On manifolds of connecting orbits in discretizations of dynamical systems

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Abstract

It is shown that one-step methods, when applied to a one-parametric dynamical system with a homoclinic orbit, exhibit a closed loop of discrete homoclinic orbits. On this loop the parameter varies periodically while the orbit shifts its index after one revolution. We show that at least two homoclinic tangencies occur on this loop. Our approach works for systems with finite smoothness and also applies to general connecting orbits. It provides an alternative to the interpolation approach by Fiedler and Scheurle [9] and it allows to recover some of their results on exponentially small splittings of separatrices by using some recent backward error analysis for the analytic case.

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1 Introduction

Time discretizations of continuous dynamical systems can lead to subtle phenomena near homoclinic and heteroclinic orbits. A fundamental result of this type was derived by Fiedler and Scheurle in [9]. They consider a homoclinic orbit $\bar{x}(t)$ $(-\infty < t < \infty)$ at $\lambda = \bar{\lambda}$ of a parameterized system

$$\dot{x} = f(x, \lambda), \quad x \in \mathbb{R}^k, \ \lambda \in \mathbb{R} \tag{1.1}$$

and an associated one-step method

$$x_{n+1} = \psi(x_n, \lambda, \varepsilon), \quad \varepsilon = \text{step-size}. \tag{1.2}$$

Assuming analyticity of f and ψ and of $\bar{x}(t)$ in some complex strip $|\mathrm{Im}\, t| \leq \eta$ they prove that the discrete system has homoclinic orbits $x_n = x_n(\varepsilon)$, $n \in \mathbb{Z}$ that occur in an exponentially narrow sector in the (λ, ε)-plane

$$|\lambda - \tilde{\lambda}(\varepsilon)| \leq ce^{-\eta/\varepsilon}. \tag{1.3}$$

Here $\tilde{\lambda}(\varepsilon) = \bar{\lambda} + O(\varepsilon^d)$ is a smooth function of ε and d is the order of the one-step method. Further quantities that measure the transversality of the stable and unstable manifolds – such as the splitting angle and the splitting distance – can also be estimated by flat terms as in (1.3). Finally, it is shown in [9] that transversal intersections occur in a generic sense.

The method of proof in [9] is to interpret the one-step method (1.2) as the ε-time map of a nonautonomous, rapidly forced perturbation of (1.1), given by

$$\dot{x} = f(x, \lambda) + \varepsilon^d g(x, \lambda, \frac{t}{\varepsilon}, \varepsilon), \tag{1.4}$$

where $g(x, \lambda, \cdot, \varepsilon)$ is smooth and 1-periodic with respect to the third variable. This system is then analyzed by a combination of Liapunov-Schmidt and Fourier techniques. Exponentially small estimates for general systems of type (1.4) have earlier been derived by Neishtadt (see [15]) and they also appear in Benettin and Giorgilli [1].

The purpose of this paper is to derive an alternative approach to splitting results for connecting orbits under discretization. In addition to the above mentioned results, we show that there are actually closed loops of discrete homoclinic orbits for fixed and small ε . Though not explicitly stated in [9], it is possible to obtain this additional information from their proofs. Our main arguments for avoiding the interpolation (1.4) are that the information extracted from (1.4) is independent of the type of interpolation and that there are many ways of constructing such an interpolation. Therefore there should be a more direct approach which works with the given one-step mapping.

In our approach we analyze (1.2) by adding to both (1.1) and (1.2) a scalar phase condition

$$\Pi(x, \varepsilon, t^*) = 0 \tag{1.5}$$

where t^* acts as a parameter. Then we show that close to the continuous homoclinic the one-step method has a curve of discrete homoclinic orbits $x_n(\varepsilon, t^*)$, $n \in \mathbb{Z}$ at parameter values $\lambda(\varepsilon, t^*)$. At fixed ε this curve is parameterized by the phase variable t^*, and it is closed in the following sense

$$\begin{aligned} x_n(\varepsilon, t^*) &= x_{n-1}(\varepsilon, t^* + \varepsilon), \quad n \in \mathbb{Z}, \\ \lambda(\varepsilon, t^*) &= \lambda(\varepsilon, t^* + \varepsilon). \end{aligned} \tag{1.6}$$

This confirms numerical findings from [5] where closed loops of homoclinics in an Euler mapping were obtained. Following a complete turn of the loop with a continuation method the discrete orbit shifts its index as (1.6) indicates. It is in fact the special choice (compare (3.9)) of the phase function Π in (1.5) which enables us to prove the periodicity (1.6) and which avoids reparametrizations as in [9].

In Figure 1.1 we illustrate the behavior of the two dimensional manifold of homoclinic orbits for (1.2) by plotting a typical function $\lambda(\varepsilon, t^*)$ which satisfies (1.3) and (1.6) with $\tilde{\lambda}(\varepsilon) = \bar{\lambda} = 0$.

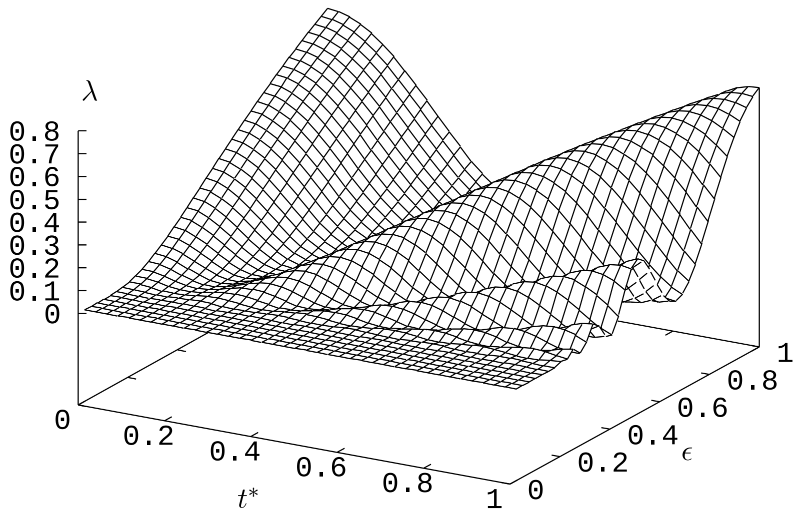

Figure 1.1 *The function* $\lambda(\varepsilon, t^*) = e^{-1/\varepsilon}(1 + \cos(2\pi\frac{t^*}{\varepsilon}))$.

Our results apply to general heteroclinic or connecting orbits with parameter λ and to systems with finite smoothness. We prove error estimates for the discrete orbit $x_n(\varepsilon, t^*)$ and the parameter $\lambda(\varepsilon, t^*)$ of type $O(\tau(\varepsilon))$ where $\tau(\varepsilon)$ is the accuracy of the method, e.g. $\tau(\varepsilon) = \varepsilon^d$ in case of a d-th order method . In this way it is still possible to retrieve exponentially small estimates of the form (1.3) by applying a recent result of Reich ([17]) on backward error analysis. He shows that in the analytic case the one-step mapping (1.2) is exponentially close to the ε-time map of a perturbed autonomous system. Applying (a parameter dependent version of) our theorem to the latter system with $\tau(\varepsilon) = e^{-c/\varepsilon}$ then yields the desired estimates (though not the precise exponent η in (1.3)).

On the manifolds of connecting orbits described by (1.6) we find at least two homoclinic tangencies where $\frac{\partial\lambda}{\partial t^*} = 0$. In general the whole curve $\lambda(\varepsilon, \cdot)$ may be constant for fixed ε. However, in a forthcoming paper [21] we show that there are transversal homoclinics for (1.2) if the function f looses its smoothness at a hyperplane transversal to the continuous homoclinic. This provides an explicit criterion which guarantees transversal intersections in (1.2) and complements the genericity result from [9].

In section 2 and 3 we show that continuous connecting orbits, when sampled at discrete times $n\varepsilon$, $n \in \mathbb{Z}$, lead to curves of tangential connecting orbits for the ε-flow map in the sense of [14]. These curves can be parameterized by the phase variable t^*. In section 4 we will prove that these curves are preserved by the discretization (1.2), where now the parameter λ is allowed to vary in an ε-periodic fashion as in (1.6). Section 5 concludes with applications of the main results (Theorem 4.3 and 4.4) to the analytic and the finitely smooth case.

2 Basic assumptions for the continuous system

In this section, we introduce the basic regularity condition for continuous connecting orbits and collect some preliminary results. We consider a parameterized dynamical system

$$\dot{x} = f(x, \lambda, \nu), \quad x \in \mathbb{R}^k, \quad \lambda \in \mathbb{R}^p, \quad \nu \in \mathbb{R}. \tag{2.1}$$

The parameter λ is used to set up a well-posed equation for connecting orbits while ν is an additional parameter used in later applications where ν corresponds to the step-size. For simplicity, we will drop the dependence on ν in most of the proofs. Our assumptions are

(H1) f is C^r $(r \geq 1)$ smooth in all variables and C^{r+1} smooth in λ, and the r-th order derivative with respect to x, i.e. $f_x^{(r)}(x, \lambda, \nu)$, is uniformly Lipschitz continuous, i.e. there exists a constant $C_1 > 0$ such that

$$\|f_x^{(r)}(x, \lambda, \nu) - f_x^{(r)}(y, \lambda, \nu)\| \leq C_1 \|x - y\|, \quad \text{for } x, y \in \mathbb{R}^k.$$

Furthermore, the $(r+1)$-th order direvative of f with respect to x exists, and the function f, together with all its derivatives, is uniformly bounded by the same constant C_1.

(H2) $\bar{x}_\pm$ are hyperbolic equilibria of equation (2.1) at $(\lambda, \nu) = (\bar{\lambda}, \bar{\nu})$.

According to these two assumptions, there exist constants $\lambda_1 > 0$, $\nu_1 > 0$, and C^r-smooth functions $\bar{x}_\pm(\lambda, \nu)$ with $\bar{x}_\pm(\bar{\lambda}, \bar{\nu}) = \bar{x}_\pm$, such that for all $|\lambda - \bar{\lambda}| < \lambda_1$ and $|\nu - \bar{\nu}| < \nu_1$, $\bar{x}_\pm(\lambda, \nu)$ are hyperbolic equilibria of equation (2.1).

By $k_{\pm s,u}$, we denote the numbers of stable (resp. unstable) eigenvalues (counting multiplicities) of the matrix $f_x(\bar{x}_\pm, \bar{\lambda}, \bar{\nu})$. By virtue of the smoothness of f, the dimension $k_{\pm s,u}$ remains constant for (λ, ν) close to $(\bar{\lambda}, \bar{\nu})$ and we have $k = k_{\pm s} + k_{\pm u}$.

(H3) At $(\lambda, \nu) = (\bar{\lambda}, \bar{\nu})$, equation (2.1) possesses a connecting orbit $\bar{x}(t)$, which satisfies

$$\bar{x}_- = \lim_{t \to -\infty} \bar{x}(t), \quad \bar{x}_+ = \lim_{t \to +\infty} \bar{x}(t).$$

We call $(\bar{x}(t), \bar{\lambda})$ a **connecting orbit pair** (COP for short).

(H4) The COP $(\bar{x}, \bar{\lambda})$ is nondegenerate with respect to the parameter λ (cf. [3, Definition 2.1] and [14, Definition 3.1]) in the following sense

(i) the numbers of stable and unstable eigenvalues are related to the number of parameters λ by $p = k + 1 - k_{-u} - k_{+s} = k_{-s} - k_{+s} + 1$,

(ii) any bounded solution $x(t)$, $\mu \in \mathbb{R}^p$ of the equation $\dot{x} = f_x(\bar{x}, \bar{\lambda}, \bar{\nu})x + f_\lambda(\bar{x}, \bar{\lambda}, \bar{\nu})\mu$ satisfies $\mu = 0$ and $x = c\dot{\bar{x}}$ for some $c \in \mathbb{R}$.

A COP $(\bar{x}(t), \bar{\lambda})$ is said to be **regular** if (H1), (H2) and (H4) are satisfied. As a consequence of the regularity, the implicit function theorem yields the following (cf. [3])

Corollary 2.1 *Assume* (H1)–(H4). *Then there exists a constant* $0 < \nu_2 < \nu_1$ *and a smooth family* $(\bar{x}(\nu, \cdot), \bar{\lambda}(\nu))$ *for* $|\nu - \bar{\nu}| < \nu_2$ *of regular COPs which converge to the hyperbolic equilibria* $\bar{x}_\pm(\bar{\lambda}(\nu), \nu)$ *and satisfy* $\bar{x}(\bar{\nu}, \cdot) = \bar{x}(\cdot)$, $\bar{\lambda}(\bar{\nu}) = \bar{\lambda}$.

Define Banach spaces

$$X^l = \{x \in C^l(\mathbb{R}, \mathbb{R}^k) : x^{(j)}(t) \text{ is bounded for } j = 0, 1, \dots, l\},$$
$$\|x\|_l = \sum_{j=0}^{l} \sup_{t\in\mathbb{R}} \|x^{(j)}(t)\|, \text{ with } \|\cdot\| \text{ some norm in } \mathbb{R}^k.$$

For any $|\nu - \bar{\nu}| < \nu_2$, consider the linear operator

$$L(\nu) : X^1 \to X^0, \quad L(\nu)x = \dot{x} - f_x(\bar{x}(\nu, \cdot), \bar{\lambda}(\nu), \nu)x, \tag{2.2}$$

and denote its solution operator, by $S(\nu, t, s)$, i.e. $x(t) = S(\nu, t, s)\xi$ is the solution of $L(\nu)x = 0$ with $x(s) = \xi$. From the semigroup property we have

$$S(\nu, t, s)^{-1} = S(\nu, s, t). \tag{2.3}$$

According to [7, 16], [3, Lemma 2.1], $L(\nu)$ has an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$ for any $|\nu - \bar{\nu}| < \nu_2$. Similar to [18] and [14, Lemma2.3],[6] we can prove that the exponential dichotomies on $\mathbb{R}^\pm$ hold uniformly in ν. More precisely, we have the following lemma.

Lemma 2.2 *Assume* (H1)–(H4). *Then $L(\nu)$ has an exponential dichotomy on $\mathbb{R}^\pm$ uniformly for $|\nu - \bar{\nu}| < \nu_2$, i.e. there exist data $(K^\pm, \alpha^\pm, P^\pm(\nu, t))$, where $K^\pm$ and $\alpha^\pm$ are constants and $P^\pm(\nu, \cdot)$ are projectors C^r smooth in ν, such that for all $|\nu - \bar{\nu}| < \nu_2$, $t \geq s$ in $\mathbb{R}^\pm$, the following holds*

$$S(\nu, t, s)P^\pm(\nu, s) = P^\pm(\nu, t)S(\nu, t, s), \tag{2.4}$$
$$\|S(\nu, t, s)P^\pm(\nu, s)\| \leq K^\pm e^{-\alpha^\pm(t-s)}, \tag{2.5}$$
$$\|S(\nu, s, t)(I - P^\pm(\nu, t))\| \leq K^\pm e^{-\alpha^\pm(t-s)}. \tag{2.6}$$

¿From (2.3) and (2.4) we see that the projectors $P^\pm(\nu, t)$ are determined by $P^\pm(\nu, 0)$ as follows

$$P^\pm(\nu, t) = S(\nu, t, 0)P^\pm(\nu, 0)S(\nu, 0, t). \tag{2.7}$$

Define the adjoint operator of $L(\nu)$ by

$$L^*(\nu) : X^1 \to X^0, \quad L^*x = \dot{x} + f_x(\bar{x}(\nu, \cdot), \bar{\lambda}(\nu), \nu)^T x,$$

and denote its solution operator by $S^*(\nu,t,s)$. Then we have $S^*(\nu,t,s) = S(\nu,s,t)^T$.

Similar to [3, Lemma 2.2], we can prove

Lemma 2.3 *Assume* (H1)–(H4), *then the adjoint operator* $L^*(\nu)$ *also has an exponential dichotomy on* $\mathbb{R}^\pm$ *with data* $(C_2K^\pm, \alpha^\pm, I - P^\pm(\nu,t)^T)$, *where* C_2 *only depends on the norm in* $\mathbb{R}^k$. *Moreover, for* $|\nu - \bar{\nu}| < \nu_2$,

$$\mathcal{N}(L(\nu)) = \{S(\nu,t,0)x_0 : x_0 \in \mathcal{R}(P^+(\nu,0)) \cap \mathcal{N}(P^-(\nu,0))\}, \quad (2.8)$$

$$\dim \mathcal{N}(L(\nu)) = \dim \mathcal{N}(L^*(\nu)) + k_{+s} - k_{-s}, \quad (2.9)$$

$$x \in \mathcal{R}(L(\nu)) \Leftrightarrow \textstyle\int_{-\infty}^{+\infty} y^T(t)x(t)dt = 0, \ \forall y(\cdot) \in \mathcal{N}(L^*(\nu)), \quad (2.10)$$

and $L(\nu) : X^1 \to X^0$ *is Fredholm of index* $k_{+s} - k_{-s} = k_{-u} - k_{+u}$.

With the aid of Lemma 2.3, we have the following characterization of the regularity, see also [3, Proposition 2.1].

Lemma 2.4 *Assume* (H1)–(H3). *Then for any* $|\nu - \bar{\nu}| < \nu_2$, $(\bar{x}(\nu), \bar{\lambda}(\nu))$ *is regular if and only if the linear operator* $L(\nu)$ *has the properties*

(1) $\dim \mathcal{N}(L(\nu)) = 1$, $\dim \mathcal{N}(L^*(\nu)) = p$,

(2) the $p \times p$ *matrix*

$$E = \int_{-\infty}^{+\infty} Y^T(\nu,t) f_\lambda(\bar{x}(\nu,t), \bar{\lambda}(\nu), \nu)dt$$

is nonsingular, where the columns $y^i(\nu,\cdot) \in X^1$ *of*

$$Y(\nu,\cdot) = (y^1(\nu,\cdot), \dots, y^p(\nu,\cdot)),$$

are C^r *smooth in* ν *and form a basis of* $\mathcal{N}(L^*(\nu))$.

Notice that $\dot{\bar{x}}(\nu,\cdot)$ is always a solution of $L(\nu)x = 0$, and by Lemma 2.3 and 2.4, there holds

Corollary 2.5

(1) $\mathcal{N}(L(\nu)) = \text{span } \{\dot{\bar{x}}(\nu,\cdot)\}$,

(2) $\mathcal{N}(L^*(\nu)) = \text{span } \{y^1(\nu,\cdot),\dots,y^p(\nu,\cdot)\}$,
where $y^i(\nu,t) = S^*(\nu,t,0)y_0^i(\nu)$, $i = 1,\dots,p$, *and* $(y_0^1(\nu),\dots,y_0^p(\nu))$ *is* C^r *smooth in* ν *and forms a basis of the space*

$$(\mathcal{R}(P^+(\nu,0)) + \mathcal{N}(P^-(\nu,0)))^\perp = \mathcal{N}(P^+(\nu,0)^T) \cap \mathcal{R}(P^-(\nu,0)^T).$$

We conclude the section with an elementary lemma on the convergence of Riemann sums which will be frequently used in the sequel.

Lemma 2.6 *Assume* $g : \mathbb{R}^+ \to \mathbb{R}^k$ *is a continuous function and satisfies the estimate* $\|g(t)\| \le Ce^{-\rho t}$ *for some* $C,\rho > 0$*. Then*

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{n=0}^{\infty} g(n\varepsilon) = \int_0^\infty g(t)dt.$$

3 Tangential heteroclinic orbits of the time-ε-flow

In this section, we study the properties of heteroclinic orbits for the time-ε-flow $\varphi(\cdot,\lambda,\nu,\varepsilon)$. We will prove that this map possesses a family of nondegenerate 1-tangential heteroclinic orbits with respect to parameter λ, which are discrete samples of the continuous regular heteroclinic orbit defined in the previous section. We also prove that these discrete samples of heteroclinic orbits are regular zeroes of a suitable operator in a space of biinfinite sequences.

We start by introducing the definition of an ℓ-tangential heteroclinic orbit for general mappings, which is studied in [14]. Then we apply the theory from [14] to our time-ε-flow to study corresponding heteroclinic tangencies.

Let $\phi(x,\lambda) : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^k$ be a family of C^r diffeomorphisms, which at $\lambda = \bar\lambda$ has a heteroclinic orbit $\bar x_n$ $(n \in \mathbb{Z})$ connecting two hyperbolic fixed points.

Definition 3.1 *The heteroclinic orbit* $\bar{x}_n$ $(n \in \mathbb{Z})$ *of the map* $\phi(\cdot, \bar{\lambda})$ *is called* ℓ-**tangential** *for some nonnegative integer* ℓ, *if the homogeneous difference equation*

$$u_{n+1} = \phi_x(\bar{x}_n, \bar{\lambda})u_n, \quad n \in \mathbb{Z}$$

has ℓ *linearly independent bounded solutions. Furthermore, an* ℓ-*tangential heteroclinic orbit* $\bar{x}_n$ $(n \in \mathbb{Z})$ *is called* **nondegenerate** *with respect to the parameter* $\lambda \in \mathbb{R}^p$ *if* $p = k_{-s} - k_{+s} + \ell$ *and if every bounded solution* $((u_n)_{n\in\mathbb{Z}}, \mu)$ *of the difference equation*

$$u_{n+1} = \phi_x(\bar{x}_n, \bar{\lambda})u_n + \phi_\lambda(\bar{x}_n, \bar{\lambda})\mu, \quad n \in \mathbb{Z}$$

satisfies $\mu = 0$.

In [14] this definition is expressed in terms of transversal intersections of parameterized stable and unstable manifolds. As in [14] we interpret Definition 3.1 in terms of operators on Banach spaces. For this purpose, we define the Banach space

$$S_{\mathbb{Z}} = \{x_{\mathbb{Z}} = (x_n)_{n\in\mathbb{Z}} : x_n \in \mathbb{R}^k, \|x_{\mathbb{Z}}\| = \sup_{n\in\mathbb{Z}} \|x_n\| < \infty\}$$

and a map

$$\Gamma : \begin{array}{l} S_{\mathbb{Z}} \times \mathbb{R}^p \to S_{\mathbb{Z}}, \\ (x_{\mathbb{Z}}, \lambda) \mapsto (x_{n+1} - \phi(x_n, \lambda))_{n\in\mathbb{Z}}. \end{array}$$

It is clear that heteroclinic orbits of the maps $\phi(\cdot, \lambda)$ are solutions of the equation $\Gamma(\cdot, \lambda) = 0$. A heteroclinic orbit is ℓ-tangential iff $\dim \mathcal{N}(\Gamma_{x_{\mathbb{Z}}}(\bar{x}_{\mathbb{Z}}, \lambda)) = \ell$, and the nondegeneracy corresponds to the dimension relations $p = k_{-s} - k_{+s} + \ell$ and $\dim \mathcal{N}(\Gamma_{(x_{\mathbb{Z}},\lambda)}(\bar{x}_{\mathbb{Z}}, \lambda)) = \ell$.

Let $\varphi(x, \lambda, \nu, t)$ with $\varphi(x, \lambda, \nu, 0) = x$ be the solution flow of equation (2.1). In our application, we replace the general mapping $\phi(x, \lambda)$ by the time-ε-flow $\varphi(x, \lambda, \nu, \varepsilon)$ and define the operator Γ as follows

$$\Gamma : \begin{array}{l} S_{\mathbb{Z}} \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} \to S_{\mathbb{Z}}, \\ (x_{\mathbb{Z}}, \lambda, \nu, \varepsilon) \mapsto (x_{n+1} - \varphi(x_n, \lambda, \nu, \varepsilon))_{n\in\mathbb{Z}}. \end{array} \tag{3.1}$$

We use the phase parameter of the COP's $(\bar{x}(\nu,\cdot),\bar{\lambda}(\nu))$ from section 2 and define for any $t^*\in\mathbb{R}$

$$\bar{x}_n(\nu,\varepsilon,t^*)=\bar{x}(\nu,t^*+n\varepsilon),\quad n\in\mathbb{R}.$$

Then $(\bar{x}_{\mathbb{Z}}(\nu,\varepsilon,t^*),\bar{\lambda}(\nu))$ is a zero of the operator Γ in (3.1) for all $|\nu-\bar{\nu}|<\nu_2$, $t^*\in\mathbb{R}$ and $\varepsilon\neq 0$.

Theorem 3.2 *Assume* (H1)–(H4). *Then for* $0<\varepsilon<\varepsilon_1$, $|\nu-\bar{\nu}|<\nu_2$, $t^*\in\mathbb{R}$, *the orbits* $\bar{x}_{\mathbb{Z}}(\nu,\varepsilon,t^*)$ *are 1-tangential heteroclinic orbits of the time-*ε*-flow* $\varphi(\cdot,\bar{\lambda}(\nu),\nu,\varepsilon)$ *and they are nondegenerate with respect to the parameter* λ.

We prove the theorem through a series of lemmata and start with studying the properties of the derivative of Γ with respect to $x_{\mathbb{Z}}$. For abbreviation we set

$$\bar{z}_n=\bar{z}_n(\nu,\varepsilon,t^*)=(\bar{x}_n(\nu,\varepsilon,t^*),\bar{\lambda}(\nu),\nu,\varepsilon,t^*). \tag{3.2}$$

Furthermore we drop the parameter ν in most of the proofs but keep it in the statements of lemmas, theorems and so on.

First, we consider the linear operator $\Gamma_{x_{\mathbb{Z}}}:S_{\mathbb{Z}}\to S_{\mathbb{Z}}$ defined by

$$\Gamma_{x_{\mathbb{Z}}}(u_{\mathbb{Z}})=(u_{n+1}-\varphi_x(\bar{z}_n)u_n)_{n\in\mathbb{Z}} \tag{3.3}$$

and the corresponding difference equation

$$u_{n+1}-\varphi_x(\bar{z}_n)u_n=0. \tag{3.4}$$

The following Lemma relates the solution map $\Phi(\nu,n,m)$ of the discrete variational equation (3.4) to the solution operator $S(\nu,t,s)$ of the continuous variational equation (2.2). It follows from an easy calculation

Lemma 3.3 *For all* $n,m\in\mathbb{Z}$ *the following relations hold*

(1) $\varphi_x(\bar{z}_n)=\Phi(\nu,n+1,n)=S(\nu,(n+1)\varepsilon+t^*,n\varepsilon+t^*)$,

(2) $\Phi(\nu,n,m)=S(\nu,n\varepsilon+t^*,m\varepsilon+t^*)$.

¿From now on we chose an open interval $\mathbb{T} = (-t_0^*, t_0^*) \subset \mathbb{R}$ with $t_0^* > 0$ and let t^* vary in $\mathbb{T}$. It will be necessary to decrease t_0^* in some results below. Take discrete samples of the projectors as

$$Q_n^{\pm}(\nu, \varepsilon, t^*) = P^{\pm}(\nu, t^* + n\varepsilon), \quad Q^{\pm}(\nu, \varepsilon, t^*) = (Q_n^{\pm}(\nu, \varepsilon, t^*))_{n \in \mathbb{Z}},$$

where $P^{\pm}(\nu, t)$ is defined in Lemma 2.2. Notice that the dichotomy on $\mathbb{R}^{\pm}$ extends to $\mathbb{R}^{\pm} \cup \mathbb{T}$ for a fixed $t_0^* > 0$.

Lemma 3.4 *Assume* (H1)–(H4). *Then for all* $|\nu - \bar{\nu}| < \nu_2$, $t^* \in \mathbb{T}$ *and* $\varepsilon > 0$, *the difference equation* (3.4) *has an exponential dichotomy on* $\mathbb{Z}^{\pm}$ *with data* $(K^{\pm}, \varepsilon\alpha^{\pm}, Q^{\pm}(\nu, \varepsilon, t^*))$, *where* $K^{\pm}$ *and* $\alpha^{\pm}$ *are defined as in* Lemma 2.2 *with* $\mathbb{R}^{\pm}$ *replaced by* $\mathbb{R}^{\pm} \cup \mathbb{T}$.

Proof. Using the linearized equation (3.4), the proof follows from Lemma 2.2, 3.3, and the definition of the projectors $Q_n^{\pm}(\nu, \varepsilon, t^*)$. ∎

Define the adjoint operator of $\Gamma_{x_{\mathbb{Z}}}$ by

$$\Gamma_{x_{\mathbb{Z}}}^* : S_{\mathbb{Z}} \to S_{\mathbb{Z}}, \quad u_{\mathbb{Z}} \mapsto (u_{n+1} - \varphi_x^{-1}(\bar{z}_n(\nu, \varepsilon, t^*))^T u_n)_{n \in \mathbb{Z}}, \tag{3.5}$$

Similar to [14, Lemma 2.6], [5], from Lemma 2.3, Corollary 2.5 and (H1) we obtain

Lemma 3.5 *Assume* (H1)–(H4). *Then for* $|\nu - \bar{\nu}| < \nu_2$, $t^* \in \mathbb{T}$ *and* $\varepsilon > 0$, *there holds*

(1) the linear operator $\Gamma_{x_{\mathbb{Z}}} : S_{\mathbb{Z}} \to S_{\mathbb{Z}}$ *defined in* (3.3) *is Fredholm of* index $k_{+s} - k_{-s} = 1 - p$, $\dim \mathcal{N}(\Gamma_{x_{\mathbb{Z}}}) = 1$, *and*

$$\mathcal{N}(\Gamma_{x_{\mathbb{Z}}}) = \operatorname{span}\{\dot{\bar{x}}_{\mathbb{Z}}(\nu, \varepsilon, t^*)\}, \quad \dot{\bar{x}}_{\mathbb{Z}}(\nu, \varepsilon, t^*) = (\dot{\bar{x}}(\nu, t^* + n\varepsilon))_{n \in \mathbb{Z}},$$

(2) the adjoint operator $\Gamma_{x_{\mathbb{Z}}}^* : S_{\mathbb{Z}} \to S_{\mathbb{Z}}$ *defined in* (3.5) *is Fredholm of* index $k_{-s} - k_{+s} = p - 1$, $\dim \mathcal{N}(\Gamma_{x_{\mathbb{Z}}}^*) = p$, *and*

$$\mathcal{N}(\Gamma_{x_{\mathbb{Z}}}^*) = \operatorname{span}\{y_{\mathbb{Z}}^1(\nu, \varepsilon, t^*), \dots, y_{\mathbb{Z}}^p(\nu, \varepsilon, t^*)\},$$

where $y_{\mathbb{Z}}^i(\nu, \varepsilon, t^*) = (y^i(\nu, t^* + n\varepsilon))_{n \in \mathbb{Z}}$ *are defined in* Corollary 2.5.

With the functions $y^i_{\mathbb{Z}}$ from this lemma, define the sequence of $k \times p$ matrices $Y_{\mathbb{Z}}(\nu, \varepsilon, t^*) = (y^1_{\mathbb{Z}}(\nu, \varepsilon, t^*), \dots, y^p_{\mathbb{Z}}(\nu, \varepsilon, t^*))$.

Lemma 3.6 *Assume* (H1)–(H4). *Then there exists $\varepsilon_1, \nu_2, t^*_0 > 0$ such that the $p \times p$ matrices*

$$\sum_{n \in \mathbb{Z}} Y_n^T(\nu, \varepsilon, t^*) \varphi_\lambda(\bar{z}_n)$$

are nonsingular and their inverses are uniformly bounded for $0 < \varepsilon < \varepsilon_1$, $|\nu - \bar{\nu}| < \nu_2$ and $t^* \in \mathbb{T}$.

Proof. ¿From the variational equation for $\varphi_\lambda(x, \lambda, t)$ we obtain

$$\begin{aligned} \varphi_\lambda(x, \lambda, t) &= \int_0^t f_x(\varphi(x, \lambda, s), \lambda) \varphi_\lambda(x, \lambda, s) ds \\ &+ \int_0^t f_\lambda(\varphi(x, \lambda, s), \lambda) ds. \end{aligned} \tag{3.6}$$

For $t^* \in \mathbb{R}$ and $\varepsilon > 0$ small, taking $t = \varepsilon$, $x = \bar{x}_n(\varepsilon, t^*)$, $\lambda = \bar{\lambda}$ in equation (3.6), leads to

$$\begin{aligned} \varphi_\lambda(\bar{z}_n) &= \int_0^\varepsilon f_x(\varphi(\bar{x}_n, \bar{\lambda}, s), \bar{\lambda}) \varphi_\lambda(\bar{x}_n, \bar{\lambda}, s) ds + \int_0^\varepsilon f_\lambda(\varphi(\bar{x}_n, \bar{\lambda}, s), \bar{\lambda}) ds \\ &= \varepsilon f_x(\varphi(\bar{x}_n, \bar{\lambda}, 0), \bar{\lambda}) \varphi_\lambda(\bar{x}_n, \bar{\lambda}, 0) + \varepsilon f_\lambda(\varphi(\bar{x}_n, \bar{\lambda}, 0), \bar{\lambda}) + O(\varepsilon^2) \\ &= \varepsilon f_\lambda(\bar{x}_n, \bar{\lambda}) + O(\varepsilon^2), \end{aligned} \tag{3.7}$$

because $\varphi(x, \lambda, 0) \equiv x$ implies $\varphi_\lambda(x, \lambda, 0) \equiv 0$.

By Lemma 2.4 the matrices $Y(t)$ converge to 0 with an exponential rate $ce^{-\alpha|t|}$ and hence $Y_n(\varepsilon, t^*)$ converges to 0 with a bound $ce^{-\alpha|n\varepsilon|}$. Multiplying equation (3.7) by $Y_n^T(\nu, \varepsilon, t^*)$ gives

$$Y_n^T(\nu, \varepsilon, t^*) \varphi_\lambda(\bar{z}_n(\nu, \varepsilon, t^*)) = \varepsilon Y_n^T(\nu, \varepsilon, t^*) f_\lambda(\bar{z}_n(\nu, \varepsilon, t^*)) + O(\varepsilon^2 e^{-\alpha|n\varepsilon|}).$$

Summing up and using Lemma 2.6 as well as $\varepsilon^2 \sum_{n \in \mathbb{Z}} e^{-\alpha|n\varepsilon|} = O(\varepsilon)$ we obtain

$$\lim_{\varepsilon \to 0} \sum_{n \in \mathbb{Z}} Y_n^T \varphi_\lambda(\bar{z}_n) = \int_{-\infty}^{\infty} Y(t)^T f_\lambda(\bar{x}(t), \bar{\lambda}) dt,$$

which is nonsingular by Lemma 2.4. The proof is finished by taking $\varepsilon_1, \nu_2, t_0^*$ sufficiently small. ■

Proof of Theorem 3.2. According to the Definition 3.1 and Lemma 3.5, it remains to prove

$$\dim \mathcal{N}(\Gamma_{(x_{\mathbb{Z}},\lambda)}(\bar{z}_{\mathbb{Z}})) \equiv 1.$$

For any $0 < \varepsilon < \varepsilon_1$, $t^* \in \mathbb{T}$, and $(u_{\mathbb{Z}}, \mu) \in S_{\mathbb{Z}} \times \mathbb{R}^p$, consider the linear equation

$$\Gamma_{x_{\mathbb{Z}}}(\bar{z}_{\mathbb{Z}})u_{\mathbb{Z}} + \Gamma_{\lambda}(\bar{z}_{\mathbb{Z}})\mu = 0. \tag{3.8}$$

Multiplying equation (3.8) from the left by $Y_{\mathbb{Z}}^T(\varepsilon, t^*)$ and using (2.10), gives

$$\sum_{n\in\mathbb{Z}} Y_n^T \varphi_\lambda(\bar{z}_{\mathbb{Z}})\mu = 0$$

By Lemma 3.6, we get $\mu = 0$. Substituting $\mu = 0$ into (3.8), we find

$$u_{\mathbb{Z}} \in \mathcal{N}(\Gamma_{x_{\mathbb{Z}}}(\bar{z}_{\mathbb{Z}})),$$

which, together with Corollary 2.5, completes the proof of theorem. ■

Obviously, we expect the solutions of equations $\Gamma(x_{\mathbb{Z}}, \lambda, \nu, \varepsilon) = 0$ to be uniquely determined by $(\bar{x}_{\mathbb{Z}}(\nu, \varepsilon, t^*), \bar{\lambda}(\nu))$ up to a phase shift controlled by t^*. We will fix this phase by a suitable condition. Define the phase function Π as

$$\Pi : \begin{array}{l} S_{\mathbb{Z}} \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x_{\mathbb{Z}}, \nu, \varepsilon, t^*) \mapsto \varepsilon \displaystyle\sum_{n=-\infty}^{\infty} (x_n - \bar{x}(\nu, t^* + n\varepsilon))^T \dot{\bar{x}}(\nu, t^* + n\varepsilon). \end{array} \tag{3.9}$$

With this and the operator Γ from (3.1) define

$$F : \begin{array}{l} S_{\mathbb{Z}} \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R} \to S_{\mathbb{Z}} \times \mathbb{R}, \\ (x_{\mathbb{Z}}, \lambda, \nu, \varepsilon, t^*) \mapsto (\Gamma(x_{\mathbb{Z}}, \lambda, \nu, \varepsilon), \Pi(x_{\mathbb{Z}}, \nu, \varepsilon, t^*)). \end{array} \tag{3.10}$$

The following Theorem contains the main properties of the operator F.

Theorem 3.7 *Assume* (H1)–(H4). *Then there exist* ε_1, ν_2, $t_0^* > 0$ *such that* $(\bar{x}_{\mathbb{Z}}(\nu,\varepsilon,t^*), \bar{\lambda}(\nu))$ *are regular solutions of the equation* $F = 0$ *in* (3.10) *for* $0 < \varepsilon < \varepsilon_1$, $|\nu - \bar{\nu}| < \nu_2$, $t^* \in \mathbb{T}$, *i.e.* $F_{(x_{\mathbb{Z}},\lambda)}(\bar{z}_{\mathbb{Z}}(\nu,\varepsilon,t^*))$ *are homeomorphisms. Furthermore, there exists a constant* $\beta > 0$ *such that*

$$\|F_{(x_{\mathbb{Z}},\lambda)}(\bar{z}_{\mathbb{Z}}(\nu,\varepsilon,t^*))(v_{\mathbb{Z}},\mu)\| \geq \beta\varepsilon\|(v_{\mathbb{Z}},\mu)\|, \quad \forall (v_{\mathbb{Z}},\mu) \in S_{\mathbb{Z}} \times \mathbb{R}^p. \tag{3.11}$$

Proof. We consider the linearization of equation (3.10) at $(\bar{x}_{\mathbb{Z}}, \bar{\lambda})$. For any $r_{\mathbb{Z}} \in S_{\mathbb{Z}}$ and $\omega \in \mathbb{R}$ we will find a solution $(v_{\mathbb{Z}}, \mu)$ of the linear equations

$$v_{n+1} - A_n v_n - g_n \mu = r_n, \quad n \in \mathbb{Z}, \tag{3.12}$$

$$\sum_{n=-\infty}^{\infty} \varepsilon v_n^T \dot{\bar{x}}_n = \omega, \tag{3.13}$$

where $A_n = \varphi_x(\bar{x}_n, \bar{\lambda}, \varepsilon)$, $g_n = \varphi_\lambda(\bar{x}_n, \bar{\lambda}, \varepsilon)$ and $\dot{\bar{x}}_n = \dot{\bar{x}}(t^* + n\varepsilon)$.

For any $y_{\mathbb{Z}} \in S_{\mathbb{Z}}$, define

$$\begin{aligned} \bar{v}_n^+(y_{\mathbb{Z}}) &= \sum_{i=1}^{n} \Phi(n,i) Q_i^+ y_{i-1} - \sum_{i=n+1}^{\infty} \Phi(n,i)(I - Q_i^+) y_{i-1}, \quad \text{for } n \geq 0, \\ \bar{v}_n^-(y_{\mathbb{Z}}) &= \sum_{-\infty}^{i=n} \Phi(n,i) Q_i^- y_{i-1} - \sum_{i=n+1}^{0} \Phi(n,i)(I - Q_i^-) y_{i-1}, \quad \text{for } n \leq 0. \end{aligned}$$

Since the equation $v_{n+1} = A_n v_n$ has an exponential dichotomy on $\mathbb{Z}_\pm$ with data $(K, \varepsilon\alpha, Q_{\mathbb{Z}_\pm})$ the sequences $\bar{v}^\pm_{\mathbb{Z}_\pm}$ are the unique bounded solutions of equation $v_{n+1} = A_n v_n + y_n$ in $\mathbb{Z}_\pm$ with initial conditions $Q_0^+ \bar{v}_0^+ = 0$ and $(I - Q_0^-)\bar{v}_0^- = 0$, respectively, and satisfy the following estimates for $K = \max\{K^+, K^-\}$ (cf. [16])

$$|\bar{v}_n^\pm| \leq K \frac{1 + e^{-\varepsilon\alpha}}{1 - e^{-\varepsilon\alpha}} \|y_{\mathbb{Z}}\|_\infty. \tag{3.14}$$

To obtain the general solution of the inhomogeneous equation (3.12) for $n \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_-$, respectively, we add solutions of the homogeneous equation as follows. For $\eta \in \mathcal{R}(Q_0^+)$, $\xi \in \mathcal{R}(I - Q_0^-)$ and $\mu \in \mathbb{R}^p$ let

$$\begin{aligned} v_n^+ &= \Phi(n,0)\eta + \bar{v}_n^+(r_{\mathbb{Z}} + g_{\mathbb{Z}}\mu), \quad n \geq 0, \\ v_n^- &= \Phi(n,0)\xi + \bar{v}_n^-(r_{\mathbb{Z}} + g_{\mathbb{Z}}\mu), \quad n \leq 0 \end{aligned} \tag{3.15}$$

and we define $v_n = v_n^+$ if $n \geq 0$ and $v_n = v_n^-$ if $n \leq -1$. Then the pair $(v_{\mathbb{Z}}, \mu)$ solves equation (3.12)–(3.13) if and only if

$$v_0^+ = v_0^-, \qquad \sum_{n=-\infty}^{\infty} \varepsilon v_n^T \dot{\bar{x}}_n = \omega. \tag{3.16}$$

In terms of ξ, η, μ equation (3.16) reads

$$\begin{aligned} \xi - \eta + \Omega(\varepsilon, t^*)\mu &= \varrho(\varepsilon, t^*), \\ \Theta(\varepsilon, t^*)\xi + \Lambda(\varepsilon, t^*)\eta + \Xi(\varepsilon, t^*)\mu &= \omega - \delta(\varepsilon, t^*), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned}
\Omega(\varepsilon, t^*) &= \sum_{-\infty}^{i=0} \Phi(0,i) Q_i^- g_{i-1} + \sum_{i=1}^{\infty} \Phi(0,i)(I - Q_i^+) g_{i-1}, \\
\Theta(\varepsilon, t^*) &= \sum_{n=-\infty}^{-1} \varepsilon \dot{\bar{x}}_n^T \Phi(n,0)(I - Q_0^-), \\
\Lambda(\varepsilon, t^*) &= \sum_{n=0}^{\infty} \varepsilon \dot{\bar{x}}_n^T \Phi(n,0) Q_0^+, \\
\Xi(\varepsilon, t^*) &= \sum_{n=-\infty}^{-1} \varepsilon \dot{\bar{x}}_n^T \bar{v}_n^-(g_{\mathbb{Z}}) + \sum_{n=0}^{\infty} \varepsilon \dot{\bar{x}}_n^T \bar{v}_n^+(g_{\mathbb{Z}}), \\
\varrho(\varepsilon, t^*) &= -\left(\sum_{-\infty}^{i=0} \Phi(0,i) Q_i^- r_{i-1} + \sum_{i=1}^{\infty} \Phi(0,i)(I - Q_i^+) r_{i-1} \right), \\
\delta(\varepsilon, t^*) &= \sum_{n=-\infty}^{-1} \varepsilon \dot{\bar{x}}_n^T \bar{v}_n^-(r_{\mathbb{Z}}) + \sum_{n=0}^{\infty} \varepsilon \dot{\bar{x}}_n^T \bar{v}_n^+(r_{\mathbb{Z}}).
\end{aligned}$$

According to (3.7) we have $g_i = \varepsilon f_\lambda(\bar{x}_n, \bar{\lambda}) + O(\varepsilon^2)$. Similar to the argument in the proof of Lemma 3.6 and by Lemma 2.6 and 3.3, we obtain that $\Omega(\varepsilon, t^*)$ converges as $\varepsilon, t^* \to 0$ to

$$\Omega_0 = \int_{-\infty}^{0} S(0,t) P^-(t) f_\lambda(\bar{x}(t), \bar{\lambda}) dt + \int_{0}^{+\infty} S(0,t)(I - P^+(t)) f_\lambda(\bar{x}(t), \bar{\lambda}) dt.$$

In an analogous fashion we have

$$\begin{aligned}
\Theta_0 = \lim_{\varepsilon,t^*\to 0}\Theta(\varepsilon,t^*) &= \int_{-\infty}^{0}\dot{\bar{x}}(t)^T S(t,0)(I-P_0^-)dt,\\
\Lambda_0 = \lim_{\varepsilon,t^*\to 0}\Lambda(\varepsilon,t^*) &= \int_{0}^{\infty}\dot{\bar{x}}(t)^T S(t,0)P_0^+dt.\\
\Xi_0 = \lim_{\varepsilon,t^*\to 0}\Xi(\varepsilon,t^*) &= \int_{0}^{\infty}\dot{\bar{x}}(t)^T\bar{v}^+(f_\lambda(\bar{x}(\cdot),\bar{\lambda}))dt\\
&+ \int_{-\infty}^{0}\dot{\bar{x}}(t)^T\bar{v}^-(f_\lambda(\bar{x}(\cdot),\bar{\lambda}))dt,
\end{aligned}$$

where as in the discrete case we set

$$\begin{aligned}
\bar{v}^+(y)(t) &= \int_0^t S(t,s)P^+(s)y(s)ds - \int_t^\infty S(t,s)(I-P^+(s))y(s)ds,\ t\ge 0\\
\bar{v}^-(y)(t) &= \int_{-\infty}^t S(t,s)P^-(s)y(s)ds - \int_t^0 S(t,s)(I-P^-(s))y(s)ds,\ t\le 0.
\end{aligned}$$

From the expressions for $(\varrho(\varepsilon,t^*),\delta(\varepsilon,t^*))$ we obtain via the dichotomies for ε small and some constant $C>0$

$$\|(\varrho(\varepsilon),\delta(\varepsilon))\| \le CK\frac{1+e^{-\varepsilon\alpha}}{1-e^{-\varepsilon\alpha}}\|r_{\mathbb{Z}}\|_\infty. \tag{3.18}$$

Let us rewrite equation (3.17) in the form $M^{\varepsilon,t^*}(\xi,\eta,\mu) = (\varrho,\omega+\delta)$, where M^{ε,t^*} is a $(k_{+s}+k_{-s}+p)\times(k+1) = (k+1)\times(k+1)$ matrix. We will prove that the matrices M^{ε,t^*} have uniformly bounded inverses for ε,t^* small. Notice that the spaces $\mathcal{R}(Q_0^+),\mathcal{R}(I-Q_0^-)$ still depend on t^*. In the limit $\varepsilon,t^*\to 0$ we obtain

$$M_0\begin{pmatrix}\xi\\ \eta\\ \mu\end{pmatrix} = \begin{pmatrix} I & -I & \Omega_0\\ \Theta_0 & \Lambda_0 & \Xi_0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\\ \mu\end{pmatrix},\quad \xi\in\mathcal{R}(I-P^-(0)),\eta\in\mathcal{R}(P^+(0)),\mu\in\mathbb{R}^p. \tag{3.19}$$

Assume that (ξ,η,μ) solves $M_0(\xi,\eta,\mu)=0$ with M_0 from (3.19). Then we construct the functions

$$\begin{aligned}
v^+(t) &= S(t,0)\eta + \bar{v}^+(f_\lambda(\bar{x}(\cdot),\bar{\lambda}))(t),\ t\ge 0\\
v^-(t) &= S(t,0)\xi + \bar{v}^-(f_\lambda(\bar{x}(\cdot),\bar{\lambda}))(t),\ t\le 0,
\end{aligned}$$

which yield bounded solution of the equation

$$\dot{x}(t) - f_x(\bar{x}(t), \bar{\lambda})x(t) = f_\lambda(\bar{x}(t), \bar{\lambda})\mu \tag{3.20}$$

for $\pm t \geq 0$ and with initial values $v^+(0) = \eta$ and $v^-(0) = \xi$, respectively.

Let $v(t) = v^+(t)$ if $t \geq 0$ and $v(t) = v^-(t)$ if $t < 0$. Similar to the arguments above for the discrete case, we find that $v^-(0) = v^+(0)$ holds and hence that $v(t)$ solves equation (3.20) for $t \in \mathbb{R}$ and satisfies the phase condition

$$\int_{-\infty}^{\infty} \dot{\bar{x}}(t)^T v(t) dt = 0. \tag{3.21}$$

¿From the nondegeneracy condition (H4) it follows that $\mu = 0$ and $v(t) = c\dot{\bar{x}}(t)$ for some constant c. Then (3.21) implies $c = 0$.

Therefore the matrix M_0 is invertible and for ε, t^* small enough the matrices M^{ε,t^*} have uniformly bounded inverses.

Combining the estimate (3.18) with equation (3.17) we find

$$|\xi| + |\eta| + |\mu| \leq C(K\frac{1+e^{-\varepsilon\alpha}}{1-e^{-\varepsilon\alpha}}\|r_{\mathbb{Z}}\|_\infty + |\omega|) \leq \frac{1}{\beta\varepsilon}(\|r_{\mathbb{Z}}\|_\infty + |\omega|) \tag{3.22}$$

for some constant $\beta > 0$. Finally, using these estimates in the definition of v_n in (3.15) and noticing that $g_n = \mathcal{O}(\varepsilon)$ we arrive at our assertion. ■

Remark 3.8 *Inequality* (3.22) *indicates that the estimate in* Theorem 3.7 *can be improved by avoiding the $\frac{1}{\varepsilon}$ factor in front of ω. In fact, even further refinements are needed in order to provide sufficient criteria for transversal intersections of the discretized orbits, see* [21].

4 Heteroclinic orbits under discretization

In this section we study the effects of discretization on nondegenerate heteroclinic orbits and prove the main theorems. Roughly speaking, we show that, for step-size sufficiently small , the one-step method possesses a curve of heteroclinic orbits which are either transversal or 1-tangential and nondegenerate. They satisfy the same phase condition (3.9) as the ε-samples of the continuous orbit. As the phase parameter t^* varies, the points on these heteroclinic orbits form an invariant manifold for a parameter suspended map, which we call a heteroclinic manifold. We will also show that this manifold contains at least two 1-tangential orbits and we will obtain the order of approximation of the heteroclinic orbits when the step-size ε tends to zero.

Consider a one-step method

$$x_{n+1} = \psi(x_n, \lambda, \nu, \varepsilon), \tag{4.1}$$

where $\varepsilon > 0$ is the step-size. Our assumptions are

(H5) All partial derivatives of $\psi(x, \lambda, \nu, \varepsilon)$ of order $\leq r + 1 (r \geq 1)$, except for ψ_x^{r+1}, exist and are continuous and bounded in $\mathbb{R}^k \times U$ for some open zero neighborhood $U \subset \mathbb{R}^{p+2}$. The r-th order derivative $\psi_x^{(r)}(x, \lambda, \nu, \varepsilon)$ is uniformly Lipschitz continuous, i.e. there exists a constant $C_3 > 0$ such that for $x, y \in \mathbb{R}^k$, $(\lambda, \nu, \varepsilon) \in U$

$$\|\psi_x^{(r)}(x, \lambda, \nu, \varepsilon) - \psi_x^{(r)}(y, \lambda, \nu, \varepsilon)\| \leq C_3 \|x - y\|.$$

(H6) ψ is d-th $(1 \leq d \leq r)$ order close to the time-ε-flow of equation (2.1) for all x, λ, ν and $\varepsilon > 0$ small, i.e. there exists a constant $C_4 > 0$ such that

$$\|\varphi(x, \lambda, \nu, \varepsilon) - \psi(x, \lambda, \nu, \varepsilon)\| \leq C_4 \varepsilon \tau(\varepsilon), \tag{4.2}$$

where $\tau(\varepsilon) = \varepsilon^d$.

We introduce the function $\tau(\varepsilon)$ here, because we will use (H6) with $\tau(\varepsilon) = e^{-c/\varepsilon}$ in section 5. An immediate consequence of the approximation properties of the one-step method are estimates of the derivatives, see [10, 19].

Lemma 4.1 *Assume* (H1), (H5) *and* (H6). *Then there exist constants* $C_5, \varepsilon_3 > 0$ *such that for all* $(x, \lambda, \nu, \varepsilon) \in \mathbb{R}^k \times U$ *and* $0 < \varepsilon < \varepsilon_3$ *the following estimates hold*

(1) $|\varphi_x(x,\lambda,\nu,\varepsilon)-\psi_x(x,\lambda,\nu,\varepsilon)|\le C_5\varepsilon^{d+1}$,

(2) $|\varphi_\lambda(x,\lambda,\nu,\varepsilon)-\psi_\lambda(x,\lambda,\nu,\varepsilon)|\le C_5\varepsilon^{d+1}$,

(3) $\psi(x,\lambda,\nu,\varepsilon)=x+\varepsilon\Delta(x,\lambda,\nu,\varepsilon)$,

where Δ has the same smoothness properties as ψ except for loosing one derivative with respect to ε.

The following version of a parameterized implicit function theorem will be used in our proof, see [8],[3].

Lemma 4.2 *Let $F:Y\times\Lambda\mapsto Z$ be a C^ℓ mapping from a Banach space $Y\times\Lambda$ into some Banach space Z. Assume there exists a function $\bar y_0:\Lambda\mapsto Y$ such that $F_y(\bar y_0(\mu),\mu)$ are homeomorphisms for all $|\mu|<\delta_2$, and there exist some constants $\kappa=\kappa(\mu)>0$, $\sigma=\sigma(\mu)>0$, such that for $|y-\bar y_0(\mu)|\le\delta_1$, $|\mu|\le\delta_2$ we have*

$$\begin{aligned}\|F_y(y,\mu)-F_y(\bar y_0(\mu),\mu)\| &\le \kappa(\mu)<\sigma(\mu)\le\|F_y(\bar y_0(\mu),\mu)^{-1}\|^{-1},\\ \|F(\bar y_0(\mu),\mu)\| &\le (\sigma(\mu)-\kappa(\mu))\delta_1.\end{aligned}$$

Then for any $|\mu|<\delta_2$, $F(\cdot,\mu)$ has a unique C^ℓ-smooth zero $\tilde y=\tilde y(\mu)$ near $\bar y_0(\mu)$, and the following estimates hold for $|y_i-\bar y_0(\mu)|\le\delta_1$, $i=1,2$

$$\begin{aligned}|\tilde y(\mu)-\bar y_0(\mu)|&\le(\sigma(\mu)-\kappa(\mu))^{-1}\|F(\bar y_0(\mu),\mu)\|, \qquad (4.3)\\ \|y_1-y_2\|&\le(\sigma(\mu)-\kappa(\mu))^{-1}\|F(y_1,\mu)-F(y_2,\mu)\|.\end{aligned}$$

In order to determine heteroclinic orbits for the map $\psi(\cdot,\lambda,\nu,\varepsilon)$ we introduce the following operators

$$\tilde\Gamma:\begin{array}{l}S_{\mathbb Z}\times\mathbb R^p\times\mathbb R^q\times\mathbb R\to S_{\mathbb Z},\\ (x_{\mathbb Z},\lambda,\nu,\varepsilon)\mapsto(x_{n+1}-\psi(x_n,\lambda,\nu,\varepsilon))_{n\in\mathbb Z},\end{array}\qquad(4.4)$$

and

$$\tilde F:\begin{array}{l}S_{\mathbb Z}\times\mathbb R^p\times\mathbb R^q\times\mathbb R\times\mathbb R\to S_{\mathbb Z}\times\mathbb R,\\ (x_{\mathbb Z},\lambda,\nu,\varepsilon,t^*)\mapsto(\tilde\Gamma(x_{\mathbb Z},\lambda,\nu,\varepsilon),\Pi(x_{\mathbb Z},\nu,\varepsilon,t^*)),\end{array}\qquad(4.5)$$

where Π is the phase function defined in (3.9). We apply Lemma 4.2 to $\tilde F$ with $\mu=(\nu,\varepsilon,t^*)$ and the given orbits $\bar y_0(\mu)=(\bar x_{\mathbb Z}(\nu,\varepsilon,t^*),\bar\lambda(\nu))$.

Theorem 4.3 *Assume* (H1)–(H6). *Then there exists constants $\varepsilon_4, \delta > 0$, such that for $0 < \varepsilon < \varepsilon_4$, $|\nu - \bar{\nu}| < \nu_2$, $t^* \in \mathbb{T}$, the function $\tilde{F}$ defined in* (4.5) *has a unique zero $(\tilde{x}_{\mathbb{Z}}(\nu, \varepsilon, t^*), \tilde{\lambda}(\nu, \varepsilon, t^*))$ in a δ neighborhood of $(\bar{x}_{\mathbb{Z}}(\nu, \varepsilon, t^*), \bar{\lambda}(\nu))$. This orbit satisfies the following estimates*

$$\begin{aligned} \sup_{n\in\mathbb{Z}, t^*\in\mathbb{T}} \|\tilde{x}_n(\nu, \varepsilon, t^*) - \bar{x}(\nu, t^* + n\varepsilon)\| &= O(\varepsilon^d), && (4.6) \\ \sup_{t^*\in\mathbb{T}} \|\tilde{\lambda}(\nu, \varepsilon, t^*) - \bar{\lambda}(\nu)\| &= O(\varepsilon^d). && (4.7) \end{aligned}$$

Proof. By virtue of Lemma 4.1, for $0 < \varepsilon < \varepsilon_3$, $|\nu - \bar{\nu}| < \nu_2$, $t^* \in \mathbb{T}$, $(u_{\mathbb{Z}}, \mu) \in S_{\mathbb{Z}} \times \mathbb{R}^p$ there holds

$$\begin{aligned} & \tilde{\Gamma}_{(x_{\mathbb{Z}},\lambda)}(x_{\mathbb{Z}}, \lambda, \nu, \varepsilon) \cdot (u_{\mathbb{Z}}, \mu) \\ = \;& \tilde{\Gamma}_{x_{\mathbb{Z}}}(x_{\mathbb{Z}}, \lambda, \nu, \varepsilon) \cdot u_{\mathbb{Z}} + \tilde{\Gamma}_{\lambda}(x_{\mathbb{Z}}, \lambda, \nu, \varepsilon) \cdot \mu \\ = \;& (u_{n+1} - \psi_x(x_n, \lambda, \nu, \varepsilon)u_n)_{n\in\mathbb{Z}} - (\psi_\lambda(x_n, \lambda, \nu, \varepsilon)\mu)_{n\in\mathbb{Z}} \\ = \;& (u_{n+1} - \varphi_x(x_n, \lambda, \nu, \varepsilon)u_n)_{n\in\mathbb{Z}} - (\varphi_\lambda(x_n, \lambda, \nu, \varepsilon)\mu)_{n\in\mathbb{Z}} + O(\varepsilon\tau(\varepsilon)(\|u_{\mathbb{Z}}\| + |\mu|)) \\ = \;& \Gamma_{(x_{\mathbb{Z}},\lambda)}(x_{\mathbb{Z}}, \lambda, \nu, \varepsilon) \cdot (u_{\mathbb{Z}}, \mu) + O(\varepsilon\tau(\varepsilon)(\|u_{\mathbb{Z}}\| + |\mu|)) \end{aligned}$$

which leads to an estimate uniformly in $x_{\mathbb{Z}}$

$$\|\tilde{\Gamma}_{(x_{\mathbb{Z}},\lambda)}(x_{\mathbb{Z}}, \lambda, \nu, \varepsilon) - \bar{\Gamma}_{(x_{\mathbb{Z}},\lambda)}(x_{\mathbb{Z}}, \lambda, \nu, \varepsilon)\| \leq C_5\varepsilon\tau(\varepsilon). \tag{4.8}$$

We show that Theorem 3.7 together with (4.8) imply that for some constant $0 < \varepsilon_4 < \varepsilon_3$ the operators $\tilde{F}_{(x_{\mathbb{Z}},\lambda)}(\bar{z}_{\mathbb{Z}}(\nu, \varepsilon, t^*))$ are homeomorphisms for $0 < \varepsilon < \varepsilon_4$, $|\nu - \bar{\nu}| < \nu_2$, $t^* \in \mathbb{T}$ and satisfy the estimate

$$\|\tilde{F}_{(x_{\mathbb{Z}},\lambda)}(\bar{z}_{\mathbb{Z}}(\nu, \varepsilon, t^*))(u_{\mathbb{Z}}, \mu)\| \geq \frac{1}{2}\beta\varepsilon\|(u_{\mathbb{Z}}, \mu)\|.$$

Let $\sigma(\varepsilon) = \|\tilde{F}_{(\bar{x}_{\mathbb{Z}},\lambda)}(\bar{z}_{\mathbb{Z}}(\nu, \varepsilon, t^*))^{-1}\|^{-1}$, $\kappa(\varepsilon) = \sigma(\varepsilon)/2$, then for $0 < \varepsilon < \varepsilon_4$, $|\nu - \bar{\nu}| < \nu_2$, $t^* \in \mathbb{T}$, there holds

$$\sigma(\varepsilon) \geq \beta\varepsilon/2, \quad \kappa(\varepsilon) \geq \beta\varepsilon/4. \tag{4.9}$$

¿From Lemma 4.1 and the definition of $\tilde{F}$ in (4.5) we have

$$\begin{aligned}
& \|(\tilde{F}_{(x_{\mathbb{Z}},\lambda)}(x_{\mathbb{Z}},\lambda,\nu,\varepsilon,t^*) - \tilde{F}_{(x_{\mathbb{Z}},\lambda)}(\bar{z}_{\mathbb{Z}}(\nu,\varepsilon,t^*)))(u_{\mathbb{Z}},\mu)\| \\
= \ & \|(\tilde{\Gamma}_{(x_{\mathbb{Z}},\lambda)}(x_{\mathbb{Z}},\lambda,\nu,\varepsilon) - \tilde{\Gamma}_{(x_{\mathbb{Z}},\lambda)}(\bar{z}_{\mathbb{Z}}(\nu,\varepsilon,t^*)))(u_{\mathbb{Z}},\mu)\| \\
\leq \ & \|((\psi_x(x_n,\lambda,\nu,\varepsilon) - \psi_x(\bar{z}_n(\nu,\varepsilon,t^*)))u_n)_{n\in\mathbb{Z}}\| \\
& + \|((\psi_\lambda(x_n,\lambda,\nu,\varepsilon) - \psi_\lambda(\bar{z}_n(\nu,\varepsilon,t^*)))\mu)_{n\in\mathbb{Z}}\| \\
\leq \ & \varepsilon\|((\Delta_x(x_n,\lambda,\nu,\varepsilon) - \Delta_x(\bar{z}_n(\nu,\varepsilon,t^*)))u_n)_{n\in\mathbb{Z}}\| \\
& + \varepsilon\|((\Delta_\lambda(x_n,\lambda,\nu,\varepsilon) - \Delta_\lambda(\bar{z}_n(\nu,\varepsilon,t^*)))\mu)_{n\in\mathbb{Z}}\| \\
\leq \ & \tilde{C}\varepsilon(\|x_{\mathbb{Z}} - \bar{x}_{\mathbb{Z}}(\nu,\varepsilon,t^*)\| + \|\lambda - \bar{\lambda}(\nu)\|)\cdot\|(u_{\mathbb{Z}},\mu)\|,
\end{aligned}$$

where $\tilde{C}$ is obtained from assumption (H5).

Because of (4.9), we can choose $\delta > 0$ small such that $\tilde{C}\varepsilon\delta \leq \kappa(\varepsilon)$. For $|(x_{\mathbb{Z}},\lambda) - (\bar{x}_{\mathbb{Z}}(\nu,\varepsilon,t^*),\bar{\lambda}(\nu))| < \delta$, we then find

$$\begin{aligned}
& \|\tilde{F}_{(x_{\mathbb{Z}},\lambda)}(x_{\mathbb{Z}},\lambda,\nu,\varepsilon) - \tilde{F}_{(x_{\mathbb{Z}},\lambda)}(\bar{z}_{\mathbb{Z}}(\nu,\varepsilon,t^*))\| \\
\leq \ & \tilde{C}\varepsilon\delta \leq \kappa(\varepsilon) < \sigma(\varepsilon) \leq \|\tilde{F}_{(x_{\mathbb{Z}},\lambda)}(\bar{z}_{\mathbb{Z}}(\nu,\varepsilon,t^*))^{-1}\|^{-1}.
\end{aligned}$$

A direct computation and (H6) yield

$$\begin{aligned}
\|\tilde{F}(\bar{z}_{\mathbb{Z}}(\nu,\varepsilon,t^*))\| &= \|(\bar{x}_{n+1}(\nu,\varepsilon,t^*) - \psi(\bar{x}_n(\nu,\varepsilon,t^*),\bar{\lambda}(\nu),\nu,\varepsilon))_{n\in\mathbb{Z}}\| \\
&= \|((\varphi-\psi)(\bar{x}_n(\nu,\varepsilon,t^*),\bar{\lambda}(\nu),\nu,\varepsilon))_{n\in\mathbb{Z}}\| \qquad (4.10) \\
&\leq C_4\varepsilon\tau(\varepsilon).
\end{aligned}$$

By (4.9) we can choose ε_4 sufficiently small such that for all $0 < \varepsilon < \varepsilon_4$ and uniformly in $t^* \in \mathbb{T}, |\nu - \bar{\nu}| < \nu_2$ we have $C_4\varepsilon\tau(\varepsilon) \leq (\sigma-\kappa)\delta$ and hence

$$\|\tilde{F}(\bar{z}_{\mathbb{Z}}(\nu,\varepsilon,t^*))\| \leq (\sigma-\kappa)\delta.$$

Now, we can apply Lemma 4.2 to the function $\tilde{F}$ near $(\bar{x}_{\mathbb{Z}}(\nu,\varepsilon,t^*),\bar{\lambda}(\nu))$ and obtain existence and smoothness of the discrete orbits. Moreover, the estimates (4.6),(4.7) follow from (4.3) and (4.10). ■

The next theorem analyzes the behavior of the discrete orbits $\tilde{x}_{\mathbb{Z}}(\nu,\varepsilon,t^*)$ and the parameter $\tilde{\lambda}(\nu,\varepsilon,t^*)$ under variation of the phase parameter t^*.

Theorem 4.4 *Assume* (H1)–(H6). *Then for* $0 < \varepsilon < \varepsilon_4$, $|\nu - \bar{\nu}| < \nu_2$, $t^*, t^* + \varepsilon \in \mathbb{T}$, $n \in \mathbb{Z}$ *there holds*

$$\begin{aligned} \tilde{x}_n(\nu, \varepsilon, t^*) &= \tilde{x}_{n-1}(\nu, \varepsilon, t^* + \varepsilon) && (4.11) \\ \tilde{\lambda}(\nu, \varepsilon, t^*) &= \tilde{\lambda}(\nu, \varepsilon, t^* + \varepsilon) && (4.12) \end{aligned}$$

Proof. For $t^*, t^* + \varepsilon \in \mathbb{T}$ define

$$\hat{y}_n(\varepsilon, t^*) = \tilde{x}_{n-1}(\varepsilon, t^* + \varepsilon), \quad \hat{\lambda}(\varepsilon, t^*) = \tilde{\lambda}(\varepsilon, t^* + \varepsilon),$$

and prove that$(\hat{y}_{\mathbb{Z}}(\varepsilon, t^*), \hat{\lambda}(\varepsilon, t^*))$ is also a solution of equation $\tilde{F} = 0$ near $(\bar{x}_{\mathbb{Z}}(\varepsilon, t^*), \bar{\lambda})$. Then the assertion follows from the uniqueness of the solution of equation $\tilde{F} = 0$. According to Theorem 4.3 and $t^* + \varepsilon \in \mathbb{T}$ we know $|\hat{\lambda}(\varepsilon, t^*) - \bar{\lambda}| = O(\varepsilon^d)$. Moreover, by (4.6),(4.7)

$$\|\hat{y}_n(\varepsilon, t^*) - \bar{x}_n(\varepsilon, t^*)\| = \|\tilde{x}_{n-1}(\varepsilon, t^* + \varepsilon) - \bar{x}((n-1)\varepsilon + t^* + \varepsilon)\| = O(\varepsilon^d).$$

This means that $(\hat{y}_{\mathbb{Z}}, \hat{\lambda})$ is sufficiently close to $(\bar{x}_{\mathbb{Z}}, \bar{\lambda})$ if ε is small enough.

Notice that

$$\hat{y}_{n+1} = \tilde{x}_n(\varepsilon, t^* + \varepsilon) = \psi(\tilde{x}_{n-1}(\varepsilon, t^* + \varepsilon), \tilde{\lambda}(\varepsilon, t^* + \varepsilon)) = \psi(\hat{y}_n, \hat{\lambda})$$

and

$$\begin{aligned} \Pi(\hat{y}_{\mathbb{Z}}, \varepsilon, t^*) &= \varepsilon \sum_{n \in \mathbb{Z}} (\tilde{x}_{n-1}(\varepsilon, t^* + \varepsilon) - \bar{x}(\varepsilon, t^* + n\varepsilon))^T \dot{\bar{x}}(t^* + n\varepsilon) \\ &= \varepsilon \sum_{n \in \mathbb{Z}} (\tilde{x}_n(\varepsilon, t^* + \varepsilon) - \bar{x}(t^* + \varepsilon + n\varepsilon))^T \dot{\bar{x}}(t^* + \varepsilon + n\varepsilon) \\ &= \Pi(\tilde{x}_{\mathbb{Z}}(\varepsilon, t^* + \varepsilon), \varepsilon, t^* + \varepsilon) = 0. \end{aligned}$$

Hence $(\hat{y}_{\mathbb{Z}}(\varepsilon, t^*), \hat{\lambda}(\varepsilon, t^*))$ is a solution of $\tilde{F} = 0$. This completes the proof of the theorem. ∎

Remark 4.5 *Equations* (4.6), (4.7) *state that we get the same orbit if we either start iterating with* $\psi(\cdot, \tilde{\lambda}(\nu, \varepsilon, t^*), \nu, \varepsilon)$ *at* $\tilde{x}_0(\nu, \varepsilon, t^*)$ *or with* $\psi(\cdot, \tilde{\lambda}(\nu, \varepsilon, t^* + \varepsilon), \nu, \varepsilon)$ *at* $\tilde{x}_{-1}(\nu, \varepsilon, t^* + \varepsilon)$.

If we restrict to $t^* \in \frac{1}{2}\mathbb{T}$ *and take* ε_4 *sufficiently small then we can guarantee* $t^*, t^* + \varepsilon \in \mathbb{T}$ *for all* $0 < \varepsilon < \varepsilon_4$ *and* (4.6), (4.7) *hold. Therefore, for any* $n \in \mathbb{Z}$ *we can extend the functions* $\tilde{x}_n, \tilde{\lambda}$ *to* $t^* \in \mathbb{R}$ *such that* (4.6), (4.7) *hold for all* $t^* \in \mathbb{R}$. *Therefore the manifold*

$$\gamma(\nu, \varepsilon) = \{(\tilde{x}_0(\nu, \varepsilon, t^*), \tilde{\lambda}(\nu, \varepsilon, t^*)) : t^* \in \mathbb{R}\}$$

can be equivalently written as

$$\gamma(\nu, \varepsilon) = \{(\tilde{x}_n(\nu, \varepsilon, t^*), \tilde{\lambda}(\nu, \varepsilon, t^*)) : n \in \mathbb{Z}, 0 \leq t^* < \varepsilon\},$$

We call $\gamma(\nu, \varepsilon)$ *a heteroclinic manifold; pictures of this manifold derived from numerical data appear in* [5] *and (for the saddle node case) in* [20]. *This manifold is invariant under the suspended one-step method*

$$(x_{n+1}, \lambda_{n+1}) = (\psi(x_n, \lambda_n, \nu, \varepsilon), \lambda_n)$$

and, according to Theorem 4.4, *it is* ε^d*-close to its continuous counterpart* $\{(\bar{x}(\nu, t), \bar{\lambda}) : t \in \mathbb{R}\}$.

In the remainder of this section, we always assume $(\tilde{x}_{\mathbb{Z}}(\nu, \varepsilon, t^*), \tilde{\lambda}(\nu, \varepsilon, t^*))$ to be the solutions of $\tilde{F} = 0$ as in Theorem 4.3. For simplicity, we write $(\tilde{x}_{\mathbb{Z}}(t^*), \tilde{\lambda}(t^*))$. We show that the discrete heteroclinic orbit $\tilde{x}_{\mathbb{Z}}(t^*)$ is either transversal or 1-tangential and nondegenerate with respect to λ.

Theorem 4.6 *Assume* (H1)–(H6) *and let* $0 < \varepsilon < \varepsilon_4$, $|\nu - \bar{\nu}| < \nu_2$, $t_0 \in \mathbb{T}$. *If* $\tilde{\lambda}'(t_0) \neq 0$, *then* $\tilde{x}_{\mathbb{Z}}(t_0)$ *is a transversal heteroclinic orbit for the map* $\psi(\cdot, \tilde{\lambda}(t_0), \nu, \varepsilon)$*; if* $\tilde{\lambda}'(t_0) = 0$, *then* $\tilde{x}_{\mathbb{Z}}(t_0)$ *is a nondegenerate* 1*-tangential heteroclinic orbit for the map* $\psi(\cdot, \tilde{\lambda}(t_0), \nu, \varepsilon)$ *with respect to* λ.

Proof. Since $(\tilde{x}_{\mathbb{Z}}(t_0), \tilde{\lambda}(t_0))$ is a regular solution of $\tilde{F}(x_{\mathbb{Z}}, \lambda, \varepsilon, t_0) = 0$, the linear operator $\tilde{\Gamma}_{(x_{\mathbb{Z}}, \lambda)}(\tilde{x}_{\mathbb{Z}}(t_0), \tilde{\lambda}(t_0), \varepsilon)$ must be onto, and

$$\dim \mathcal{N}(\tilde{\Gamma}_{(x_{\mathbb{Z}}, \lambda)}(\tilde{x}_{\mathbb{Z}}(t_0), \tilde{\lambda}(t_0), \varepsilon)) = 1. \tag{4.13}$$

Differentiating $\tilde{F} = 0$ leads to

$$\tilde{\Gamma}_{x_{\mathbb{Z}}}(\tilde{x}_{\mathbb{Z}}(t_0), \tilde{\lambda}(t_0), \varepsilon)\tilde{x}'_{\mathbb{Z}}(t_0) + \tilde{\Gamma}_{\lambda}(\tilde{x}_{\mathbb{Z}}(t_0), \tilde{\lambda}(t_0), \varepsilon)\tilde{\lambda}'(t_0) = 0.$$

If $\tilde{\lambda}' \neq 0$, then $\tilde{\Gamma}_{\lambda}(\tilde{z}_{\mathbb{Z}}(\varepsilon, t_0)) \in \mathcal{R}(\tilde{\Gamma}_{x_{\mathbb{Z}}}(\tilde{z}_{\mathbb{Z}}(\varepsilon, t_0)))$, Noticing that $\tilde{\Gamma}_{(x_{\mathbb{Z}}, \lambda)} = (\tilde{\Gamma}_{x_{\mathbb{Z}}}, \tilde{\Gamma}_{\lambda})$, we can conclude that $\tilde{\Gamma}_{x_{\mathbb{Z}}}(\tilde{x}_{\mathbb{Z}}(t_0), \tilde{\lambda}(t_0), \varepsilon)$ is also onto. By the bordering lemma [3, Lemma 2.3] $\tilde{\Gamma}_{x_{\mathbb{Z}}}(\tilde{x}_{\mathbb{Z}}(t_0), \tilde{\lambda}(t_0), \varepsilon)$ is Fredholm of index 0 and hence is a homeomorphism on $S_{\mathbb{Z}}$. Then $\tilde{x}_{\mathbb{Z}}(t_0)$ is a transversal heteroclinic orbit of the map $\psi(\cdot, \tilde{\lambda}(t_0), \nu, \varepsilon)$, as shown in [5, 14].

If $\tilde{\lambda}'(t_0) = 0$ then

$$\tilde{\Gamma}_{\lambda}(\tilde{x}_{\mathbb{Z}}(t_0), \tilde{\lambda}(t_0), \varepsilon) \notin \mathcal{R}(\tilde{\Gamma}_{x_{\mathbb{Z}}}(\tilde{x}_{\mathbb{Z}}(t_0), \tilde{\lambda}(t_0), \varepsilon)).$$

¿From equation (4.13), we infer that $\tilde{x}_{\mathbb{Z}}(t_0)$ is a non-degenerate 1-tangential heteroclinic orbit for the map $\psi(\cdot, \tilde{\lambda}(t_0), \nu, \varepsilon)$. ■

Corollary 4.7 *Under the assumptions of* Theorem 4.3*, for any* $0 < \varepsilon < \varepsilon_4$, $|\nu - \bar{\nu}| < \nu_2$*, among the family of heteroclinic orbits* $\tilde{x}_{\mathbb{Z}}(t^*)$*,* $t^* \in [0, \varepsilon]$ *of the map* $\psi(\cdot, \tilde{\lambda}(t^*), \nu, \varepsilon)$*, there exist at least two nondegenerate* 1*-tangential heteroclinic orbits with respect to parameter* λ*.*

Proof. Because $\tilde{\lambda}(t^*)$ is C^r smooth and periodic in t^* with period ε, there exist at least one maximal and one minimal value for the function $\tilde{\lambda}(t^*)$ at some $t^* \in [0, \varepsilon]$, and necessarily $\tilde{\lambda}'(t^*) = 0$. ■

5 Exponentially small estimates

In the previous section we already proved that if the dynamical system (2.1) possesses a nondegenerate heteroclinic orbit $\bar{x}(t)$ with respect to λ at $(\bar{\lambda}, \bar{\nu})$, then the associated one-step method (4.1) contains a family of COP's $(\tilde{x}_{\mathbb{Z}}(\bar{\nu}, \varepsilon, t^*), \tilde{\lambda}(\bar{\nu}, \varepsilon, t^*))$, $t^* \in [0, \varepsilon]$, if the step-size $\varepsilon > 0$ is small enough. In this section we will estimate the splitting distance (cf. [9])

$$D(\varepsilon) = \sup_{0 \leq t_1^*, t_2^* \leq \varepsilon} |\tilde{\lambda}(\bar{\nu}, \varepsilon, t_1^*) - \tilde{\lambda}(\bar{\nu}, \varepsilon, t_2^*)|,$$

which measures the size of the parameter region in which the map (4.1) has heteroclinic orbits. If f is smooth enough then we get estimates for $D(\varepsilon)$ of arbitrary order while in the real analytic case we obtain exponentially small estimates by using the backward error analysis of Reich [17].

Consider a C^r $(r \geq 1)$ smooth dynamical system

$$\dot{x} = f(x, \lambda), \quad x \in \mathbb{R}^k, \ \lambda \in \mathbb{R}^p, \tag{5.1}$$

with solution flow $\varphi(\cdot, \lambda, t)$. Let

$$x_{n+1} = \psi(x_n, \lambda, \varepsilon) \tag{5.2}$$

be a one-step method of smoothness C^r and of order d where $1 \leq d \leq r$. That is, for any compact subset $K \subset \mathbb{R}^k \times \mathbb{R}^p$ there exists a constant $C = C_K$ such that

$$\sup_{x \in K} \|\varphi(x, \lambda, \varepsilon) - \psi(x, \lambda, \varepsilon)\| \leq C\varepsilon^{d+1}. \tag{5.3}$$

In the following theorem we summarize basic results from the theory of backward error analysis (see [11],[12],[17]).

Theorem 5.1 *Under the above assumptions, for any index $d \leq \ell \leq r$ and any compact set $K \subset \mathbb{R}^k \times \mathbb{R}^p$ there exists a modified equation*

$$\dot{x} = \tilde{f}(x, \lambda, \varepsilon) \tag{5.4}$$

such that $\tilde{f}(x, \lambda, \varepsilon)$ is C^r smooth in some open neighborhood of $K \times \{0\}$ and satisfies

$$\sup_{(x,\lambda) \in K} \|f(x, \lambda) - \tilde{f}(x, \lambda, \varepsilon)\| = O(\varepsilon^{d+1}).$$

Moreover, the solution flow $\tilde{\varphi}(x, \lambda, \varepsilon, t)$ of the modified system (5.4) interpolates the one-step mapping $\psi(x, \lambda, \varepsilon)$ to the order ℓ, i.e.

$$\sup_{x \in K} \|\tilde{\varphi}(x, \lambda, \varepsilon, \varepsilon) - \psi(x, \lambda, \varepsilon)\| \leq C\varepsilon\tau(\varepsilon), \quad \tau(\varepsilon) = \varepsilon^{\ell}. \tag{5.5}$$

If, in addition, $f(x, \lambda)$ and $\psi(x, \lambda, \varepsilon)$ are real analytic functions in $\mathbb{R}^{k+p}$ and $\mathbb{R}^{k+p} \times (-\varepsilon_0, \varepsilon_0)$ respectively then the estimate in (5.5) holds with $\tau(\varepsilon) = e^{-c/\varepsilon}$ for a suitable constant $c = c(K)$.

Remark 5.2 *The statements of* Theorem 5.1 *are given in* [11], [12], [17] *for the parameter independent case. Working through the proofs it can, however, be seen that their results extend to parameterized systems with the estimates being uniform in a compact parameter set.*

Now we are going to apply Theorems 4.3 and 4.4 to the modified equation (5.4) with ε taking the role of the parameter ν in (2.1). Given a nondegenerate COP $(\bar{x}(t), \bar{\lambda})$ of system (5.1) we first select a compact set $K \subset \mathbb{R}^{k+p}$ which contains the COP in its interior. Then by Corollary 2.1 we obtain a smooth branch of COP's $(\bar{x}(t,\varepsilon), \bar{\lambda}(\varepsilon))$ for the modified equation (5.4) which still lies in the interior of K and which are nondegenerate with respect to λ. Since $\psi(x, \lambda, \varepsilon)$ is a method of order $\tau(\varepsilon)$ for the modified equation by (5.2), Theorem 4.3 yields the estimate

$$\|\tilde{\lambda}(\varepsilon, t^*) - \bar{\lambda}(\varepsilon)\| \leq C\tau(\varepsilon). \tag{5.6}$$

Here we use the fact that the estimates (4.6) and (4.7) improve to $O(\tau(\varepsilon))$ if (H6) holds with a general $\tau(\varepsilon)$ (compare (4.10) and the last line in the proof of theorem 4.3). By the triangle inequality we obtain

$$D(\varepsilon) = O(\tau(\varepsilon)).$$

Theorem 5.3 *Let the system* (5.1) *satisfy the assumptions* (H1)-(H4) *and let the one-step method* (5.2) *satisfy* (H5), (H6) *(without parameter ν). Then the connecting orbits $(\tilde{x}_{\mathbb{Z}}(\varepsilon, t^*), \tilde{\lambda}(\varepsilon, t^*))$ of the one-step map as given by* Theorem 4.3 *satisfy*

$$\sup_{0 \leq t_1^*, t_2^* \leq \varepsilon} \|\tilde{\lambda}(\varepsilon, t_1^*) - \tilde{\lambda}(\varepsilon, t_2^*)\| = O(\varepsilon^r). \tag{5.7}$$

If f and ψ are real analytic in all their variables then the right hand side in (5.7) *can be replaced by $O(e^{-c/\varepsilon})$ with a suitable constant c.*

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