatrix} s \\ 0 \\ w \end{pmatrix} \in V(0) \right\}. \tag{22}$$

Here $V(0)$ is a sufficiently small neighborhood of 0 and $\phi^{sc} : X^{sc} \to X^u$ denotes a smooth function, fulfilling $\phi^{sc}(0) = 0$ and $D\phi^{sc}(0) = 0$. Set $\tilde{\phi}^{sc} := \big(\phi^{sc}\big)_u$.

Lemma 9 *Let the assumptions, described above be fulfilled. Assume* $(\bar{s}, \bar{u}, \bar{w}) \in W^{sc}_{\mathrm{loc}}(0)$ *and* $\big(0, 0, h^n(\bar{w})\big) \in V(0)$ *holds for all* $n \in \mathbb{Z}^+$. *Then* $\bar{u} = 0$.

Proof: Since $(\bar{s}, \bar{u}, \bar{w}) \in W^{sc}_{\mathrm{loc}}(0)$, the local graph representation (22) of $W^{sc}_{\mathrm{loc}}(0)$ yields $\bar{u} = \tilde{\phi}^{sc}(\bar{s}, 0, \bar{w})$.

Iterating the center-stable component gives us

$$\begin{pmatrix} \bar{s}_n \\ 0 \\ \bar{w}_n \end{pmatrix} = f^n \begin{pmatrix} \bar{s} \\ 0 \\ \bar{w} \end{pmatrix} = \begin{pmatrix} A^n \bar{s} \\ 0 \\ h^n(\bar{w}) \end{pmatrix} \in V(0).$$

On the one hand the unstable component can be determined using local graph representation

$$\bar{u}_n = \tilde{\phi}^{sc}\big(A^n \bar{s}, 0, h^n(\bar{w})\big).$$

On the other hand $\bar{u}_n = B^n(\bar{u}) = B^n\big(\tilde{\phi}^{sc}(\bar{s}, 0, \bar{w})\big)$ is unbounded for $\bar{u} \neq 0$, since B is unstable. Thus $\tilde{\phi}^{sc}(\bar{s}, 0, \bar{w}) = \bar{u} = 0$. ■

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