Connecting paracontractivity and convergence of products

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Abstract

In [2] the LCP-property of a finite set Σ of square complex matrices was introduced and studied. Σ is an LCP-set if all left infinite products formed from matrices in Σ are convergent. It had been shown earlier in [3] that a set Σ paracontracting with respect to a fixed norm is an LCP-set. Here we prove a converse statement: If Σ is an LCP-set with a continuous limit function then there exists a norm such that all matrices in Σ are paracontracting with respect to this norm.

In addition we introduce the stronger property of l-paracontractivity. It is shown that common l-paracontractivity of a set of matrices has a simple characterization. It turns out that in the above mentioned converse statement the norm can be chosen such that all matrices are l-paracontracting.

It is shown that for Σ consisting of two projectors the LCP-property is equivalent to l-paracontractivity, even without requiring continuity.

1 Introduction

In the investigation of chaotic iteration procedures for linear consistent systems matrices which are paracontracting with respect to some vector norm play an important role. It was shown in [3], that if $A_1, \ldots, A_m$ are finitely many $k \times k$ complex matrices which are paracontracting with respect to the same norm, then for any sequence $d_i,\ 1 \le d_i \le m,\ i = 1, 2, \dots$ and any x_0 the sequence

$$x_i = A_{d_i} x_{i-1} \quad i = 1, 2, \dots \tag{1}$$

is convergent. In particular $A^{(d)} = \lim_{i\to\infty} A_{d_i} \dots A_{d_1}$ exists for all sequences $\{d_i\}_{i=1}^{\infty} = d$. Hence those sets are examples of sets of matrices all infinite products of which converge. Such sets have been studied in [2]. Following [2], we call them LCP–sets.

In this note we investigate the question of necessity. As our main result we show that under the additional assumption that the mapping

$$d = \{d_i\}_{i=1}^{\infty} \to A^{(d)} = \lim_{i\to\infty} A_{d_i} A_{d_{i-1}} \dots A_{d_1} \tag{2}$$

is continuous (which is equivalent to the set of fixed points of A_i being the same for all $1 \le i \le m$), an LCP–set is necessarily paracontracting with respect to some norm. In this sense paracontractivity is equivalent to the LCP–property. We will show in addition that continuity implies even the stronger property of l-paracontractiveness.

In the last section we consider the case $m = 2$. It is shown that for two projectors the equivalence of LCP-property and l-paracontractivity holds even without continuity.

Some parts of this paper are contained in [7].

2 Notations and known results

Let $||\ ||$ denote a vector norm in C^k. A $k \times k$ matrix P is **paracontracting** with respect to $||\ ||$, if for all x

$$Px \ne x \Leftrightarrow ||Px|| < ||x||. \tag{3}$$

We denote by $\mathcal{N}(||\ ||)$ the set of all $k \times k$ matrices paracontracting w.r.t. $||\ ||$. We call P **l-paracontracting** w.r.t. $||\ ||$, if there exists $\gamma > 0$ such that

$$||Px|| \le ||x|| - \gamma ||Px - x|| \tag{4}$$

holds for all $x \in C^k$ and denote this set of matrices by $\mathcal{N}_\gamma(||\ ||)$. Obviously

$$\mathcal{N}_\gamma(||\ ||) \subset \mathcal{N}(||\ ||). \tag{5}$$

The example of an orthogonal projection $P, P \neq I, P \neq 0$ which is paracontracting w.r.t. the Euclidean vector norm but never l-paracontracting shows that in (5) equality does not hold in general.

For a bounded set $\Sigma = \Sigma_1$ of complex $k \times k$ - matrices define $\Sigma_0 = \{I\}$ and for $n \geq 1$

$$\Sigma_n = \{M_1\ M_2 \ldots M_n : M_i \in \Sigma\},$$

the set of all products of matrices in Σ of length n. Let $\Sigma = \{A_1, \ldots, A_m\}$ be finite. For $d = (d_1, d_2, \ldots) \in \{1, \ldots, m\}^N$, i.e. $1 \leq d_i \leq m$ for $i \in N$ define

$$A^{(d)} = \lim_{n \to \infty} A_{d_n} A_{d_{n-1}} \ldots A_{d_1}, \tag{6}$$

if the limit exists. Σ is an LCP–set (left–convergent–product), if for all $d \in \{1, \ldots, m\}^N$ the limit $A^{(d)}$ exists. The function $d \to A^{(d)}$ mapping $\{1, \ldots, m\}^N$ into the space of $k \times k$ - matrices is called the **limit function**.

We note in passing that in [2] also the right–convergent–product property (RCP) was introduced. For convenience we restrict our considerations to the left convergence case. Introducing in $\{1, \ldots, m\}^N$ the metric

$$\text{dist}(d, d') = m^{-r} \qquad r \text{ smallest index such that } d_r \neq d'_r,$$

we define the concept of a **continuous limit function** in the standard way. Σ is **product bounded**, if there exists $\Delta > 0$ such that

$$||A|| \leq \Delta \quad \text{for all} \quad A \in \Sigma_n,\ n = 1, 2, \ldots$$

Here $||\ ||$ denotes any matrix norm. Obviously this concept is independent of the norm. G. Schechtman has proved that LCP–sets are product bounded (see [1, Theorem I]). We have the following statement.

Lemma 1 *For a set Σ of $k \times k$ - matrices the following are equivalent:*
(i) Σ is product bounded.
(ii) $\exists$ vector norm $||\ ||$ such that $||Ax|| \leq ||x||$ for all $A \in \Sigma,\ x \in C^k$.
(iii) $\exists$ multiplicative matrix norm $||\ ||$ such that $||A|| \leq 1$ for all $A \in \Sigma$.

Proof As $(ii) \Longrightarrow (iii)$ (the operator norm is multiplicative) and $(iii) \Longrightarrow (i)$ are obvious, only $(i) \Longrightarrow (ii)$ has to be shown.
For some vector norm ν define the norm

$$||x|| = \sup_{n \geq 0} \{ \sup_{A \in \Sigma_n} \nu(Ax)\} \tag{7}$$

which is finite by (i). Then $||Ax|| \leq ||x||$ for all $A \in \Sigma$. $\square$

Remark: This result could also be derived from [5]. For a given matrix norm $||\ ||$ and bounded Σ let

$$\widehat{\rho}_n = \widehat{\rho}_n(\Sigma) = \max\{||A||, A \in \Sigma_n\}$$

and

$$\widehat{\rho} = \widehat{\rho}(\Sigma) = \lim_{n\to\infty} \widehat{\rho}_n^{1/n}. \tag{8}$$

$\widehat{\rho}$ is called the joint spectral radius of Σ. It has been introduced in [5] for general bounded sets in a normed algebra. In [5] and in [2] the limit is replaced by lim sup, however, it is implicitly shown in [2] (see there (3.12)), that the limit exists.
We give a characterization of $\widehat{\rho}(\Sigma)$, which can be found essentially in [5].

Lemma 2 *For any bounded set Σ of $k \times k$ - matrices*

$$\widehat{\rho}(\Sigma) = \inf_{\nu \quad operatornorm} \sup_{A\in\Sigma} \nu(A). \tag{9}$$

Proof For any $\epsilon > 0$ the set

$$\Sigma_\epsilon = \{\frac{1}{\widehat{\rho}+\epsilon}\ A,\ A \in \Sigma\}$$

is product bounded, as for any $B \in (\Sigma_\epsilon)_n$

$$||B|| \leq \frac{1}{(\widehat{\rho}+\epsilon)^n}\ \widehat{\rho}_n \to 0 \quad \text{as} \quad n \to \infty.$$

Hence by Lemma 1, (2) there exists a norm ν_ϵ such that

$$\nu_\epsilon(\frac{A}{\widehat{\rho}+\epsilon}\ x) \leq \nu_\epsilon(x) \quad \text{for all} \quad A \in \Sigma,\ x \in C^k.$$

and therefore

$$\nu_\epsilon(Ax) \leq (\widehat{\rho}(\Sigma)+\epsilon)\nu_\epsilon(x) \quad \text{for all} \quad A \in \Sigma,\ x \in C^k.$$

$\square$

In the following the subspaces

$$M_i = \{x : A_i x = x\} = N(I - A_i) \qquad i = 1, \dots, m$$

play a fundamental role. If Σ has the LCP–property, then in particular $\lim_{n\to\infty} A_i^n$ exist, and hence

$$C^k = N(I - A_i) \oplus R(I - A_i) \qquad i = 1, \dots, m.$$

The same holds if $A_i \in \mathcal{N}(\|\ \|)$ for some norm, see [3] and [4]. This is not surprising in view of the following result, which is just a restatement of the Theorem in [3].

Theorem 3 *Let $\Sigma \subset \mathcal{N}(\|\ \|)$ for some vector norm $\|\ \|$, Σ finite. Then Σ has the LCP–property.*

We finish this section by pointing out that if in addition $\Sigma \subset \mathcal{N}_\gamma(\|\ \|)$ for some positive γ then the proof of Theorem 3 is very simple. This is outlined below. It is a consequence of the following characterization of l-paracontractivity of the set Σ.
Let $\Sigma = \{A_i\}_{i\in I}$ be a set of matrices, not necessarily finite. Let $d = (d_1, \dots, d_r) \in I^r$, ν a vector norm. Define

$$\nu_d(x) = \nu(x_r) + \sum_{k=1}^{r} \nu(x_k - x_{k-1}) \tag{10}$$

where the vectors x_i are defined as in (1) and $x = x_0$. Then obviously, for any $i \in I$ and $d' = (i, d_1, \dots, d_r)$

$$\nu_d(A_i x) = \nu_{d'}(x) - \nu(A_i x - x). \tag{11}$$

We define now

$$\nu_*(x) = \sup\{\nu_d(x) : \text{d finite}\} \tag{12}$$

This is a vector norm provided that $\nu_*(x) < \infty$ for all x.

Theorem 4 *For a set of $k \times k$ - matrices $\{A_i\}_{i\in I}$ t.f.a.e.*
(i) There exists a norm ν and a positive γ such that

$$A_i \in \mathcal{N}_\gamma(\nu) \qquad \textit{for all} \qquad i \in I.$$

(ii) There exists a vector norm μ such that

$$\mu_*(x) < \infty \qquad \textit{for all} \qquad x \in C^k$$

(iii) For all vector norms μ

$$\mu_*(x) < \infty \qquad \textit{for all} \qquad x \in C^k$$

,

Proof We show $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.
Assume that (i) holds. Then from

$$\nu(A_i x - x) \leq \gamma^{-1}\{\nu(x) - \nu(A_i x)\} \qquad \forall i \in I, \forall x \tag{13}$$

we have, using the notation in (10) and assuming (w.l.o.g.) $\gamma \leq 1$

$$\begin{aligned} \nu_d(x) &\leq \nu(x_r) + \gamma^{-1}\sum_{k=1}^{r}(\nu(x_{k-1}) - \nu(x_k)) \\ &= \nu(x_r) + \gamma^{-1}\{\nu(x) - \nu(x_r)\} \\ &\leq \gamma^{-1}\nu(x). \end{aligned} \tag{14}$$

If μ is a fixed vector norm, then due to the compatibility of any two norms we have a constant κ such that $\mu(x) \leq \kappa\nu(x)$ and hence also $\mu_d(x) \leq \kappa\nu_d(x)$. (14) gives that $\mu_*(x)$ exists, hence we have (iii).
Obviously (iii) implies (ii).
Now we assume (ii). From (11) we have

$$\mu_*(A_i x) \leq \mu_*(x) - \mu(A_i x - x) \leq \mu_*(x) - \gamma\mu_*(A_i x - x) \tag{15}$$

where we have chosen γ such that $\mu(\xi) \geq \gamma\mu_*(\xi)$ for all ξ. Hence (i) holds with $\nu = \mu_*$.
□
We indicate now the easy proof of the fact that a finite set $\Sigma = \{A_1, \ldots, A_m\} \subset \mathcal{N}_\gamma(\nu)$ has the LCP-property. It suffices to show that for any x_0 and any $d = (d_1, d_2, \ldots) \in \{1, ..., m\}^N$ the sequence $\{x_i\}_{i=1}^{\infty}$ defined by (1) is convergent. By Theorem 4 we have $\nu_*(x_0) < \infty$, hence the sequence $\sum_{i=1}^{\infty} \nu(x_i - x_{i-1})$ is convergent. This implies that the sequence of the $x_i's$ is a Cauchy sequence.

3 Main result

It is tempting to conjecture that the converse statement of Theorem 3 also holds, namely that if Σ is an LCP-set, then there exists a vector norm $|| \; ||$ such that $\Sigma \subset \mathcal{N}(|| \; ||)$. We were unable to decide this question in general.

However, the converse is true if Σ is an LCP-set with a continuous limit function. More precisely, the following holds:

Theorem 5 *Let $\Sigma = \{A_1, \ldots, A_m\}$ be a finite set of $k \times k$ - matrices and $M_i = N(I - A_i)$, $i = 1, \ldots, m$. Then the following are equivalent:*
(i) Σ has the LCP–property and for $i, j = 1, \ldots, m$

$$M_i = M_j.$$

(ii) Σ has the LCP–property with continuous limit function.
(iii) There exists a vector norm $|| \; ||$ in C^k and a positive γ such that $\Sigma \subset \mathcal{N}_\gamma(|| \; ||)$ and for $i, j = 1, \ldots, m$

$$M_i = M_j.$$

(iv) There exists a vector norm $|| \; ||$ in C^k such that $\Sigma \subset \mathcal{N}(|| \; ||)$ and for $i, j = 1, \ldots, m$

$$M_i = M_j.$$

Proof We will show $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (i)$.
To prove $(i) \Longrightarrow (ii)$, we are going to show that

$$||A^{(d)} - A^{(d')}|| \leq (2 + \Delta)||A_{(r)} - A^{(d)}|| \tag{16}$$

where $|| \; ||$ is a fixed operator norm, $(d), (d') \in \{1, \ldots, m\}^N$, $d_i = d'_i$ for $i \leq r$ and Δ the bound in the definition of product boundedness. Here we use the fact that by [1] Σ is product bounded. Also we use the notation

$$A_{(r)} = A_{d_r} A_{d_{r-1}} \ldots A_{d_1}, \; A'_{(s)} = A_{d'_s} \ldots A_{d'_1}.$$

Let $M_0 = N(I - A_i)$, $i = 1, \ldots, m$ the common pointwise invariant subspace of the matrices A_i.
If $i \in \{1, \ldots, m\}$ occurs infinitely often in the sequence $d_1, d_2, \ldots$, then by the usual reasoning

$$A_i A^{(d)} = A^{(d)}$$

and hence all columns of $A^{(d)}$ are in M_0. Hence $A_j A^{(d)} = A^{(d)}$ for all $A_j \in \Sigma$. This implies the relation

$$A'_{(r+s)} - A_{(r)} = (A_{d'_{r+s}} \ldots A_{d'_{r+1}} - I)(A_{(r)} - A^{(d)}) \qquad s > 0$$

and hence $||A'_{(r+s)} - A_{(r)}|| \leq (1+\Delta)||A_{(r)} - A^{(d)}||$. Taking $s \to \infty$, we get

$$||A^{(d')} - A_{(r)}|| \leq (1+\Delta)||A_{(r)} - A^{(d)}||,$$

from which (16) follows. This implies continuity: Given $\epsilon > 0$, as $A_{(r)} \to A^{(d)}$, there exists r_0 such that

$$||A_{(r_0)} - A^{(d)}|| \leq (2+\Delta)^{-1}\epsilon.$$

Now, if (d') is such that

$$\mathrm{dist}(d, d') \leq m^{-r_0-1}$$

then $d_i = d'_i$ for $i \leq r_0$ and hence by (16)

$$||A^{(d')} - A^{(d)}|| \leq (2+\Delta)||A_{(r_0)} - A^{(d)}|| \leq \epsilon.$$

We remark that this step is not directly contained in [2], we used however tools and ideas from this paper.
Finally we show $(ii) \Longrightarrow (iii)$.
Assume that (ii) holds. By Theorem 4.2 in [2] the subspaces M_i are the same for $i = 1, \ldots, m$. By a similarity transformation, i.e.

$$\Sigma \to S^{-1}\Sigma S = \{S^{-1}A_iS : \ i = 1, \ldots, m\}$$

which does not change the properties involved, we can assume that M_i is spanned by the first r unit vectors $e_1, \ldots, e_r$ so that for $i = 1, \ldots, m$

$$A_i = \begin{pmatrix} I_r & C_i \\ 0 & \widetilde{A}_i \end{pmatrix}.$$

Obviously $\widetilde{\Sigma} = \{\widetilde{A}_1, \ldots, \widetilde{A}_m\}$ has the LCP–property also and its limit function is identically zero. Otherwise if $\widetilde{A}^{(d)} \neq 0$, for some $d \in \{1, \ldots, m\}^N$ we would have $\widetilde{A}_r\widetilde{A}^{(d)} = \widetilde{A}^{(d)}$ for at least one r and $\widetilde{A}_r$ would have 1 as an eigenvalue. This contradicts our assumptions. But then, from Theorem 4.1 in [2], it follows that $\widehat{\rho}(\widetilde{\Sigma}) < 1$. We select some q in $(\widehat{\rho}(\widetilde{\Sigma}), 1)$. By Lemma 2 we find a norm $||\ ||$ on C^{k-r} such that

$$||\widetilde{A}_i x|| \leq q||x|| \quad \text{for all} \quad x \in C^{k-r} \quad \text{and all} \quad i = 1, \ldots, m. \tag{17}$$

Denoting by $|| \, ||_2$ the Euclidean norm in C^r, we introduce for positive ϵ the following vector norm in C^k:

$$\mu_\epsilon(x) = \mu_\epsilon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \epsilon ||x_1||_2 + ||x_2||.$$

Then we observe

$$\begin{aligned} \mu_\epsilon(A_i x) &= \mu_\epsilon \begin{pmatrix} x_1 + C_i x_2 \\ \widetilde{A}_i x_2 \end{pmatrix} \\ &= \epsilon ||x_1 + C_i x_2||_2 + ||\widetilde{A}_i x_2|| \\ &\leq \epsilon ||x_1||_2 + (\epsilon ||C_i|| + q) ||x_2|| \end{aligned} \tag{18}$$

where $||C_i|| = \max \left\{ \frac{||C_i x||_2}{||x||}, x \in C^{k-r} \right\}$. Choose $\epsilon > 0$ such that $\tilde{q} = \max_i (\epsilon ||C_i|| + q) < 1$ and let $\gamma = (1 - \tilde{q})/(1 + \tilde{q})$. Then we get after some manipulations using (17) and (18) the inequality

$$\mu_\epsilon(A_i x) \leq \mu_\epsilon(x) - \gamma \mu_\epsilon(A_i x - x).$$

Hence $\Sigma \subset \mathcal{N}_\gamma(\mu_\epsilon)$ and (iii) is proved.
$(iii) \Longrightarrow (iv)$ is trivial, while $(iv) \Longrightarrow (i)$ is Theorem 3.
□

4 Final remarks

The conjecture at the beginning of the previous section is unsolved even in the case $m = 2$. A related result has been proved in [6]:

Theorem 6 *For $\Sigma = \{A_1, A_2\}$ the following are equivalent.*
(i) Σ is an LCP-set.
(ii) (a) there exist a vector norm $|| \, ||$ such that

$$||A_i x|| \leq ||x||, \qquad i = 1, 2 \qquad \textit{for all} \qquad x \in C^k$$

$$||A_1 A_2 x|| = ||x|| \Longrightarrow A_1 x = A_2 x = x.$$

(b) For $i = 1, 2$ if λ is an eigenvalue of A_i, $|\lambda| = 1$, then $\lambda = 1$.

Notice that here we have finitely many conditions characterising the LCP-property. Nevertheless (ii) seems not to imply paracontractivity of Σ.
In the case of two projectors $P_i, i = 1, 2$, not necessarily orthogonal, the conjecture can be proved.

Theorem 7 *Let $P_i, i = 1, 2$ be projectors, i.e. $P_i^2 = P_i, i = 1, 2$. Then the following are equivalent.*
(i) $\{P_1, P_2\}$ is an LCP-set.
(ii) There exists a vector norm $||\ ||$ and a positive γ such that

$$\{P_1, P_2\} \subset \mathcal{N}_\gamma(||\ ||)$$

.

The proof is given after the following auxiliary result.

Lemma 8 *Let A, B be complex $k \times k$ -matrices such that*
(i) B is convergent, i.e. the powers of B converge, and
(ii) $\lim_{n\to\infty} AB^n = 0$.
Then there exists $\alpha \in (0, 1)$ such that for any norm $||\ ||$

$$||AB^n|| \leq C\alpha^n \qquad \text{for all} \qquad n \in N.$$

with $C > 0$ a constant depending on the norm.

Proof By eventually changing the basis accordingly, we have by (i) that B is of the form

$$B = \begin{pmatrix} I_r & 0 \\ 0 & B_0 \end{pmatrix}$$

with $\alpha = ||B_0|| < 1$ for a suitable norm. Here r is the dimension of $N(I - B)$ and we assume $r > 0$. Otherwise nothing has to be proved. Partitioning $A = (A_1, A_2)$, where A_1 contains the first r columns of A, we get $AB^n = (A_1, A_2B_0^n)$, and we see from (ii) that $A_1 = 0$. But then clearly

$$||AB^n|| = ||(0, A_2B_0^n)|| \leq C\alpha^n$$

for a suitable C. □
Proof of Theorem 7. Obviously we need only to show the implication $(i) \Longrightarrow (ii)$.

Let $|| \; ||$ denote a vector norm satisfying $||P_i x|| \leq ||x||, i = 1, 2, x \in C^k$ (See Lemma 1, (ii)) and define for $n \geq 0$

$$\begin{aligned} a_n(x) &= ||(P_1 - I)(P_2 P_1)^n x|| \\ b_n(x) &= ||(P_2 - I)P_1(P_2 P_1)^n x|| \\ c_n(x) &= ||(P_2 - I)(P_1 P_2)^n x|| \\ d_n(x) &= ||(P_1 - I)P_2(P_1 P_2)^n x|| \end{aligned}$$

By (i) the sequence

$$x_0 = x, x_{2i+1} = P_1 x_{2i}, x_{2i+2} = P_2 x_{2i+1}, i = 0, \ldots$$

is convergent, which gives that $a_n(x) = ||x_{2n+1} - x_{2n}|| \to 0$ and $b_n(x) = ||x_{2n+2} - x_{2n+1}|| \to 0$. The analogous result holds for c_n and d_n. Similarly we prove that the matrices $P_1 P_2$ and $P_2 P_1$ are convergent. Hence by the previous Lemma $r_n(x) \leq C\alpha^n$ for suitable $C > 0, \alpha \in (0, 1)$ and $r = a, b, c, d$. This shows that the following expression

$$||x||_* = ||x|| + \max(\Sigma_{n=0}^{\infty}(a_n(x) + b_n(x)), \Sigma_{n=0}^{\infty}(c_n(x) + d_n(x))$$

is finite, and it is easy to see that $||x||_* = 0$ if and only if $x = 0$. Hence it is a norm in C^k. (This is essentially the same construction as in (12), but in this special case we can give a closed expression for the norm). By some simple manipulations we get

$$||P_1 x||_* \leq ||x||_* - a_0(x) = ||x||_* - ||P_1 x - x||$$

and the same result for P_2. As there is a $\gamma > 0$ satisfying $||x|| \geq \gamma ||x||_*$ we see that $\{P_1, P_2\} \subset \mathcal{N}_\gamma(|| \; ||_*)$. $\square$

References

[1] Marc A. Berger and Yang Wang *Bounded Semigroups of Matrices* Linear Algebra Appl., 166, pp. 21–27 (1992)

[2] I. Daubechies and J.C. Lagarias *Sets of Matrices All Infinite Products of Which Converge* Linear Algebra Appl., 161, pp. 227–263(1992)

[3] L. Elsner, I. Koltracht and M. Neumann *On the Convergence of Asynchronous Paracontractions with Applications to Tomographic Reconstruction from Incomplete Data* Linear Algebra Appl., 130, pp. 65–82,(1990)

[4] S. Nelson and M. Neumann *Generalization of the Projection Method with Applications to SOR Method for Hermitian Positive Semidefinite Linear Systems* Numer. Math.,51, pp. 123–141, (1987)

[5] G.–C. Rota and W.G. Strang *A Note on the Joint Spectral Radius* Indagationes, 22, pp. 379–381, (1960)

[6] L. Elsner and S. Friedland *Norm Conditions for Convergence of infinite products* Preprint 95 - 004, Sonderforschungsbereich 343 "Diskrete Strukturen in der Mathematik", Universität Bielefeld, 1995

[7] L. Elsner *Infinite products and paracontracting matrices* Preprint 93-058, Sonderforschungsbereich 343 "Diskrete Strukturen in der Mathematik", Universität Bielefeld, 1993.