A note on symbolic dynamics near connecting orbits of maps

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Abstract

In this note we consider the dynamics of a diffeomorphism on a maximal, invariant set in the neighborhood of a transversal homoclinic orbit. It is shown that the map conjugates to a topological Markov chain on a finite number of symbols defined by the transition matrix

$$\begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 \end{bmatrix}$$

The theorem collects in a unified way various well–known results of Smale, Shilnikov, Palmer and others on symbolic dynamics.

Generalizations of the result are indicated for finite sets of hyperbolic fixed points with a given finite collection of transversal connecting orbits between them.

Keywords: Dynamical systems, symbolic dynamics, homoclinic points for maps.

AMS subject classification: 58 F03, 58 F13, 58 F15, 58 F08

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1 Introduction and Main Result

The main purpose of this paper is to reconsider (and correct some details of) Palmer's symbolic version [Pa95] of Smale's Homoclinic Theorem.

We formulate the result in terms of topological Markov chains, a special class of subshifts of finite type where the transitions in the symbol sequence are specified by a $0-1$ matrix (see [KH95] for an introduction).

To be specific, consider the space of biinfinite sequences on N symbols

$$S_N = \{0, \dots, N-1\}^{\mathbb{Z}}, \tag{1.1}$$

with elements $s = (s_i)_{i \in \mathbb{Z}}$ and equipped with the metric

$$d(s,t) = \sum_{i \in \mathbb{Z}} 2^{-|i|} \, |s_i - t_i|, \quad s, t \in S_N. \tag{1.2}$$

This generates the product topology on S_N and turns it into a compact metric space.

Any closed subset $\Sigma \subset S_N$ which is invariant under the Bernoulli shift

$$\sigma(s)_i = s_{i+1}, \ i \in \mathbb{Z}, \ s \in S_N \tag{1.3}$$

is called a **subshift.**

Any $N \times N$ binary matrix $A = (A_{ij})_{i,j=0,\dots,N-1}$, $A_{ij} \in \{0,1\}$ generates a special subshift

$$\Sigma_A = \{s \in S_N : A_{s_i, s_{i+1}} = 1 \quad \forall i \in \mathbb{Z}\} \tag{1.4}$$

which is called the **topological Markov chain associated with** A. Σ_A is obviously closed and shift invariant.

For the special case

$$A = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 \end{bmatrix} \in \{0,1\}^{N \times N} \tag{1.5}$$

we obtain

$$\Sigma_A = \{s \in S_N : s_i = 0 \Rightarrow s_{i+1} \in \{0,1\}, \ s_i \geq 1 \Rightarrow s_{i+1} = s_i + 1 \pmod{N}\}. \tag{1.6}$$

Now we consider a C^1–diffeomorphism

$$f \in C^1(U, \mathbb{R}^m), \ U \subset \mathbb{R}^m \ \text{ open}$$

such that f has a hyperbolic fixed point $\xi \in U$ and a transversal homoclinic orbit $\{x_n\}_{n\in\mathbb{N}} \subset U$

$$x_{n+1} = f(x_n) \quad \text{for} \quad n \in \mathbb{Z}, \quad \lim_{n\to\pm\infty} x_n = \xi. \tag{1.7}$$

Transversality can be characterized by either transversal intersection of the stable and unstable manifold of ξ at x_0 (and hence at any $x_n,\ n \in \mathbb{Z}$) or by exponential dichotomy of the associated variational equation (cf. [Pa88], [BK97b]).

Assuming transversality it is well–known that

$$H = \{\xi\} \cup \{x_n\}_{n\in\mathbb{Z}} \tag{1.8}$$

is a compact hyperbolic set (see [Pa88]) to which the shadowing lemma can be applied (see [Pa88], [Pa95] for the following version).

Proposition 1.1 *(**Shadowing Lemma**) Let $H \subset U$ be a compact, invariant hyperbolic set of a diffeomorphism $f \in C^1(U,\mathbb{R}^m)$. Then there exists $\epsilon_0 > 0$ and a function $\delta : (0,\epsilon_0] \to (0,\infty)$ with $\delta(\epsilon) \le \epsilon$ and the following property. For any $0 < \epsilon \le \epsilon_0$ and any $\delta(\epsilon)$ –pseudo orbit $\{y_n\}_{n\in\mathbb{Z}} \subset H$ (i.e. $|f(y_n) - y_{n+1}| \le \delta(\epsilon) \quad \forall n \in \mathbb{Z}$) there exists a unique ϵ –shadowing orbit $\{z_n\}_{n\in\mathbb{Z}}$ (i.e. $z_{n+1} = f(z_n),\ |z_n - y_n| \le \epsilon$ for all $n \in \mathbb{Z}$). The true orbit depends continuously on the pseudo orbit in the sense, that to any $0 < \epsilon_1 \le \epsilon$ there exists $N = N(\epsilon,\epsilon_1) \in \mathbb{N}$ such that the ϵ –shadowing orbit $\{\overline{z}_n\}_{n\in\mathbb{Z}}$ of some $\delta(\epsilon)$ –pseudo orbit $\{\overline{y}_n\}_{n\in\mathbb{Z}}$ with $y_n = \overline{y}_n$ for $|n| \le N$ satisfies $|z_0 - \overline{z}_0| \le \epsilon_1$.*

In the following $\delta(\epsilon)$ will always denote the shadowing function of Proposition 1.1 when applied to H in (1.8). We now formulate the main result.

Theorem 1.2 *Let $\{x_n\}_{n\in\mathbb{Z}} \subset U \subset \mathbb{R}^m$ be a transversal homoclinic orbit of some diffeomorphism $f \in C^1(U,\mathbb{R}^m)$. Then there exists a neighborhood $\mathcal{O}$ of $H = \{\xi\} \cup \{x_n\}_{n\in\mathbb{Z}}$ and an integer N such that the maximal invariant set*

$$\Lambda = \{x \in \mathcal{O} : f^n(x) \in \mathcal{O} \quad \forall n \in \mathbb{Z}\} \tag{1.9}$$

is compact, invariant and hyperbolic and there exists a homeomorphism

$$h : \Lambda \to \Sigma_A \tag{1.10}$$

with the topological Markov chain Σ_A defined by (1.5) such that

$$h \circ f(x) = \sigma \circ h(x) \quad \forall x \in \Lambda. \tag{1.11}$$

Relation (1.11) shows that f conjugates on Λ to a subshift Σ_A given by (1.6). The proof will show that the assertion of the Theorem holds for N sufficiently large with the neighborhood $\mathcal{O}$ as well as Λ and h depending on N .

Generalizations of the Theorem to non–diffeomorphisms appear in cf. [HL86], [SW90], [Pa95]. Rather than mapping a subset Λ in phase space one then has to consider subsets in the space of f –orbits.

In [BK97a], [BK97b] a numerical procedure is proposed to evaluate the symbolic coding in the case of a conjugacy of f^N to a full shift on two symbols. It is easy to adapt this recipe to the current setting by using the pseudo orbits constructed in the proof below as starting vectors for a suitable Newton method (cf. [BK97b]).

2 Proof of the Main Theorem

We proceed along the lines of [Pa95], but it will be necessary to modify the construction of the neighborhood.

Step 1: Construction of neighborhood $\mathcal{O}$ and integer N.

We set

$$\delta_1 = \text{Min}\left(\frac{\delta(\epsilon_0)}{2},\ \frac{\epsilon_0}{4}\right). \tag{2.1}$$

Then choose $n_+ \geq 0 \geq n_-$ such that

$$|x_n - \xi| \leq \delta_1 \quad \text{for} \quad n \geq n_+ \quad \text{and} \quad n \leq n_-. \tag{2.2}$$

and let $J = n_+ - n_- - 1$.

By $B(x,\epsilon) = \{y \in \mathbb{R}^m : |y - x| < \epsilon\}$ we denote open balls of radius ϵ. Because of the continuity of f and $\lim\limits_{n\to\pm\infty} x_n = \xi$ we can choose $0 < \epsilon \leq \frac{\epsilon_0}{2}$ such that the open sets

$$V_0 = B(\xi,\epsilon)\ \cup \bigcup_{n\geq n_+} B(x_n,\epsilon)\ \cup \bigcup_{n\leq n_-} B(x_n,\epsilon) \tag{2.3}$$

$$B_j := B(x_{n_-+j},\epsilon), \quad j = 1,\dots,J \tag{2.4}$$

are mutually disjoint and in addition

$$f(V_0) \cap B_j = \emptyset \quad \text{for} \quad j = 2,\dots,J. \tag{2.5}$$

Notice that (2.1), (2.2) imply

$$|x - \xi| \leq \epsilon + \delta_1 \leq \frac{3}{4}\,\epsilon_0 \quad \forall x \in V_0. \tag{2.6}$$

Using Proposition 1.1 we define

$$\delta_2 = \delta(\frac{\epsilon}{2}) \tag{2.7}$$

and choose $n_* \in \mathbb{N}$ such that

$$|x_{n_- - n} - \xi|,\ |x_{n_+ + n} - \xi| \leq \frac{1}{2}\,\delta_2 \quad \forall n \geq n_*. \tag{2.8}$$

With this choice the number N in (1.5) is finally defined by

$$N = J + M, \quad \text{where} \quad M = 2n_* + 2. \tag{2.9}$$

Next we define a neighborhood of H by

$$\mathcal{O} = \bigcup_{j=0}^{J} V_j \tag{2.10}$$

where the $V_j,\ j=1,\ldots,J$ are mutually disjoint nonempty open neighborhoods of x_{n_-+j} given recursively by

$$V_J = B_J \cap \bigcup_{n=1}^{M} f^{-n}(V_0) \tag{2.11}$$

$$V_j = B_j \cap f^{-1}(V_{j+1}),\ j = J-1,\ldots,1. \tag{2.12}$$

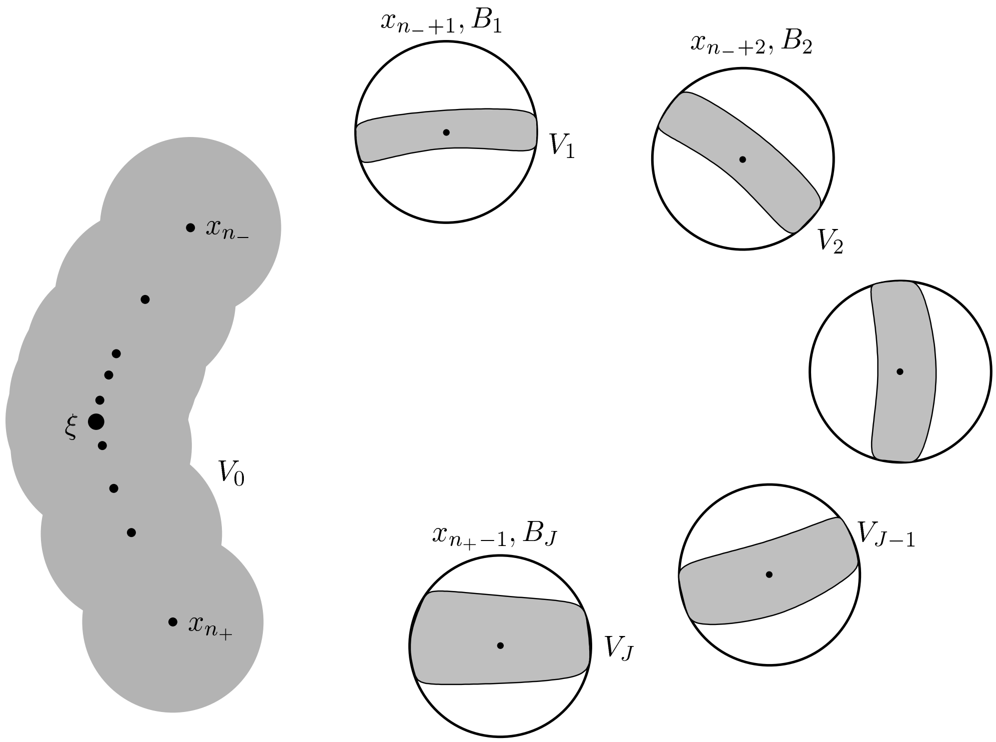

Figure 1: Construction of the neighborhood $\mathcal{O} = \bigcup_{j=0}^{J} V_j$ of the homoclinic orbit.

This construction implies that for any f –orbit $\{y_n\}_{n\in\mathbb{Z}} \subset \mathcal{O}$ we have the following properties

$$y_n \in V_0,\ y_{n+1} \notin V_0 \Rightarrow y_{n+1} \in V_1, \tag{2.13}$$

$$y_n \in V_j \ \text{ for some } \ 1 \le j \le J-1 \Rightarrow y_{n+1} \in V_{j+1}, \tag{2.14}$$

$$y_n \in V_J \Rightarrow y_{n+\ell} \in V_0 \ \text{ for } \ \ell = 1,\ldots,M. \tag{2.15}$$

Assertion (2.13) follows from (2.5) and (2.10), (2.14) from (2.12) and (2.15) from (2.11).

Step 2: Construction of the map h

Consider an orbit $y_n = f^n(y)$ with $y \in \Lambda$ (see (1.9)) and the set of indices

$$I = I(y) = \{n \in \mathbb{Z} : y_n \in V_1\}. \tag{2.16}$$

This is a finite (or even void) or infinite subset of $\mathbb{Z}$ which we order as

$$\ldots < n_{-1} < n_0 < n_1 < \ldots \ .$$

To be precise we write

$$I(y) = \{n_k : k \in [k_-, k_+]\} \tag{2.17}$$

where n_k is strictly monotone increasing and the interval $[k_-, k_+] \subset \mathbb{Z}$ is defined for $k_\pm \in \mathbb{Z} \cup \{\pm\infty\}$ as follows

$$[k_-, k_+] = \begin{cases} \{k \in \mathbb{Z} : k_- \le k \le k_+\} & \text{if } -\infty < k_- \le k_+ < \infty \\ \{k \in \mathbb{Z} : k \le k_+\} & \text{if } -\infty = k_- < k_+ < \infty \\ \{k \in \mathbb{Z} : k_- \le k\} & \text{if } -\infty < k_- < k_+ = \infty \\ \mathbb{Z} & \text{if } -\infty = k_-,\ k_+ = \infty \\ \emptyset & \text{if } k_+ < k_- \end{cases} .$$

Note that $I(y)$ determines the interval $[k_-, k_+]$ up to a shift.

It will be convenient to consider also

$$[k_-, k_+) = \begin{cases} [k_-,\ k_+ - 1] & \text{if } k_+ < \infty \\ [k_-,\ \infty] & \text{if } k_+ = \infty \end{cases} .$$

We notice that (2.13) and (2.16) imply

$$y_{n_k + j} \in V_{j+1} \quad \text{for} \quad j = 0, \ldots, J-1,\ k \in [k_-, k_+]. \tag{2.18}$$

Then (2.15) applies to $y_{n_k + J - 1}$ and yields

$$y_{n_k + J - 1 + \ell} \in V_0 \quad \text{for} \quad \ell = 1, \ldots, M, k \in [k_-, k_+], \tag{2.19}$$

hence we obtain

$$n_k + N = n_k + J + M \le n_{k+1} \quad \text{for} \quad k \in [k_-, k_+) \tag{2.20}$$

and with the help of (2.19), (2.13)

$$y_{n_k + j} \in V_0 \quad \text{for} \quad j = J, \ldots, n_{k+1} - n_k - 1,\ k \in [k_-, k_+). \tag{2.21}$$

Next we notice that by (1.6) we can associate to any $s \in \Sigma_A$ a subset of $\mathbb{Z}$ by

$$\widetilde{I} = \widetilde{I}(s) = \{n \in \mathbb{Z} : s_n = 1\}.$$

As in (2.17) the set $\widetilde{I}(s)$ may be written as

$$\widetilde{I}(s) = \{n_k : k \in [k_-, k_+]\} \tag{2.22}$$

where n_k is strictly monotone increasing and satisfies

$$n_k + N \le n_{k+1}, \quad \text{for} \quad k \in [k_-, k_+). \tag{2.23}$$

Moreover, (1.6) implies

$$s_{n_k+j} = j+1 \quad \text{for} \quad j = 0, \dots, N-2, \; k \in [k_-, k_+] \quad \text{and} \quad s_i = 0 \quad \text{otherwise} . \tag{2.24}$$

Conversely, for any strictly increasing sequence of type n_k, $k \in [k_-, k_+]$ with (2.23) we can define $s \in \Sigma_A$ by (2.24) and obtain the relation (2.22). If the sequence is empty then we put $s = 0$.

Therefore, we can define h formally by

$$h(y) = \widetilde{I}^{-1}(I(y)), \; y \in \Lambda. \tag{2.25}$$

From the above construction it follows that $s = h(y)$ may be written as

$$s_n = \begin{cases} j, & \text{if} \;\; f^n(y) \in V_j \;\; \text{and} \;\; j \in \{1, \dots, J\} \\ J+j, & \text{if} \;\; f^n(y) \in V_0 \;\; \text{and} \;\; f^{n-j}(y) \in V_J \;\; \text{for} \;\; j \in \{1, \dots, M-1\} \\ 0 & \text{otherwise} . \end{cases} \tag{2.26}$$

Step 3: h is one to one

Suppose that $s = h(y)$, $y \in \Lambda$ is given and let $\{n_k : k \in [k_-, k_+]\} = I(y) = \widetilde{I}(s)$.

Consider the pseudo orbit $\{z_n\}_{n \in \mathbb{Z}} \subset \Lambda$ defined by

$$z_{n_k+j} = \begin{cases} x_{n_-+j+1}, & 0 \le j \le J-1 \\ \xi, & J \le j \le n_{k+1} - n_k - 2 \\ x_{n_-}, & j = n_{k+1} - n_k - 1 \end{cases} \tag{2.27}$$

for $k \in [k_-, k_+)$ and

$$z_{n_{k_-}-1} = x_{n_-} \quad \text{if} \quad -\infty < k_- \quad \text{and} \quad z_n = \xi \quad \text{otherwise} . \tag{2.28}$$

This is a $\delta(\epsilon_0)$ –pseudo orbit since by (2.2)

$$|f(x_{n_-+J}) - \xi| \;=\; |x_{n_+} - \xi| \le \delta_1$$

and

$$|f(\xi) - x_{n_-}| = |\xi - x_{n_-}| \le \delta_1.$$

Moreover, $y_n = f^n(y)$ is an ϵ_0 –shadowing orbit since

$$y_{n_k+j} \in V_{j+1} \subset B(x_{n_-+j+1}\epsilon) \quad \text{for} \quad 0 \le j \le J-1 \quad \text{by (2.12), (2.18),}$$

$|y_{n_k+j} - \xi| \;\le\; \frac{3}{4}\,\epsilon_0$ for $J \le j \le n_{k+1} - n_k - 2$ by (2.6), (2.19), (2.21).

$|y_{n_{k+1}-1} - x_{n_-}| \;\le\; |y_{n_{k+1}-1} - \xi| \;+\; |\xi - x_{n_-}| \;\le\; \frac{3}{4}\,\epsilon_0 + \frac{1}{4}\,\epsilon_0$ by (2.2), (2.6) and (2.21),

$|y_{n_{k_-}-1}| \;\le\; |y_{n_{k_-}-1} - \xi| \;+\; |\xi - x_{n_-}| \;\le\; \epsilon$ by (2.2), (2.6) and (2.21) if $-\infty < k_-$,

$$|y_n - \xi| \ \le \ \epsilon \ \text{ for } \ n \le n_{k_-} - 1 \ \text{ and } \ n \ge n_{k_+} \ \text{ by (2.6).}$$

The uniqueness of the orbit $\{y_n\}_{n\in\mathbb{Z}} \subset \mathcal{O}$ and hence the uniqueness of $y = y_0$ now follows from Proposition 1.1. This proof also applies to $s = 0$ in which case we take $z_n = 0 \quad \forall n \in \mathbb{Z}$ and obtain $y_n = \xi \quad \forall n \in \mathbb{Z}$ from Proposition 1.1.

Step 4: h is onto

Given $s \in \Sigma_A$ we define $\widetilde{I}(s) = \{n_k, k \in [k_-, k_+]\}$ as in (2.22) and consider the pseudo orbit $\{z_n\}_{n\in\mathbb{Z}} \subset \Lambda$ defined by

$$z_{n_k+j} = \begin{cases} x_{n_-+j+1}, & 0 \le j \le J - 1 + n_* \\ \xi, & J + n_* \le j \le n_{k+1} - n_k - 2 - n_* \\ x_{n_-+n_{k+1}-n_k-1-j-n_*}, & n_{k+1} - n_k - 1 - n_* \le j \le n_{k+1} - n_k - 1 \end{cases} \tag{2.29}$$

for $k \in [k_-, k_+)$. The definition is completed by setting

$$\begin{aligned} z_{n_{k_-}+j} &= x_{n_-+j+1} \ \text{ for } \ j \le -1 \ \text{ if } \ -\infty < k_- \\ z_{n_{k_+}+j} &= x_{n_-+j+1} \ \text{ for } j \ge 1 \text{ if } \ k_+ < \infty. \end{aligned} \tag{2.30}$$

Notice that at least one ξ appears in (2.29) due to (2.20). We claim that $\{z_n\}_{n\in\mathbb{Z}}$ is a δ_2 –pseudo orbit. This follows from (2.8) since

$$\begin{aligned} |f(x_{n_-+J+n_*}) - \xi| \ &= \ |x_{n_++n_*} - \xi| \ \le \delta_2 \\ |f(\xi) - x_{n_--M+2+n_*}| \ &= \ |\xi - x_{n_--n_*}| \ \le \delta_2. \end{aligned}$$

Therefore, there is a unique $\frac{\epsilon}{2}$ –shadowing orbit $\{y_n\}_{n\in\mathbb{Z}}$ for $\{z_n\}_{n\in\mathbb{Z}}$ and $s = h(y_0)$ follows if we show

$$\{y_n\}_{n\in\mathbb{Z}} \subset \mathcal{O} \ \text{ and } \ I(y_0) = \widetilde{I}(s). \tag{2.31}$$

From the definition (2.29) we obtain

$$|y_{n_k+j} - x_{n_-+j+1}| \ \le \ \frac{\epsilon}{2}, \quad 0 \le j \le J - 1 + n_*. \tag{2.32}$$

In particular, for $J \le j \le J - 1 + n_*$ we have $n_- + j + 1 \ge n_- + J + 1 = n_+$ and hence by (2.3)

$$y_{n_k+j} \in V_0 \ \text{ for } \ J \le j \le J - 1 + n_*. \tag{2.33}$$

Similarly, from the $\frac{\epsilon}{2}$ –shadowing property and (2.29) we find

$$y_{n_k+j} \in V_0 \ \text{ for } \ J + n_* \le j \le n_{k+1} - n_k - 1. \tag{2.34}$$

Since (2.32) implies $y_{n_k+j} \in B_{j+1}$ for $0 \le j \le J-1$ we obtain from (2.11), (2.33) and (2.34) that $y_{n_k+J-1} \in V_J$ and then by induction from (2.12) that

$$y_{n_k+j} \in V_{j+1}, \ j = J - 1, \dots, 0. \tag{2.35}$$

First these conclusions hold for $k \in [k_-, k_+)$ but in case $k_+ < \infty$ we can use the same argument together with (2.30) to find (2.35) for $k = k_+$ as well as $y_{n_{k_+}+j} \in V_0$ for all $j \geq J$.

Finally, $y_{n_{k_-}+j} \in V_0$ for $j \leq -1$ in case $-\infty < k_-$ follows directly from (2.30), the shadowing property and the definition of V_0 in (2.3).

Thus we have shown $\{y_n\}_{n\in\mathbb{Z}} \subset \mathcal{O}$ and simultaneously that $I(y_0)$ consists of the subsequence n_k as can be read off from (2.33)–(2.35).

Step 5: Continuity of h and $\mathbf{h}^{-1}$

Let $y \in \Lambda$ and and $s = h(y)$ be the corresponding symbolic sequence. For any $\eta > 0$ we find $i_0 = i_0(\eta) \in \mathbb{N}$ such that

$$d(s,t) \leq \eta \ \text{ if } \ s_i = t_i \ \text{ for } \ |i| \leq i_0.$$

Since f is continuous and the V_j are disjoint there exists a $\rho > 0$ such that $f^n(z)$, $|z-y| \leq \rho$ lies in the same V_j as $f^n(y)$ for all $|n| \leq i_0 + M$. Then with $t = h(z)$ we obtain $s_i = t_i$ for $|i| \leq i_0$ from (2.26) and hence $d(h(y), h(z)) \leq \eta$.

Given $s \in \Sigma_A$ the inverse $y = h^{-1}(s)$ was determined in Step 4. For a given $0 < \epsilon_1 \leq \epsilon$ choose $\nu = \nu(\epsilon, \epsilon_1)$ as in the shadowing lemma. For $t \in \Sigma_A$ with $t_i = s_i$, $|i| \leq \nu$ we find that the pseudo orbits $\{z_n\}_{n\in\mathbb{Z}}$ and $\{w_n\}_{n\in\mathbb{N}}$ associated with s and t via (2.29), (2.30) agree for $|n| \leq \nu$. Let $\{y_n\}_{n\in\mathbb{Z}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be the corresponding $\frac{\epsilon}{2}$ –shadowing orbits. Then Proposition 1.1 shows

$$|h^{-1}(s) - h^{-1}(t)| = |y_0 - v_0| \leq \epsilon_1.$$

Step 6: The conjugacy

For $y \in \Lambda$ let $s = h(y)$ and $t = h(f(y))$. From (2.26) we immediately see $t_{n-1} = s_n$ for all $n \in \mathbb{Z}$ hence $t = \sigma(s) = \sigma(h(y)) = h(f(y))$. Λ is invariant by definition and compact since $\Lambda = h^{-1}(\Sigma_A)$ and Σ_A is compact.

Hyperbolicity follows from the last part of Proposition 1.1 because Step 4 shows that all points in Λ lie on appropriate shadowing orbits and because exponential dichotomy of variational equations with uniform data and with projectors of the constant rank is sufficient for hyperbolicity (see cf. [Pa88], § 3).

3 Extensions to several connecting orbits

It is quite straightforward to generalize Theorem 1.2 to a finite set of mutually disjoint transversal homoclinic orbits

$$\{x_n^\ell\}_{n\in\mathbb{Z}}, \quad \lim_{n\to\pm\infty} x_n^\ell = \xi \quad \text{for} \quad \ell = 1,\dots,L. \tag{3.1}$$

For example, this situation typically occurs with $L = 4$ in certain parameter regions for the Henon example, see cf. [BK97b].

In this situation it follows from [Pa88] that

$$H = \{\xi\} \cup \{x_n^\ell : n \in \mathbb{Z},\ \ell = 1,\dots,L\} \tag{3.2}$$

is a hyperbolic set. Hence the shadowing lemma applies and we can mimic the proof of Theorem 1.2.

For some large N the defining matrix of the topological Markov chain is now of dimension $1 + L(N-1)$ and has the form

$$A_0 = \begin{bmatrix} 1 & e_1^T & e_1^T & \dots & e_1^T \\ e_{N-1} & D & 0 & \dots & 0 \\ e_{N-1} & 0 & D & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ e_{N-1} & 0 & \dots & 0 & D \end{bmatrix} \tag{3.3}$$

where $D = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \in \{0,1\}^{(N-1)\times(N-1)}$.

and e_j is the j–th Cartesian basis vector in $\mathbb{R}^{N-1}$. We indicate for this case the construction of the neighborhoods V_j^ℓ in

$$\mathcal{O} = V_0 \cup \bigcup_{\substack{1\le j\le J \\ 1\le \ell\le L}} V_j^\ell .$$

First choose n_-, n_+ such that

$$|x_n^\ell - \xi| \ \le\ \delta_1 \quad \text{for} \quad n \ge n_+, n \le n_-,\ \ell = 1,\dots,L$$

and let $J = n_+ - n_- - 1$. Then take $\epsilon_1 \le \frac{1}{2}\,\epsilon_0$ such that

$$f(B(\xi,\epsilon_1)) \ \cap\ B(x_n^\ell,\epsilon_1) = \emptyset \quad \text{for} \quad \ell = 1,\dots,L, n = n_- + 1,\dots,n_+ - 1$$

and $m_- \le n_-,\ m_+ \ge n_+$ such that

$$x_n^\ell \in B(\xi,\ \frac{1}{2}\,\epsilon_1) \quad \text{for} \quad n \le m_-, n \ge m_+, \ell = 1,\dots,L.$$

Finally choose $\epsilon \leq \frac{\epsilon_1}{2}$ with the following two properties

$f(B(x_n^\ell,\epsilon)) \cap B(x_\nu^\ell,\epsilon) = \emptyset$ for $n = m_-{+}1,\dots,n_-{-}1,\ \nu = n_-{+}1,\dots,n_+{-}1$ and $\ell = 1,\dots,L.$

$B(x_n^\ell,\epsilon)$ are mutually disjoint for $\ell = 1,\dots,L,\ n = n_-,\dots,n_+$.

With this choice define

$$V_0 = B(\xi,\epsilon_1) \ \cup \bigcup_{\substack{n\leq n_-,n\geq n_+\\ \ell=1,\dots,L}} B(x_n^\ell,\epsilon)$$

and obtain (using $B(x_n^\ell,\epsilon) \subset B(\xi,\epsilon_1)$ for $n \leq m_-, n \geq m_+$)

$$f(V_0) \ \cap \ B(x_n^\ell,\epsilon) = \emptyset \ \text{ for } \ n = n_- + 2,\dots,n_- + J,\ \ell = 1,\dots,L.$$

With this property we can proceed as in Step 1 above and define for $\ell = 1,\dots,L$

$$\begin{aligned} V_J^\ell &= B(x_{n_+-1}^\ell,\epsilon) \ \cap \ \bigcup_{n=1}^{M} f^{-1}(V_0), \\ V_j^\ell &= B(x_{n_-+j},\epsilon) \ \cap \ f^{-1}(V_{j+1}^\ell) \ \text{ for } \ j = J-1,\dots,1. \end{aligned}$$

If a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{O}$ leaves V_0 it does so through one of the sets $V_1^\ell,\ \ell \in \{1,\dots,L\}$.

For the next J steps it stays in V_j^ℓ with the same ℓ and then enters V_0 again where it stays for at least M steps. This pattern is captured by a topological Markov chain which belongs to the martix A_0 from (3.3).

Of course, by changing the neighborhood $\mathcal{O}$ we could also work with matrices D of varying dimension $N_\ell - 1$ for $\ell = 1,\dots,L$. Then the topological Markov chain has

$$1 + \sum_{\ell=1}^{L} (N_\ell - 1)$$

symbols.

If $\{x_n\}_{n\in\mathbb{Z}}$ is a transversal heteroclinic orbit connecting two different hyperbolic fixed points

$$\lim_{n\to\pm\infty} x_n = \xi_\pm,\ \xi_- \neq \xi_+$$

then we obtain a Markov chain on $N+1$ symbols $0,\dots,N$ which belongs to the matrix

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}. \tag{3.4}$$

Notice that now $H = \{\xi_-\} \ \cup \ \{x_n\}_{n\in\mathbb{Z}} \ \cup \ \{\xi_+\}$ and the maximal invariant set

$$\Lambda = \{y \in \mathcal{O} : f^n(y) \in \mathcal{O}\ \forall n \in \mathbb{Z}\}$$

coincides with H . Up to a shift there are only three different sequences of symbols

$$(\dots 0,0,0,\dots),\ (\dots,0,1,\dots,N-1,N,N,\dots),\ (\dots N,N,N,\dots)$$

which belong to the two fixed points and the connecting orbit.

Finally, it becomes clear how to find the appropriate Markov chain for a finite set of transversal connecting orbits between a finite number of hyperbolic fixed points.

We view this as a graph (with the fixed points taken as vertices and the connecting orbits taken as edges) and assume that the graph is connected. Then all fixed points have the same number of stable dimensions and hence the union of the fixed points and the connecting orbits forms a hyperbolic set.

The matrix of the emerging Markov chain is then obtained by piecing together matrices of type A_0 in (3.3) and A_1 in (3.4) in an appropriate way.

For example, if we have the heteroclinic orbit from ξ_- to ξ_+ as above and for each fixed point an additional transversal homoclinic orbit then the appropriate matrix reads (using the Jordan block D form (3.3))

$$A = \begin{bmatrix} 1 & e_1^T & 0 & 0 & e_1^T \\ e_{N-1} & D & 0 & 0 & 0 \\ 0 & 0 & 1 & e_1^T & 0 \\ 0 & 0 & e_{N-1} & D & 0 \\ 0 & 0 & e_{N-1} & 0 & D \end{bmatrix}.$$

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