Diploma Thesis
Applied Mathematics

Solitary Waves in Infinite Cylindrical Domains

Exponential Dichotomies and Numerical Computation

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Contents

Introduction

During the last decades the theory of solitary waves is an important subject in applied mathematics and natural science. The theory plays a significant role in the study of nonlinear partial differential equations. In this diploma thesis we analyse solitary waves in infinite cylindrical domains. A solitary wave is heuristically characterised by the properties of

- spatial locality,
- constant shape and velocity
- and stability against small perturbations,

confer [16]. If the wave is additionally

- stable against scattering and collision among one another

the wave is usually called a soliton. A travelling wave[1] is defined by the properties of spatial locality and of constant shape and velocity. These properties make solitary waves very special compared to other waves which often dissolve and are unstable against perturbations.

Differential equations with accurate boundary conditions provide a mathematical description of wave phenomena. In many cases solutions are given by wave packets. Dissolving wave packets are a consequence of dispersion, i.e. the phase velocity depends on the wave length and the different superposed parts of the packet move away from each other. However, a nonlinear structure of the differential equations can compensate the dissolution and lead to solitary waves under certain conditions.

The efforts to describe solitary waves with differential equations brought also new insights in the study of nonlinear partial differential equations. For linear equations there are established concepts and theories such as the Fourier method, which enable to prove and compute solutions. However, the case of nonlinear differential equations is faced with much more difficulties. The theory of solitary waves contributes many interesting ideas and aspects to solve such nonlinear problems. This theory subdivides into many different subjects such as inverse scattering method, symmetry and numerical study of nonlinear waves. Its foundation consists of many branches of mathematics, confer [6], such as classical and functional analysis, dynamical systems, topology, computational mathematics and differential geometry.

After John Scott Russell first discovered the phenomenon of solitary waves in a narrow channel in 1834 mathematicians and physicists tried to explain this appearance, confer [16]. Roughly 60 years were needed until Korteweg and de Vries could introduce a nonlinear partial differential equation

$$u_t + u_{xxx} + 6u_x u = 0,$$

[1] The notions of solitary waves, solitons and travelling waves are not uniform in the literature.

which describes solitary waves on shallow water surfaces. Besides hydrodynamics more fields of physics were found such as nonlinear optics and quantum theory, where solitary waves play an important role. Moreover, there are examples of solitary waves in biology and chemistry such as nerve impulses, bloodflows in arteries and chemical kinetics, see [15] and [16]. Since the emergence of the theory of solitary waves mathematicians try to keep up with the desire for an exact understanding of the phenomenon and for obtaining new ideas in the study of nonlinear partial differential equations. Whereas some subjects such as small perturbations are well understood there are still many open problems.

In this thesis the concept of exponential dichotomies is one of the major tools for proving the existence of solitary waves in infinite cylindrical domains, confer [19]. For the study of differential equations exponential dichotomies have become a promising subject. They describe important properties of the solutions such as uniqueness and exponential decay. In the seventies the notion of exponential dichotomies was introduced and applied to questions of asymptotic behaviour for non-autonomous differential equations, confer [7]. In the first chapter we show that exponential dichotomies can be successfully employed in conjunction with linear operators which have unbounded spectra in both the positive and the negative half plane. Under certain conditions such operators define analytic semigroups on subspaces of the underlying Banach space. These semigroups lead to solutions of the considered differential equations which can be characterised by exponential dichotomies. First, we discuss the case of a linear and autonomous differential equation given by a possibly unbounded operator. Hereupon we perturb the equation by a linear, non-autonomous but bounded part and obtain a roughness[2] theorem for exponential dichotomies. At the end of the first chapter we consider some important implications of the roughness theorem and analyse in particular the case of inhomogeneous linear equations and nonlinear equations. To prove the results we use integral equations as mild formulation for the differential equations and we show that Fredholm's alternative applies to the setting.

When we started with the thesis exponential dichotomies were only supposed to be a major tool for the subject of solitary waves and we followed closely [19]. However, right at the beginning we had critical concerns regarding major assumptions for the evolution equations. We came to the conclusion that a certain resolvent estimate on the imaginary axis is not sufficient to obtain some needed sectorial operators. After we had spent some time and had consulted the authors of [19] and [15] we decided to change some hypotheses and to present the subject of exponential dichotomies in a more detailed way. We also needed to correct some parts of the proof of the roughness theorem.

However, the main issue of this diploma thesis is still the study of solitary waves which are described by semilinear elliptic equations[3] with appropriate boundary conditions:

$$\begin{aligned} u_{xx} + \Delta_y u + g(y, u, u_x, \nabla_y u) = 0, \quad (x, y) \in \mathbb{R} \times \Omega,\ u \in \mathbb{R}^m, \\ R\left((u, u_x, \nabla_y u)|_{\mathbb{R}\times\partial\Omega}\right) = 0 \quad \text{on } \mathbb{R} \times \partial\Omega, \end{aligned} \tag{0.1}$$

where $\mathbb{R} \times \Omega$ is an infinite cylinder with $\Omega \subset \mathbb{R}^n$ open and bounded. In this context a solitary wave is a solution h of the boundary value problem which satisfies

$$\lim_{x\to\pm\infty} h(x, y) = p_\pm(y)$$

[2]Here, roughness has the meaning of small perturbations.

[3]$\Delta_y u := \sum_{k=1}^n \frac{\partial^2 u}{\partial y_k^2}$, $\nabla_y u := (\frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_n})$.

uniformly for $y \in \Omega$ and for some functions $p_\pm$. They describe the profile of travelling waves $u(x-ct, y)$ for parabolic equations

$$u_t = u_{xx} + \Delta_y u + \tilde{g}(y, u, u_x, \nabla_y u), \quad (x, y) \in \mathbb{R} \times \Omega.$$

We suppose the existence of a solitary wave. In order to determine the wave numerically we truncate the cylinder and adjust the boundary conditions. We examine whether the truncated system has a unique solution close to h and we prove estimates for the truncation error.

An important idea of our procedure is rewriting the differential equation (0.1) to a first order system of the form

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} + f(u, v) \quad \text{with } A = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta_y & 0 \end{pmatrix}.$$

Here, A is a densely defined and closed operator and f is a smooth function. After having merged u, v into one variable, called u again, and after having added a real parameter μ we analyse differential equations of the form

$$\frac{\partial}{\partial x} u = Au + f(u, \mu).$$

The variable u is now an element of some function space which incorporates the boundary conditions. A solitary wave solution corresponds to a homoclinic or heteroclinic solution which we call h again.

In the second chapter we discretize the cross-section Ω by introducing the Galerkin projection

$$\frac{\partial}{\partial x} u = Au + Q_\rho f(u, \mu), \quad u \in R(Q_\rho),$$

where $\{Q_\rho\}_{\rho > 0}$ is a family of projections[4]. The projections Q_ρ map the function space in Ω onto a subspace that is typically finite-dimensional. We prove the persistence of a hyperbolic equilibrium and of a homoclinic orbit under the Galerkin approximation. This theorem is the main result of the second chapter besides statements regarding the truncated boundary value problem and projection boundary conditions. For the proof of the results we consider exponential dichotomies for the linearization

$$\frac{\partial}{\partial x} v = (A + D_u f(h(x), 0))v$$

and apply a version of the contraction mapping theorem.

In the second chapter we follow closely [15]. Again, we had to adapt some important assumptions in order to obtain the desired results. Furthermore, we had to correct and complete some aspects of the proofs. In particular, questions of regularity had to be considered thoughtfully such as joining solutions which are given on different semiaxes.

In the third chapter we consider a concrete numerical example in order to compare theoretical and numerical results. We analyse a truncated boundary value problem with elliptic differential equations and projection boundary conditions. Since we use a Galerkin approximation

[4] Actually one applies the operators Q_ρ to $\frac{\partial}{\partial x} u = Au + f(u, \mu)$ with $u \in R(Q_\rho)$ and requires that A and Q_ρ commute.

with projections of finite-dimensional ranges we obtain a finite-dimensional system of ordinary differential equations with two-point boundary conditions. For the boundary value problem we use a solver which is based on a collocation method with a C^1-piecewise cubic polynomial. More details of the computations which lead to the finite-dimensional system of differential equations are given in the appendix.

Further interesting issues are the case of time-dependent differential equations and the stability of solitary waves. Moreover, a more detailed view on the aspects of regularity and on the spectra of linearizations relating to the differential equations could be an interesting prospect.

1 Exponential Dichotomies

In this chapter we consider evolution equations with linear operators which have unbounded spectra in both the positive and negative half plane. Under certain assumptions they define analytic semigroups on some subspaces of the underlying Banach space. The notion of an exponential dichotomy is the major issue of this chapter which describes important properties of the solutions such as uniqueness and exponential decay. Exponential dichotomies also constitute a primal tool for the following chapters. Exponential dichotomies were first used for ordinary differential equations and applied to questions of asymptotic behaviour for non-autonomous differential equations, confer [7] and [18].

First, we define exponential dichotomies for the general case of a linear, autonomous and unbounded partial differential equation perturbed by a linear, non-autonomous but bounded part. We consider the unperturbed situation in the first section. Here, the concept of sectorial operators, analytic semigroups and fractional powers of operators plays the central role. The following section discusses the case of a bounded perturbation and establishes a roughness theorem for exponential dichotomies. In the third section we prove the roughness theorem which occupies most of this chapter. The important ideas of the proof consist of integral equations used as mild formulation of the corresponding differential equations and of employing Fredholm's alternative. Finally, we state in the last section some important implications of the roughness theorem. In particular, we consider inhomogeneous linear equations and nonlinear equations whose linear parts consist of the previously analysed differential equations.

We follow closely [19] but we prove the statements in a more detailed way and give some important corrections. In particular, we have to correct some assumptions in order to keep the desired theorems. The spectral properties of Hypothesis (H1) in [19] have to be sharpened significantly to obtain analytic semigroups which lead to the existence of exponential dichotomies.

Initial situation

Let

$$A : D(A) \subset X \to X$$

be a densely defined and closed operator on a reflexive Banach space $(X, ||\cdot||_X)$. Let Z be some Banach space and regard $X^1 := D(A)$ as a Banach space with a norm so that there exist continuous embeddings

$$X^1 \hookrightarrow Z \hookrightarrow X.$$

Let $J \subset \mathbb{R}$ be some closed interval and let

$$B \in C^0(J, L(Z, X))$$

be a continuous family of operators.

In this chapter we consider differential equations of the form

$$\frac{\partial}{\partial x} u = (A + B(x))u, \quad x \in J. \tag{1.1}$$

First, we specify requirements for a solution of this equation. We focus on the cases with $J = \mathbb{R}$, $J = \mathbb{R}^+$ and $J = \mathbb{R}^-$.

Definition 1.0.1 *A solution of (1.1) is a function u defined on J with the properties*

- $u \in C^0(\mathring{J}, X^1) \cap C^1(\mathring{J}, X)$,
- $u \in C^0(J, Z)$,
- *(1.1) holds as an equation in $C^0(\mathring{J}, X)$*

We also call the function u a strong solution of (1.1).

In the following we define exponential dichotomies of (1.1), which is the central subject of this chapter.

Definition 1.0.2 ***(Exponential dichotomy)***
The differential equation (1.1) has an exponential dichotomy in Z on the interval J if there exists a family of projections $\{P(x)\}_{x \in J}$ so that

$$P(x) \in L[Z], \quad (P(x))^2 = P(x), \quad P(\cdot)w \in C^0(J, Z) \quad \forall w \in Z$$

and so that there exist constants[1] $C, \eta > 0$ with the properties:

- *Stability. There exists a unique solution $u^s(x; x_0, w)$ of (1.1) for any $x_0 \in J$, $w \in Z$ and defined for $x \in J \cap [x_0, \infty)$ with $u^s(x_0; x_0, w) = P(x_0)w$. The solution u^s satisfies*

$$||u^s(x; x_0, w)||_Z \le Ce^{-\eta|x-x_0|}\,||w||_Z \quad \forall x \in J \cap [x_0, \infty).$$

- *Instability. There exists a unique solution $u^u(x; x_0, w)$ of (1.1) for any $x_0 \in J$, $w \in Z$ and defined for $x \in J \cap (-\infty, x_0]$ with $u^u(x_0; x_0, w) = (id - P(x_0))w$. The solution u^u satisfies*

$$||u^u(x; x_0, w)||_Z \le Ce^{-\eta|x-x_0|}\,||w||_Z \quad \forall x \in J \cap (-\infty, x_0].$$

- *Invariance. For $w \in Z$,*

$$\begin{aligned} u^s(x; x_0, w) &\in R(P(x)) \quad \forall x \in J \cap [x_0, \infty), \\ u^u(x; x_0, w) &\in N(P(x)) \quad \forall x \in J \cap (-\infty, x_0]. \end{aligned}$$

Note that the initial value problem can only be solved uniquely in forward or backward time direction if the initial data w is in a certain subspace. Otherwise, different initial data w can lead to the same solution. Moreover, the solutions are marked by an exponential decay in the corresponding time direction.

[1] In all chapters, C is generally used to denote a constant determined only by the assumptions made on each occasion.

1.1 A Class of Abstract Differential Equations

In this section we consider the equation

$$\frac{\partial}{\partial x}u = Au \tag{1.2}$$

and analyse possible exponential dichotomies in X on $\mathbb{R}$. Requiring certain sectorial properties of the spectrum of A results in the existence of exponential dichotomies. Such properties are taken into account by the following hypothesis **(H1)**. Here, note the differences to [19] and [15]. To formulate **(H1)** we need the definition of sectorial operators. Confer Appendix A.3.

Hypothesis (H1)
The operator $A : D(A) \subset X \to X$ is densely defined, closed and zero is an element of[2] $\rho(A)$. There exists a projection $P_- \in L[X]$ and constants $\delta > 0, \phi_\pm \in (0, \pi/2), M \geq 1$ with the following properties:

(i) $[A^{-1}, P_-] = 0$,

(ii) $A_+ := (\mathrm{id} - P_-)A$ and $A_- := -P_-A$ are sectorial on the Banach spaces $X_+ := R(\mathrm{id} - P_-)$ and $X_- := R(P_-)$, respectively,

(iii)

$$S_{\delta,\phi_\pm} = \{\lambda \in \mathbb{C} \,|\, \phi \leq |\arg(\lambda - a)| \leq \pi,\, \lambda \neq a\} \subset \rho(A_\pm),$$

$$\left|\left|(\lambda - A_\pm)^{-1}\right|\right|_{L[X_\pm]} \leq \frac{M}{|\lambda - \delta|} \quad \forall \lambda \in S_{\delta,\phi_\pm}.$$

Moreover, one defines $P_+ := \mathrm{id} - P_-$.

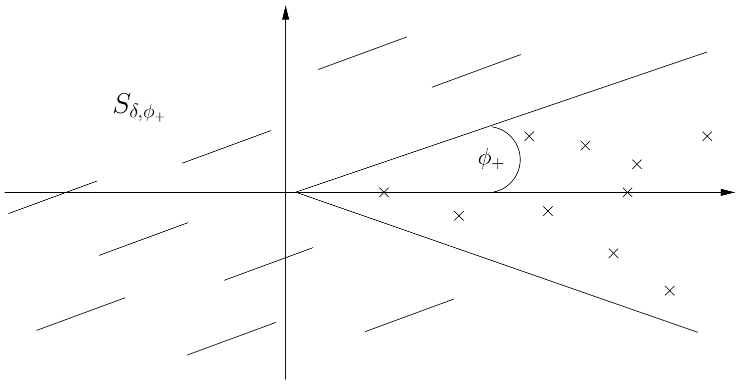

Figure 1.1: The resolvent set of A_+ contains S_{δ,ϕ_+}. A cross indicates an element of the spectrum which is restrained by the lines starting from the origin. A similar situation holds for A_-.

In [19] and [15] the authors consider only the operator A and demand a resolvent estimate solely on the imaginary axis. However, this does not suffice to make $A_\pm$ sectorial and therefore does not ensure the existence of analytic semigroups. That is why we sharpened the assumptions relating to A in order to obtain the needed sectorial properties.

[2] $\rho(A)$ is the resolvent set of A. We refer to definition A.2.2.

Theorem 1.1.1 *Under the assumption **(H1)** the operators A_+ and A_- generate analytic semigroups*

$$e^{-A_+x} = \frac{1}{2\pi i}\int_{\Gamma_+} e^{\lambda x}(\lambda + A_+)^{-1}\,d\lambda, \quad x \geq 0,$$
$$e^{-A_-x} = \frac{1}{2\pi i}\int_{\Gamma_-} e^{\lambda x}(\lambda + A_-)^{-1}\,d\lambda, \quad x \geq 0,$$

on X_+ and X_-, respectively. Γ_+ and Γ_- are contours in $\rho(-A_+)$ and $\rho(-A_-)$ with $\arg(\lambda) \to \pm\theta_+$ and $\arg(\lambda) \to \pm\theta_-$ as $|\lambda| \to \infty$ for some $\theta_+, \theta_- \in \left(\frac{\pi}{2}, \pi\right)$, respectively. Furthermore, there is some constant $C > 0$ so that

$$\frac{d}{dx}e^{-A_\pm x} = -A_\pm e^{-A_\pm x}, \quad ||e^{-A_\pm x}||_{L[X_\pm]} \leq Ce^{-\delta x}, \quad x \geq 0.$$

Proof These results are consequences of **(H1)** and Theorem A.3.3. ∎

Note that δ is a positive constant defined in Hypothesis **(H1)**. Next, we define the interpolation spaces X_+^α and X_-^α. Here, we need the concept of fractional powers of operators which is explained in Appendix A.3.

Definition 1.1.2 *We define for $\alpha \geq 0$:*

$$X_+^\alpha := D(A_+^\alpha) = R(A_+^{-\alpha}), \quad X_-^\alpha := D(A_-^\alpha) = R(A_-^{-\alpha}), \quad X^\alpha := X_+^\alpha \oplus X_-^\alpha.$$

We also define the norms $||v||_{X^\alpha} := ||v||_{X_+^\alpha} + ||v||_{X_-^\alpha}$ on X^α and $||w||_\oplus := ||w||_{X_+} + ||w||_{X_-}$ on $X = X_+ \oplus X_-$. Here, $||v||_{X_\pm^\alpha} := ||v_\pm||_{X_\pm^\alpha}$, where v can be uniquely written as $v = v_+ + v_-$ with $v_\pm \in X_\pm^\alpha$, and $||w||_{X_\pm} := ||w_\pm||_X$, where w can be uniquely written as $w = w_+ + w_-$ with $w_\pm \in X_\pm$.

Remark 1.1.3 *Theorem A.3.13 leads to $X_\pm^\alpha \subset X_\pm$. Therefore, $X_+ \cap X_- = \{0\}$ results in $X_+^\alpha \cap X_-^\alpha = \{0\}$.*

In the following we consider the Banach space $X = X_+ \oplus X_-$ equipped with the norm $||\cdot||_\oplus$ and $X^1 = D(A)$ equipped with the norm defined by $||v||_{X^1} = ||A_+v||_X + ||A_-v||_X$, $v \in D(A)$.

Lemma 1.1.4 *If $\alpha \in [0, 1)$, the canonical embeddings $X^1 \hookrightarrow X^\alpha \hookrightarrow X = X_+ \oplus X_-$ are continuous.*

Proof This statement follows directly from Theorem A.3.13.

First, we show $X^\alpha \hookrightarrow X = X_+ \oplus X_-$. Let $v \in X^\alpha$, then

$$||v||_\oplus = ||v||_{X_+} + ||v||_{X_-} \leq C||v||_{X_+^\alpha} + C||v||_{X_-^\alpha} \leq C||v||_{X^\alpha}.$$

Finally, we prove $X^1 \hookrightarrow X^\alpha$. Let $v \in X^1 = D(A)$, then

$$||v||_{X^\alpha} = ||v||_{X_+^\alpha} + ||v||_{X_-^\alpha} \leq C||v||_{X_+^1} + C||v||_{X_-^1} = C(||A_+v||_X + ||A_-v||_X) \leq C||v||_{X^1}.$$ ∎

Lemma 1.1.5
$$P_{\pm} \in L[X^\alpha]$$

Proof First, we prove: $v \in X^\alpha \Rightarrow P_- v \in X^\alpha$.

Let $v \in X^\alpha = R(A_+^{-\alpha}) \oplus R(A_-^{-\alpha})$. Then there are $w_\pm \in X$ with $v = A_+^{-\alpha} w_+ + A_-^{-\alpha} w_-$. It follows $P_- v = P_- A_+^{-\alpha} w_+ + P_- A_-^{-\alpha} w_- = A_-^{-\alpha} P_- w_-$ from **(H1)**, so $P_- v \in R(A_-^{-\alpha})$.

Finally, we obtain from **(H1)**

$$\begin{aligned} ||P_-||_{X^\alpha} &= \sup_{||v||_{X^\alpha} \le 1} ||P_- v||_{X^\alpha} = \sup_{||v||_{X^\alpha} \le 1} ||A_-^\alpha P_- v||_{X_-} = \sup_{||v||_{X^\alpha} \le 1} ||P_- A_-^\alpha v||_{X_-} \\ &\le ||P_-||_{L[X_-]} \sup_{||v||_{X^\alpha} \le 1} ||A_-^\alpha v||_{X_-} \le ||P_-||_{L[X]}. \quad \blacksquare \end{aligned}$$

Now we summarise the main result of this section:

Theorem 1.1.6 *Provided that the assumption **(H1)** is satisfied, equation (1.2) has an exponential dichotomy on any closed interval $J \subset \mathbb{R}$ in X. The corresponding projections $P(x) = P_- \in L[X]$ are independent of x. For any $x_0 \in J$ and $w \in X^\alpha$ the solution is given by $u^s(x; x_0, w) = e^{-A_-(x-x_0)} P_- w$ defined for $x \in J \cap [x_0, \infty)$ and by $u^u(x; x_0, w) = e^{-A_+(x_0-x)} P_+ w$ defined for $x \in J \cap (-\infty, x_0]$.*

Proof Due to Lemma 1.1.4 the canonical embeddings

$$X^1 = D(A) \hookrightarrow Z := X^\alpha = X_+^\alpha \oplus X_-^\alpha \hookrightarrow X = X_+ \oplus X_-.$$

are continuous. The operators $P(x) = P_-$, $x \in J$, form a family of projections with $P_- \in L[X^\alpha]$, $P_-^2 = P_-$ and $P(\cdot)w \in C^0(J, X^\alpha)$ for all $w \in X^\alpha$.

Stability: For any $x_0 \in J$ and $w \in X^\alpha$ there exists a unique solution given by $u^s(x; x_0, w) = e^{-A_-(x-x_0)} P_- w$ for $x \in [x_0, \infty) \cap J$. Consider $e^{-A_-(x_0-x_0)} P_- w = P_- w$ and

$$\begin{aligned} \frac{\partial}{\partial x} u^s(x; x_0, w) &= -A_- e^{-A_-(x-x_0)} P_- w = P_- A e^{-A_-(x-x_0)} P_- w \overbrace{=}^{[P_-, A]=0} A e^{-A_-(x-x_0)} P_- w \\ &= A u^s(x; x_0, w) \quad \forall x \in [x_0, \infty) \cap \mathring{J}. \end{aligned}$$

$u \in C^0(\mathring{J} \cap (x_0, \infty), X^1) \cap C^1(\mathring{J} \cap (x_0, \infty), X)$, $u \in C^0(J \cap [x_0, \infty), X^\alpha)$ and the uniqueness can be shown by using Theorem A.3.3 and the statements at the beginning of Section 3.2 in [11]. Moreover, u^s satisfies

$$\begin{aligned} ||u^s(x; x_0, w)||_{X^\alpha} &= ||e^{-A_-(x-x_0)} P_- w||_{X^\alpha} = ||A_-^\alpha e^{-A_-(x-x_0)} P_- w||_{X_-} = ||e^{-A_-(x-x_0)} A_-^\alpha P_- w||_{X_-} \\ &\le ||e^{-A_-(x-x_0)}||_{L[X_-]} ||A_-^\alpha P_- w||_{X_-} \le C e^{-\delta(x-x_0)} ||P_- w||_{X^\alpha} \\ &\le C e^{-\delta|x-x_0|} ||w||_{X^\alpha} \quad \forall x \in [x_0, \infty) \cap J. \end{aligned}$$

Instability: For any $x_0 \in \mathbb{R}$ and $w \in X^\alpha$ there exists a unique solution given by $u^u(x; x_0, w) = e^{-A_+(x_0-x)}P_+w$ for $x \in (-\infty, x_0] \cap J$. Consider $e^{-A_+(x_0-x_0)}P_+w = P_+w$ and

$$\begin{aligned}\frac{\partial}{\partial x}u^u(x; x_0, w) &= A_+e^{-A_+(x_0-x)}P_+w = P_+Ae^{-A_+(x_0-x)}P_+w \overbrace{=}^{[P_+,A]=0} Ae^{-A_+(x_0-x)}P_+w \\ &= Au^u(x; x_0, w) \quad \forall x \in (-\infty, x_0] \cap \mathring{J}.\end{aligned}$$

$u \in C^0(\mathring{J} \cap (-\infty, x_0), X^1) \cap C^1(\mathring{J} \cap (-\infty, x_0), X)$, $u \in C^0(J \cap (-\infty, x_0], X^\alpha)$ and the uniqueness are shown as in the case of stability. Moreover, u^u satisfies

$$\begin{aligned}||u^u(x; x_0, w)||_Z &= ||A_+^\alpha e^{-A_+(x_0-x)}P_+w||_{X_+} \\ &\leq ||e^{-A_+(x_0-x)}||_{L[X_+]}||A_+^\alpha P_+z||_{X_+} = ||e^{-A_+(x_0-x)}||_{L[X_+]}||P_+z||_{X^\alpha} \\ &\leq Ce^{-\delta|x-x_0|}||w||_Z \quad \forall x \in (-\infty, x_0] \cap J.\end{aligned}$$

Invariance: For $w \in X^\alpha$,

$$\begin{aligned}u^s(x; x_0, w) &= e^{-A_-(x-x_0)}P_-w = P_-e^{-A_-(x-x_0)}P_-w \in R(P_-) \quad \forall x \in [x_0, \infty) \cap J, \\ u^u(x; x_0, w) &= e^{-A_+(x_0-x)}P_+w = P_+e^{-A_+(x_0-x)}P_+w \in N(P_-) \quad \forall x \in (-\infty, x_0] \cap J.\end{aligned}$$

Consider $N(P_-) = R(id - P_-) = R(P_+)$. ∎

Corollary 1.1.7 *Let A satisfy **(H1)**, $B = 0$ and let u be a bounded solution of (1.1) on a closed interval $J \subset \mathbb{R}$. If there is some $x_0 \in J$ with $P_-u(x_0) = 0$, then $u = 0$ on J.*

Proof Due to the properties of u and Theorem 1.1.6 we obtain

$$u(x) = e^{-A_+(x_0-x)}P_+u(x_0) = e^{-A_+(x_0-x)}u(x_0), \quad x, x_0 \in J,\ x \leq x_0.$$

We can continue $u(x)$ with $e^{-A_+(x_0-x)}u(x_0)$, where $x, x_0 \in \mathbb{R}$ and $x \leq x_0$. This results in

$$||u(x)||_{X^\alpha} \leq Ce^{-\eta(x_0-x)}||u(x_0)||_{X^\alpha} \leq Ce^{-\eta(x_0-x)} \quad x_0 \geq x.$$

Here, we used $\sup_{x \in J}||u(x)||_{X^\alpha} < \infty$. Because of $e^{-\eta(x_0-x)} \to 0$ as $x_0 \to \infty$ for every $x \in J$ we obtain $u(x) = 0$ for all $x \in J$. ∎

1.2 A Roughness Theorem for Exponential Dichotomies

In this section we analyse exponential dichotomies for the perturbation

$$\frac{\partial}{\partial x}u = (A + B(x))u \tag{1.3}$$

of $\frac{\partial}{\partial x}u = Au$. For the notion of solution we refer to Definition 1.0.1. In order to obtain a roughness theorem for exponential dichotomies we have to require certain Hölder and compactness properties of the operators $B(x)$ and A, respectively, additionally to the assumptions of **(H1)**. Moreover, we have to introduce a uniqueness hypothesis regarding (1.3) and the adjoint equation of (1.3). From now on we choose $J \in \{\mathbb{R}, \mathbb{R}^+, \mathbb{R}^-\}$. For the results of this section confer [19].

The constant $\varepsilon > 0$ contained in the next hypothesis will be specified in the roughness Theorem 1.2.1 for exponential dichotomies.

Hypothesis (H2)
There exist $\alpha \in [0, 1)$, $\vartheta > 0$, $x_* \geq 0$ and[3] $S, K \in C^{0,\vartheta}(J, L[X^\alpha, X])$ so that

$$B(x) = S(x) + K(x), \quad ||S(x)||_{L[X^\alpha,X]} \leq \varepsilon$$

for $x \in J$ and $K(x) = 0$ for all $x \in J$ with $|x| \geq x_*$.

The constant ε has to ensure a small perturbation for all sufficiently large x so that important resolvent properties will be kept. Moreover, we note that **(H2)** results in

$$\sup_{x \in J} ||B(x)||_{L[X^\alpha,X]} < \infty.$$

For some needed compactness properties we require **(H3)** or **(H4)**:

Hypothesis (H3)
A^{-1} is a compact operator in X.

Hypothesis (H4)
There is a Banach space Y with $Y \hookrightarrow X$ compact so that $K \in C^{0,\vartheta}(J, L[X^\alpha, Y])$. Moreover, the restriction of A to Y is a densely defined and closed operator $A : D(A) \subset Y \to Y$ which satisfies **(H1)** with X replaced by Y.

In this thesis we will restrict to **(H3)** and refer to [19] for applications of Hypothesis **(H4)**. These applications include semilinear elliptic equations on $\mathbb{R} \times \mathbb{R}^n$ with localized solutions $u(x, y)$ which are marked by an exponential decay in $|y|$ uniformly in x. Finally, forward and backward uniqueness of solutions of equation (1.3) is assumed on the interval J. The continuation of exponential dichotomies from a strict subinterval of J to J itself seems to require this assumption.

Hypothesis (H5)
The only bounded solution of (1.3) or its adjoint equation

$$\frac{\partial}{\partial x}\xi = -(A' + B(x)')\xi, \quad \xi \in X' \tag{1.4}$$

[3] Hölder spaces $C^{m,\vartheta}$ are defined in Definition A.1.4.

on J with $u(0) = 0$ is the trivial solution $u = 0$.

The main result of this section is given by the following roughness theorem for exponential dichotomies.

Theorem 1.2.1 ***(Roughness theorem for exponential dichotomies)***
*Let $J = \mathbb{R}^+$ and let **(H1)** be satisfied. Choose η so that[4] $0 \leq \eta < \delta$. Then, there exist positive constants ε_0 and C with the following properties: Assume that **(H2)**, **(H5)**, and either **(H3)** or **(H4)** are satisfied for some $\varepsilon \leq \varepsilon_0$. Differential equation (1.3) then has an exponential dichotomy in X^α on $J = \mathbb{R}^+$ with rate η.*

Moreover, the corresponding projections $P(\cdot)$ are Hölder continuous in $J = \mathbb{R}^+$ with values in $L[X^\alpha]$ and $E^s := R(P(0))$ is uniquely determined. The statement

$$w \in E^s \quad \Rightarrow \quad w = P_- w + P_+(S_0 + K_0)w \tag{1.5}$$

holds for some operators $S_0 \in L[X^\alpha]$ and[5] $K_0 \in K[X^\alpha]$ with $||S_0||_{L[X^\alpha]} \leq C\varepsilon$. Furthermore, for any closed complement E^u of E^s there exists a unique exponential dichotomy so that $R(P(0)) = E^s$ and $N(P(0)) = E^u$. In particular, there exist closed complements of E^s.

An analogous theorem holds for $J = \mathbb{R}^-$. In the next section we prove the roughness theorem and in Section 1.4 we consider some important implications.

[4] δ is defined in **(H1)**
[5] $K[X]$ is the set of compact operators on a normed vector space X, see [26] section II.3.

1.3 Proof of the Roughness Theorem

At the beginning of the proof we consider an integral equation of the evolution operators $u^s(x; x_0, w)$ and $u^u(x; x_0, w)$. This integral equation constitutes a mild formulation of (1.3) which is equivalent to the strong formulation. Here, the strong formulation is given by Definition 1.0.1. Having proven the equivalence we use the integral equation and Fredholm's alternative to construct the subspace $E^s = R(P(0))$ which includes the bounded solutions of (1.3) on $J = \mathbb{R}^+$. Hereupon we choose a fixed complement E^u of E^s. Then we prove that the mild formulation has a unique solution[6] $(u^s(\cdot, x_0), u^u(\cdot, x_0))$ for any fixed $x_0 \geq 0$ which meets $u^u(0, x_0) \in E^u$. We conclude the proof by showing the strong continuity and the semigroup properties of these solutions. In this section we follow closely [19].

Mild Formulation

Definition 1.3.1 ***(Mild Formulation)***
The integral equations

$$\begin{aligned}
&e^{-A_-(x-x_0)}P_-w \\
&= u^s(x,x_0) + e^{-A_-x}P_-u^u(0,x_0) + \int_x^\infty e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma,x_0)d\sigma \\
&\quad - \int_{x_0}^x e^{-A_-(x-\sigma)}P_-B(\sigma)u^s(\sigma,x_0)d\sigma + \int_0^{x_0} e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma,x_0)d\sigma, \\
&\text{for} \quad x \geq x_0 \geq 0, \\
&e^{A_+(x-x_0)}P_+w \\
&= u^u(x,x_0) - e^{-A_-x}P_-u^u(0,x_0) - \int_{x_0}^x e^{A_+(x-\sigma)}P_+B(\sigma)u^u(\sigma,x_0)d\sigma \\
&\quad + \int_x^0 e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma,x_0)d\sigma - \int_{x_0}^\infty e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma,x_0)d\sigma, \\
&\text{for} \quad x_0 \geq x \geq 0
\end{aligned} \tag{1.6}$$

with $w \in X^\alpha$ are called the mild formulation of $\frac{\partial}{\partial x}u = (A + B(x))u$. A solution (u^s, u^u) of the integral equations is an element of $C^0([x_0, \infty), X^\alpha) \times C^0([0, x_0], X^\alpha)$ and satisfies (1.6).

Lemma 1.3.2 *The mild formulation is well-defined.*

Proof We show exemplary that

$$\int_x^\infty e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma,x_0)d\sigma$$

exists in X_+^α for every $x \geq x_0 \geq 0$. Theorem A.3.3 and A.3.13 yield

$$R(e^{A_+(x-\sigma)}) \subset D(A) = D(A_+) = X_+^1 \subset X_+^\alpha$$

[6] In this section we write $u^s(x, x_0) = u^s(x; x_0, w)$ and $u^u(x, x_0) = u^u(x; x_0, w)$ if we can avoid confusion.

because of $\alpha \in [0,1)$. Furthermore, it follows[7]

$$\begin{aligned}
&\int_x^\infty ||e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma,x_0)||_{X_+^\alpha}d\sigma \\
&\le \int_x^\infty ||e^{A_+(x-\sigma)}||_{L[X,X_+^\alpha]}||P_+||_{L[X]}||B(\sigma)||_{L[X^\alpha,X]}||u^s(\sigma,x_0)||_{X^\alpha}d\sigma \\
&\le C\int_x^\infty ||A_+^\alpha e^{-A_+(\sigma-x)}||_{L[X]}d\sigma \\
&\le C\int_x^\infty (\sigma-x)^{-\alpha}e^{-\delta(\sigma-x)}d\sigma \\
&\le C\lim_{R\to\infty}\left[\frac{1}{1-\alpha}(\sigma-x)^{1-\alpha}e^{-\delta(\sigma-x)}\right]_x^R + \frac{C}{(1-\alpha)\delta}\int_x^\infty (\sigma-x)^{1-\alpha}e^{-\delta(\sigma-x)}d\sigma \\
&\le C\Gamma(2-\alpha) \\
&< \infty
\end{aligned}$$

for every $x \ge 0$ from $1-\alpha > 0$ and Lemma A.3.9. Note that C depends on α. ∎

We will conclude that the solutions (u^s, u^u) of the mild formulation are the evolution operators which are given in the definition of exponential dichotomies. Furthermore, we will show that the projections of the exponential dichotomy are determined by $P(x)w = u^s(x;x,w)$ and $(\mathrm{id} - P(x))w = u^u(x;x,w)$. The operator $u^u(0;0,\cdot)$ is given by the choice of the complement E^u. Note that we cannot use the contraction mapping theorem as a major tool for the proof since the integrands of (1.6) are not small. This is a consequence of **(H2)** which does not exclude large values of the norm of the operator B.

Lemma 1.3.3

(i) *If (u^s, u^u) is a bounded solution of (1.6) for some $w \in X^\alpha$, then $u^s(\cdot, x_0)$ and $u^u(\cdot, x_0)$ is a bounded solution of (1.3) on $J = [x_0, \infty)$ and $J = [0, x_0]$, respectively.*

(ii) *If $u^1(\cdot)$ and $u^2(\cdot)$ are bounded solutions of (1.3) on $J_1 = [x_0, \infty)$ and $J_2 = [0, x_0]$, respectively, then $u^1(\cdot)$ and $u^2(\cdot)$ are bounded solutions of (1.6) with $u^s(x, x_0) = u^1(x)$, $u^u(x, x_0) = u^2(x)$ and $w = u^1(x_0) + u^2(x_0)$.*

Proof (i) Suppose that (u^s, u^u) satisfies the mild formulation (1.6) for some $w \in X^\alpha$. This results directly in $(u^s, u^u) \in C^0([x_0,\infty), X^\alpha) \times C^0([0,x_0], X^\alpha)$. The Hölder continuity of B and Lemma 3.5.1 in [11] yield $(u^s, u^u) \in C^1((x_0,\infty), X) \times C^1((0,x_0), X)$. Differentiating (1.6) with respect to x results in

$$\begin{aligned}
\frac{\partial}{\partial x}u^s(x,x_0) &= (A + B(x))u^s(x,x_0), \\
\frac{\partial}{\partial x}u^u(x,x_0) &= (A + B(x))u^u(x,x_0).
\end{aligned}$$

[7] The Gamma function is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$.

Because of

$$
\begin{aligned}
Au^s(x,x_0) &= \frac{\partial}{\partial x}u^s(x,x_0) - B(x)u^s(x,x_0) \in C^0((x_0,\infty),X),\\
Au^u(x,x_0) &= \frac{\partial}{\partial x}u^u(x,x_0) - B(x)u^u(x,x_0) \in C^0((0,x_0),X)
\end{aligned}
$$

we obtain $(u^s,u^u) \in C^0((x_0,\infty),X^1) \times C^0((0,x_0),X^1)$, too.

(ii) We assume that $u^1(x)$ and $u^2(x)$ are bounded solutions of (1.3). At first, we show that u^1 and u^2 are also solutions of

$$
\begin{aligned}
u^1(x) =& e^{-A_-(x-x_0)}P_-u^1(x_0) + \int_{x_0}^{x} e^{-A_-(x-\sigma)}P_-B(\sigma)u^1(\sigma)d\sigma\\
&- \int_{x}^{\infty} e^{A_+(x-\sigma)}P_+B(\sigma)u^1(\sigma)d\sigma, \quad x \geq x_0,\\
u^2(x) =& e^{-A_-x}P_-u^2(0) + e^{A_+(x-x_0)}P_+u^2(x_0) + \int_{x_0}^{x} e^{A_+(t-\sigma)}P_+B(\sigma)u^2(\sigma)d\sigma\\
&+ \int_{0}^{x} e^{-A_-(x-\sigma)}P_-B(\sigma)u^2(\sigma)d\sigma, \quad 0 \leq x \leq x_0.
\end{aligned}
\tag{1.7}
$$

Defining

$$
v_1(x) := e^{-A_-(x-x_0)}P_-u^1(x_0) + \int_{x_0}^{x} e^{-A_-(x-\sigma)}P_-B(\sigma)u^1(\sigma)d\sigma - \int_{x}^{\infty} e^{A_+(x-\sigma)}P_+B(\sigma)u^1(\sigma)d\sigma
$$

yields

$$
\begin{aligned}
\frac{\partial}{\partial x}v_1(x) =& -A_-e^{-A_-(x-x_0)}P_-u^1(x_0) + P_-B(x)u^1(x) - \int_{x_0}^{x} A_-e^{-A_-(x-\sigma)}P_-B(\sigma)u^1(\sigma)d\sigma\\
&+ P_+B(x)u^1(x) - \int_{x}^{\infty} A_+e^{A_+(x-\sigma)}P_+B(\sigma)u^1(\sigma)d\sigma\\
=& Ae^{-A_-(x-x_0)}P_-u^1(x_0) + B(x)u^1(x) + A\int_{x_0}^{x} e^{-A_-(x-\sigma)}P_-B(\sigma)u^1(\sigma)d\sigma\\
&- A\int_{x}^{\infty} e^{A_+(x-\sigma)}P_+B(\sigma)u^1(\sigma)d\sigma\\
=& Av_1(x) + B(x)u^1(x).
\end{aligned}
$$

Because of $\frac{\partial}{\partial x}u^1(x) = Au^1(x) + B(x)u^1(x)$ we obtain $\frac{\partial}{\partial x}(u^1 - v_1)(x) = A(u^1 - v_1)(x)$. Thus $\eta := u^1 - v_1$ is a strong solution of $\frac{\partial}{\partial x}\eta = A\eta$ with $P_+\eta(x_0) = \eta(x_0)$. Corollary 1.1.7 results in $\eta = 0$ on $\mathbb{R}^+$. Therefore, u^1 is a solution of (1.7). The proof for u^2 is similar.

Now we set $w = u^1(x_0) + u^2(x_0)$, $u^s(x,x_0) = u^1(x)$ and $u^u(x,x_0) = u^2(x)$. Considering $u^s(x_0,x_0) = w - u^u(x_0,x_0)$ we obtain from (1.7)

$$
\begin{aligned}
&u^s(x,x_0) - \int_{x_0}^{x} e^{-A_-(x-\sigma)}P_-B(\sigma)u^s(\sigma,x_0)d\sigma + \int_{x}^{\infty} e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma,x_0)d\sigma\\
&+ e^{-A_-(x-x_0)}P_-u^u(x_0,x_0) = e^{-A_-(x-x_0)}P_-w,\\
&u^u(x,x_0) - \int_{x_0}^{x} e^{A_+(x-\sigma)}P_+B(\sigma)u^u(\sigma,x_0)d\sigma + \int_{x}^{0} e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma,x_0)d\sigma\\
&- e^{-A_-x}P_-u^u(0,x_0) + e^{A_+(x-x_0)}P_+u^s(x_0,x_0) = e^{A_+(x-x_0)}P_+w.
\end{aligned}
\tag{1.8}
$$

It follows from the equations[8]

$$\int_0^{x_0} e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma,x_0)d\sigma = e^{-A_-(x-x_0)}P_-u^u(x_0,x_0) - e^{-A_-x}P_-u^u(0,x_0),$$
$$\int_{x_0}^{\infty} e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma,x_0)d\sigma = -e^{A_+(x-x_0)}P_+u^s(x_0,x_0)$$

that (1.8) is the mild formulation (1.6). ∎

Construction of the stable eigenspace

Now we determine the initial values for which we obtain bounded solutions of (1.3) on $\mathbb{R}^+$. Therefore, we set $x_0 = 0$ in the integral equations. In the following steps we omit $x_0 = 0$ in u^s and u^u. As we analyse the case of initial values with $u^s(0;w) = w$ we set $u^u(0) = 0$. Considering this in the mild formulation (1.6) yields

$$\begin{aligned} \tilde{\varphi}_0 z &= \tilde{T}_0 x^s, \\ P_+ w &= -\int_0^\infty e^{-A_+\sigma}P_+B(\sigma)u^s(\sigma)d\sigma, \end{aligned} \tag{1.9}$$

where

$$(\tilde{T}_0u^s)(x) := u^s(x) - \int_0^x e^{-A_-(x-\sigma)}P_-B(\sigma)u^s(\sigma)d\sigma + \int_x^\infty e^{A_+(x-\sigma)}P_+B(\sigma)x^s(\sigma)d\sigma, \quad x \geq 0,$$
$$(\tilde{\varphi}_0 w)(x) := e^{-A_-x}P_-w, \quad x \geq 0.$$

The following definition deals with the spaces in which we will solve (1.9).

Definition 1.3.4 *For a fixed constant $\eta \in [0,\delta)$ and $x_0 \geq 0$ we set*

$$\begin{aligned} \mathscr{X}^s_{x_0} &:= \left\{ u \in C^0([x_0,\infty), X^\alpha) \; : \; ||u||_{\mathscr{X}^s_{x_0}} := \sup_{x \geq x_0} e^{\eta|x-x_0|}||u(x)||_{X^\alpha} < \infty \right\}, \\ \mathscr{X}^u_{x_0} &:= \left\{ u \in C^0([0,x_0], X^\alpha) \; : \; ||u||_{\mathscr{X}^u_{x_0}} := \sup_{0 \leq x \leq x_0} e^{\eta|x-x_0|}||u(x)||_{X^\alpha} < \infty \right\}, \end{aligned} \tag{1.10}$$

where $||\cdot||_{\mathscr{X}^s_{x_0}}$ and $||\cdot||_{\mathscr{X}^u_{x_0}}$ are norms. We also set $\mathscr{X}_{x_0} := \mathscr{X}^s_{x_0} \oplus \mathscr{X}^u_{x_0}$.

Lemma 1.3.5 *$\tilde{\varphi}_0 : X^\alpha \to \mathscr{X}^s_0$ is bounded and $R(\tilde{\varphi}_0)$ is closed.*

Proof Let $w \in X^\alpha$, then

$$\begin{aligned} ||\tilde{\varphi}_0 w||_{\mathscr{X}^s_0} &= \sup_{x\geq 0}\{e^{\eta x}||e^{-A_-x}P_-w||_{X^\alpha}\} = \sup_{x \geq 0}\{e^{\eta x}(||e^{-A_-x}P_-w||_{X^\alpha_+} + ||e^{-A_-x}P_-w||_{X^\alpha_-})\} \\ &= \sup_{x\geq 0}\{e^{\eta x}||A^\alpha_- e^{-A_-x}P_-w||_{X_-}\} = \sup_{x\geq 0}\{e^{\eta x}||e^{-A_-x}A^\alpha_- P_-w||_{X_-}\} \end{aligned}$$

[8]Antiderivatives of the integrands are given by $e^{-A_-(x-\sigma)}P_-u^u(\sigma,x_0)$ and $e^{A_+(x-\sigma)}P_+u^s(\sigma,x_0)$, respectively.

$$\leq \sup_{x\geq 0}\{e^{\eta x}||e^{-A_- x}||_{L[X_-]}||A_-^\alpha P_- w||_{X_-}\} \leq \sup_{x\geq 0}\{e^{\eta x}Ce^{-\delta x}||P_- w||_{X_-^\alpha}\}$$
$$= C\sup_{x\geq 0}\{e^{-x(\delta-\eta)}||P_- w||_{X^\alpha}\} \leq C||P_-||_{L[X^\alpha]}||w||_{X^\alpha} \leq C||w||_{X^\alpha}.$$

Now we show the closedness of $R(\tilde{\varphi}_0)$. Let $(v_n)_{n\in\mathbb{N}} \subset R(\tilde{\varphi}_0)$ converge to $v \in \mathscr{X}_0^s$. We must prove $v \in R(\tilde{\varphi}_0)$. Because of $v_n \in R(\tilde{\varphi}_0)$ there exists $w_n \in X^\alpha$ with $v_n(x) = e^{-A_- x}P_- w_n$ for each $n \in \mathbb{N}$. Moreover,

$$v(x) = \lim_{n\to\infty} e^{-A_- x}P_- w_n \quad \forall x \in [0,\infty).$$

In particular, there exists $\hat{w} := v(0) = \lim_{n\to\infty} P_- w_n \in X^\alpha$. Consider $P_- \in L[X^\alpha]$. Since $e^{-A_- x}$ is a continuous operator, we obtain

$$v(x) = e^{-A_- x}\lim_{n\to\infty} P_- w_n = e^{-A_- x}P_-\hat{w} = (\tilde{\varphi}_0\hat{w})(x) \quad \forall x \in [0,\infty). \quad \blacksquare$$

For the following lemma recall the definition of a Fredholm operator and confer Definition A.2.15

Lemma 1.3.6 *$\tilde{T}_0$ is an element of $L[\mathscr{X}_0^s]$ and Fredholm with index zero.*

Proof We can write $\tilde{T}_0 = \mathrm{id} + I_1 + I_2$ where I_1 and I_2 are the integral operators

$$(I_1 u^s)(x) = -\int_0^x e^{-A_-(x-\sigma)}P_- B(\sigma)u^s(\sigma)d\sigma,$$
$$(I_2 u^s)(x) = \int_x^\infty e^{A_+(x-\sigma)}P_+ B(\sigma)u^s(\sigma)d\sigma.$$

We will now show that $I_j = S_j + K_j$ for $j = 1,2$ such that $||S_j||_{L[\mathscr{X}_0^s]} < 1/4$ and K_j is compact for $j = 1,2$. It follows that $\mathrm{id} + S_1 + S_2$ is invertible, and hence Fredholm with index zero. If we add the compact operators K_1 and K_2 the Fredholm property is preserved with the same index, see Theorem A.2.16. Along the way we obtain $\tilde{T}_0 \in L[\mathscr{X}_0^s]$.

We decompose $I_1 = S_1 + K_1$ with

$$(K_1 u^s)(x) = \begin{cases} -\int_0^x e^{-A_-(x-\sigma)}P_- B(\sigma)u^s(\sigma)d\sigma, & x \leq x^*, \\ -e^{-A_-(x-x^*)}\int_0^{x^*} e^{-A_-(x^*-\sigma)}P_- B(\sigma)u^s(\sigma)d\sigma, & x \geq x^*, \end{cases}$$

$$(S_1 u^s)(x) = \begin{cases} 0, & x \leq x^*, \\ -\int_{x^*}^x e^{-A_-(x-\sigma)}P_- B(\sigma)u^s(\sigma)d\sigma, & x \geq x^*, \end{cases}$$

for any $x^* \geq 0$. The operators S_1 and K_1 map $\mathscr{X}_0^s$ into itself because $S_1 u^s$ and $K_1 u^s$ are continuous at $x = x^*$. Due to hypothesis **(H2)** and Lemma A.3.9 we have

$$
\begin{aligned}
&||S_1 u^s||_{\mathscr{X}_0^s} = \sup_{x\geq 0}\{e^{\eta x}||S_1 u^s(x)||_{X^\alpha}\}\\
&\leq \sup_{x\geq x^*}\left\{e^{\eta x}\int_{x^*}^{x}\left|\left|e^{-A_-(x-\sigma)}P_-B(\sigma)u^s(\sigma)\right|\right|_{X^\alpha}d\sigma\right\}\\
&\leq C\sup_{x\geq x^*}\left\{e^{\eta x}\int_{x^*}^{x}\left|\left|A_-^\alpha e^{-A_-(x-\sigma)}\right|\right|_{L[X_-]}\sup_{\tilde{x}\geq x^*}\left\{||B(\tilde{x})||_{L[X^\alpha,X]}\right\}||u^s(\sigma)||_{X^\alpha}d\sigma\right\}\\
&\leq C\sup_{x\geq x^*}\left\{e^{\eta x}\int_{x^*}^{x}(x-\sigma)^{-\alpha}e^{-\delta(x-\sigma)}e^{-\eta\sigma}||u^s||_{\mathscr{X}_0^s}d\sigma\right\}\sup_{\tilde{x}\geq x^*}\left\{||B(\tilde{x})||_{L[X^\alpha,X]}\right\} \qquad (1.11)\\
&\leq C\sup_{x\geq x^*}\left\{\int_{x^*}^{x}(x-\sigma)^{-\alpha}e^{-(x-\sigma)(\delta-\eta)}d\sigma\right\}\sup_{\tilde{x}\geq x^*}\left\{||B(\tilde{x})||_{L[X^\alpha,X]}\right\}||u^s||_{\mathscr{X}_0^s}\\
&\leq C\sup_{x\geq x^*}\left\{\int_{0}^{(x-x^*)(\delta-\eta)}t^{-\alpha}e^{-t}(\delta-\eta)^{\alpha-1}dt\right\}\sup_{\tilde{x}\geq x^*}\left\{||B(\tilde{x})||_{L[X^\alpha,X]}\right\}||u^s||_{\mathscr{X}_0^s}\\
&\leq C\,\Gamma(1-\alpha)\sup_{\tilde{x}\geq x^*}\left\{||B(\tilde{x})||_{L[X^\alpha,X]}\right\}||u^s||_{\mathscr{X}_0^s}.
\end{aligned}
$$

For large x^* and $\varepsilon > 0$ small enough we have $||S_1||_{L[\mathscr{X}_0^s]} < \frac{1}{4}$. Here, ε_0 of the roughness theorem is, inter alia, specified.

Next, we show the compactness of K_1. First, we restrict $K_1 u^s$ to the interval $[0, x^*]$ and $K_1|_{[0,x^*]}$ denotes the restriction. The proof depends on whether hypothesis **(H3)** or **(H4)** is met. We only handle the case of Hypothesis **(H3)**. For **(H4)** confer [19].
First, we show that $K_1|_{[0,x^*]}$ maps $\mathscr{X}_0^s$ continuously into $C^{0,\kappa}([0,x^*], X^{\alpha+\kappa})$ for some small $\kappa > 0$. Because of Lemma A.3.9 and estimates similar to those in (1.11) $(K_1u^s)(x)$ exists in $X^{\alpha+\kappa}$ for all $0 \leq x \leq x^*$ and for some small $\kappa > 0$. The Hölder continuity is also a consequence of Lemma A.3.9:

$$
\begin{aligned}
&||(K_1u)^s(x) - (K_1u)^s(\xi)||_{X^{\alpha+\kappa}}\\
&= \left|\left|-\int_0^x e^{-A_-(x-\sigma)}P_-B(\sigma)u^s(\sigma)d\sigma + \int_0^\xi e^{-A_-(\xi-\sigma)}P_-B(\sigma)u^s(\sigma)d\sigma\right|\right|_{X_-^{\alpha+\kappa}}\\
&\leq \left|\left|\int_\xi^x e^{-A_-(x-\sigma)}P_-B(\sigma)u^s(\sigma)d\sigma\right|\right|_{X_-^{\alpha+\kappa}} + \left|\left|\int_0^\xi \left(e^{-A_-(\xi-\sigma)} - e^{-A_-(x-\sigma)}\right)P_-B(\sigma)u^s(\sigma)d\sigma\right|\right|_{X_-^{\alpha+\kappa}}\\
&\leq C\int_\xi^x (x-\sigma)^{-(\alpha+\kappa)}d\sigma + \left|\left|\int_0^\xi \left(\mathrm{id} - e^{-A_-(x-\xi)}\right)e^{-A_-(\xi-\sigma)}P_-B(\sigma)u^s(\sigma)d\sigma\right|\right|_{X_-^{\alpha+\kappa}}\\
&\leq \left[\frac{-1}{1-\alpha}(x-\sigma)^{1-(\alpha+\kappa)}\right]_\xi^x + \int_0^\xi \left|\left|\left(\mathrm{id} - e^{-A_-(x-\xi)}\right)A_-^{\alpha+\kappa}e^{-A_-(\xi-\sigma)}P_-B(\sigma)u^s(\sigma)\right|\right|_{X_-}d\sigma\\
&\leq C(x-\xi)^{1-(\alpha+\kappa)} + C(x-\xi)^\kappa\int_0^\xi \left|\left|A_-^{\alpha+\kappa}e^{-A_-(\xi-\sigma)}P_-B(\sigma)u^s(\sigma)\right|\right|_{X_-}d\sigma\\
&\leq C(x-\xi)^{1-(\alpha+\kappa)} + C(x-\xi)^\kappa\int_0^\xi (\xi-\sigma)^{-(\alpha+\kappa)}d\sigma\\
&\leq C(x-\xi)^\kappa
\end{aligned}
$$

for $0 \leq \xi \leq x$, $|x-\xi| \leq 1$ and for some small $\kappa > 0$.

Hereupon we prove that the canonical inclusion $X^{\alpha+\kappa} \hookrightarrow X^{\alpha}$ is compact:
Due to **(H1)** we have $0 \in \rho(A_+)$ and so it exists A_+^{-1} on X_+. Moreover, $A^{-1} = A_+^{-1}P_+$ on X_+ and A_+^{-1} is compact on X_+:
Let $(u_n)_{n\in\mathbb{N}} \subset X_+$ be bounded. Since $A_+^{-1}u_n = A_+^{-1}P_+u_n = A^{-1}u_n$ and A^{-1} is compact the sequence $(A_+^{-1}u_n)_{n\in\mathbb{N}}$ has a convergent subsequence. In the same way one shows the compactness of A_-^{-1} on X_-. Finally, the compactness of the above inclusion follows from the Theorems A.3.10 and A.3.13.

Now we show that the compactness of $X^{\alpha+\kappa} \hookrightarrow X^{\alpha}$ and Arzela's Theorem A.2.7 result in the compact embedding $C^{0,\kappa}([0,x^*],X^{\alpha+\kappa}) \hookrightarrow C^0([0,x^*],X^{\alpha})$:

Let $\mathcal{F} \subset C^{0,\kappa}([0,x^*],X^{\alpha+\kappa})$ be bounded, i.e.

$$\sup_{f\in\mathcal{F}} ||f||_{C^{0,\alpha+\kappa}([0,x^*],X^{\alpha+\kappa})} \overset{(*)}{=} \sup_{f\in\mathcal{F}} \left\{ \sum_{|s|\le 0} ||\partial^s f||_{C^0([0,x^*],X^{\alpha+\kappa})} + \sum_{|s|=0} \text{Höl}_{\alpha+\kappa}(\partial^s f,[0,x^*]) \right\} < \infty.$$

Therefore, $\{f(x)|f \in \mathcal{F}\}$ is bounded in $X^{\alpha+\kappa}$ for all $x \in [0,x^*]$. Because $X^{\alpha+\kappa}$ is compactly embedded in X^{α} the set $\{f(x)|f \in \mathcal{F}\}$ is relatively compact in X^{α} for all $x \in [0,x^*]$. The equicontinuity of $\mathcal{F}$ is a consequence of

$$||f(x)-f(y)||_{X^\alpha} \overset{X^{\alpha+\kappa}\overset{\text{cont.}}{\hookrightarrow} X^{\alpha}}{\le} C||f(x)-f(y)||_{X^{\alpha+\kappa}} \overset{\text{see }(*)}{\le} C\sup_{f\in\mathcal{F}} ||f||_{C^{0,\alpha+\kappa}([0,x^*],X^{\alpha+\kappa})}|x-y|^{\alpha-\kappa}.$$

Considering

$$K_1|_{[0,x^*]} : \mathscr{X}_0^s \overset{\text{bounded}}{\longrightarrow} C^{0,\kappa}([0,x^*],X^{\alpha+\kappa}) \overset{\text{compact}}{\hookrightarrow} C^0([0,x^*],X^{\alpha})$$

we see that $K_1|_{[0,x^*]} : \mathscr{X}_0^s \to \mathscr{X}_0^s$ is compact.

Finally, we define the bounded operator

$$\begin{cases} \text{id} & 0 \le x \le x^*, \\ e^{-A_-(x-x^*)}P_- & x^* \le x. \end{cases} \tag{1.12}$$

Since K_1 is the composition of $K_1|_{[0,x^*]}$ with the multiplication operator (1.12) we obtain the compactness of K_1. ∎

The following subspace includes all initial values which lead to a bounded solution on $\mathbb{R}^+$.

Definition 1.3.7

$$E^s := (\tilde{T}_0^{-1}(R(\tilde{\varphi}_0)))(0) = \{w \in X^{\alpha} \,:\, \exists u^s \in \mathscr{X}_0^s \textit{ with } u^s(0) = w \textit{ and } \tilde{T}_0 u^s = \tilde{\varphi}_0 w\}.$$

This subspace is closed as $\tilde{T}_0$ is Fredholm and $R(\tilde{\varphi}_0)$ is closed.

Lemma 1.3.8

$$\dim N\left(P_-|_{E_s}\right) = \dim N(\tilde{T}_0) = \operatorname{codim} R(\tilde{T}_0) = \operatorname{codim}_{X_-^\alpha}(P_- E^s) = k^s$$

for some $k^s < \infty$*.*

Proof At the beginning we prove $\dim N\left(P_-|_{E_s}\right) = \dim N(\tilde{T}_0)$. Define the linear map

$$F : N(\tilde{T}_0) \to N\left(P_-|_{E^s}\right), \quad u^s(\cdot) \mapsto u^s(0).$$

F is well-defined, i.e. $Fu^s = u^s(0) \in N(P_-|_{E^s})$ for all $u^s \in N(\tilde{T}_0)$:

Let $u^s \in N(\tilde{T}_0)$, then it follows from $\tilde{T}_0 u^s = 0$:

$$u^s(0) = -\int_0^\infty e^{-A_+\sigma} P_+ B(\sigma) u^s(\sigma) d\sigma \quad \Rightarrow \quad P_- u^s(0) = 0.$$

This results in $\tilde{T}_0 u^s = 0 = e^{-A_- \cdot} P_- u^s(0) = \tilde{\varphi}_0 u^s(0)$ and so $u^s(0) \in N(P_-|_{E^s})$. By the way we obtained another term for Fu^s:

$$Fu^s = u^s(0) = -\int_0^\infty e^{-A_+\sigma} P_+ B(\sigma) u^s(\sigma) d\sigma. \tag{1.13}$$

F is continuous due to $\alpha \in [0,1)$:

$$\begin{aligned}
||Fu^s||_{X^\alpha} &\overset{(1.13)}{=} \left|\left| \int_0^\infty e^{-A_+\sigma} P_+ B(\sigma) u^s(\sigma) d\sigma \right|\right|_{X^\alpha} \\
&\le \int_0^\infty \left|\left| A_+^\alpha e^{-A_+\sigma} \right|\right|_{L[X]} ||P_+||_{L[X]} ||B(\sigma)||_{L[X^\alpha, X]} e^{-\eta\sigma} ||e^{\eta\sigma} u^s(\sigma)||_{X^\alpha} d\sigma \\
&\le \int_0^\infty C\sigma^{-\alpha} e^{-(\delta+\eta)\sigma} ||u^s||_{\mathscr{X}_0^s} d\sigma \\
&\le C||u^s||_{\mathscr{X}_0^s}.
\end{aligned}$$

F is injective:

$$Fu^s = u^s(0) = 0 \overset{\textbf{(H5)}}{\Rightarrow} u^s = 0 \Rightarrow N(F) = \{0\}.$$

F is surjective:

Choose $w \in E^s = (\tilde{T}_0^{-1}(R(\tilde{\varphi}_0)))(0) = \{w \in X^\alpha : \exists u^s \in \mathscr{X}_0^s \text{ with } u^s(0) = w \text{ and } \tilde{T}_0 u^s = \tilde{\varphi}_0 w\}$ with $P_- w = 0$. According to the construction of E^s there is a $u^s \in \mathscr{X}_0^s$ with $u^s(0) = w$ and $\tilde{T}_0 u^s = \tilde{\varphi}_0 w$. Therefore,

$$P_- w = 0 \Rightarrow \tilde{\varphi}_0 w = 0 \Rightarrow \tilde{T}_0 u^s = 0 \Rightarrow u^s \in N(\tilde{T}_0) \Rightarrow Fu^s = w.$$

We can now conclude that there is an isomorphism between $N(\tilde{T}_0)$ and $N\left(P_-|_{E_s}\right)$. So the first equation is valid.

The second equation $\dim N(\tilde{T}_0) = \operatorname{codim} R(\tilde{T}_0)$ is valid because $\tilde{T}_0$ is a Fredholm operator with index zero.

To verify the last equation $\operatorname{codim} R(\tilde{T}_0) = \operatorname{codim}_{X_-^\alpha}(P_-E^s)$ one chooses a complement V_- of P_-E^s in X_-^α, i.e.

$$X_-^\alpha = V_- \oplus P_-E^s.$$

Because of the construction we obtain $\tilde{\varphi}_0 w \notin R(\tilde{T}_0)$ for all $w \in V_-$. Hereupon we define $G : V_- \to \mathscr{X}_0^s$ by $w \mapsto \tilde{\varphi}_0 w = e^{-A_- x} P_- w$. This map is injective:
Let w_1 and w_2 be arbitrary elements of V_- with $G(w_1) = G(w_2)$, i.e. $e^{-A_- x} P_- w_1 = e^{-A_- x} P_- w_2$. For $x = 0$ it follows $P_- w_1 = P_- w_2$ and finally $w_1 = w_2$ because of $w_1, w_2 \in X_-^\alpha \subset R(P_-)$.
As G maps the complement V_- of P_-E^s in X_-^α one-to-one into a complement of $R(\tilde{T}_0)$ in $\mathscr{X}_0^s$ we obtain

$$\operatorname{codim}_{X_-^\alpha}(P_-E^s) \leq \operatorname{codim} R(\tilde{T}_0) = k. \tag{1.14}$$

In the following we consider the adjoint equation

$$\frac{\partial}{\partial x}\xi = -(A' + B(x)')\xi, \quad \xi \in (X')^\alpha. \tag{1.15}$$

The previous results apply also to the adjoint equation. Let ξ and u be arbitrary solutions of (1.15) and $\frac{\partial}{\partial x}u = (A + B(x))u$, respectively. Then[9]

$$\begin{aligned}\frac{\partial}{\partial x}\langle \xi(x), u(x)\rangle &= \left\langle \frac{\partial}{\partial x}\xi(x), u(x)\right\rangle + \left\langle \xi(x), \frac{\partial}{\partial x}u(x)\right\rangle \\ &= \langle -(A' + B(x)')\xi(x), u(x)\rangle + \langle \xi(x), (A + B(x))u(x)\rangle \\ &= 0.\end{aligned} \tag{1.16}$$

We now claim that any bounded solution ξ of (1.15) satisfies $\langle \xi(0), w\rangle = 0$ for all $w \in E^s$: Because of $w \in E^s$ there exists a $u^s \in \mathscr{X}_0^s$ with $u^s(0) = w$ and $\tilde{T}_0 u^s = \tilde{\varphi}_0 w$. It follows from (1.16) that $\langle \xi(x), u^s(x)\rangle$ is constant. Moreover,

$$\begin{aligned}|\langle \xi(0), w\rangle| = |\langle \xi(0), u^s(0)\rangle| = |\langle \xi(x), u^s(x)\rangle| &\overset{(*)}{\leq} \|\xi(x)\|_{L[X^\alpha,\mathbb{K}]}\, \|u^s(x)\|_{X^\alpha} \leq C\|u^s(x)\|_{X^\alpha} \\ &\leq Ce^{-\eta x}\|u^s\|_{\mathscr{X}_0^s} \to 0\end{aligned}$$

for $x \to \infty$. For $(*)$ consider that $\xi(x)$ is a bounded linear functional.

We define E_*^s as the subspace of $(X')^\alpha$ which consists of initial values $\xi(0)$ of bounded solutions for (1.15). We can apply the previous arguments to the adjoint equation and write $(X')^\alpha = (X')_+^\alpha \oplus (X')_-^\alpha$. Using above arguments again yields

$$\infty > \dim N(P'_+|_{E_*^s}) = k^* \geq \operatorname{codim}_{(X')_+^\alpha}\left(P'_+ E_*^s\right).$$

Since we have proven above that E_*^s annihilates E^s, we can conclude

$$\begin{aligned}k^* = \dim N(P'_+|_{E_*^s}) &\leq \dim N\left(P'_+|_{\text{Annih.}(E^s)}\right) \qquad (\text{consider } E_*^s \subset \text{Annih.}(E^s)) \\ &\overset{(\star)}{=} \dim\{(\xi_-, 0) \in (X')_-^\alpha \oplus (X')_+^\alpha \;:\; \langle \xi_-, w_-\rangle = 0\, \forall w_- \in P_-E^s\} \\ &\overset{(\dagger)}{=} \operatorname{codim}_{X_-^\alpha}(P_-E^s) \leq k.\end{aligned}$$

[9] $\langle \cdot, \cdot \rangle$ denotes the dual pair, see Definition A.2.8.

For $(\star)$ consider that no elements of the kernel of P'_+ has a contribution in the $(X')^\alpha_+$-direction. The requirement $\langle \xi_-, w_- \rangle = 0$ for all $w_- \in P_- E^s$ is sufficient because any P_+-direction of an element of E^s will be annihilated anyway by an element of $(X')^\alpha_-$. $(\dagger)$ is a consequence of the Hahn-Banach theorem, see Theorem A.2.14:
As P_- is a projection and E^s is closed $P_- E^s$ is also closed. Furthermore, $\operatorname{codim}_{X^\alpha_-}(P_- E^s) \leq k$ due to (1.14). Therefore, $X^\alpha_- / P_- E^s$ is finite-dimensional and its dimension equals the dimension of $(X^\alpha_- / P_- E^s)'$. Finally, Theorem A.2.14 leads to $(\star)$.

If we employ the same argument for the adjoint system and make use of the reflexivity of X, we obtain

$$\begin{aligned} k^{**} = \dim N\left(P''_-|_{E^s_{**}}\right) &= k = \dim N\left(P_-|_{E^s}\right) \\ k = k^{**} \leq \operatorname{codim}_{(X^*)^\alpha_+}\left(P''_+ E^s_*\right) &\leq k^* \leq k. \end{aligned}$$

We have strict inequality if and only if $\dim N(P_-|_{E^s}) > \operatorname{codim}_{X^\alpha_-}(P_- E^s)$. ∎

Existence of $u^s(\cdot; x_0, w)$ and $u^u(\cdot; x_0, w)$ for fixed x_0

To construct solutions $u^s(\cdot; x_0, w)$ and $u^u(\cdot; x_0, w)$ for fixed x_0 we have to include a fixed complement E^u of the stable subspace E^s. Therefore, choose any closed complement E^u of E^s in X^α with

$$\operatorname{codim}_{X^\alpha_+}(P_+ E^u) = \dim N(P_+|_{E^u}) = k^u < \infty. \tag{1.17}$$

We outline the following exemplary construction of E^u:
Because of $\operatorname{codim}_{X^\alpha_-}(P_- E^s) < \infty$ and $\dim N(P_-|_{E^s}) < \infty$, see Lemma 1.3.8, we can choose closed complements E^u_- of $P_- E^s$ in X^α_- and E^u_+ of $N(P_-|_{E^s})$ in X^α_+:

$$E^u_- \oplus P_- E^s = X^\alpha_-, \quad E^u_+ \oplus N(P_-|_{E^s}) = X^\alpha_+. \tag{1.18}$$

$E^u = E^u_- \oplus E^u_+ \subset X^\alpha_- \oplus X^\alpha_+$ is then a complement of E^s in X^α satisfying (1.17) with $k^u = k^s$ where k^s appears in Lemma 1.3.8. This can be shown by

$$\begin{aligned} \operatorname{codim}_{X^\alpha_+} P_+ E^u = \operatorname{codim}_{X^\alpha_+} E^u_+ \overset{(1.18)}{=} \dim N(P_-|_{E^s}) \overset{(\text{Lemma 1.3.8})}{=} k^s = \operatorname{codim}_{X^\alpha_-} P_- E^s \overset{(1.18)}{=} \dim E^u_- \\ = \dim R(P_-|_{E^u_-}) = \dim N(P_+|_{E^u_-}) = \dim N(P_+|_{E^u}) = k^u < \infty. \end{aligned}$$

Other complements can also be considered, confer [19].

Definition 1.3.9 *Let E be a closed subspace of X^α. One defines*

$$\mathscr{X}^E_{x_0} := \{(u^s, u^u) \in \mathscr{X}^s_{x_0} \oplus \mathscr{X}^u_{x_0} \; : \; u^u(0) \in E\}.$$

Remark 1.3.10 *$\mathscr{X}^E_{x_0}$ is a closed subspace of $\mathscr{X}^s_{x_0} \oplus \mathscr{X}^u_{x_0}$.*

Definition 1.3.11 *For fixed $x_0 \geq 0$ let*

$$
\begin{aligned}
(T_{x_0}u)^s(x) :=& u^s(x) + e^{-A_-x}P_-u^u(0) + \int_x^\infty e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma)d\sigma \qquad x \geq x_0 \\
&- \int_{x_0}^x e^{-A_-(x-\sigma)}P_-B(\sigma)u^s(\sigma)d\sigma + \int_0^{x_0} e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma)d\sigma, \\
(T_{x_0}u)^u(x) :=& u^u(x) - e^{-A_-x}P_-u^u(0) - \int_{x_0}^x e^{A_+(x-\sigma)}P_+B(\sigma)u^u(\sigma)d\sigma \qquad x_0 \geq x \geq 0 \\
&+ \int_x^0 e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma)d\sigma - \int_{x_0}^\infty e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma)d\sigma
\end{aligned}
$$

and

$$
\begin{aligned}
(\varphi_{x_0}w)^s(x) &:= e^{-A_-(x-x_0)}P_-w, \quad x \geq x_0 \geq 0, \quad w \in X^\alpha, \\
(\varphi_{x_0}w)^u(x) &:= e^{A_+(x-x_0)}P_+w, \quad x_0 \geq x \geq 0, \quad w \in X^\alpha.
\end{aligned}
$$

Lemma 1.3.12 $\varphi_{x_0} \in L[X^\alpha, \mathscr{X}_{x_0}^{X_+}]$ *holds for any $x_0 \geq 0$ and with bound independent of x_0.*

Proof See proof of Proposition 1.3.13. ∎

Proposition 1.3.13 *If T_{x_0} is considered as a map $T_{x_0} : \mathscr{X}_{x_0}^{E^u} \to \mathscr{X}_{x_0}^{X_+}$ the operator T_{x_0} is an isomorphism for any fixed $x_0 \geq 0$. Moreover, the bound of T_{x_0} is independent of x_0.*

Proof At the beginning we show T_{x_0} is well-defined, an element of $L[\mathscr{X}_{x_0}^{E^u}, \mathscr{X}_{x_0}^{X_+}]$ and bounded independently of x_0:

$$
\begin{aligned}
(T_{x_0}u)^u(0) &= u^u(0) - P_-u^u(0) - \int_{x_0}^0 e^{-A_+\sigma}P_+B(\sigma)u^u(\sigma)d\sigma - \int_{x_0}^\infty e^{-A_+\sigma}P_+B(\sigma)u^s(\sigma)d\sigma \\
&= P_+u^u(0) - \int_{x_0}^0 e^{-A_+\sigma}P_+B(\sigma)u^u(\sigma)d\sigma - \int_{x_0}^\infty e^{-A_+\sigma}P_+B(\sigma)u^s(\sigma)d\sigma
\end{aligned}
$$

is an element of X_+. Furthermore,

$$
||T_{x_0}u||_{\mathscr{X}_{x_0}^{X_+}} = ||(T_{x_0}u)^s||_{\mathscr{X}_{x_0}^s} + ||(T_{x_0}u)^u||_{\mathscr{X}_{x_0}^u}
$$

Consider $||(T_{x_0}u)^s||_{\mathscr{X}_{x_0}^s}$:

$$
\begin{aligned}
&||(T_{x_0}u)^s||_{\mathscr{X}_{x_0}^s} \\
&= \sup_{x \geq x_0} \left\{ e^{\eta(x-x_0)}||(T_{x_0}u)^s(x)||_{X^\alpha} \right\} \\
&\leq ||u^s||_{\mathscr{X}_{x_0}^s} + \sup_{x \geq x_0} \Bigg\{ e^{\eta(x-x_0)} \Bigg(\left|\left|e^{-A_-x}P_-u^u(0)\right|\right|_{X^\alpha} + \int_x^\infty \left|\left|e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma)\right|\right|_{X^\alpha} d\sigma \\
&\quad + \int_{x_0}^x \left|\left|e^{-A_-(x-\sigma)}P_-B(\sigma)u^s(\sigma)\right|\right|_{X^\alpha} d\sigma + \int_0^{x_0} \left|\left|e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma)\right|\right|_{X^\alpha} d\sigma \Bigg) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq ||u^s||_{\mathscr{X}^s_{x_0}} + C||u^u||_{\mathscr{X}^u_{x_0}} + \sup_{x\geq x_0}\Big\{ e^{\eta(x-x_0)}\int_x^\infty \Big|\Big|A_+^\alpha e^{-A_+(\sigma-x)}\Big|\Big|_{L[X_+]} Ce^{-\eta(\sigma-x_0)}||u^s||_{\mathscr{X}^s_{x_0}}d\sigma \\
&\quad + e^{\eta(x-x_0)}\int_{x_0}^x \Big|\Big|A_-^\alpha e^{-A_-(x-\sigma)}\Big|\Big|_{L[X_-]} Ce^{-\eta(\sigma-x_0)}||u^s||_{\mathscr{X}^s_{x_0}}d\sigma \\
&\quad + e^{\eta(x-x_0)}\int_0^{x_0} \Big|\Big|A_-^\alpha e^{-A_-(x-\sigma)}\Big|\Big|_{L[X_-]} Ce^{-\eta(x_0-\sigma)}||u^u||_{\mathscr{X}^u_{x_0}}d\sigma\Big\} \\
&\leq ||u^s||_{\mathscr{X}^s_{x_0}} + C||u^u||_{\mathscr{X}^u_{x_0}} + \sup_{x\geq x_0}\Big\{ e^{\eta x}\int_x^\infty (\sigma-x)^{-\alpha}e^{-\delta(\sigma-x)}Ce^{-\eta\sigma}||u^s||_{\mathscr{X}^s_{x_0}}d\sigma \\
&\quad + e^{\eta x}\int_{x_0}^x (x-\sigma)^{-\alpha}e^{-\delta(x-\sigma)}Ce^{-\eta\sigma}||u^s||_{\mathscr{X}^s_{x_0}}d\sigma \\
&\quad + e^{\eta x}e^{-2\eta x_0}\int_0^{x_0} (x-\sigma)^{-\alpha}e^{-\delta(x-\sigma)}Ce^{\eta\sigma}||u^u||_{\mathscr{X}^u_{x_0}}d\sigma\Big\} \\
&\leq ||u^s||_{\mathscr{X}^s_{x_0}} + C||u^u||_{\mathscr{X}^u_{x_0}} + \sup_{x\geq x_0}\Big\{ \int_x^\infty (\sigma-x)^{-\alpha}e^{-(\sigma-x)(\delta+\eta)}d\sigma\, C||u^s||_{\mathscr{X}^s_{x_0}} \\
&\quad + \int_{x_0}^x (x-\sigma)^{-\alpha}e^{-(x-\sigma)(\delta-\eta)}d\sigma\, C||u^s||_{\mathscr{X}^s_{x_0}} + e^{-x(\delta-\eta)}\int_0^{x_0}(x-\sigma)^{-\alpha}e^{\sigma(\delta-\eta)}d\sigma\, C||u^u||_{\mathscr{X}^u_{x_0}}\Big\} \\
&\leq ||u^s||_{\mathscr{X}^s_{x_0}} + C||u^u||_{\mathscr{X}^u_{x_0}} + C\Gamma(1-\alpha)||u^s||_{\mathscr{X}^s_{x_0}} + C\Gamma(1-\alpha)||u^s||_{\mathscr{X}^s_{x_0}} \\
&\quad + \sup_{x\geq x_0}\Big\{\int_0^{x_0}(x-\sigma)^{-\alpha}e^{-(x-\sigma)(\delta-\eta)}d\sigma\, C||u^u||_{\mathscr{X}^u_{x_0}}\Big\} \\
&\leq ||u^s||_{\mathscr{X}^s_{x_0}} + C||u^u||_{\mathscr{X}^u_{x_0}} + C\Gamma(1-\alpha)||u^s||_{\mathscr{X}^s_{x_0}} + C\Gamma(1-\alpha)||u^s||_{\mathscr{X}^s_{x_0}} + C\Gamma(1-\alpha)||u^u||_{\mathscr{X}^u_{x_0}} \\
&\leq C||(u^s,u^u)||_{\mathscr{X}^{E^u}_{x_0}}.
\end{aligned}
$$

We emphasize that C can be chosen as a constant which does not depend on x_0. In a similar way one can show $||(T_{x_0}u)^u||_{\mathscr{X}^u_{x_0}} \leq C||(u^s,u^u)||_{\mathscr{X}^{E^u}_{x_0}}$ where C is independent of x_0, too. Finally, we can conclude

$$
||T_{x_0}u||_{\mathscr{X}^{X_+}_{x_0}} = ||(T_{x_0}u)^s||_{\mathscr{X}^s_{x_0}} + ||(T_{x_0}u)^u||_{\mathscr{X}^u_{x_0}} \leq C||(u^s,u^u)||_{\mathscr{X}^{E^u}_{x_0}}
$$

with C bound independent of x_0.

Hereupon we prove the statements

(a) $N(T_{x_0}) = \{0\}$,

(b) T_{x_0} is a Fredholm operator with index zero for $B = 0$.

(a) Let $(u^s,u^u) \in N(T_{x_0}) \subset \mathscr{X}^{E^u}_{x_0}$ be arbitrary. Adding the equations $(T_{x_0}u)^s(x_0) = 0$ and $(T_{x_0}u)^u(x_0) = 0$, see Definition 1.3.11, we obtain $u^u(x_0,x_0) = -u^s(x_0,x_0)$. Therefore,

$$
\tilde{u}^s(x,0) = \begin{cases} u^u(x,x_0), & 0 \leq x \leq x_0 \\ -u^s(x,x_0), & x_0 \leq x \leq \infty \end{cases} \tag{1.19}
$$

is continuous. We can show $T_0(\tilde{u}^s,0) = \varphi_0(\tilde{u}^s(0,0)) = \varphi_0(u^u(0,x_0))$ considering the definition of φ_{x_0}:

Because of

$$T_0(\tilde{u}^s,0)(x)$$
$$= \begin{cases} \tilde{u}^s(x,0) + \int_x^\infty e^{A_+(x-\sigma)} P_+ B(\sigma)\tilde{u}^s(\sigma,0)d\sigma - \int_0^x e^{-A_-(x-\sigma)} P_- B(\sigma)\tilde{u}^s(\sigma,0)d\sigma, & x \geq 0, \\ -\int_0^\infty e^{-A_+\sigma} P_+ B(\sigma)\tilde{u}^s(\sigma,0)d\sigma, & x = 0, \end{cases}$$
$$\varphi_0(\tilde{u}^s(0,0))(x) = \varphi_0(u^u(0,x_0))(x)$$
$$= \begin{cases} e^{-A_- x} P_- u^u(0,x_0), & 0 \leq x, \\ P_+ u^u(0,x_0), & x = 0 \end{cases}$$

the corresponding equality follows from $T_{x_0}(u^s,u^u) = 0$, (1.19) and from distinguishing the cases $x \leq x_0$ and $x \geq x_0$.

Due to $(u^s,u^u) \in \mathscr{X}_{x_0}^{E^u}$ we get $\tilde{u}^s(0,0) = u^u(0,x_0) \in E^u$. Moreover, $\tilde{u}^s(0,0) \in E^s$ since $\tilde{u}^s$ is a bounded solution of (1.6) at $x_0 = 0$. That is why $\tilde{u}^s(0,0)$ is zero as E^u is a complement of E^s in X^α. Finally, Hypothesis **(H5)** yields $\tilde{u}^s(x,0) = 0$ for all $x \geq 0$ which implicates $(u^s,u^u) = 0$.

(b) For $B = 0$, $T_{x_0}(u^s,u^u) = (g^s,g^u) \in \mathscr{X}_{x_0}^{X_+}$ yields

$$(T_{x_0}u)^s(x) = u^s(x) + e^{-A_- x} P_- u^u(0) = g^s(x), \quad (T_{x_0}u)^u(x) = u^u(x) - e^{-A_- x} P_- u^u(0) = g^u(x).$$

$$\begin{aligned} \Longrightarrow \quad & P_+ u^s(x,x_0) = P_+ g^s(x,x_0), \quad P_- u^s(x,x_0) = P_- g^s(x,x_0) - e^{-A_- x} P_- u^u(0,x_0), \\ & P_+ u^u(x,x_0) = P_+ g^u(x,x_0), \quad P_- u^u(x,x_0) = P_- g^u(x,x_0) + e^{-A_- x} P_- u^u(0,x_0). \end{aligned} \tag{1.20}$$

Note that the P_+-contributions of (u^s,u^u) and (g^s,g^u) must coincide.
First, analyse the dimension of the kernel of T_{x_0}. So let (g^s,g^u) be zero and find $(u^s,u^u) \in \mathscr{X}_{x_0}^{E^u}$ with $T_{x_0}(u^s,u^u) = 0$. It follows from (1.20)

$$\begin{aligned} & P_+ u^s(x,x_0) = 0, \quad P_- u^s(x,x_0) = -e^{-A_- x} P_- u^u(0,x_0), \\ & P_+ u^u(x,x_0) = 0, \quad P_- u^u(x,x_0) = e^{-A_- x} P_- u^u(0,x_0). \end{aligned}$$

Therefore, for any $u^u(0,x_0) \in E^u$ with $u^u(0,x_0) \in P_- E^u = N(P_+|_{E^u})$ we get a unique solution of (1.20) in $\mathscr{X}_{x_0}^{E^u}$. According to (1.17) $\dim N(P_+|_{E^u}) = k^u < \infty$.
Finally, we consider the dimension of $R(T_{x_0})$ for $B = 0$. We can solve (1.20) for any (g^s,g^u) provided $P_+ g^u(0,x_0) \in P_+ E^u$ which defines a subspace of $\mathscr{X}_{x_0}^{X_+}$ of codimension k^u, see (1.17).

As in Lemma 1.3.6, we can even conclude from (b) that T_{x_0} is Fredholm with index zero for any perturbation B which satisfies **(H2)** for sufficiently small ε. We decompose T_{x_0} according to

$$T_{x_0} = F_{x_0} + K_{x_0},$$

where F_{x_0} is the above contribution of T_{x_0} with $B = 0$ and K_{x_0} consists of the integrals of T_{x_0}. Having proven the compactness of K_{x_0} we see that T_{x_0} stays Fredholm with index zero.

Finally, we conclude that T_{x_0} is onto and one-to-one. By Theorem A.2.17 the operator T_{x_0} is continuously invertible. ∎

Lemma 1.3.14 *Let $(u^s,u^u) \in \mathscr{X}_{x_0}^{E^u}$ be the unique solution of $T_{x_0}(x^s,x^u) = \varphi_{x_0} w$ where $x_0 \geq 0$ and $w \in X^\alpha$ are arbitrary. Denoting the solution by $(u^s(x;x_0,w), u^u(x;x_0,w))$ one obtains the identity*

$$u^s(x_0;x_0,w) = w - u^u(x_0;x_0,w).$$

Proof Adding the two equations

$$(T_{x_0}u)^s(x_0) = u^s(x_0; x_0, z) + e^{-A_- x_0} P_- u^u(0; x_0, z) + \int_{x_0}^{\infty} e^{A_+(x_0-\sigma)} P_+ B(\sigma) u^s(\sigma; x_0, z) d\sigma$$
$$+ \int_0^{x_0} e^{-A_-(x_0-\sigma)} P_- B(\sigma) u^u(\sigma; x_0, z) d\sigma = (\varphi_{x_0} z)^s(x_0) = P_- w,$$
$$(T_{x_0}u)^u(x_0) = u^u(x_0; x_0, z) - e^{-A_- x_0} P_- u^u(0; x_0, z) + \int_{x_0}^{0} e^{-A_-(x_0-\sigma)} P_- B(\sigma) u^u(\sigma; x_0, z) d\sigma$$
$$- \int_{x_0}^{\infty} e^{A_+(x_0-\sigma)} P_+ B(\sigma) u^s(\sigma; x_0, z) d\sigma = (\varphi_{x_0} z)^u(x_0) = P_+ w$$

yields the assertion of the lemma. ∎

Proof of the roughness theorem for exponential dichotomies

The following spaces and maps are similar to the previous ones but x_0 is not fixed any more.

Definition 1.3.15

$$\mathscr{X}^s := \left\{ u \in C^0(D^s, X^\alpha) \,:\, ||u||_{\mathscr{X}^s} := \sup_{(x,x_0)\in D^s} e^{\eta|x-x_0|} ||u(x,x_0)||_{X^\alpha} < \infty \right\},$$
$$\mathscr{X}^u := \left\{ u \in C^0(D^u, X^\alpha) \,:\, ||u||_{\mathscr{X}^u} := \sup_{(x,x_0)\in D^u} e^{\eta|x-x_0|} ||u(x,x_0)||_{X^\alpha} < \infty \right\}$$
$$\textit{with} \quad D^s := \{(x,x_0) \,:\, x \geq x_0 \geq 0\} \quad \textit{and} \quad D^u := \{(x,x_0) \,:\, x_0 \geq x \geq 0\}.$$

Definition 1.3.16 *For $E \subset X^\alpha$ closed subspace one defines*

$$\mathscr{X}^E := \{(u^s, u^u) \in \mathscr{X}^s \oplus \mathscr{X}^u \,:\, u^u(0, x_0) \in E \; \forall x_0 \geq 0\}.$$

The following definition takes (1.6) into consideration:

Definition 1.3.17

$$(\varphi w)^s(x,x_0) := e^{-A_-(x-x_0)} P_- w, \quad (x,x_0) \in D^s,$$
$$(\varphi w)^u(x,x_0) := e^{A_+(x-x_0)} P_+ w, \quad (x,x_0) \in D^u.$$

Lemma 1.3.18 *$\varphi : X^\alpha \to \mathscr{X}^{X_+}$ is a bounded operator.*

Proof See Lemma 1.3.5. ∎

Definition 1.3.19

$$
\begin{aligned}
&T(u^s,u^u)(x,x_0)\\
&:= u^s(x,x_0) + e^{-A_- x}P_- u^u(0,x_0) + \int_x^\infty e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma,x_0)d\sigma\\
&\quad - \int_{x_0}^x e^{-A_-(x-\sigma)}P_-B(\sigma)u^s(\sigma,x_0)d\sigma + \int_0^{x_0} e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma,x_0)d\sigma, \quad (x,x_0)\in D^s,\\
&T(u^s,u^u)(x,x_0)\\
&:= u^u(x,x_0) - e^{-A_- x}P_- u^u(0,x_0) - \int_{x_0}^x e^{A_+(x-\sigma)}P_+B(\sigma)u^u(\sigma,x_0)d\sigma\\
&\quad + \int_x^0 e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma,x_0)d\sigma - \int_{x_0}^\infty e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma,x_0)d\sigma, \quad (x,x_0)\in D^u.
\end{aligned}
$$

Lemma 1.3.20 *$T : \mathscr{X}^{E^u} \to \mathscr{X}^{X_+}$ is an isomorphism and continuously invertible.*

Proof The well-definedness and the continuity of T can be proven in a similar way as in the proof of the Proposition 1.3.13.

Next, we show $N(T) = \{0\}$. Choose an arbitrary $u \in N(T)$. Then $u(\cdot, x_0) \in N(T_{x_0})$ for any $x_0 \geq 0$ and $u(\cdot, x_0) = 0$ by Proposition 1.3.13. This leads to $N(T) = \{0\}$.

Hereupon we prove that $T : \mathscr{X}^{E^u} \to \mathscr{X}^{X_+}$ is surjective. Proposition 1.3.13 states that there is an unique family $u(\cdot, x_0)$ that meets $T_{x_0}u(\cdot, x_0) = \varphi_{x_0}w$ for any fixed $x_0 \geq 0$ and $w \in X^\alpha$. The equation $Tu = \varphi w$ holds for $u \in \mathscr{X}^{E^u}$. In the following we must prove the continuity of $u(\cdot, x_0)$ in x_0 and the exponential decay of $u(\cdot, x_0)$ uniformly in x_0.

Let (u^s, u^u) be the unique solution of

$$T_{x_0}(u^s,u^u) = \varphi_{x_0}w \tag{1.21}$$

which is denoted by $(u^s(x;x_0,w), u^u(x;x_0,w))$. In the following we will prove the statements

(a) Invariance and semigroup properties:

$$
\begin{aligned}
u^s(x;\sigma,u^s(\sigma;x_0,w)) &= u^s(x;x_0,w), \quad x\geq\sigma\geq x_0, \quad u^s(x;\sigma,u^u(\sigma;x_0,w)) = 0, \quad \sigma\leq x,x_0,\\
u^u(x;\sigma,u^u(\sigma;x_0,w)) &= u^u(x;x_0,w), \quad x\leq\sigma\leq x_0, \quad u^u(x;\sigma,u^s(\sigma;x_0,w)) = 0, \quad \sigma\geq x,x_0.
\end{aligned}
$$

(b) Continuity:

$$u^s(\cdot;\cdot,w)\,\text{and}\,u^u(\cdot;\cdot,w)\text{are continuous.}$$

(c) Exponential decay:

$$
\begin{aligned}
||u^s(x;x_0,w)||_{X^\alpha} &\leq Ce^{-\eta|x-x_0|}||w||_{X^\alpha}, \quad x\geq x_0,\\
||u^u(x;x_0,w)||_{X^\alpha} &\leq Ce^{-\eta|x-x_0|}||w||_{X^\alpha}, \quad x\leq x_0.
\end{aligned}
$$

Proof of (a),(b) and (c):

(a) We define $\hat{w} := u^s(\sigma; x_0, w)$ for $\sigma \geq x_0$ and

$$\begin{aligned} v^s(x) &:= u^s(x; \sigma, \hat{w}) = u^s(x; \sigma, u^s(\sigma; x_0, w)), \quad x \geq \sigma, \\ v^u(x) &:= u^u(x; \sigma, \hat{w}) = u^u(x; \sigma, u^s(\sigma; x_0, w)), \quad x \leq \sigma. \end{aligned} \tag{1.22}$$

Then $(v^s, v^u) = (u^s, u^u)(\cdot; \sigma, \hat{w})$ results in $T_\sigma(v^s, v^u) = \varphi_\sigma \hat{w}$, i.e.

$$\begin{aligned} e^{-A_-(x-\sigma)} P_- \hat{w} &= (T_\sigma(v^s, v^u))^s(x), \quad x \geq \sigma, \\ e^{A_+(x-\sigma)} P_+ \hat{w} &= (T_\sigma(v^s, v^u))^u(x), \quad x \leq \sigma. \end{aligned} \tag{1.23}$$

Here, $(T_\sigma v)^s$ and $(T_\sigma v)^u$ are the components of $T_\sigma v$ in $\mathscr{X}_\sigma = \mathscr{X}_\sigma^s \oplus \mathscr{X}_\sigma^u$.
$\hat{w} = u^s(\sigma; x_0, w)$ and $(T_{x_0}(u^s, u^u))^s(\sigma) = e^{-A_-(\sigma - x_0)} P_- w$ lead to

$$\begin{aligned} \hat{w} =& e^{-A_-(\sigma-x_0)} P_- w - e^{-A_-\sigma} P_- u^u(0; x_0, w) - \int_0^{x_0} e^{-A_-(\sigma-\rho)} P_- B(\rho) u^u(\rho; x_0, w) d\rho \\ &- \int_\sigma^\infty e^{A_+(\sigma-\rho)} P_+ B(\rho) u^s(\rho; x_0, w) d\rho + \int_{x_0}^\sigma e^{-A_-(\sigma-\rho)} P_- B(\rho) u^s(\rho; x_0, w) d\rho. \end{aligned} \tag{1.24}$$

If we substitute (1.24) into (1.23) we obtain

$$\begin{aligned} e^{-A_-(x-x_0)} P_- w =& \int_0^{x_0} e^{-A_-(x-\rho)} P_- B(\rho) u^u(\rho; x_0, w) d\rho \\ &- \int_{x_0}^\sigma e^{-A_-(x-\rho)} P_- B(\rho) u^s(\rho; x_0, w) d\rho \\ &+ e^{-A_- x} P_- u^u(0; x_0, w) + (T_\rho(v^s, v^u))^s(x), \quad x \geq \sigma, \\ 0 =& \int_\sigma^\infty e^{A_+(x-\rho)} P_+ B(\rho) u^s(\rho; x_0, w) d\rho + (T_\sigma(v^s, v^u))^u(x), \quad x \leq \sigma. \end{aligned} \tag{1.25}$$

Because T_σ is invertible equations (1.25) can be uniquely solved when (v^s, v^u) are considered as unknowns. We already know the unique solution by (1.22). Moreover,

$$\begin{aligned} &v^s(x) = u^s(x; x_0, w), \quad x \geq \sigma, \\ &v^u(x) = 0, \quad x \leq \sigma \end{aligned} \tag{1.26}$$

is also a solution of (1.25).

Putting (1.26) into (1.25) leads to

$$\begin{aligned} &e^{-A_-(x-x_0)} P_- w \\ &= \int_0^{x_0} e^{-A_-(x-\rho)} P_- B(\rho) u^u(\rho; x_0, w) d\rho - \int_{x_0}^\sigma e^{-A_-(x-\rho)} P_- B(\rho) u^s(\rho; x_0, w) d\rho \\ &\quad + e^{-A_- x} P_- u^u(0; x_0, w) + (T_\rho(v^s, v^u))^s(x) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{x_0} e^{-A_-(x-\rho)} P_- B(\rho) u^u(\rho; x_0, w) d\rho - \int_{x_0}^{\sigma} e^{-A_-(x-\rho)} P_- B(\rho) u^s(\rho; x_0, w) d\rho \\
&\quad + e^{-A_- x} P_- u^u(0; x_0, w) + u^s(x; x_0, w) + e^{-A_- x} P_- \cdot 0 + \int_x^{\infty} e^{A_+(x-\rho)} P_+ B(\rho) u^s(\rho; x_0, w) d\rho \\
&\quad - \int_{\sigma}^{x} e^{-A_-(x-\rho)} P_- B(\rho) u^s(\rho; x_0, w) d\rho + \int_0^{\sigma} e^{-A_-(x-\rho)} P_- B(\rho) \cdot 0 \, d\rho \\
&= u^s(x; x_0, w) + e^{-A_- x} P_- u^u(0; x_0, w) + \int_0^{x_0} e^{-A_-(x-\rho)} P_- B(\rho) u^u(\rho; x_0, w) d\rho \\
&\quad - \int_{x_0}^{x} e^{-A_-(x-\rho)} P_- B(\rho) u^s(\rho; x_0, w) d\rho + \int_x^{\infty} e^{A_+(x-\rho)} P_+ B(\rho) u^s(\rho; x_0, w) d\rho
\end{aligned}
$$

This equation coincides with the first one of $T_{x_0}(u^s, u^u) = \varphi_{x_0} w$, see (1.21). The uniqueness of the solution results in two of the four identities in (a). Similarly one can show the two remaining identities.

(b) We compare the solutions $u(\cdot, x_0 + h)$ and $u(\cdot, x_0)$ for small h. Choose $h > 0$ and $w \in X^\alpha$ with $||w||_{X^\alpha} = 1$. For $h < 0$ one can proceed in similar way. We define

$$
\begin{aligned}
v_h^s(x) &= \begin{cases} u^s(x, x_0 + h), & x_0 + h \leq x, \\ w - u^u(x, x_0 + h), & x_0 + h \geq x \geq x_0, \end{cases} \\
v_h^u(x) &= u^u(x, x_0 + h), \quad x \leq x_0.
\end{aligned}
$$

Considering Lemma 1.3.14 we see that v_h^s is continuous at $x = x_0 + h$ and therefore $v_h \in \mathscr{X}_{x_0}^{E^u}$.

The key for the proof is the assertion that

$$
||T_{x_0} v_h - T_{x_0} u(\cdot, x_0)||_{\mathscr{X}_{x_0}^{X_+}} \leq o(1) \tag{1.27}
$$

is met for some function $o(1)$ with $o(1) \to 0$ as $h \to 0$. Since T_{x_0} is continuously invertible, see Proposition 1.3.13, we obtain

$$
||v_h - u(\cdot, x_0)||_{\mathscr{X}_{x_0}^{E^u}} = ||T_{x_0}^{-1} T_{x_0} v_h - T_{x_0}^{-1} T_{x_0} u(\cdot, x_0)||_{\mathscr{X}_{x_0}^{E^u}} \leq C ||T_{x_0} v_h - T_{x_0} u(\cdot, x_0)||_{\mathscr{X}_{x_0}^{X_+}}
$$

where C is a positive constant independent of h. Due to (1.27) we obtain $||v_h - u(\cdot, x_0)||_{\mathscr{X}_{x_0}^{E^u}} \to 0$ for $h \to 0$ what proves statement (b).

Proof of (1.27):
Consider $T_{x_0+h} u(\cdot, x_0) = \varphi_{x_0+h} w$. To compare $T_{x_0} v_h$ with $T_{x_0} u(\cdot, x_0)$ we compute $T_{x_0} v_h$. For $x \leq x_0$ we have

$$
\begin{aligned}
&(T_{x_0} v_h)^u(x) \\
&= v_h^u(x) - e^{-A_- x} P_- v_h^u(0) - \int_{x_0}^{x} e^{A_+(x-\sigma)} P_+ B(\sigma) v_h^u(\sigma) d\sigma \\
&\quad + \int_x^0 e^{-A_-(x-\sigma)} P_- B(\sigma) v_h^u(\sigma) d\sigma - \int_{x_0}^{\infty} e^{A_+(x-\sigma)} P_+ B(\sigma) v_h^s(\sigma) d\sigma \\
&= u^u(x, x_0+h) - e^{-A_- x} P_- u^u(0, x_0+h) - \int_{x_0}^{x} e^{A_+(x-\sigma)} P_+ B(\sigma) u^u(\sigma, x_0+h) d\sigma \\
&\quad + \int_x^0 e^{-A_-(x-\sigma)} P_- B(\sigma) u^u(\sigma, x_0+h) d\sigma - \int_{x_0}^{\infty} e^{A_+(x-\sigma)} P_+ B(\sigma) v_h^s(\sigma) d\sigma \\
&= u^u(x, x_0+h) - e^{-A_- x} P_- u^u(0, x_0+h) - \int_{x_0+h}^{x} e^{A_+(x-\sigma)} P_+ B(\sigma) u^u(\sigma, x_0+h) d\sigma \\
&\quad - \int_{x_0}^{x_0+h} e^{A_+(x-\sigma)} P_+ B(\sigma) u^u(\sigma, x_0+h) d\sigma + \int_x^0 e^{-A_-(x-\sigma)} P_- B(\sigma) u^u(\sigma, x_0+h) d\sigma \\
&\quad - \int_{x_0+h}^{\infty} e^{A_+(x-\sigma)} P_+ B(\sigma) u^s(\sigma, x_0+h) d\sigma - \int_{x_0}^{x_0+h} e^{A_+(x-\sigma)} P_+ B(\sigma) (w - u^u(\sigma, x_0+h)) d\sigma \\
&= (T_{x_0+h} u(\cdot, x_0+h))^u(x) - \int_{x_0}^{x_0+h} e^{A_+(x-\sigma)} P_+ B(\sigma) u^u(\sigma, x_0+h) d\sigma \\
&\quad - \int_{x_0}^{x_0+h} e^{A_+(x-\sigma)} P_+ B(\sigma) (w - u^u(\sigma, x_0+h)) d\sigma \\
&\overset{(*)}{=} (\varphi_{x_0+h} w)^u(x) + o(1) \\
&= e^{A_+(x-x_0-h)} P_+ w + o(1),
\end{aligned}
$$

where $(*)$ is a consequence of

$$
\begin{aligned}
&\left\| \int_{x_0}^{x_0+h} e^{A_+(x-\sigma)} P_+ B(\sigma) u^u(\sigma, x_0+h) d\sigma + \int_{x_0}^{x_0+h} e^{A_+(x-\sigma)} P_+ B(\sigma) (w - u^u(\sigma, x_0+h)) d\sigma \right\|_{\mathscr{X}^u_{x_0}} \\
&= \left\| \int_{x_0}^{x_0+h} e^{A_+(x-\sigma)} P_+ B(\sigma) w \, d\sigma \right\|_{\mathscr{X}^u_{x_0}} \\
&= \sup_{0 \le x \le x_0} \left\{ e^{\eta |x-x_0|} \left\| \int_{x_0}^{x_0+h} e^{A_+(x-\sigma)} P_+ B(\sigma) w \, d\sigma \right\|_{X^\alpha} \right\} \\
&\le \sup_{0 \le x \le x_0} \left\{ C e^{\eta(x_0-x)} \int_{x_0}^{x_0+h} \left\| A_+^\alpha e^{A_+(x-\sigma)} \right\|_{L[X]} d\sigma \right\} \\
&\le \sup_{0 \le x \le x_0} \left\{ C e^{\eta(x_0-x)} \int_{x_0}^{x_0+h} (\sigma - x)^{-\alpha} e^{-\delta(\sigma-x)} \, d\sigma \right\} \\
&\le \sup_{0 \le x \le x_0} \left\{ C e^{\eta x_0} \left[(1-\alpha)^{-1} (\sigma - x)^{1-\alpha} e^{-\delta\sigma} \right]_{x_0}^{x_0+h} - C e^{\eta x_0} \int_{x_0}^{x_0+h} (1-\alpha)^{-1} (\sigma - x)^{1-\alpha} e^{-\delta\sigma} \, d\sigma \right\} \\
&\to 0 \quad \text{as} \quad h \to 0.
\end{aligned}
$$

In a similar way we obtain for $x_0 + h \le x$

$$(T_{x_0} v_h)^s(x) = (T_{x_0+h} u(\cdot, x_0 + h))^s(x) - \int_{x_0}^{x_0+h} e^{-A_-(x-\sigma)} P_- B(\sigma)(w - u^u(\sigma, x_0 + h)) d\sigma$$
$$- \int_{x_0}^{x_0+h} e^{-A_-(x-\sigma)} P_- B(\sigma) u^u(\sigma, x_0 + h) d\sigma = e^{-A_-(x-x_0-h)} P_- w + o(1)$$

and for $x_0 \le x \le x_0 + h$

$$(T_{x_0} v_h)^s(x) = w - (T_{x_0+h} u(\cdot, x_0 + h))^u(x) - \int_{x_0}^{x} e^{-A_-(x-\sigma)} P_- B(\sigma)(w - u^u(\sigma, x_0 + h)) d\sigma$$
$$+ \int_{x}^{x_0+h} e^{A_+(x-\sigma)} P_+ B(\sigma) w d\sigma + \int_{x}^{x_0} e^{-A_-(x-\sigma)} P_- B(\sigma) u^u(\sigma, x_0 + h) d\sigma$$
$$= w - e^{A_+(x-x_0-h)} P_+ w + o(1).$$

If we summarise the inequalities we get

$$(T_{x_0} v_h)^s(x) - (T_{x_0} u(\cdot, x_0))^s(x) = (T_{x_0} v_h)^s(x) - (\varphi_{x_0} w)^s(x) = (T_{x_0} v_h)^s(x) - e^{-A_-(x-x_0)} P_- w$$
$$= \begin{cases} e^{-A_-(x-x_0-h)}(P_- - e^{-A_- h} P_-) w + o(1), & x \ge x_0 + h, \\ w - e^{A_+(x-x_0-h)} P_+ w - e^{-A_-(x-x_0)} P_- w + o(1), & x_0 + h \ge x \ge x_0, \end{cases}$$
$$(T_{x_0} v_h)^u(x) - (T_{x_0} x(\cdot, x_0))^u(x) = (T_{x_0} v_h)^u(x) - (\varphi_{x_0} w)^u(x)$$
$$= e^{A_+(x-x_0-h)} P_+ w + o(1) - e^{A_+(x-x_0)} P_+ w$$
$$= e^{A_+(x-x_0)}(e^{-A_+ h} P_+ - P_+) w + o(1), \quad x \le x_0.$$

This results in assertion (1.27).

(c) We show the first estimate. Considering $x \ge x_0 \ge x^*$ for some x^* large is sufficient since one can use $u^s(x; x_0, z) = u^s(x; x^*, u^s(x^*; x_0, w))$ for $x > x^* > x_0$ and the boundedness of $u^s(x; x_0, z)$ on $x, x_0 \le x^*$ to obtain the general result. On the smaller interval $[x^*, \infty)$ the operator T is continuously invertible since we can write $T = \mathrm{id} + I$ for some operator I which is small in norm on $[x^*, \infty)$. Confer also [19]. The latter can be achieved because B is small on $[x^*, \infty)$, see the proof of Lemma 1.3.6. Finally, this yields uniform exponential bounds of $u^s(x; x_0, \cdot)$ for $x \ge x_0 \ge x^*$. In a similar way one can show $||u^u(x; x_0, w)||_{X^\alpha} \le Ce^{-\eta|x-x_0|}||w||_{X^\alpha}$ for $x \le x_0$.

(b) and (c) result in the surjectivity of T because $(u^s, u^u) \in \mathscr{X}^{E^u}$: (b) yields $u^s \in C^0(D^s, X^\alpha)$ and $u^u \in C^0(D^u, X^\alpha)$. (c) yields $||u^s||_{\mathscr{X}^s} := \sup\left\{e^{\eta|x-x_0|}||u^s(x, x_0)||_{X^\alpha} : (x, x_0) \in D^s\right\} < \infty$ and $||u^u||_{\mathscr{X}^u} := \sup\left\{e^{\eta|x-x_0|}||u^u(x, x_0)||_{X^\alpha} : (x, x_0) \in D^u\right\} < \infty$.

We can now conclude that T is an isomorphism and continuously invertible. The latter is a consequence of Theorem A.2.17. ∎

Finally, we can construct the exponential dichotomy employing the previous lemma. At first we must specify the family of projections $\{P(x)\}_{x \in \mathbb{R}^+}$ with the demanded properties:

$$P(x) w = u^s(x; x, w), \quad x \in \mathbb{R}^+,\ w \in X^\alpha, \tag{1.28}$$

where u is a solution of $Tu = \varphi w$.

$P(x)$ is a projection on X^α:

$$(P(x))^2 w = P(x)P(x)w = u^s(x;x,P(x)w) = u^s(x;x,u^s(x;x,w)) \stackrel{(a)}{=} u^s(x;x,w) = P(x)w.$$

$P(x)$ is bounded:

$$||P(x)w||_{X^\alpha} = ||u^s(x;x,w)||_{X^\alpha} \stackrel{(c)}{\leq} Ce^{-\eta|x-x|}||w||_{X^\alpha} = C||w||_{X^\alpha}.$$

Moreover, $P(\cdot)w = u^s(\cdot;\cdot,w) \in C^0(\mathbb{R}^+, X^\alpha)$ because of the continuity property (b).

In the following we have to show the properties stability, instability and invariance that characterize an exponential dichotomy, recall Definition 1.0.2:

- *Stability.* There exists a unique solution $u^s(x;x_0,w)$ of (1.1) for any $x_0 \in \mathbb{R}^+$, $w \in X^\alpha$ and defined for $x \in \mathbb{R}^+ \cap [x_0, \infty)$ with $u^s(x_0;x_0,w) = P(x_0)w$. The solution u^s satisfies
$$||u^s(x;x_0,w)||_{X^\alpha} \leq Ce^{-\eta|x-x_0|}\,||w||_{X^\alpha} \quad \forall x \in \mathbb{R}^+ \cap [x_0,\infty).$$
- *Instability.* There exists a unique solution $u^u(x;x_0,w)$ of (1.1) for any $x_0 \in \mathbb{R}^+$, $w \in X^\alpha$ and defined for $x \in \mathbb{R}^+ \cap (-\infty, x_0]$ with $u^u(x_0;x_0,w) = (id - P(x_0))w$. The solution u^u satisfies
$$||u^u(x;x_0,w)||_{X^\alpha} \leq Ce^{-\eta|x-x_0|}\,||w||_{X^\alpha} \quad \forall x \in \mathbb{R}^+ \cap (-\infty,x_0].$$
- *Invariance.* For $w \in X^\alpha$,
$$\begin{aligned} u^s(x;x_0,w) &\in R(P(x)) \quad \forall x \in \mathbb{R}^+ \cap [x_0,\infty),\\ u^u(x;x_0,w) &\in N(P(x)) \quad \forall x \in \mathbb{R}^+ \cap (-\infty,x_0]. \end{aligned}$$

Consider that u is a solution of $Tu = \varphi w$ and that $(\mathrm{id} - P(x))w = u^u(x;x,w)$ is well-defined because of Lemma 1.3.14. The estimates follow from (c) in proof of Lemma 1.3.20. To show the invariance properties we choose arbitrary $x, x_0 \in \mathbb{R}^+$ with $x \geq x_0$ and $w \in X^\alpha$. Defining $\tilde{w} = u^s(x;x_0,w) \in X^\alpha$ leads to

$$P(x)\tilde{w} = u^s(x;x,\tilde{w}) = u^s(x;x,u^s(x;x_0,w)) \stackrel{(a)}{=} u^s(x;x_0,w) \in R(P(x)).$$

Similarly one shows $u^u(x;x_0,w) \in N(P(x)) = R(\mathrm{id} - P(x))$ for $x \leq x_0$ with $x, x_0 \in \mathbb{R}^+$.

According to $E^s = \{w \in X^\alpha : \exists u^s \in \mathscr{X}_0^s \textit{ with } u^s(0;0,w) = w \textit{ and } \tilde{T}_0 u^s = \tilde{\varphi}_0 w\}$ and $u^s(0;0,w) = P(0)w$ we get $E^s = R(P(0))$. So E^s is uniquely determined. (1.6) and equation (3.20) in [19] lead to

$$\begin{aligned} w \in E^s = R(P(0)) \Rightarrow w = u^s(0;0,w) & \\ &= (\varphi w)^s(0) - P_- u^u(0;0,w) - \int_0^\infty e^{-A_+\sigma} P_+ B(\sigma) u^s(\sigma;0,w)d\sigma \\ &= P_- w - \int_0^\infty e^{-A_+\sigma} P_+ B(\sigma) u^s(\sigma;0,w)d\sigma \stackrel{(*)}{=} P_- w + P_+(S_0 + K_0)w \end{aligned}$$

for some operators S_0 and K_0 in $L[X^\alpha]$ with $||S_0||_{L[X^\alpha]} \leq C\varepsilon$ and K_0 compact. Consider that $u^u(0;0,z) = 0$ holds and that $(*)$ has been proven in Lemma 1.3.6. Confer also [22]. This completes the proof of the roughness theorem. The next section deals with important implications of this theorem.

1.4 Implications of the Roughness Theorem

In this section we will outline some important implications of the roughness theorem. Confer [19] and [15]. The following statements and the roughness theorem itself are major tools for the next chapters.

Let $J \in \{\mathbb{R}, \mathbb{R}^+, \mathbb{R}^-\}$ and recall Theorem 1.2.1, Definition 1.0.2 and Hypothesis **(H1)**, where $\{P(x)\}_{x\in J}$ and P_- are specified. It is a consequence of the roughness theorem that the space $R(P(0)) = E^s$ is close to $R(P_-)$ up to factoring a finite-dimensional subspace of E^s. This leads to the following corollary which can easily be proven by employing the characterization of the stable subspaces in Theorem 1.2.1.

Corollary 1.4.1 *Let A and $B(x)$ satisfy the assumptions of Theorem 1.2.1 on $J = \mathbb{R}^+$ and $J = \mathbb{R}^-$. If $P(x)$ and $Q(x)$ are the projections of the associated exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively, the intersection $R(P(0)) \cap R(Q(0))$ is finite-dimensional.*

Corollary 1.4.2 *Let A and $B(x)$ satisfy the assumptions of Theorem 1.2.1 on $J = \mathbb{R}^+$ and suppose*

$$||B(x)||_{L[X^\alpha,X]} \leq Ce^{-\theta x} \quad \forall x \in \mathbb{R}^+$$

for some positive constants C and θ. Then, the rate η appearing in the roughness theorem can be chosen from the closed interval $[0,\delta]$ and the estimate

$$||P(x) - P_-||_{L[X^\alpha]} \leq C\left(e^{-2\delta x} + e^{-\theta x}\right) \quad \forall x \in \mathbb{R}^+$$

holds for some $C > 0$. An analogous statement is true for $J = \mathbb{R}^-$.

Proof We take only complements E^u into account which satisfy (1.17). Under the condition that $B(x)$ decays exponentially it is straightforward to prove that the right hand side of (1.6) is well-defined and an isomorphism from the spaces $\mathscr{X}^{E^u}$ to $\mathscr{X}^{X_+}$ even for $\eta = \delta$. The asserted estimate of the corollary is a consequence of

$$P(x)w = u^s(x;x,w) = (T_x u)^s(x) - e^{-A_- x}P_- u^u(0;x,w) - \int_0^x e^{-A_-(x-\sigma)}P_- B(\sigma)u^u(\sigma;x,w)d\sigma$$
$$+ \int_x^\infty e^{A_+(x-\sigma)}P_+ B(\sigma)u^s(\sigma;x,w)d\sigma.$$

Considering the assumption $||B(x)||_{L[X^\alpha,X]} \leq Ce^{-\theta x}$ for $x \geq 0$ and $(T_x u)^s(x) = (\varphi_x w)^s(x) =$

P_-w we obtain

$$
\begin{aligned}
||P(x)w - P_-w||_{X^\alpha} \le\; & ||e^{-A_-x}P_-u^u(0;x,w)||_{X^\alpha} + \left|\left|\int_0^x e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma;x,w)d\sigma\right|\right|_{X^\alpha} \\
& + \left|\left|\int_x^\infty e^{A_+(x-\sigma)}P_+B(\sigma)u^s(\sigma;x,w)d\sigma\right|\right|_{X^\alpha} \\
& \overset{(*)}{\le} Ce^{-\delta x}e^{-\eta x}||w||_{X^\alpha} + C\int_0^x (x-\sigma)^{-\alpha}e^{-\delta(x-\sigma)}e^{-\theta\sigma}e^{-\eta(x-\sigma)}d\sigma\,||w||_{X^\alpha} \\
& + C\int_x^\infty (\sigma-x)^{-\alpha}e^{-\delta(\sigma-x)}e^{-\theta\sigma}e^{-\eta(\sigma-x)}d\sigma\,||w||_{X^\alpha} \\
& \overset{(\dagger)}{\le} C(e^{-(\delta+\eta)x}+e^{-\theta x})||w||_{X^\alpha}.
\end{aligned}
$$

$(*)$ follows from

$$
\begin{aligned}
& \int_0^x ||e^{-A_-(x-\sigma)}P_-B(\sigma)u^u(\sigma;x,w)||_{X^\alpha}d\sigma \\
& \le \int_0^x ||A_-^\alpha e^{-A_-(x-\sigma)}||_{L[X]}C||B(\sigma)u^u(\sigma;x,w)||_X d\sigma \\
& \le \int_0^x C(x-\sigma)^{-\alpha}e^{-\delta(x-\sigma)}||B(\sigma)||_{L[X^\alpha,X]}||u^u(\sigma;x,w)||_{X^\alpha}d\sigma \\
& \le \int_0^x C(x-\sigma)^{-\alpha}e^{-\delta(x-\sigma)}e^{-\theta x}e^{-\eta(x-\sigma)}d\sigma
\end{aligned}
$$

and $(\dagger)$ from

$$
0 < \int_0^y x^{1-\alpha-1}e^{-x}dx < \int_0^\infty x^{1-\alpha-1}e^{-x}dx = \Gamma(1-\alpha) < \infty, \quad y > 0,\ 0 \le \alpha < 1. \quad \blacksquare
$$

The previous corollary includes the expected behaviour of $P(x)$ converging to the projection P_- as $x \to \infty$. In the following theorem we give a characterization of equations which have exponential dichotomies on $\mathbb{R}$.

Theorem 1.4.3 *Let the assumptions of the roughness theorem hold for $J = \mathbb{R}^+$ and $J = \mathbb{R}^-$. Then $u = 0$ is the only bounded solution of the differential equation $\frac{\partial}{\partial x}u = (A + B(x))u$ on $\mathbb{R}$ if and only if the equation has an exponential dichotomy on $\mathbb{R}$.*

Proof At first we assume that $\frac{\partial}{\partial x}u = (A + B(x))u$ has an exponential dichotomy $\{P(x)\}_{x\in\mathbb{R}}$ on $\mathbb{R}$. Then any bounded solution u meets $P(0)u(0) = u(0)$ due to the boundedness of u on $\mathbb{R}^+$. Consider that $u(0)$ is an element of the stable subspace $E^s = R(P(0))$. In a similar way we obtain $P(0)u(0) = 0$ due to the boundedness of u on $\mathbb{R}^-$. Hence $u(0) = 0$ which results in $u = 0$ because of **(H5)**.

Conversely, we suppose that $u = 0$ is the only bounded solution of $\frac{\partial}{\partial x}u = (A + B(x))u$ on $\mathbb{R}$. We can write the mild formulation (1.6) in the form

$$
\begin{aligned}
T^-u &= \varphi^-\xi, \quad x \in \mathbb{R}^+, \\
T^+u &= \varphi^+\xi, \quad x \in \mathbb{R}^-,
\end{aligned}
$$

where $T^{\pm}$ and $\varphi^{\pm}$ are the right and left hand side of (1.6), respectively. Furthermore, we call the associated projections of the exponential dichotomies $\{P(x)\}_{x\in\mathbb{R}^+}$ and $\{Q(x)\}_{x\in\mathbb{R}^-}$, respectively. As $\frac{\partial}{\partial x}u = (A + B(x))u$ has no bounded non-trivial solution we obtain

$$R(P(0)) \cap R(\mathrm{id} - Q(0)) = \{0\}.$$

Hence $R(\mathrm{id} - Q(0))$ is a complement of $R(P(0))$ so that we have an exponential dichotomy on $\mathbb{R}^+$ which is associated with projections $\{\tilde{P}(x)\}_{x\in\mathbb{R}^+}$ and with $R(\tilde{P}(0)) = R(P(0))$ and $N(\tilde{P}(0)) = R(\mathrm{id} - Q(0))$. Moreover, there is an exponential dichotomy on $\mathbb{R}^-$ where the associated projection at $x = 0$ is again given by $\tilde{P}(0)$. This results in the continuity of the projections at $x = 0$ and therewith in an exponential dichotomy on $\mathbb{R}$. ∎

The next theorem is taken from Lemma 3.3 in [15] and compares the evolution operators for different equations. Because the formulation in [15] has a certain lack of precision we had to change it slightly.

Theorem 1.4.4 *Consider equation $\frac{\partial}{\partial x}u = (A + B(x))u$ and require the assumptions of the roughness theorem for $J = \mathbb{R}^+$. Moreover, let a second differential equation be given by*

$$\frac{\partial}{\partial x}v = (A + \tilde{B}(x))v,$$

where $\tilde{B} \in C^{0,\vartheta}(\mathbb{R}^+, L[X^\alpha, X])$. Then, there are constants $C, \eta_0 > 0$ so that the estimate

$$\sup_{x\geq 0} ||B(x) - \tilde{B}(x)||_{L[X^\alpha,X]} < \eta$$

for some $\eta < \eta_0$ results in

$$\sup_{x\geq 0} ||P(x) - \tilde{P}(x)||_{L[X^\alpha,X]} < C\eta,$$

where $P(x)$ and $\tilde{P}(x)$ are the corresponding projections to the differential equations.

Proof Confer [15] and [11]. ∎

Previously we analysed $u^s(x; x_0, w)$ and $u^u(x; x_0, w)$ for fixed $w \in X^\alpha$. Now we consider w as a variable and stress the operator-point-of-view what is more associated with the semigroup theory.

Definition 1.4.5 *For $w \in X^\alpha$ and x, $x_0 \in J$ define*

$$\begin{aligned} \Phi^s(x, x_0)w &:= u^s(x; x_0, w), \quad x \geq x_0, \\ \Phi^u(x, x_0)w &:= u^u(x; x_0, w), \quad x \leq x_0. \end{aligned}$$

Theorem 1.4.6 *Let A and $B(x)$ satisfy the assumptions of the roughness theorem on the interval $J = \mathbb{R}^+$. Then the following statements hold for x, $x_0 \in J$ with $x \geq x_0$:*

(i) *$\Phi^s(x,x_0)$ has a bounded extension to X with $\Phi^s(x,x) = P(x)$ and the equation*

$$\Phi^s(x,\sigma)\Phi^s(\sigma,x_0)w = \Phi^s(x,x_0)w$$

holds for all $\sigma \in [x,x_0]$ and any $w \in X$.

(ii) *For fixed $0 \le \beta < 1$ the evolution operator $\Phi^s(x,x_0)$ is strongly continuous in (x,x_0) with values in $L[X^\beta]$.*

(iii) *For any $0 \le \gamma,\ \beta < 1$, there exists $C > 0$ so that $\Phi^s(x,x_0) \in L[X^\gamma, X^\beta]$ for $x > x_0$ and*

$$||\Phi^s(x,x_0)||_{L[X^\gamma,X^\beta]} \le C\max(1,(x-x_0)^{\gamma-\beta})e^{-\eta(x-x_0)}$$

There are analogous properties for $\Phi^u(x,x_0)$ with $x,\ x_0 \in J$ and $x \le x_0$.

Proof Confer [19] Section 4. ∎

Employing the previous theorem and the roughness theorem one can show the existence of solutions of inhomogeneous linear equations

$$\frac{\partial}{\partial x}u = (A+B(x))u + f(x), \quad f \in C^{0,\vartheta}(\mathbb{R}^+, X), \quad \vartheta > 0 \tag{1.29}$$

as well as nonlinear equations

$$\frac{\partial}{\partial x}u = (A+B(x))u + G(x,u), \quad G \in C^{1,1}(\mathbb{R}^+ \oplus X^\alpha, X) \tag{1.30}$$

with $G(x,0) = 0$ and $DG(x,0) = 0$. In the case of (1.29) and (1.30) define $F = f$ and $F = G$, respectively. Hereupon one obtains the corresponding mild formulation:

$$\begin{aligned}
e^{-A_-(x-x_0)}P_-w =& u^s(x,x_0) + e^{-A_-x}P_-u^u(0,x_0)\\
&+ \int_x^\infty e^{A_+(x-\sigma)}P_+(B(\sigma)u^s(\sigma,x_0) + F(\sigma,u^s(\sigma,x_0)))d\sigma\\
&- \int_{x_0}^x e^{-A_-(x-\sigma)}P_-(B(\sigma)u^s(\sigma,x_0) + F(\sigma,u^s(\sigma,x_0)))d\sigma\\
&+ \int_0^{x_0} e^{-A_-(x-\sigma)}P_-(B(\sigma)u^u(\sigma,x_0) + F(\sigma,u^u(\sigma,x_0)))d\sigma,\\
e^{A_+(x-x_0)}P_+w =& u^u(x,x_0) - e^{-A_-x}P_-u^u(0,x_0)\\
&- \int_{x_0}^x e^{A_+(x-\sigma)}P_+(B(\sigma)u^u(\sigma,x_0) + F(\sigma,u^u(\sigma,x_0)))d\sigma\\
&+ \int_x^0 e^{-A_-(x-\sigma)}P_-(B(\sigma)u^u(\sigma,x_0) + F(\sigma,u^u(\sigma,x_0)))d\sigma\\
&- \int_{x_0}^\infty e^{A_+(x-\sigma)}P_+(B(\sigma)u^s(\sigma,x_0) + F(\sigma,u^s(\sigma,x_0)))d\sigma,
\end{aligned} \tag{1.31}$$

where $w \in X^\alpha$. For $F = f$ the proof complies with [11] Theorem 7.1.4. For $F = G$ the right hand side of (1.31) is a differentiable map when considered as a map from $\mathscr{X}^{E^u}$ to $\mathscr{X}^{X_+}$ with $\eta = 0$. Because the linear part is invertible as the operator T is, see proof of the roughness theorem, one can use an implicit function theorem in order to get solution operators $\Phi^s(x;x_0,w)$ and $\Phi^u(x;x_0,w)$ for $x \ge x_0$ and $0 \le x \le x_0$, respectively, which are defined for small $w \in X^\alpha$ and depend smoothly on w. Confer also [19].

2 Numerical Computation of Solitary Waves in Infinite Cylindrical Domains

Solitary waves are a phenomenon in nature which can be found in many fields of physics, biology and chemistry. Examples are nonlinear optics, hydrodynamics, quantum theory, nerve impulses, bloodflows in arteries and chemical kinetics, confer [15] and [16]. From a heuristic-experimental point of view[1] a wave is called solitary if

- it is spatially local,
- its shape does not change and moves with constant velocity,
- it is stable against small perturbations.

If the wave is additionally

- stable against scattering and collision among one another

it is called a soliton. For dissipative systems this property does not necessarily hold, see [24], and cancellation can occur. In general, the velocity depends on the shape of the soliton. These listed properties make solitary waves very special since in many cases waves dissolve and are unstable against perturbations. Moreover, one can define a travelling wave[2] by the properties of spatial locality and of constant shape and velocity.

Wave phenomena are mathematically described by differential equations with accurate boundary conditions. Very often solutions of these equations are given by wave packets. Dissolving wave packets are a result of dispersion, i.e. the phase velocity depends on the wave length and the different superposed parts of the packet move away from each other. Exceptional cases occur when the angular velocity is proportional to the wavenumber. The propagation of electromagnetic and acoustic waves are famous examples. To describe non-dissolving waves linear differential equations cannot be adapted. But a nonlinear structure of the equations can provide an effect that compensates the dissolution and can thus lead to solitary waves.

The Korteweg and de Vries (KdV) equation[3]

$$u_t + u_{xxx} + 6u_x u = 0, \tag{2.1}$$

confer [6], is the first equation which was analysed regarding solitons. This nonlinear equation describes waves on shallow water surfaces. It was introduced long after John Scott Russell first discovered the phenomenon of solitary waves in a narrow channel.

[1] The definition is based on [16]. However, we add the distinction between a solitary wave and a soliton.
[2] The notions of solitary waves, solitons and travelling waves are not uniform in the literature.
[3] We define $u_x = \frac{\partial u}{\partial x}$.

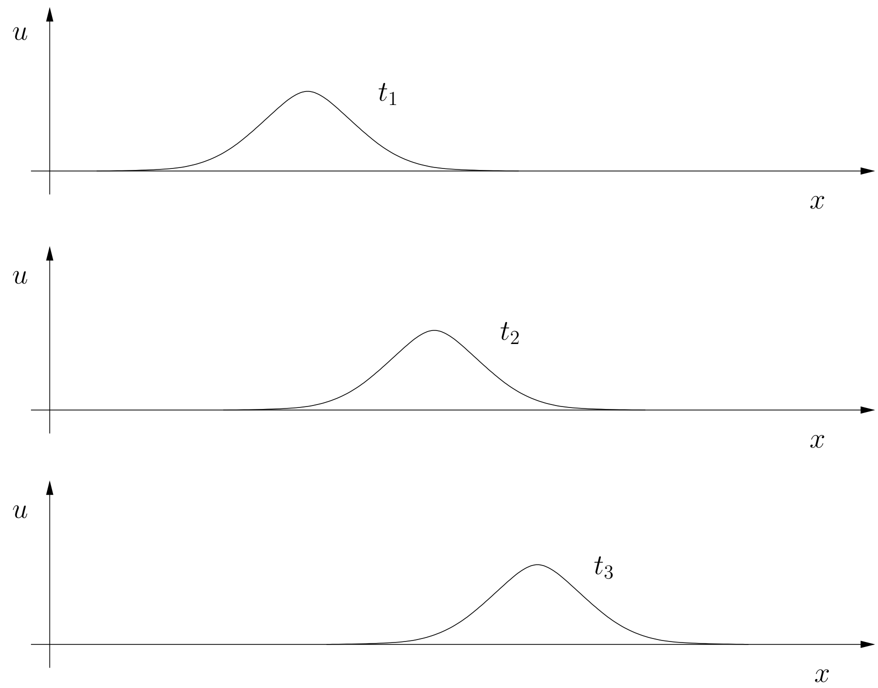

Figure 2.1: A soliton solution of the KdV equation (2.1) for different times $t_1 < t_2 < t_3$ moving in the x-direction.

If one considers only the linear part $u_t + u_{xxx} = 0$ of (2.1) and regards the elementary solution $u(x,t) = e^{i(kx-\omega t)}$ one obtains the dispersion relation $\omega = -k^3$. A solution of (2.1) considering the nonlinear part is given by $u(x,t) = \frac{1}{2}\,\alpha\,\mathrm{sech}^2\left(\sqrt{\frac{\alpha}{4}}(x - \alpha t + \varphi_0)\right)$ for some constant $\alpha > 0$ and some integration constant φ_0, confer [16]. In Figure 2.1 a soliton solution of the KdV equation is sketched for three different times $t_1 < t_2 < t_3$. The wave moves with constant shape and velocity in the x-direction. See also Figure 2.7 in [16].

To illustrate the different behaviour of the solutions of the linear and nonlinear KdV equation we refer to Figure 2.2 which is similar to Figure 2.6 in [16]. For the linear case (above sequence of pictures) the box-shaped distribution of the beginning dissolves. However, in the nonlinear case a soliton comes into existence.

In this thesis we consider solitary waves which are described by semilinear elliptic equations[4] with appropriate boundary conditions:

$$\begin{aligned} u_{xx} + \Delta_y u + g(y, u, u_x, \nabla_y u) = 0, &\quad (x,y) \in \mathbb{R} \times \Omega,\, u \in \mathbb{R}^m, \\ R\left((u, u_x, \nabla_y u)|_{\mathbb{R}\times\partial\Omega}\right) = 0 &\quad \text{on } \mathbb{R} \times \partial\Omega, \end{aligned} \tag{2.2}$$

where $\mathbb{R} \times \Omega$ is an infinite cylinder with $\Omega \subset \mathbb{R}^n$ open and bounded. In this context a solitary wave is a solution h of (2.2) satisfying

$$\lim_{x\to\pm\infty} h(x,y) = p_\pm(y)$$

[4] We define $\Delta_y u := \sum_{k=1}^n \frac{\partial^2 u}{\partial y_k^2}$ and $\nabla_y u := (\frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_n})$.

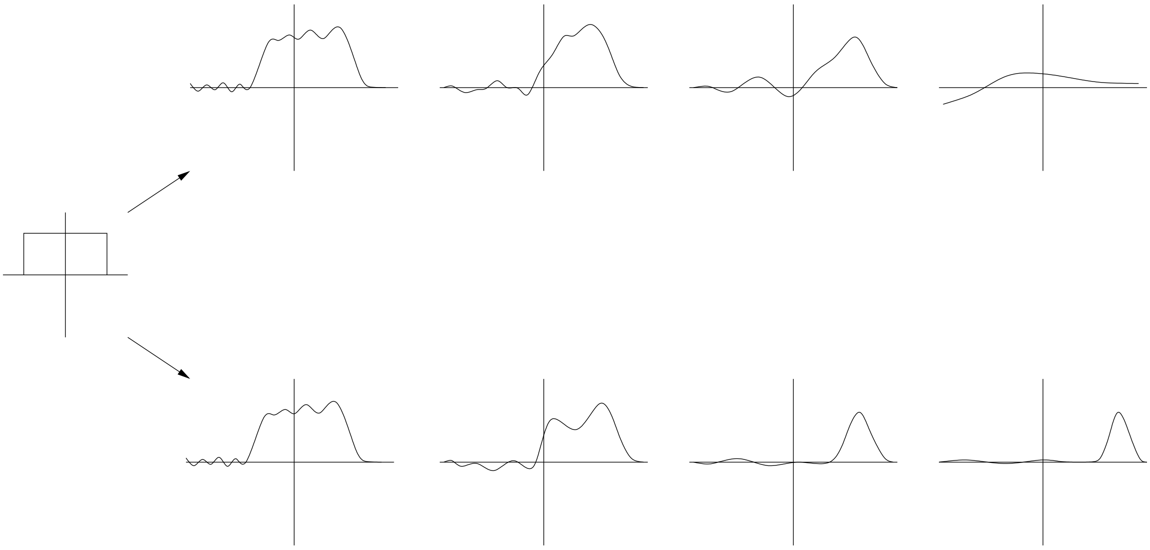

Figure 2.2: A box-shaped distribution dissolves for the linear KdV equation, but it evolves into a soliton for the nonlinear case.

uniformly for $y \in \Omega$ and for some functions $p_{\pm}$. For a drawing of an exemplary solitary wave we refer also to Figure 3.1 in Chapter 3.

They describe the profile of travelling waves $u(x - ct, y)$ for parabolic equations

$$u_t = u_{xx} + \Delta_y u + \tilde{g}(y, u, u_x, \nabla_y u), \quad (x, y) \in \mathbb{R} \times \Omega.$$

Analytically, the existence of solitons with a nontrivial form in the cross-section Ω is a difficult problem. In some cases proofs are possible using center-manifold theory [17], maximum principles [3], [4], variational structure [20] and topological methods [9].

In this chapter we follow closely [15] and suppose the existence of a solitary wave $h(x, y)$ which satisfies (2.2). In order to determine h numerically one truncates the cylinder and considers

$$\begin{aligned} u_{xx} + \Delta_y u + g(y, u, u_x, \nabla_y u) = 0, \quad (x, y) \in (T_-, T_+) \times \Omega,\ u \in \mathbb{R}^m, \\ R\left((u, u_x, \nabla_y u)|_{[T_-,T_+]\times\partial\Omega}\right) = 0 \end{aligned} \tag{2.3}$$

for some $T_1 < T_2$. At the end of the cylinder axis we have to add boundary conditions of the form

$$\begin{aligned} R_-((u, u_x, \nabla_y u)|_{\{T_-\}\times\Omega}) = 0, \\ R_+((u, u_x, \nabla_y u)|_{\{T_+\}\times\Omega}) = 0. \end{aligned}$$

We will examine if this truncated system has a unique solution close to h and we will give estimates for the truncation error.

An important part of our procedure is writing (2.2) as a first order system

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta_y & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{g}(u, v) \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + f(u, v), \tag{2.4}$$

where

$$\hat{g}(u,v)(y) = -g(y,u(y),v(y),\nabla_y u(y)), \quad A = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta_y & 0 \end{pmatrix}, \quad f(u,v) = \begin{pmatrix} 0 \\ \hat{g}(u,v) \end{pmatrix}.$$

$(u,v)(x)$ is a function of $y \in \Omega$ for every $x \in \mathbb{R}$. This function is an element of some function space which incorporates the boundary conditions on $\partial\Omega$. A solitary wave solution of (2.2) corresponds to a homoclinic or heteroclinic solution of (2.4) with $(h(x), h_x(x)) \to (p_\pm, 0)$ as $x \to \pm\infty$, where $(p_\pm, 0)$ is an equilibrium of equation (2.4). Having replaced these limiting values by

$$R_-((u,v)(T_-)) = 0, \quad R_+((u,v)(T_+)) = 0$$

we analyse the resulting truncated system.

In the following delineation we merge u, v into one variable and denote it again by u. Furthermore, we add a parameter $\mu \in \mathbb{R}$. Thus we examine differential equations of the form

$$\frac{\partial}{\partial x} u = Au + f(u,\mu), \tag{2.5}$$

where A is a densely defined and closed operator on a reflexive Banach space X and f has some smoothness property.

In the first section we discretize the cross-section Ω by introducing the Galerkin projection

$$\frac{\partial}{\partial x} u = Au + Q_\rho f(u,\mu), \quad u \in R(Q_\rho), \tag{2.6}$$

where $\{Q_\rho\}_{\rho>0} \subset L[X]$ is a family of projections[5]. These operators Q_ρ project the function space in Ω onto a subspace that is typically finite-dimensional. Then, we obtain a finite-dimensional system of ordinary differential equations. The main result of this chapter is the persistence of a hyperbolic equilibrium and of a homoclinic orbit under the Galerkin approximation. Moreover, we present theorems dealing with the truncated boundary value problem and with projection boundary conditions. For the proof of the following results relating to the Galerkin approximation the main aspects are exponential dichotomies for the linearization $\frac{\partial}{\partial x} v = (A + D_u f(h(x),0))v$ and applying a version of the contraction mapping theorem.

Hereupon we analyse in the forth section of this chapter the truncated boundary value problem

$$\begin{pmatrix} \frac{\partial}{\partial x} u - Au - Q_\rho f(u,\mu) \\ R_\rho(u(T_+), u(T_-), \mu) \\ J_{T,\rho}(u,\mu) \end{pmatrix} = 0,$$

where $x \in (T_-, T_+)$. The functional $J_{T,\rho}$ represents a phase condition and R_ρ describes the boundary conditions. In the last section of this chapter we examine the case of projection boundary conditions and in the following chapter we consider a concrete numerical example.

Now we outline the initial situation which is very similar to that one of the previous chapter and introduce the main hypotheses.

[5]Actually one applies the operators Q_ρ to (2.5) and requires that A and Q_ρ commute.

Initial situation

Let $(A, D(A))$ be the densely defined and closed operator of Chapter 1. Therefore, let A satisfy the corresponding hypothesis **(H1)**, **(H3)** and[6] consider a reflexive Banach space X. Let

$$f \in C^2(X^\alpha \times \mathbb{R}, X)$$

for some fixed $\alpha \in [0, 1)$, where again the interpolation spaces[7] X^α are used.

In this chapter we analyse abstract evolution equations of the form

$$\frac{\partial}{\partial x} u = Au + f(u, \mu), \quad (u, \mu) \in X^\alpha \times \mathbb{R}. \tag{2.7}$$

In the following we give the definition of a solution and require additional hypotheses regarding equation (2.7).

Definition 2.0.7 *A solution of (2.7) is a function u defined on $[0, T)$ for some $T > 0$ with the following properties:*

(i) $u \in C^0((0, T), X^1) \cap C^1((0, T), X)$,

(ii) $u \in C^0([0, T), X^\alpha)$,

(iii) *(2.7) holds as an equation in* $C^0((0, T), X)$.

We also call u a strong solution of (2.7).

The following hypotheses **(H6)** and **(H7)** postulate the existence of a solitary wave in a cylindrical domain. The wave is given by a homoclinic solution h of (2.7). Moving along the x-axis of the cylinder the wave reaches the final state p_0 which is a hyperbolic equilibrium of the evolution equation (2.7).

Hypothesis (H6)
The evolution equation (2.7) has a hyperbolic equilibrium[8] $p_0 \in D(A) = X^1$ for $\mu = 0$. Moreover, **(H1)** is satisfied with A replaced by[9] $A + D_u f(p_0, 0)$.

Hypothesis (H7)
Let the function $h \in C^0(\mathbb{R}, X^1) \cap C^1(\mathbb{R}, X)$ be a homoclinic solution of (2.7) for $\mu = 0$ with $h(x) \to p_0$ as $|x| \to \infty$. Furthermore, $\frac{\partial}{\partial x} h$ is the only bounded solution, up to constant multiples, of the variational equation

$$\frac{\partial}{\partial x} v = (A + D_u f(h(x), 0))v. \tag{2.8}$$

One says that h is nondegenerate.

[6]In this chapter we will not consider hypothesis **(H4)** of Section 1.2.
[7]Consider Appendix A.3.
[8]Here, hyperbolicity is given by $\Re(\sigma(A + D_u f(p_0, 0))) \neq 0$, i.e. $\Re(\lambda) \neq 0 \, \forall \lambda \in \sigma(A + D_u f(p_0, 0))$.
[9]Consider that $A + D_u f(p_0, 0) : D(A) \subset X^\alpha \subset X \to X$ is a densely defined and closed operator.

In order to describe the asymptotic behavior of solutions of (2.8) and of the adjoint variational equation

$$\frac{\partial}{\partial x}v = -(A' + D_u f(h(x), 0)')v, \tag{2.9}$$

we require forward and backward uniqueness:

Hypothesis (H8)
The trivial solution $v = 0$ is the only bounded solution of (2.8) and (2.9) on $\mathbb{R}^+$ or $\mathbb{R}^-$ with $v(0) = 0$.

Remark 2.0.8 *Hypotheses **(H7)** and **(H8)** result in the existence of a bounded and unique solution ψ of (2.9) on $\mathbb{R}$, up to scalar multiples.*

The following hypothesis is important for applying implications of the contraction mapping theorem which lead to the existence of the solutions of the considered differential equations.

Hypothesis (H9)
The Melnikov integral satisfies

$$\int_{-\infty}^{\infty} \langle \psi(x), D_\mu f(h(x), 0)\rangle dx \neq 0,$$

where ψ is the bounded function of Remark 2.0.8.

2.1 Galerkin Approximation and Main Result

In this section we consider the persistence of the hyperbolic equilibrium and of the homoclinic solution under Galerkin approximation (2.6). The Galerkin approximation is given by projections $\{Q_\rho\}_{\rho>0} \subset L[X]$ with $Q_0 = \text{id}$. Typically $R(Q_\rho)$ is finite-dimensional for every $\rho > 0$. The results are summarised in Theorem 2.1.6. Confer also [15]. Main aspects of the proof are implications of the contraction mapping theorem and exponential dichotomies for the linearization $\frac{\partial}{\partial x} v = (A + D_u f(h(x), 0))v$ which lead to an appropriate mild formulation of the evolution equation. For the following hypothesis recall the definitions $A_- := -P_- A$ and $A_+ := (\text{id} - P_-)A$, confer **(H1)**.

Hypothesis (Q)

(i) $[A_\pm, Q_\rho] = 0$ on $D(A)$.

(ii) There is constant C so that $||Q_\rho||_{L[X]} \leq C$ uniformly in ρ.

(iii) $||Q_\rho u - u||_{X^0} \to 0$ as $\rho \to 0$ for any $u \in X$.

Lemma 2.1.1

(i) $[A^\alpha, Q_\rho] = 0$ *on* X^α.

(ii) $Q_\rho \in L[X^\alpha]$ *and* $||Q_\rho||_{L[X^\alpha]} \leq C$ *independently of* $\rho > 0$.

(iii) $||Q_\rho u - u||_{X^\alpha} \to 0$ *as* $\rho \to 0$ *for any* $u \in X^\alpha$.

Proof (i) Because of $[A_\pm, Q_\rho] = 0$ on $D(A)$ and $A_\pm^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-A_\pm t} dt$ we obtain $[A_\pm^{-\alpha}, Q_\rho] = 0$ on $D(A)$. Since $[A_\pm^{-\alpha}, Q_\rho]$ is continuous and $D(A)$ is dense we obtain $[A_\pm^{-\alpha}, Q_\rho] = 0$ on X. This results in

$$\begin{aligned} & A_\pm^{-\alpha} Q_\rho u = Q_\rho A_\pm^{-\alpha} u && \forall u \in X \\ \Leftrightarrow \quad & A_\pm^{-\alpha} Q_\rho A_\pm^\alpha v = Q_\rho v, && v = A_\pm^{-\alpha} u, \forall u \in X \\ \Leftrightarrow \quad & Q_\rho A_\pm^\alpha v = A_\pm^\alpha Q_\rho v && \forall v \in R(A_\pm^{-\alpha}) = D(A_\pm^\alpha). \end{aligned}$$

Therefore,

$$Q_\rho A^\alpha u = Q_\rho A_+^\alpha u_+ + Q_\rho A_-^\alpha u_- = A_+^\alpha Q_\rho u_+ + A_-^\alpha Q_\rho u_- = A^\alpha Q_\rho u$$

for $u \in X^\alpha$, where we have to take into account

$$\begin{aligned} u = u_+ + u_- \in X_+^\alpha \oplus X_-^\alpha \Rightarrow u = A_+^{-\alpha} v_+ + A_-^{-\alpha} v_- \quad \text{for some } v_\pm \in X \\ \Rightarrow Q_\rho u = Q_\rho A_+^{-\alpha} v_+ + Q_\rho A_-^{-\alpha} v_- = A_+^{-\alpha} Q_\rho v_+ + A_-^{-\alpha} Q_\rho v_- \in X_+^\alpha \oplus X_-^\alpha = X^\alpha. \end{aligned}$$

(ii) It follows from $X^\alpha \subset X$ that $Q_\rho z$ is well-defined for $z \in X^\alpha$ with $1 > \alpha \geq 0$ and $\rho > 0$. Because of (i) we have $Q_\rho X^\alpha \subset X^\alpha$. Furthermore, we obtain

$$\begin{aligned} ||Q_\rho||_{L[X^\alpha]} &= \sup_{||z||_{X^\alpha} \leq 1} \{||A_+^\alpha Q_\rho z_+||_X + ||A_-^\alpha Q_\rho z_-||_X\} = \sup_{||z||_{X^\alpha} \leq 1} \{||Q_\rho A_+^\alpha z_+||_X + ||Q_\rho A_-^\alpha z_-||_X\} \\ &\leq ||Q_\rho||_{L[X]} \sup_{||z||_{X^\alpha} \leq 1} \{||A_+^\alpha z_+||_X + ||A_-^\alpha z_-||_X\} \overset{\textbf{(Q)}\text{(ii)}}{\leq} C. \end{aligned}$$

(iii) Let $u \in X^\alpha$. Then

$$\begin{aligned} ||Q_\rho u - u||_{X^\alpha} &= ||A_-^\alpha(Q_\rho u - u)||_X + ||A_+^\alpha(Q_\rho u - u)||_X \\ &= ||Q_\rho A_-^\alpha u - A_-^\alpha u||_X + ||Q_\rho A_+^\alpha u - A_+^\alpha u||_X \to 0 \quad \text{as } \rho \to 0 \end{aligned}$$

because of **(Q)(iii)**. ∎

The following hypothesis intends to ensure the uniform convergence of the Galerkin approximation.

Hypothesis (K)
If $Q_\rho \neq \mathrm{id}$ for some $\rho > 0$ we suppose that $f(\cdot, 0) : X^\alpha \to X$ is a compact[10] map for $\mu = 0$.

Remark 2.1.2 *$D_u f(u, 0) : X^\alpha \to X$ is a compact operator for all $u \in X^\alpha$ if $f(\cdot, 0) : X^\alpha \to X$ is compact.*

Lemma 2.1.3 *Provided that **(Q)** is met and $K \in L[X^\alpha, X]$ is compact, then*

$$||(id - Q_\rho)K||_{L[X^\alpha, X]} \to 0 \; as \; \rho \to 0.$$

Proof We assume that there is a sequence $(v_n)_{n\in\mathbb{N}} \subset X^\alpha$ and $(\rho_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $||v_n||_{X^\alpha} = 1$ and $\rho_n \to 0$ as $n \to \infty$ so that $||(\mathrm{id} - Q_{\rho_n})Kv_n||_{X^0} \geq \delta$ for some positive constant δ. Because K is compact there exists a convergent subsequence $(Kv_{n'})_{n'\in\mathbb{N}}$ with $Kv_{n'} \to w$ in X as $n' \to \infty$. Due to Lemma 2.1.1 we obtain

$$\begin{aligned} ||(id - Q_{\rho_{n'}})Kv_{n'}||_{X^0} &\leq ||(id - Q_{\rho_{n'}})w||_{X^0} + ||(id - Q_{\rho_{n'}})(Kv_{n'} - w)||_{X^0} \\ &\leq ||(id - Q_{\rho_{n'}})w||_{X^0} + (1 + C)||Kv_{n'} - w||_{X^0} \to 0 \end{aligned}$$

as $n' \to \infty$ which leads to a contradiction. ∎

The following theorem deals with the Galerkin approximation

$$\frac{\partial}{\partial x} u = Au + Q_\rho f(u, \mu), \quad (u, \mu) \in X^\alpha \times \mathbb{R} \tag{2.10}$$

of the evolution equation (2.7).

Definition 2.1.4 *h_ρ is a homoclinic solution of (2.10) if*

(i) *$h_\rho(\cdot) \in C^1(\mathbb{R}, X) \cap C^0(\mathbb{R}, X^1)$,*

(ii) *(2.10) holds as an equation in $C^0(\mathbb{R}, X)$,*

(iii) *$h_\rho(x) \to p_\rho$ as $|x| \to \infty$ for some $p_\rho \in X$.*

[10] See Definition A.2.5 in Appendix A.2.

Remark 2.1.5 *Furthermore, the Galerkin approximation (2.10) reduces to* $\frac{\partial}{\partial x}u = Au$ *for initial data in* $(id - Q_\rho)X^\alpha$ *because of* ***(Q)****. The only bounded solution of* $\frac{\partial}{\partial x}u = Au$ *on* $\mathbb{R}$ *is* $u = 0$*. If* $\dim R(Q_\rho) < \infty$ *the norms on* $Q_\rho X^\alpha$ *and* $Q_\rho X$ *are equivalent. Because the equivalence constants tend to infinity as* $\rho \to 0$*, estimates which are uniform with regard to* ρ *can only be expected in the norm of the space* X^α*.*

The following theorem deals with the persistence of the hyperbolic equilibrium and of the homoclinic orbit under the Galerkin approximation. This theorem is the main result of this chapter next to Theorem 2.4.2 of Section 2.4.

Theorem 2.1.6 ***(Persistence of dynamics under Galerkin approximation)***
Provided that the assumptions ***(H1), (H3), (H6)-(H9), (K)*** *and* ***(Q)*** *are satisfied, there are constants* $\rho_0, \delta_0, C > 0$ *so that the following statements are true for any* $0 \leq \rho < \rho_0$ *and* $|\mu| < \delta_0$*:*

(i) *The Galerkin approximation (2.10) has a hyperbolic equilibrium* $p_\rho(\mu) \in R(Q_\rho)$ *with* $p_0(0) = p_0$ *and*

$$||p_\rho(\mu) - p_0||_{X^\alpha} \leq C(||(id - Q_\rho)p_0||_{X^\alpha} + |\mu|).$$

(ii) *For every* ρ *there is a* $\mu_\rho \in \mathbb{R}$ *so that the Galerkin approximation (2.10) has a nondegenerate homoclinic orbit* $h_\rho(x) \in Q_\rho X^\alpha$ *with* $h_\rho(x) \to p_\rho(\mu_\rho)$ *as* $|x| \to \infty$*. Moreover,*

$$|\mu_\rho| + \sup_{x\in\mathbb{R}} ||h_\rho(x) - h(x)||_{X^\alpha} \leq C \sup_{x\in\mathbb{R}} ||(id - Q_\rho)h(x)||_{X^\alpha}.$$

(iii) $p_\rho(\mu)$ *is the only equilibrium and* h_ρ *the only homoclinic solution of (2.10) with*

$$(h_\rho(x), \mu) \in \{(u, \mu) \in X^\alpha \times \mathbb{R} : \ |\mu| + \inf_{\tilde{x}\in\mathbb{R}} ||u - h(\tilde{x})||_{X^\alpha} < \delta_0\} \quad \forall x \in \mathbb{R}.$$

In the next two sections we prove this theorem.

2.2 Persistence of the Hyperbolic Equilibrium

To prove Theorem 2.1.6 we use some classical results from analysis. The next theorem is a consequence of the contraction mapping Theorem A.1.3 with parameters and constitutes a crucial part of the proof. Confer [5] and [15].

Theorem 2.2.1 *Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces and let*

$$G : U \subset X \times \mathbb{R}^p \to Y, \quad (u, \mu) \mapsto G(u, \mu)$$

be continuously differentiable, where U is open and $p \in \mathbb{N}$. Moreover, let $L \in L[X, Y]$ be an invertible operator and assume that there exist $u_0 \in X$, a neighbourhood $V \subset \mathbb{R}^p$ of zero and constants $0 < r$, $0 < \kappa < q < 1$ so that

$$\begin{aligned} &S = \{(u, \mu) \in X \times \mathbb{R}^p : ||u - u_0||_X \leq r, \mu \in V\} \subset U, \\ &||id - L^{-1} D_u G(u, \mu)||_{L[X]} \leq \kappa \quad \forall (u, \mu) \in S, \\ &||L^{-1} G(u_0, \mu)||_X \leq r(1 - q) \quad \forall \mu \in V. \end{aligned}$$

Then, there are a neighbourhood $\bar{V} \subset \mathbb{R}^p$ of zero and $\bar{r} > 0$ so that for every $\mu \in \bar{V}$ the equation $G(u, \mu) = 0$ has a unique solution $\bar{u} = \bar{u}(\mu)$ in $\{u \in X : ||u - u_0|| \leq \bar{r}\}$. The function $\bar{u} : \bar{V} \to X$ is continuously differentiable and

$$||u_0 - \bar{u}(\mu)||_X \leq (1 - q)^{-1} ||L^{-1} G(u_0, \mu)||_X \tag{2.11}$$

holds for $\mu \in \bar{V}$. Finally, $\bar{u} \in C^k(\bar{V}, X)$ if $G \in C^k(U, Y)$.

Proof We define the map $F : X \times \mathbb{R}^p \to X$, $F(u, \mu) = u - L^{-1} G(u, \mu)$. Hereupon we choose the neighbourhood $\bar{V} \subset \mathbb{R}^p$ of zero and $\bar{r} > 0$ sufficiently small so that

$$\begin{aligned} ||F(w, \mu) - F(u, \mu)||_X &= ||D_u F(u, \mu)(w - u) + r_u(w)||w - u||_X||_X \\ &\leq (||D_u F(u, \mu)||_{L[X]} + ||r_u(w)||_Y)||w - u||_X \\ &\leq \left(||\mathrm{id} - L^{-1} D_u G(u, \mu)||_{L[X]} + ||r_u(w)||_X\right) ||w - u||_X \\ &\leq (\kappa + ||r_u(w)||_X)||w - u||_X \\ &\leq q||w - u||_X \end{aligned}$$

holds for all $(u, \mu), (w, \mu) \in \bar{S} = \{(u, \mu) \in X \times \mathbb{R}^p : ||u - u_0||_X \leq \bar{r}, \mu \in \bar{V}\}$ and so that for $g_0 : \bar{V} \to X$, $g_0(\mu) = u_0$, we have the estimate

$$||F(g_0(\mu), \mu) - g_0(\mu)||_Y = ||F(u_0, \mu) - u_0||_Y = ||L^{-1} G(u_0, \mu)||_Y \leq r(1 - q) \quad \forall \mu \in \bar{V}.$$

Then, it follows from Theorem A.1.3 that for every $\mu \in \bar{V}$ the fixed point problem $F(u, \mu) = u$ has a unique solution $\bar{u} = \bar{u}(\mu)$ in $\{u \in X : ||u - u_0|| \leq r\}$ and $\bar{u} : \bar{V} \to X$ is continuous. If $G \in C^k(U, X)$ we obtain $\bar{u} \in C^k(\bar{V}, X)$. Moreover, $G(\bar{u}(\mu), \mu) = 0$ for $\mu \in \bar{V}$ is equivalent to $F(\bar{u}(\mu), \mu) = \bar{u}(\mu) - L^{-1} G(\bar{u}(\mu), \mu) = \bar{u}(\mu)$ for $\mu \in \bar{V}$. Finally, the estimate (A.1) of Theorem A.1.3 results in

$$||u_0 - \bar{u}(\mu)||_X \leq \frac{1}{1 - q} ||u_0 - F(u_0, \mu) - (\bar{u}(\mu) - F(\bar{u}(\mu), \mu))||_X \leq (1 - q)^{-1} ||L^{-1} G(u_0, \mu)||_X$$

for $\mu \in \bar{V}$. ∎

Proof of the equilibrium's persistence

We define for $\rho > 0$ the maps $G_\rho,\, F_\rho :\, X^\alpha \times \mathbb{R} \to X^\alpha$ as follows

$$\begin{aligned} G_\rho(u,\mu) &:= (A + D_u f(p_0,0))^{-1}[A(p_0+u) + Q_\rho f(p_0+u,\mu)], \\ F_\rho(u,\mu) &:= -(A + D_u f(p_0,0))^{-1}\left[Q_\rho(f(p_0+u,\mu) - f(p_0,0) - D_u f(p_0,u)u)\right. \\ &\qquad \left. -(\mathrm{id} - Q_\rho)(f(p_0,0) + D_u f(p_0,0)u)\right]. \end{aligned} \tag{2.12}$$

Note that $(A + D_u f(p_0,0))^{-1} \in L[X, X^\alpha]$ due to **(H6)** and that

$$\begin{aligned} (A + D_u f(p_0,0))^{-1}A &= (A + D_u f(p_0,0))^{-1}(A + D_u f(p_0,0) - D_u f(p_0,0)) \\ &= \mathrm{id} - (A + D_u f(p_0,0))^{-1} D_u f(p_0,0). \end{aligned}$$

That is why we can extend $(A + D_u f(p_0,0))^{-1}A$ to a bounded operator in $L[X^\alpha]$. Consider $D(A) = D(A_\pm) \subset D(A^\alpha_\pm) \Rightarrow D(A) \oplus D(A) \subset D(A^\alpha_+) \oplus D(A^\alpha_-) = X^\alpha$ because of $\alpha \in [0,1)$. We obtain

$$G_\rho(u,\mu) = u - F_\rho(u,\mu). \tag{2.13}$$

To find zeros of $G_\rho(u,\mu)$ near the origin we apply Theorem 2.2.1:

The map $G_\rho : X^\alpha \times \mathbb{R} \to X^\alpha$ is smooth because of $f \in C^2[X^\alpha \times \mathbb{R}, X]$. We set $L = \mathrm{id}$. Considering **(H6)**, $(A + D_u f(p_0,0))^{-1} \in L[X, X^\alpha]$ and $(A + D_u f(p_0,0))^{-1}A \in L[X^\alpha]$ leads to

$$\begin{aligned} &||G_\rho(0,\mu)||_{X^\alpha} \overset{(2.13)}{=} ||0 - F_\rho(0,\mu)||_{X^\alpha} \\ &\le C||(A + D_u f(p_0,0))^{-1}[Q_\rho(f(p_0,\mu) - f(p_0,0)) - (\mathrm{id} - Q_\rho)f(p_0,0)]||_{X^\alpha} \\ &\le C(||(A + D_u f(p_0,0))^{-1}Q_\rho(f(p_0,0) + D_\mu f(p_0,0)\mu + o(|\mu|) - f(p_0,0))||_{X^\alpha} \\ &\quad + ||(A + D_u f(p_0,0))^{-1}A(\mathrm{id} - Q_\rho)p_0||_{X^\alpha}) \\ &\le C(|\mu| + ||(\mathrm{id} - Q_\rho)p_0||_{X^\alpha}) \to 0 \text{ as } \mu, \rho \to 0 \end{aligned} \tag{2.14}$$

and to

$$\begin{aligned} &||\mathrm{id} - D_u G_\rho(u,\mu)||_{X^\alpha} \overset{(2.13)}{=} ||\mathrm{id} - (\mathrm{id} - D_u F_\rho(u,\mu))||_{X^\alpha} \\ &= ||(A + D_u f(p_0,0))^{-1}[Q_\rho(D_u f(p_0+u,\mu) - D_u f(p_0,0)) - (\mathrm{id} - Q_\rho)D_u f(p_0,0)]||_{X^\alpha} \\ &= ||(A + D_u f(p_0,0))^{-1}[Q_\rho D_u f(p_0+u,\mu) - D_u f(p_0,0)]||_{X^\alpha} \\ &\le C||Q_\rho D_u f(p_0+u,\mu) - D_u f(p_0,0)||_{X^\alpha} \\ &\le C\left|\left|Q_\rho\left(D_u f(p_0,0) + DD_u f(p_0,0)\begin{pmatrix} u \\ \mu \end{pmatrix} + o\left(\left|\left|\begin{pmatrix} u \\ \mu \end{pmatrix}\right|\right|_{X^\alpha\times\mathbb{R}}\right)\right) - D_u f(p_0,0)\right|\right|_{X^\alpha} \\ &\le C||(\mathrm{id} - Q_\rho)D_u f(p_0,0)||_{X^\alpha} + C(|\mu| + ||u||_{X^\alpha}) \\ &< \frac{1}{2} \end{aligned}$$

for all u, μ and ρ sufficiently close to zero. Consider that $(u,\mu) \mapsto D_u f(p_0+u,\mu)$ is continuously differentiable because of $f \in C^2(X^\alpha \times \mathbb{R}, X)$ and that $DD_u f(p_0,0)$ is its Frechet derivative at the origin, confer [27]. Moreover, note that we employed the compactness of $D_u f(p_0,0)$ and Lemma 2.1.3.

Theorem 2.2.1 implies that $G_\rho(u,\mu)=0$ has a unique solution $\tilde{p}_\rho(\mu)$ in $\{u \in X^\alpha : ||u||_{X^\alpha} \leq \bar{r}\}$ for every sufficiently small ρ, for some $\bar{r} > 0$ and for every $\mu \in \bar{V}$, where $\bar{V}$ is some neighbourhood of zero. Therefore, $G_\rho(\tilde{p}_\rho(\mu),\mu)=0$ for all $\mu \in \bar{V}$. Defining

$$p_\rho(\mu) := p_0 + \tilde{p}_\rho(\mu) \tag{2.15}$$

we obtain

$$Ap_\rho(\mu) + Q_\rho f(p_\rho(\mu),\mu) = 0.$$

$p_\rho : \bar{V} \to X^\alpha$ is smooth and satisfies

$$||p_\rho(\mu) - p_0||_{X^\alpha} = ||\tilde{p}_\rho(\mu) - 0||_{X^\alpha} \overset{(2.11)}{\leq} C||G_\rho(0,\mu)||_{X^\alpha} \overset{(2.14)}{\leq} C(|\mu| + ||(\mathrm{id} - Q_\rho)p_0||_{X^\alpha}).$$

It follows from $p_\rho(\mu) = -Q_\rho A^{-1} f(p_\rho(\mu),\mu)$ that $p_\rho(\mu)$ is an element of $R(Q_\rho)$. Due to uniqueness we have $p_0(0) = p_0$. Furthermore, we have

$$\begin{aligned}
&\lambda i - (A + Q_\rho D_u f(p_\rho(\mu),\mu)) \\
&= [\lambda i - (A + D_u f(p_0,0))] \left[\mathrm{id} + (\lambda i - (A + D_u f(p_0,0)))^{-1}(D_u f(p_0,0) - Q_\rho D_u f(p_\rho(\mu),\mu))\right]
\end{aligned}$$

and

$$\begin{aligned}
&||D_u f(p_0,0) - Q_\rho D_u f(p_\rho(\mu),\mu)||_{L[X^\alpha,X]} \\
&\leq \left|\left| D_u f(p_0,0) - Q_\rho \left[D_u f(p_0,0) + DD_u f(p_0,0) \begin{pmatrix} \tilde{p}_\rho(\mu) \\ \mu \end{pmatrix} + o\left(\left|\left| \begin{pmatrix} \tilde{p}_\rho(\mu) \\ \mu \end{pmatrix} \right|\right|_{X^\alpha \times \mathbb{R}} \right) \right] \right|\right|_{L[X^\alpha,X]} \\
&\leq ||(\mathrm{id} - Q_\rho) D_u f(p_0,0)||_{L[X^\alpha,X]} \\
&\quad + C \left|\left| DD_u f(\tilde{p}_\rho(\mu),\mu) \begin{pmatrix} \tilde{p}_\rho(\mu) \\ \mu \end{pmatrix} + o\left(\left|\left| \begin{pmatrix} \tilde{p}_\rho(\mu) \\ \mu \end{pmatrix} \right|\right|_{X^\alpha \times \mathbb{R}} \right) \right|\right|_{L[X^\alpha,X]} \to 0 \quad \text{as } \rho,\mu \to 0.
\end{aligned}$$

This and hypothesis **(H6)** result in the invertibility of $\lambda i - (A + Q_\rho D_u f(p_\rho(\mu),\mu))$ for all $\lambda \in \mathbb{R}$ and for sufficiently small $\rho, |\mu|$. Note that $\lambda i - (A + D_u f(p_0,0))$ is invertible for every $\lambda \in \mathbb{R}$ with $||(\lambda i - (A + D_u f(p_0,0)))^{-1}||$ bounded independently of λ because p_0 is a hyperbolic equilibrium of (2.7). Thus, we can conclude that $\Re(\sigma(A + Q_\rho D_u f(p_\rho(\mu),\mu))) \neq 0$ and that $p_\rho(\mu)$ is a hyperbolic equilibrium for sufficiently small $\rho, |\mu|$. ∎

The following lemma is needed for later purposes.

Lemma 2.2.2

$$\frac{\partial}{\partial x} v = (A + Q_\rho D_u f(p_\rho(\mu),\mu))v \tag{2.16}$$

has an exponential dichotomy on $\mathbb{R}$ for sufficiently small μ. The associated projections are denoted by $P_{+,\rho}(\mu)$ and $P_{-,\rho}(\mu)$.

Proof We apply the roughness Theorem 1.2.1 to the following rewritten form of (2.16):

$$\frac{d}{dx} v = (A + D_u f(p_0,0))v + (Q_\rho D_u f(p_\rho(\mu),\mu) - D_u f(p_0,0))v.$$

The operator $A + D_u f(p_0, 0)$ satisfies **(H1)** and **(H3)**. Defining the linear operators $B(x) := Q_\rho D_u f(p_\rho(\mu), \mu) - D_u f(p_0, 0)$ results in $B(\cdot) \in C^{0,\vartheta}(\mathbb{R}, L[X^\alpha, X])$ for every $\vartheta > 0$ and[11] in

$$
\begin{aligned}
&||B(x)||_{L[X^\alpha,X]} \\
&= ||Q_\rho D_u f(p_\rho(\mu), \mu) - D_u f(p_0, 0)||_{L[X^\alpha,X]} \\
&\leq \left|\left| Q_\rho \left(D_u f(p_0,0) + DD_u f(p_0,0) \begin{pmatrix} \tilde{p}_\rho(\mu) \\ \mu \end{pmatrix} + o\left(\left|\left| \begin{pmatrix} \tilde{p}_\rho(\mu) \\ \mu \end{pmatrix} \right|\right|_{X^\alpha \times \mathbb{R}} \right) \right) - D_u f(p_0,0) \right|\right|_{L[X^\alpha,X]} \\
&\leq ||(\mathrm{id} - Q_\rho) D_u f(p_0,0)||_{L[X^\alpha,X]} + C \left|\left| \begin{pmatrix} \tilde{p}_\rho(\mu) \\ \mu \end{pmatrix} \right|\right|_{X^\alpha \times \mathbb{R}} \to 0 \quad \text{as } \rho, \mu \to 0.
\end{aligned}
$$

For the latter we used $f \in C^2[X^\alpha \times \mathbb{R}, X]$, Lemma 2.1.3 and Hypothesis **(Q)**. Moreover, the problem $\frac{d}{dx} v = (A + B)v$, $v(0) = 0$, is uniquely satisfied by $v = 0$. The same is true for the adjoint equation. Finally, all conditions of Theorem 1.2.1 are satisfied. So for all μ in a small neighbourhood of zero (2.16) has an exponential dichotomy on $\mathbb{R}$ with projections $P_{+,\rho}(\mu)$ and $P_{-,\rho}(\mu)$. ∎

2.3 Persistence of the Homoclinic Orbit

In this section we prove Theorem 2.1.6 (ii) and use again Theorem 2.2.1. We follow [15] but we describe the proof in more detail and give some new arguments which are in particular needed for joining solutions of different semiaxes.

Definition 2.3.1 *For $\rho > 0$ define the maps F_ρ, $\hat{F}_\rho : \mathbb{R} \times X^\alpha \times \mathbb{R} \to X$ by*

$$
\begin{aligned}
F_\rho(x, v, \mu) &:= D_\mu f(h(x), 0)\mu + \hat{F}_\rho(x, v, \mu), \\
\hat{F}_\rho(x, v, \mu) &:= -(id - Q_\rho)[D_u f(h(x), 0)v + D_\mu f(h(x), 0)\mu + f(h(x), 0)] \\
&\quad + Q_\rho(f(h(x) + v, \mu) - f(h(x), 0) - D_u f(h(x), 0)v - D_\mu f(h(x), 0)\mu).
\end{aligned}
\tag{2.17}
$$

Substituting

$$
u(x) = h(x) + v(x), \quad x \in \mathbb{R}, \tag{2.18}
$$

in $\frac{\partial}{\partial x} u = Au + Q_\rho f(u, \mu)$ leads to

$$
\begin{aligned}
\frac{\partial}{\partial x} v &= (A + D_u f(h(x), 0))v + F_\rho(x, v, \mu) \\
&= (A + D_u f(h(x), 0))v + D_\mu f(h(x), 0)\mu + \hat{F}_\rho(x, v, \mu).
\end{aligned}
\tag{2.19}
$$

In the following we search a strong solution v of this differential equation, where strong is defined by $v \in C^1(\mathbb{R}, X) \cap C^0(\mathbb{R}, X^1)$.

Lemma 2.3.2

$$
\frac{\partial}{\partial x} v = (A + D_u f(h(x), 0))v \tag{2.20}
$$

has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$.

[11] Consider definition (2.15).

Proof First, we write

$$(A + D_u f(h(x), 0))v = (A + D_u f(p_0, 0))v + (D_u f(h(x), 0) - D_u f(p_0, 0))v.$$

Because of **(H6), (H8)** and because of $D_u f(h(x), 0) - D_u f(p_0, 0) \in C^{0,\vartheta}(\mathbb{R}, L[X^\alpha, X])$ converging to zero as $|x| \to \infty$ the roughness Theorem 1.2.1 ensures that the equation (2.20) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. ∎

Definition 2.3.3 *We call the associated projections*[12] *of (2.20) P_- and $P_+ = id - P_-$. Moreover, we define*[13] *the solution operators of (2.20) by $\Phi^s(x, x_0)$ for $x \geq x_0$ and $\Phi^u(x, x_0)$ for $x \leq x_0$. Finally, we set $\Phi^s_+(x, x_0) := \Phi^s(x, x_0)$ and $\Phi^u_+(x_0, x) := \Phi^u(x, x_0)$ for $x \geq x_0 \geq 0$ and $\Phi^s_-(x_0, x) := \Phi^s(x, x_0)$ and $\Phi^u_-(x, x_0) := \Phi^u(x, x_0)$ for $x \leq x_0 \leq 0$.*

Lemma 2.3.4

$$\frac{\partial}{\partial x_0}\Phi^s_\pm(x, x_0)w = -\Phi^s_\pm(x, x_0)(A + D_u f(h(x_0), 0))w, \quad x \geq x_0.$$

Proof On the one hand, for $\sigma \leq x_0 \leq x$ we obtain $\Phi^s_+(x, x_0)\Phi^s_+(x_0, \sigma) = \Phi^s_+(x, \sigma)$ and therefore

$$\begin{aligned} 0 &= \frac{\partial}{\partial x_0}[\Phi^s_+(x, x_0)\Phi^s_+(x_0, \sigma)]w \\ &= \left(\frac{\partial}{\partial x_0}\Phi^s_+(x, x_0) + \Phi^s_+(x, x_0)[A + D_u f(h(x_0), 0)]\right)\Phi^s_+(x_0, \sigma)w. \end{aligned}$$

On the other hand, for $x_0 \leq x$ and $x_0 \leq \sigma$ we obtain $\Phi^s_+(x, x_0)\Phi^u_+(x_0, \sigma) = 0$ and therefore

$$\begin{aligned} 0 &= \frac{\partial}{\partial x_0}[\Phi^s_+(x, x_0)\Phi^u_+(x_0, \sigma)]w \\ &= \left(\frac{\partial}{\partial x_0}\Phi^s_+(x, x_0) + \Phi^s_+(x, x_0)[A + D_u f(h(x_0), 0)]\right)\Phi^u_+(x_0, \sigma)w. \end{aligned}$$

Setting $\sigma = x_0$, combining the equations and considering $\Phi^s_+(x_0, x_0) + \Phi^u_+(x_0, x_0) = \mathrm{id}$ leads to the assertion of the lemma. ∎

As in Chapter 1 we introduce a mild formulation of the considered differential equation (2.19). But here we will employ Theorem 2.2.1, an implication of the contraction mapping theorem, in order to prove the main results of this chapter.

[12] **(H6)** assumes that $A + D_u f(p_0, 0)$ satisfies **(H1)**
[13] Recall Definition 1.4.5.

Definition 2.3.5 ***(Mild formulation)***
The equations

$$\begin{aligned} v_+(x) &= \Phi^s_+(x,0)b_+ + \int_0^x \Phi^s_+(x,x_0)F_\rho(x_0,v_+(x_0),\mu)dx_0 \\ &\quad + \int_\infty^x \Phi^u_+(x,x_0)F_\rho(x_0,v_+(x_0),\mu)dx_0, \qquad\qquad x\in\mathbb{R}^+, \\ v_-(x) &= \Phi^u_-(x,0)b_- + \int_0^x \Phi^u_-(x,x_0)F_\rho(x_0,v_-(x_0),\mu)dx_0 \\ &\quad + \int_{-\infty}^x \Phi^s_-(x,x_0)F_\rho(x_0,v_-(x_0),\mu)dx_0, \qquad\qquad x\in\mathbb{R}^-, \\ v_+(0) &= v_-(0) \end{aligned} \tag{2.21}$$

with $(b_+,b_-)\in R(\Phi^s_+(0,0))\times R(\Phi^u_-(0,0))$*and* $\mu\in\mathbb{R}$ *are called the mild formulation of the nonlinear equation (2.19). We call a solution*

$$(v_+,v_-)\in C^0(\mathbb{R}^+,X^\alpha)\times C^0(\mathbb{R}^-,X^\alpha)$$

of (2.21) a mild solution of (2.19).

Remark 2.3.6 *In the following it suffices to find mild solutions due to equivalence of bounded mild and bounded strong solutions of (2.19), see the following Theorem 2.3.8. Before proving this equivalence we need Lemma 2.3.7.*

Let $J\subset\mathbb{R}$ be a closed interval and consider the differential equation

$$\frac{\partial}{\partial x}u(x) = Au(x) + r(x), \quad x\in J. \tag{2.22}$$

We use the notion of a solution corresponding to (1.1):

- $u\in C^0(\mathring{J},X^1)\cap C^1(\mathring{J},X)$,
- $u\in C^0(J,X^\alpha)$,
- (2.22) holds as an equation in $C^0(\mathring{J},X)$.

Lemma 2.3.7 *Let $A: D(A)\subset X\to X$ be a densely defined and closed operator on a Banach space $(X,||\cdot||)$ satisfying* ***(H1)****. Moreover, let $r\in C^{0,\vartheta}(\mathbb{R},X)$, $\vartheta>1$. Then, the following statements hold:*

(i) *u_+ is a bounded solution of (2.22) on $\mathbb{R}^+$ if and only if*

$$u_+(x) = -e^{-A_-x}b_- + \int_0^x e^{A_-(\sigma-x)}P_-r(\sigma)d\sigma - \int_x^\infty e^{A_+(x-\sigma)}P_+r(\sigma)d\sigma \tag{2.23}$$

for some $b_-\in X_-$ and for all $x\geq 0$.

(ii) *u_- is a bounded solution of (2.22) on $\mathbb{R}^-$ if and only if*

$$u_-(x) = e^{-A_+ x} b_+ + \int_0^x e^{A_+(x-\sigma)} P_+ r(\sigma) d\sigma + \int_{-\infty}^x e^{A_-(\sigma - x)} P_- r(\sigma) d\sigma \qquad (2.24)$$

for some $b_+ \in X_+$ and for all $x \leq 0$.

(iii) *u is a bounded solution of (2.22) on $\mathbb{R}$ if and only if*

$$u(x) = -\int_x^\infty e^{A_+(x-\sigma)} P_+ r(\sigma) d\sigma + \int_{-\infty}^x e^{A_-(\sigma - x)} P_- r(\sigma) d\sigma \qquad (2.25)$$

for all $x \in \mathbb{R}$.

(iv) *If u_+ and u_- are given by (2.23) and (2.24), respectively, and satisfy $u_-(0) = u_+(0)$ for some $b_- \in X_-$ and $b_+ \in X_+$, the function*

$$u(x) := \begin{cases} u_+(x), & x \in \mathbb{R}^+ \\ u_-(x), & x \in \mathbb{R}^- \end{cases}$$

satisfies

$$u(x) = -\int_x^\infty e^{A_+(x-\sigma)} P_+ r(\sigma) d\sigma + \int_{-\infty}^x e^{-A_-(x-\sigma)} P_- r(\sigma) d\sigma, \quad x \in \mathbb{R}.$$

The function u is also differentiable at the origin and is a bounded solution of (2.22) on $\mathbb{R}$.

Proof (i) Let u_+ be a bounded solution of (2.22). Defining

$$v_+(x) := -e^{-A_- x} b_- + \int_0^x e^{A_-(\sigma - x)} P_- r(\sigma) d\sigma - \int_x^\infty e^{A_+(x-\sigma)} P_+ r(\sigma) d\sigma, \quad x \geq 0,$$

yields $\frac{\partial}{\partial x} v_+(x) = A v_+(x) + r(x)$ for $x > 0$. Consider [11] Lemma 3.5.1 and the local Hölder continuity of r. Furthermore, we obtain $\frac{\partial}{\partial x}(v_+ - u_+)(x) = A(v_+ - u_+)(x)$ for $x > 0$ by subtraction. If we set $b_- := -P_- u_+(0)$ and $\eta := v_+ - u_+$ we get $P_- \eta(0) = 0$. Finally, it follows from Corollary 1.1.7 that $\eta = 0$ on $\mathbb{R}^+$.

Conversely, let

$$u_+(x) = -e^{-A_- x} b_- + \int_0^x e^{A_-(\sigma - x)} P_- r(\sigma) d\sigma - \int_x^\infty e^{A_+(x-\sigma)} P_+ r(\sigma) d\sigma, \quad x \geq 0.$$

This results directly in $u_+ \in C^0([0, \infty), X^\alpha)$. Considering [11] Lemma 3.5.1 and the local Hölder continuity of r we have $u_+ \in C^1((0, \infty), X)$. Differentiating u_+ with respect to x yields

$$\frac{\partial}{\partial x} u_+ = A u_+ + r.$$

Moreover, writing

$$A u_+ = \frac{\partial}{\partial x} u_+ - r \in C^0((0, \infty), X)$$

shows $u_+ \in C^0((0,\infty), X^1)$.

(ii)-(iii) are similar to (i).

(iv) $u_-(0) = u_+(0)$ results in

$$-\int_0^\infty e^{-A_+\sigma} P_+(\sigma) d\sigma - b_+ = b_- + \int_{-\infty}^0 e^{A_-\sigma} P_- r(\sigma) d\sigma \in X_- \cap X_+ = \{0\}.$$

Hence

$$\begin{aligned} e^{A_+x} b_+ &= -\int_0^\infty e^{A_+(x-\sigma)} P_+ r(\sigma) d\sigma \\ &= -\int_0^x e^{A_+(x-\sigma)} P_+ r(\sigma) d\sigma - \int_x^\infty e^{A_+(x-\sigma)} P_+ r(\sigma) d\sigma, \quad x \le 0, \\ -e^{-A_-x} b_- &= \int_{-\infty}^0 e^{-A_-(x-\sigma)} P_- r(\sigma) d\sigma \\ &= \int_x^0 e^{-A_-(x-\sigma)} P_- r(\sigma) d\sigma + \int_{-\infty}^x e^{-A_-(x-\sigma)} P_- r(\sigma) d\sigma, \quad x \ge 0. \end{aligned}$$

Putting this into the expressions for u_- and u_+, respectively, yields

$$\begin{aligned} u_+(x) &= -\int_x^\infty e^{A_+(x-\sigma)} P_+ r(\sigma) d\sigma + \int_{-\infty}^x e^{-A_-(x-\sigma)} P_- r(\sigma) d\sigma, \quad x \ge 0, \\ u_-(x) &= -\int_x^\infty e^{A_+(x-\sigma)} P_+ r(\sigma) d\sigma + \int_{-\infty}^x e^{-A_-(x-\sigma)} P_- r(\sigma) d\sigma, \quad x \le 0. \end{aligned}$$

Finally, the strong solution properties on $\mathbb{R}$ follow as shown in (i) for $\mathbb{R}^\pm$. ∎

Theorem 2.3.8 $(v_+, v_-) \in C^0(\mathbb{R}^+, X^\alpha) \times C^0(\mathbb{R}^-, X^\alpha)$ *is a bounded mild solution of (2.19) if and only if the function v defined by*

$$v(x) = \begin{cases} v_+(x), & x \in \mathbb{R}^+ \\ v_-(x), & x \in \mathbb{R}^- \end{cases}$$

is a bounded strong solution of (2.19) on $\mathbb{R}$.

Proof According to Section 4 in [19] $v_\pm \in C^0(\mathbb{R}^\pm, X^\alpha)$ are bounded mild solutions of (2.19) on $\mathbb{R}^\pm$ if and only if $v_\pm$ are bounded strong solutions of (2.19) on $\mathbb{R}^\pm$.

Here, we show how to join the mild solutions on the semiaxes to obtain a strong solution on $\mathbb{R}$. Let $v_\pm \in C^0(\mathbb{R}^\pm, X^\alpha)$ be bounded mild solutions of (2.19). So they are strong solutions on $\mathbb{R}^\pm$ and satisfy

$$\frac{\partial}{\partial x} v_\pm(x) = (A + D_u f(h(x), 0)) v_\pm(x) + F_\rho(x, v_\pm(x), \mu), \quad x \in \mathring{\mathbb{R}}^\pm.$$

Because $D_u f(h(x), 0) v_\pm(x) + F_\rho(x, v_\pm(x), \mu)$ is locally Hölder continuous on $\mathring{\mathbb{R}}^\pm$ we can apply Lemma 2.3.7. Part (i), $v_+(0) = v_-(0)$ and part (iv) lead to the differentiability of

$$v(x) = \begin{cases} v_+(x), & x \in \mathbb{R}^+ \\ v_-(x), & x \in \mathbb{R}^- \end{cases}$$

and the strong solution properties of v.

Conversely, if v is a bounded strong solution of (2.19) on $\mathbb{R}$ decompose v into the restrictions $v_+ := v|_{\mathbb{R}^+}$ and $v_- := v|_{\mathbb{R}^-}$ and use again the results of Section 4 in [19]. ∎

In order to apply Theorem 2.2.1 we have to define a map and Banach spaces which are determined by the mild formulation.

Definition 2.3.9 *Let*

$$Y := R(\Phi_+^s(0,0)) \times R(\Phi_-^u(0,0)) \times C_b^0(\mathbb{R}^+, X^\alpha) \times C_b^0(\mathbb{R}^-, X^\alpha) \times \mathbb{R},$$
$$\hat{Y} := C_b^0(\mathbb{R}^+, X^\alpha) \times C_b^0(\mathbb{R}^-, X^\alpha) \times X^\alpha \times \mathbb{R},$$

$$G_\rho(b_+, b_-, v_+, v_-, \mu) :=$$
$$\begin{pmatrix} v_+(\cdot) - \Phi_+^s(\cdot,0)b_+ - \int_0^{\cdot} \Phi_+^s(\cdot,x_0)F_\rho(x_0,v_+(x_0),\mu)dx_0 - \int_\infty^{\cdot} \Phi_+^u(\cdot,x_0)F_\rho(x_0,v_+(x_0),\mu)dx_0 \\ v_-(\cdot) - \Phi_-^u(\cdot,0)b_- - \int_0^{\cdot} \Phi_-^u(\cdot,x_0)F_\rho(x_0,v_-(x_0),\mu)dx_0 - \int_{-\infty}^{\cdot} \Phi_-^s(\cdot,x_0)F_\rho(x_0,v_-(x_0),\mu)dx_0 \\ b_+ - b_- - \int_0^\infty \Phi_+^u(0,x_0)F_\rho(x_0,v_+(x_0),\mu)dx_0 - \int_{-\infty}^0 \Phi_-^s(0,x_0)F_\rho(x_0,v_-(x_0),\mu)dx_0 \\ \langle \varphi, \Phi_+^s(0,0)b_+ - \int_0^\infty \Phi_+^u(0,x_0)F_\rho(x_0,v_+(x_0),\mu)dx_0 \rangle \end{pmatrix}$$

for $\rho > 0$ and $\varphi \in (X^\alpha)'$ with $\langle \varphi, \frac{\partial}{\partial x} h(0) \rangle = 1$.

Remark 2.3.10 *The forth component of G_ρ takes the translational invariance of the Galerkin approximation $\frac{\partial}{\partial x} u = Au + Q_\rho f(u,\rho)$ into consideration and ensures the uniqueness of the solution.*

Lemma 2.3.11 *G_ρ can be considered as a map defined on a sufficiently small neighbourhood U of the origin in Y with values in $\hat{Y}$.*

Proof We prove exemplary $-\int_\infty^{\cdot} \Phi_+^u(\cdot,x_0)F_\rho(x_0,v_+(x_0),\mu)dx_0 \in C_b^0(\mathbb{R}^+, X^\alpha)$ which is a term of the first component of G_ρ. It follows from Theorem 1.4.6:

$$\Phi_+^u(x,x_0) \in L[X,X^\alpha], \quad ||\Phi_+^u(x,x_0)||_{L[X,X^\alpha]} \le C \max\{1,(x_0-x)^{-\alpha}\} e^{-\eta|x-x_0|}, \quad x_0 > x \ge 0.$$

Furthermore, we get $F_\rho(x_0, v_+(x_0), \mu) \in X$ and $||F_\rho(x_0,v_+(x_0),\mu)||_X < C$ for all $x_0 \in [x,\infty)$ if U is sufficiently small. To prove this we proceed as follows:
Consider the Definition (2.17) of F_ρ. Because of $h(x) \to p_0$ as $|x| \to \infty$ we only have to treat the term $f(h(x_0)+v_+(x_0),\mu) - f(h(x_0),0)$ for $x_0 \in [x,\infty)$.
First, we show that the set $K := \{h(x) : x \in \mathbb{R}\}$ is compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If it is not bounded there exists a subsequence $(x_{n'})_{n'\in\mathbb{N}}$ with $|x_{n'}| \to \infty$ and $h(x_{n'}) \to p_0$. If it is bounded there is a convergent subsequence $(x_{n'})_{n'\in\mathbb{N}}$ with $x_0 := \lim_{n'\to\infty} x_{n'} \in \mathbb{R}$ and $h(x_{n'}) \to h(x_0)$.
Because $f(\cdot,\cdot)$ is continuous and $K \times \{0\}$ is compact there is an open neighbourhood $\tilde{O}$ of $K \times \{0\}$ so that $f(\cdot,\cdot)$ is uniformly continuous on $\tilde{O}$ (consider proof of Theorem 9 in Paragraph 3 of [10]). Hence for every $C > 0$ there is a $\delta > 0$ such that

$$||f(h(x_0)+v_+(x_0),\mu) - f(h(x_0),0)|| \le C, \quad \forall x_0 \in [0,\infty)$$

if $||(v_+,\mu)||_{C_b^0\times\mathbb{R}} < \delta$. This explains why we need to choose U sufficiently small.

In the following we regard a fixed $x \in \mathbb{R}^+$ and choose $\tilde{x}$ so that $\tilde{x} > x$ and $\tilde{x} - x \geq 1$. Then,

$$\begin{aligned}
&\left|\left|\int_x^\infty \Phi_+^u(x,x_0)F_\rho(x_0,v_+(x_0),\mu)dx_0\right|\right|_{X^\alpha}\\
&\leq \int_x^\infty ||\Phi_+^u(x,x_0)||_{L[X,X^\alpha]}||F_\rho(x_0,v_+(x_0),\mu)||_X dx_0\\
&\leq C\int_x^\infty \max\{1,(x_0-x)^{-\alpha}\}e^{-\eta(x_0-x)}dx_0\\
&\leq C\int_x^{\tilde{x}} (x_0-x)^{-\alpha}e^{-\eta(x_0-x)}dx_0 + C\int_{\tilde{x}}^\infty e^{-\eta(x_0-x)}dx_0.\\
&= C\int_0^{\tilde{x}-x} s^{1-\alpha-1}e^{-s}ds + Ce^{-\eta(\tilde{x}-x)}\\
&\leq C\Gamma(1-\alpha)+C.
\end{aligned}$$

The bound‘s independence of x yields the assertion. ∎

In the following we apply Theorem 2.2.1 to G_ρ. We now split G_ρ into a linear and a quadratic part using equation (2.17):

$$G_\rho(b_+,b_-,v_+,v_-,\mu) = L(b_+,b_-,v_+,v_-,\mu) - \hat{G}_\rho(b_+,b_-,v_+,v_-,\mu), \tag{2.26}$$

where L and $\hat{G}_\rho$ are given by

$$L(b_+,b_-,v_+,v_-,\mu) \;=$$
$$\begin{pmatrix}
v_+(\cdot) - \Phi_+^s(\cdot,0)b_+ - \mu(\int_0^\cdot \Phi_+^s(\cdot,x_0)D_\mu f(h(x_0),0)dx_0 + \int_\infty^\cdot \Phi_+^u(\cdot,x_0)D_\mu f(h(x_0),0)dx_0)\\
v_-(\cdot) - \Phi_-^u(\cdot,0)b_- - \mu(\int_0^\cdot \Phi_-^u(\cdot,x_0)D_\mu f(h(x_0),0)dx_0 + \int_{-\infty}^\cdot \Phi_-^s(\cdot,x_0)D_\mu f(h(x_0),0)dx_0)\\
b_+ - b_- - \mu(\int_0^\infty \Phi_+^u(0,x_0)D_\mu f(h(x_0),0)dx_0 + \int_{-\infty}^0 \Phi_-^s(0,x_0)D_\mu f(h(x_0),0)dx_0)\\
\langle\varphi, \Phi_+^s(0,0)b_+ - \mu\int_0^\infty \Phi_+^u(0,x_0)D_\mu f(h(x_0),0)dx_0\rangle
\end{pmatrix},$$

$$\hat{G}_\rho(b_+,b_-,v_+,v_-,\mu) =$$
$$\begin{pmatrix}
\int_0^\cdot \Phi_+^s(\cdot,x_0)\hat{F}_\rho(x_0,v_+(x_0),\mu)dx_0 + \int_\infty^\cdot \Phi_+^u(\cdot,x_0)\hat{F}_\rho(x_0,v_+(x_0),\mu)dx_0\\
\int_0^\cdot \Phi_-^u(\cdot,x_0)\hat{F}_\rho(x_0,v_-(x_0),\mu)dx_0 + \int_{-\infty}^\cdot \Phi_-^s(\cdot,x_0)\hat{F}_\rho(x_0,v_-(x_0),\mu)dx_0\\
\int_0^\infty \Phi_+^u(0,x_0)\hat{F}_\rho(x_0,v_+(x_0),\mu)dx_0 + \int_{-\infty}^0 \Phi_-^s(0,x_0)\hat{F}_\rho(x_0,v_-(x_0),\mu)dx_0\\
\langle\varphi, \int_0^\infty \Phi_+^u(0,x_0)\hat{F}_\rho(x_0,v_+(x_0),\mu)dx_0\rangle
\end{pmatrix}.$$

The linear part $L : Y \to \hat{Y}$ is bounded and its last two components are independent of v_+ and v_-. In particular, L is continuously invertible, which will be proven by using the following map and lemma.

Definition 2.3.12

$$\Psi_0 : R(\Phi_+^s(0,0)) \times R(\Phi_-^u(0,0)) \to X^\alpha, \quad \Psi_0(b_+,b_-) = b_+ - b_- \tag{2.27}$$

Ψ_0 is a Fredholm operator with index zero[14] and satisfies:

Lemma 2.3.13

(i) $N(\Psi_0) = \operatorname{span}\left\{\left(\frac{\partial}{\partial x}h(0), \frac{\partial}{\partial x}h(0)\right)\right\}$,

(ii) $R(\Psi_0) = \{w \in X^\alpha : \langle \psi(0), w\rangle = 0\}$,

(iii) $\langle \varphi, \Phi_+^s(0,0)\frac{\partial}{\partial x}h(0)\rangle = \langle \varphi, \frac{\partial}{\partial x}h(0)\rangle = 1$,

(iv) $\left\langle \psi(0), \int_0^\infty \Phi_+^u(0,x_0)D_\mu f(h(x_0),0)dx_0 + \int_{-\infty}^0 \Phi_-^s(0,x_0)D_\mu f(h(x_0),0)dx_0 \right\rangle$
$= \int_{-\infty}^\infty \langle \psi(x_0), D_\mu f(h(x_0),0)\rangle dx_0 \overset{\textbf{(H9)}}{\neq} 0.$

Proof **(i)** Let $(b_+, b_-) \in N(\Psi_0)$, i.e. $b_+ = b_- \in R(\Phi_+^s(0,0)) \cap R(\Phi_-^u(0,0))$ and

$$u(x) := \begin{cases} \Phi_+^s(x,0)b_+, & x \in \mathbb{R}^+ \\ \Phi_-^u(x,0)b_-, & x \in \mathbb{R}^-. \end{cases}$$

Considering Lemma 2.3.7 the function u is a bounded solution of the differential equation $\frac{\partial}{\partial x}v = (A + D_u f(h(x),0))v$. Then, it follows from Hypothesis **(H7)** that there is a constant c so that $u(x) = c\frac{\partial h(x)}{\partial x}$ for all $x \in \mathbb{R}$. Therefore, $(b_+, b_-) \in \operatorname{span}\left\{\left(\frac{\partial}{\partial x}h(0), \frac{\partial}{\partial x}h(0)\right)\right\}$. Conversely, let $(b_+, b_-) \in \operatorname{span}\left\{\left(\frac{d}{dx}h(0), \frac{d}{dx}h(0)\right)\right\}$. Then, (b_+, b_-) is an element of $R(\Phi_+^s(0,0)) \times R(\Phi_-^u(0,0))$ with $\Psi(b_+, b_-) = 0$.

(ii) Let $(b_+, b_-) \in R(\Phi_+^s(0,0)) \times R(\Phi_-^u(0,0))$ so there are w_+ and w_- with $b_+ = \Phi_+^s(0,0)w_+$ and $b_- = \Phi_-^u(0,0)w_-$. Since $\Phi_+^s(\cdot,0)w_+$ solves (2.20) and ψ solves its adjoint equation (2.9), see Remark 2.0.8, we get $\frac{d}{dx}\langle \psi(x), \Phi_+^s(x,0)w_+\rangle = 0$. The boundedness of ψ and the exponential decay of $\Phi_+^s(\cdot,0)w_+$ yields $\int_0^\infty |\langle \psi(x), \Phi_+^s(x,0)w_+\rangle| dx < \infty$ which results in $\langle \psi(x), \Phi_+^s(x,0)w_+\rangle = 0$ and $\langle \psi(0), b_+\rangle = 0$. In a similar way we can prove $\langle \psi(0), b_-\rangle = 0$. Therefore, $\langle \psi(0), b_+ - b_-\rangle = 0$. Since Ψ_0 is a Fredholm operator with index zero, $\dim N(\Psi_0) = 1$ and $R(\Psi_0) \subset \{w \in X^\alpha : \langle \psi(0), w\rangle = 0\}$ we get $R(\Psi_0) = \{w \in X^\alpha : \langle \psi(0), w\rangle = 0\}$.

(iii) There is a $v \in X^\alpha$ and a constant c so that $\frac{d}{dx}h(0) = c\,\Phi_+^s(0,0)v$. Theorem 1.4.6 (i) yields $\Phi_+^s(0,0)\frac{d}{dx}h(0) = \Phi_+^s(0,0)c\,\Phi_+^s(0,0)v = c\,\Phi_+^s(0,0)v = \frac{d}{dx}h(0)$. The choice of $\langle \varphi, \frac{d}{dx}h(0)\rangle = 1$ completes the proof of (iii).

(iv)

$$\begin{aligned} &\int_{-\infty}^\infty \langle \psi(x_0), D_\mu f(h(x_0),0)\rangle dx_0 \\ &= \int_0^\infty \langle \psi(x_0), D_\mu f(h(x_0),0)\rangle dx_0 + \int_{-\infty}^0 \langle \psi(x_0), D_\mu f(h(x_0),0)\rangle dx_0 \end{aligned}$$

[14] This property is stated in [15] but without proof. However, a proof is not obvious and requires new arguments. Conducting a complete proof would have exceeded the duration of the thesis and so we decided to state it without proof.

$$\overset{(*)}{=} \int_0^\infty \langle \Phi_+^u(0,x_0)'\psi(0), D_\mu f(h(x_0),0)\rangle dx_0 + \int_{-\infty}^0 \langle \Phi_-^s(0,x_0)'\psi(0), D_\mu f(h(x_0),0)\rangle dx_0$$
$$= \left\langle \psi(0), \int_0^\infty \Phi_+^u(0,x_0) D_\mu f(h(x_0),0) dx_0 + \int_{-\infty}^0 \Phi_-^s(0,x_0) D_\mu f(h(x_0),0) dx_0 \right\rangle,$$

where $(*)$ is a consequence of

$$\Phi_+^u(0,x_0)'\psi(0) = \psi(x_0), \quad x_0 \in \mathbb{R}^+, \quad \Phi_-^s(0,x_0)'\psi(0) = \psi(x_0), \quad x_0 \in \mathbb{R}^-. \tag{2.28}$$

(2.28) follows from Remark 2.0.8 and Lemma 2.3.4. ∎

Theorem 2.3.14 *The linear part $L : Y \to \hat{Y}$ of (2.26) is continuously invertible.*

Proof In order to prove the continuous invertibility of L it suffices to show that L is injective and surjective. This is a consequence of the closed graph Theorem A.2.17. In this proof the numbers (i)-(iv) relate to the previous lemma.

Injectivity: We consider $L(b_+, b_-, v_+, v_-, \mu) = 0$. Because of (iv) and (ii)

$$\int_0^\infty \Phi_+^u(0,x_0) D_\mu f(h(x_0),0) dx_0 + \int_{-\infty}^0 \Phi_-^s(0,x_0) D_\mu f(h(x_0),0) dx_0 \notin R(\Psi_0). \tag{2.29}$$

Therefore, the third row of $L(b_+, b_-, v_+, v_-, \mu) = 0$ yields $\mu = 0$ and $b_+ - b_- = 0$. Thus $(b_+, b_-) \in N(\Psi_0) = \operatorname{span}\left\{\frac{\partial}{\partial x}h(0), \frac{\partial}{\partial x}h(0)\right\}$ and the forth row of $L(b_+, b_-, v_+, v_-, \mu) = 0$ becomes

$$\langle \varphi, \Phi_+^s(0,0) b_+\rangle = \left\langle \varphi, \Phi_+^s(0,0) c \frac{\partial}{\partial x} h(0)\right\rangle \overset{(iii)}{=} \left\langle \varphi, c\frac{\partial}{\partial x}h(0)\right\rangle \overset{(iii)}{=} c = 0,$$

where $b_+ = c\frac{\partial}{\partial x}h(0)$ for some constant c. This leads to $b_- = b_+ = 0$. Finally, we obtain $v_+ = 0$ and $v_- = 0$ considering the first and second row of $L(b_+, b_-, v_+, v_-, \mu) = 0$. Therefore, L is injective.

Surjectivity: Let $(w_+, w_-, d, s) \in \hat{Y}$ be arbitrary and solve $L(b_+, b_-, v_+, v_-, \mu) = (w_+, w_-, d, s)$. First, consider the third row. Because of

$$0 \overset{(iv)}{\neq} \left\langle \psi(0), \int_0^\infty \Phi_+^u(0,x_0) D_\mu f(h(x_0),0) dx_0 + \int_{-\infty}^0 \Phi_-^s(0,x_0) D_\mu f(h(x_0),0) dx_0 \right\rangle$$

we can multiply the equation with $\langle \psi(0),$ from the left side and obtain

$$\mu = \frac{-\langle \psi(0), d\rangle}{\left\langle \psi(0), \int_0^\infty \Phi_+^u(0,x_0) D_\mu f(h(x_0),0) dx_0 + \int_{-\infty}^0 \Phi_-^s(0,x_0) D_\mu f(h(x_0),0) dx_0 \right\rangle},$$
$$\tilde{d} := \mu \left(\int_0^\infty \Phi_+^u(0,x_0) D_\mu f(h(x_0),0) dx_0 + \int_{-\infty}^0 \Phi_-^s(0,x_0) D_\mu f(h(x_0),0) dx_0 \right) + d \in R(\Psi_0).$$

Therefore, we can find $(b_+^0, b_-^0) \in R(\Phi_+^s(0,0)) \times R(\Phi_-^u(0,0))$ with

$$\Psi_0 \left((b_+^0, b_-^0) + c\left(\frac{\partial}{\partial x}h(0), \frac{\partial}{\partial x}h(0) \right) \right) = \tilde{d}$$

for any $c \in \mathbb{R}$. We put $b_+ = b_+^0 + c\frac{\partial}{\partial x}h(0)$ into the forth row and obtain

$$\left\langle \varphi, \Phi_+^s(0,0)\left(b_+^0 + c\frac{\partial}{\partial x}h(0)\right) - \mu\int_0^\infty \Phi_+^u(0,x_0)D_\mu f(h(x_0),0)dx_0 \right\rangle = s$$
$$\Rightarrow c\left\langle \varphi, \Phi_+^s(0,0)\frac{\partial}{\partial x}h(0)\right\rangle \overset{(iii)}{=} c = \mu\left\langle \varphi, \int_0^\infty \Phi_+^u(0,x_0)D_\mu f(h(x_0),0)dx_0\right\rangle + s.$$

Considering the first and second row of $L(b_+, b_-, v_+, v_-, \mu) = (w_+, w_-, d, s)$ we can adjust v_+ and v_- to the given w_+, w_- and to the chosen b_+, b_- and μ. ∎

Hereupon we prove two important estimates regarding partial derivatives of

$$\begin{aligned}\hat{F}_\rho(x,v,\mu) = &-(\mathrm{id} - Q_\rho)(D_u f(h(x),0)v + D_\mu f(h(x),0)\mu + f(h(x),0))\\ &+ Q_\rho(f(h(x)+v,\mu) - f(h(x),0) - D_u f(h(x),0)v - D_\mu f(h(x),0)\mu)\end{aligned}$$

and regarding $G_\rho(0,0,0,0,0)$, see (2.17) and Definition 2.3.9, respectively.

Lemma 2.3.15
(a)

$$||D_{(v,\mu)}\hat{F}_\rho(x,v,\mu)||_{L[X^\alpha\times\mathbb{R},X]} \le g(\rho) + C(|\mu| + ||v||_{X^\alpha})$$

for v and μ sufficiently small and for some $g(\rho) \to 0$ with $\rho \to 0$.

(b)

$$||G_\rho(0,0,0,0,0)||_{\hat{Y}} \le C\sup_{x\in\mathbb{R}}||(id - Q_\rho)h(x)||_{X^\alpha}.$$

Proof (a) If we set $g(\rho) := 2\sup_{x\in\mathbb{R}}||(\mathrm{id} - Q_\rho)D_u f(h(x),0)||_X$ we obtain the estimate

$$\begin{aligned}
&||D_{(v,\mu)}\hat{F}_\rho(x,v,\mu)||_{L[X^\alpha\times\mathbb{R},X]}\\
&= ||[-D_u f(h(x),0) + Q_\rho D_u f(h(x)+v,\mu),\\
&\qquad -(\mathrm{id} - Q_\rho)D_\mu f(h(x),0) + Q_\rho[D_\mu f(h(x)+v,\mu) - D_\mu f(h(x),0)]]||_{L[X^\alpha\times\mathbb{R},X]}\\
&\le ||D_u f(h(x),0) - Q_\rho D_u f(h(x)+v,\mu)||_{L[X^\alpha,X]}\\
&\quad + ||(\mathrm{id} - Q_\rho)D_u f(h(x),0) - Q_\rho[D_\mu f(h(x)+v,\mu) - D_\mu f(h(x),0)]||_X\\
&\le ||D_u f(h(x),0) - Q_\rho D_u f(h(x)+v,\mu)||_{L[X^\alpha,X]}\\
&\quad + \frac{1}{2}g(\rho) + C||[D_\mu f(h(x)+v,\mu) - D_\mu f(h(x),0)]||_X\\
&\le ||D_u f(h(x),0) - Q_\rho(D_u f(h(x),0) + DD_u f(h(x),0)[(v,\mu)]\\
&\quad + R_1(D_u f(h(x),0),(v,\mu)))||_{L[X^\alpha,X]} + \frac{1}{2}g(\rho)\\
&\quad + C||[D_\mu f(h(x),0) + DD_\mu f(h(x),0)[(v,\mu)] + R_1(D_\mu f(h(x),0),(v,\mu)) - D_\mu f(h(x),0)]||_X\\
&\le C||DD_\mu f(h(x),0)[(v,\mu)] + R_1(D_u f(h(x),0),(v,\mu))||_{L[X^\alpha,X]}\\
&\quad + g(\rho) + C||[DD_\mu f(h(x),0)[(v,\mu)] + R_1(D_\mu f(h(x),0),(v,\mu))]||_X.
\end{aligned}$$

Due to Theorem A.1.2 we have

$$\begin{aligned}&||R_1(D_uf,(h(x),0),(v,\mu))||_{L[X^\alpha,X]}\\&\le \max_{t\in[0,1]}||DD_uf(h(x)+tv,t\mu)-DD_uf(h(x),0)||_{L[X^\alpha\times\mathbb{R},L[X^\alpha,X]]}\;||(v,\mu)||_{X^\alpha\times\mathbb{R}}.\end{aligned}\tag{2.30}$$

We set $K:=\{h(x):x\in\mathbb{R}\}$. Since $DD_uf(\cdot,\cdot)$ is continuous and $K\times\{0\}$ is compact there is an open neighbourhood $\tilde{O}$ of $K\times\{0\}$ such that[15] $DD_uf(\cdot,\cdot)$ is uniformly continuous in $\tilde{O}$. Hence for $C>0$ there is a $\delta>0$ so that

$$\max_{t\in[0,1]}||DD_uf(h(x)+tv,t\mu)-DD_uf(h(x),0)||_{L[X^\alpha\times\mathbb{R},L[X^\alpha,X]]}<C\quad\text{for}\quad ||(v,\mu)||_{X^\alpha\times\mathbb{R}}<\delta.$$

This and similar considerations lead to

$$||D_{(v,\mu)}\hat{F}_\rho(x,v,\mu)||_{L[X^\alpha\times\mathbb{R},X]}\le g(\rho)+C(||v||_{X^\alpha}+|\mu|)$$

for v and μ sufficiently small and with $g(\rho)\to 0$ as $\rho\to 0$.

(b) Because of $\hat{F}_\rho(x_0,0,0)=-(\mathrm{id}-Q_\rho)f(h(x_0),0)$ we get the expression

$$\begin{aligned}&G_\rho(0,0,0,0,0)\\&=\begin{pmatrix}\int_0^{\cdot}\Phi^s_+(\cdot,x_0)(\mathrm{id}-Q_\rho)f(h(x_0),0)dx_0+\int_\infty^{\cdot}\Phi^u_+(\cdot,x_0)(\mathrm{id}-Q_\rho)f(h(x_0),0)dx_0\\ \int_0^{\cdot}\Phi^u_-(\cdot,x_0)(\mathrm{id}-Q_\rho)f(h(x_0),0)dx_0+\int_{-\infty}^{\cdot}\Phi^s_-(\cdot,x_0)(\mathrm{id}-Q_\rho)f(h(x_0),0)dx_0\\ \int_0^{\infty}\Phi^u_+(0,x_0)(\mathrm{id}-Q_\rho)f(h(x_0),0)dx_0+\int_{-\infty}^{0}\Phi^s_-(0,x_0)(\mathrm{id}-Q_\rho)f(h(x_0),0)dx_0\\ \langle\varphi,\int_0^{\infty}\Phi^u_+(0,x_0)(\mathrm{id}-Q_\rho)f(h(x_0),0)dx_0\rangle\end{pmatrix}.\end{aligned}$$

In the following we prove the estimate

$$\left|\left|\int_0^x\Phi^s_+(x,x_0)(\mathrm{id}-Q_\rho)f(h(x_0),0)dx_0\right|\right|_{X^\alpha}\le C\sup_{x\in\mathbb{R}}||(\mathrm{id}-Q_\rho)h(x)||_{X^\alpha},\tag{2.31}$$

where C is a constant independent of x. Using **(H7)** and **(Q)(i)** yields

$$\begin{aligned}&\int_0^x\Phi^s_+(x,x_0)(\mathrm{id}-Q_\rho)f(h(x_0),0)dx_0=\int_0^x\Phi^s_+(x,x_0)(\mathrm{id}-Q_\rho)\left(\frac{\partial}{\partial x_0}h(x_0)-Ah(x_0)\right)dx_0\\&=\left[\Phi^s_+(x,x_0)(\mathrm{id}-Q_\rho)h(x_0)\right]_0^x\\&\quad-\int_0^x\left(\frac{\partial}{\partial x_0}\Phi^s_+(x,x_0)(\mathrm{id}-Q_\rho)h(x_0)+\Phi^s_+(x,x_0)A(\mathrm{id}-Q_\rho)h(x_0)\right)dx_0\\&\overset{(*)}{=}\Phi^s_+(x,x)(\mathrm{id}-Q_\rho)h(x)-\Phi^s_+(x,0)(\mathrm{id}-Q_\rho)h(0)\\&\quad-\int_0^x\left(-\Phi^s_+(x,x_0)(A+D_uf(h(x_0),0)(\mathrm{id}-Q_\rho)h(x_0)+\Phi^s_+(x,x_0)A(\mathrm{id}-Q_\rho)h(x_0)\right)dx_0\\&=\Phi^s_+(x,x)(\mathrm{id}-Q_\rho)h(x)-\Phi^s_+(x,0)(\mathrm{id}-Q_\rho)h(0)-\int_0^x\Phi^s_+(x,x_0)D_uf(h(x_0),0)(\mathrm{id}-Q_\rho)h(x_0)dx_0,\end{aligned}$$

where in $(*)$ we used Lemma 2.3.4. The estimate (2.31) is a direct consequence of the definition of exponential dichotomies and of Theorem 1.4.6. Similar estimates for the other integrals of $G_\rho(0,0,0,0,0)$ are met which lead to the statement (b). ∎

[15] Confer proof of Lemma 2.3.11.

Theorem 2.3.16 *The nonlinear part $\hat{G}_\rho : U \subset Y \to \hat{Y}$ of (2.26) is smooth on a sufficiently small neighbourhood U of the origin and satisfies*

$$||D\hat{G}_\rho(b_+, b_-, v_+, v_-, \mu)||_{L[Y,\hat{Y}]} \to 0 \quad \textit{as} \quad (b_+, b_-, v_+, v_-, \mu) \to 0, \quad \rho \to 0.$$

Proof In the following we show that

$$\begin{aligned}
&\hat{G}_{\rho,C^0} : \tilde{U} \subset C^0(\mathbb{R}^+, X^\alpha) \times \mathbb{R} \to C^0(\mathbb{R}^+, X^\alpha),\\
&\hat{G}_{\rho,C^0}(v_+, \mu)(x) = \int_\infty^x \Phi_+^u(x, x_0)\hat{F}_\rho(x_0, v_+(x_0), \mu)dx_0
\end{aligned}$$

is smooth with $||D\hat{G}_{\rho,C^0}(v_+, \mu)||_{L[C^0(\mathbb{R}^+,X^\alpha)\times\mathbb{R},C^0(\mathbb{R}^+,X^\alpha)]} \to 0$ as $(v_+, \mu) \to 0$, $\rho \to 0$. Here, $\tilde{U}$ is a sufficiently small neighbourhood of the origin. Note that $\hat{G}_{\rho,C^0}(v_+, \mu)(x)$ is the second summand of the first row of $\hat{G}_\rho(b_+, b_-, v_+, v_-, \mu)(x)$. All the summands of the rows of $\hat{G}_\rho(b_+, b_-, v_+, v_-, \mu)(x)$ resemble themselves. That is why it suffices to show the above mentioned statement in order to show the smoothness of $\hat{G}_\rho$ and $||D\hat{G}_\rho(b_+, b_-, v_+, v_-, \mu)||_{L[Y,\hat{Y}]} \to 0$ as $(b_+, b_-, v_+, v_-, \mu) \to 0$, $\rho \to 0$.

Consider

$$\begin{aligned}
&\hat{G}_{\rho,C^0}(v_+ + u_+, \mu + \nu)(x) - \hat{G}_{\rho,C^0}(v_+, \mu)(x)\\
&= \int_\infty^x \Phi_+^u(x, x_0)(\hat{F}_\rho(x_0, v_+(x_0) + u_+(x_0), \mu + \nu) - \hat{F}_\rho(x_0, v_+(x_0), \mu))dx_0\\
&= \int_\infty^x \Phi_+^u(x, x_0)D_{(u,\mu)}\hat{F}_\rho(x_0, v_+(x_0), \mu)\begin{pmatrix} u_+(x_0) \\ \nu \end{pmatrix} dx_0\\
&\quad + \int_\infty^x \Phi_+^u(x, x_0)o\left(\left|\left|\begin{pmatrix} u_+(x_0) \\ \nu \end{pmatrix}\right|\right|_{X^\alpha\times\mathbb{R}}\right) dx_0.
\end{aligned}$$

We claim that the Frechet derivative of $\hat{G}_{\rho,C^0}$ is given by

$$T_\rho(v_+, \mu)\begin{pmatrix} u_+ \\ \nu \end{pmatrix}(x) = \int_\infty^x \Phi_+^u(x, x_0)D_{(u,\mu)}\hat{F}_\rho(x_0, v_+(x_0), \mu)\begin{pmatrix} u_+(x_0) \\ \nu \end{pmatrix} dx_0.$$

The linearity of T_ρ is a consequence of the linear structure of the integrand and of the integral's linearity. Furthermore,

$$\begin{aligned}
&\left|\left|T_\rho(v_+, \mu)\begin{pmatrix} u_+ \\ \nu \end{pmatrix}\right|\right|_{C^0}\\
&= \sup_{x\in\mathbb{R}^+}\left|\left|\int_\infty^x \Phi_+^u(x, x_0)D_{(u,\mu)}\hat{F}_\rho(x_0, v_+(x_0), \mu)\begin{pmatrix} u_+(x_0) \\ \nu \end{pmatrix} dx_0\right|\right|_{X^\alpha}\\
&\le \sup_{x\in\mathbb{R}^+}\int_\infty^x ||\Phi_+^u(x, x_0)||_{L[X,X^\alpha]}||D_{(u,\mu)}\hat{F}_\rho(x_0, v_+(x_0), \mu)||_{L[X^\alpha\times\mathbb{R},X]}\left|\left|\begin{pmatrix} u_+(x_0) \\ \nu \end{pmatrix}\right|\right|_{X^\alpha\times\mathbb{R}} dx_0\\
&\overset{(a)}{\le} \sup_{x\in\mathbb{R}^+}\int_\infty^x ||\Phi_+^u(x, x_0)||_{L[X,X^\alpha]}(g(\rho) + C(||v_+||_{C^0} + |\mu|))dx_0\left|\left|\begin{pmatrix} u_+ \\ \nu \end{pmatrix}\right|\right|_{C^0\times\mathbb{R}}\\
&\le C(g(\rho) + ||v_+||_{C^0} + |\mu|)\left|\left|\begin{pmatrix} u_+ \\ \nu \end{pmatrix}\right|\right|_{C^0\times\mathbb{R}},
\end{aligned}$$

where (a) relates to Lemma 2.3.15 and where in the last step we used Theorem 1.4.6. So we obtain the boundedness of T_ρ with

$$||T_\rho(v_+,\mu)||_{L[C^0\times\mathbb{R},C^0]} \leq C(g(\rho)+||v_+||_{C^0}+|\mu|) \to 0 \quad \text{as} \quad (v_+,\mu)\to 0, \quad \rho\to 0. \tag{2.32}$$

Finally[16],

$$\frac{\left|\left|\int_\infty^x \Phi_+^u(x,x_0) o\left(\left|\left|\begin{pmatrix} u_+(x_0) \\ \nu \end{pmatrix}\right|\right|_{X^\alpha\times\mathbb{R}}\right) dx_0\right|\right|_{C^0}}{\left|\left|\begin{pmatrix} u_+ \\ \nu \end{pmatrix}\right|\right|_{C^0\times\mathbb{R}}}$$

$$\leq \sup_{x\in\mathbb{R}^+}\int_x^\infty ||\Phi_+^u(x,x_0)||_{L[X,X^\alpha]}dx_0 \frac{\sup_{x\in\mathbb{R}^+}\left|\left|o\left(\left|\left|\begin{pmatrix} u_+(x) \\ \nu \end{pmatrix}\right|\right|_{X^\alpha\times\mathbb{R}}\right)\right|\right|_{X\times\mathbb{R}}}{\left|\left|\begin{pmatrix} u_+ \\ \nu \end{pmatrix}\right|\right|_{C^0\times\mathbb{R}}}$$

$$\leq C\sup_{x\in\mathbb{R}^+}\left|\left|\frac{o\left(\left|\left|\begin{pmatrix} u_+(x) \\ \nu \end{pmatrix}\right|\right|_{X^\alpha\times\mathbb{R}}\right)}{\left|\left|\begin{pmatrix} u_+(x) \\ \nu \end{pmatrix}\right|\right|_{X^\alpha\times\mathbb{R}}}\right|\right|_{X\times\mathbb{R}} \to 0 \quad \text{as} \quad \left|\left|\begin{pmatrix} u_+ \\ \nu \end{pmatrix}\right|\right|_{C^0\times\mathbb{R}} \to 0,$$

which proves that $\hat{G}_{\rho,C^0}$ is Frechet differentiable with

$$D\hat{G}_{\rho,C^0}(v_+,\mu) = T_\rho(v_+,\mu) \quad \text{and} \quad D\hat{G}_{\rho,C^0}(v_+,\mu)\to 0 \quad \text{as} \quad (v_+,\mu)\to 0, \quad \rho\to 0.$$

The continuity of $(v_+,\mu)\mapsto T_\rho(v_+,\mu)$ is a direct consequence of $f\in C^2(X^\alpha\times\mathbb{R},X)$. ∎

Now we can conclude that there are constants $0<r$ and $0<\kappa<q<1$ so that

$$\begin{aligned}
&||\mathrm{id} - L^{-1}DG_\rho(b_+,b_-,v_+,v_-,\mu)||_{L[Y]} \leq C||D\hat{G}_\rho(b_+,b_-,v_+,v_-,\mu)||_{L[Y]} \leq \kappa \\
&\quad \forall (b_+,b_-,v_+,v_-,\mu)\in S = \{(b_+,b_-,v_+,v_-,\mu)\in Y : ||(b_+,b_-,v_+,v_-,\mu)||_Y \leq r\}, \\
&||L^{-1}G_\rho(0,0,0,0,0)||_Y \leq r(1-q)
\end{aligned}$$

for every sufficiently small $\rho>0$. Thus, it follows from Theorem 2.2.1 that

$$G_\rho(b_+,b_-,v_+,v_-,\mu) = 0$$

has a unique solution $(b_{\rho,+},b_{\rho,-},\tilde{h}_{\rho,+},\tilde{h}_{\rho,-},\mu_\rho)$ in a sufficiently small ball in Y centered at the origin and for every sufficiently small $\rho>0$. Moreover, the function $\tilde{h}_\rho$ defined by

$$\tilde{h}_\rho(x) = \begin{cases} \tilde{h}_{\rho,+}(x), & x\in\mathbb{R}^+ \\ \tilde{h}_{\rho,-}(x), & x\in\mathbb{R}^- \end{cases}$$

[16]Consider the following statement for $A,B\subset\mathbb{R}^+$:
$\sup(A\cdot B) = \sup(A)\cdot\sup(B), \quad \sup(A^{-1}) = (\inf(A))^{-1} \Rightarrow \sup(\frac{A}{B}) = \sup(A)\cdot\sup(B^{-1}) = \frac{\sup(A)}{\inf(B)} \geq \frac{\sup(A)}{\sup(B)}$.

is a strong solution of equation (2.19)

$$\frac{d}{dx}v = (A + D_u f(h(x), 0))v + F_\rho(x, v, \mu)$$

due to Theorem 2.3.8.

Considering the relation (2.18) we obtain the solution $h_\rho = h + \tilde{h}_\rho$ of the Galerkin approximation

$$\frac{\partial}{\partial x}u = Au + Q_\rho f(u, \mu_\rho).$$

Estimate (2.11), Lemma 2.3.15 (b) and Theorem 2.3.14 yield

$$\begin{aligned}
&||(b_{\rho,+}, b_{\rho,-}, \tilde{h}_+, \tilde{h}_-, \mu_\rho)||_Y \\
&= ||b_{\rho,+}||_{X^\alpha} + ||b_{\rho,-}||_{X^\alpha} + \sup_{x\in\mathbb{R}^+} ||\tilde{h}_{\rho,+}(x)||_{X^\alpha} + \sup_{x\in\mathbb{R}^-} ||\tilde{h}_{\rho,-}(x)||_{X^\alpha} + |\mu_\rho| \\
&\le (1-q)^{-1}||L^{-1}G_\rho(0,0,0,0,0)||_Y \\
&\le C \sup_{x\in\mathbb{R}} ||(\mathrm{id} - Q_\rho)h(x)||_{X^\alpha}.
\end{aligned}$$

This leads to the estimate

$$|\mu_\rho| + \sup_{x\in\mathbb{R}} ||h_\rho(x) - h(x)||_{X^\alpha} \le C \sup_{x\in\mathbb{R}} ||(\mathrm{id} - Q_\rho)h(x)||_{X^\alpha}.$$

Theorem 2.3.17

$$h_\rho(x) \in Q_\rho X^\alpha \quad \forall x \in \mathbb{R}.$$

Proof The equation

$$\frac{\partial}{\partial x}h_\rho(x) = Ah_\rho(x) + Q_\rho f(h_\rho(x), \mu_\rho), \quad x \in \mathbb{R},$$

leads to

$$\frac{\partial}{\partial x}(\mathrm{id} - Q_\rho)h_\rho(x) = A(\mathrm{id} - Q_\rho)h_\rho(x), \quad x \in \mathbb{R}.$$

Because $(\mathrm{id} - Q_\rho)h_\rho(\cdot)$ is a bounded solution of $\frac{\partial}{\partial x}u = Au$ on $\mathbb{R}$ Theorem 1.1.6 and Theorem 1.4.3 yield $(\mathrm{id} - Q_\rho)h_\rho = 0$. Therefore, $h_\rho(x) \in Q_\rho X^\alpha$ for all $x \in \mathbb{R}$. ∎

Theorem 2.3.18

$$h_\rho(x) \to p_\rho(\mu_\rho) \quad \textit{as} \quad |x| \to \infty.$$

Proof By subtraction we obtain from the equations

$$\frac{\partial}{\partial x}h_\rho(x) = Ah_\rho(x) + Q_\rho f(h_\rho(x), \mu_\rho),\ x \in \mathbb{R}, \quad 0 = Ap_\rho(\mu_\rho) + Q_\rho f(p_\rho(\mu_\rho), \mu_\rho)$$

the differential equation

$$\frac{\partial}{\partial x} y_\rho(x) = Ay_\rho(x) + Q_\rho(f(h_\rho(x), \mu_\rho) - f(p_\rho(\mu_\rho), \mu_\rho)),$$

where $y_\rho(x) := h_\rho(x) - p_\rho(\mu_\rho)$ and $x \in \mathbb{R}$. The assumption $f \in C^2(X^\alpha \times \mathbb{R}, X)$ results in

$$\begin{aligned} & Q_\rho(f(h_\rho(x), \mu_\rho) - f(p_\rho(\mu_\rho), \mu_\rho)) \\ &= Q_\rho \left(D_u f(p_\rho(\mu_\rho), \mu_\rho) + \int_0^1 (D_u f(p_\rho(\mu_\rho) + t(h_\rho(x) - p_\rho(\mu_\rho)), \mu_\rho) - D_u f(p_\rho(\mu_\rho)\mu_\rho))dt \right) \\ &\quad (h_\rho(x) - p_\rho(\mu_\rho)) \\ &= B_\rho(x) y_\rho(x), \end{aligned}$$

where we defined $B_\rho(\cdot) \in C^{0,\vartheta}(\mathbb{R}, L[X^\alpha, X])$ by

$$\begin{aligned} & B_\rho(x) := \\ & Q_\rho \left(D_u f(p_\rho(\mu_\rho), \mu_\rho) + \int_0^1 (D_u f(p_\rho(\mu_\rho) + t(h_\rho(x) - p_\rho(\mu_\rho)), \mu_\rho) - D_u f(p_\rho(\mu_\rho)\mu_\rho))dt \right). \end{aligned}$$

Using arguments of the above proofs we can also show that for every $\tilde{\varepsilon}$ there exists a ρ_0 such that $\sup_{x\in\mathbb{R}} \left\{ ||D_u f(h(x), 0) - B(x)||_{L[X^\alpha, X]} \right\} \leq \tilde{\varepsilon}$ for all $\rho \in [0, \rho_0)$. Therefore, Theorem 1.4.4 yields that

$$\frac{\partial}{\partial x} y_\rho = (A + B(x)) y_\rho$$

has an exponential dichotomy on $\mathbb{R}$. From the exponential behavior we obtain $y_\rho(x) \to 0$ as $|x| \to \infty$ which proves the assertion. ∎

Theorem 2.3.19 *The solutions h_ρ are nondegenerate.*

Proof We apply Theorem 1.4.4 to

$$\frac{\partial}{\partial x} v = (A + Q_\rho f(h_\rho(x), \mu_\rho))v. \tag{2.33}$$

Because of the already proven estimate of Theorem 2.1.6 (ii) and because of Lemma 2.1.3 there exists a constant $\eta_0 > 0$ such that

$$\begin{aligned} & \sup_{x\in\mathbb{R}^+} ||D_u f(h(x), 0) - Q_\rho D_u f(h_\rho(x), \mu_\rho)||_{L[X^\alpha, X]} \\ & \leq \sup_{x\in\mathbb{R}^+} \Bigg|\Bigg| D_u f(h(x), 0) \\ & \quad - Q_\rho \left[D_u f(h(x), 0) + DD_u f(h(x), 0) \begin{pmatrix} \tilde{h}_\rho(x) \\ \mu_\rho \end{pmatrix} + o\left(\left|\left| \begin{pmatrix} \tilde{h}_\rho \\ \mu_\rho \end{pmatrix} \right|\right|_{C^0 \times \mathbb{R}^+} \right) \right] \Bigg|\Bigg|_{L[X^\alpha, X]} \\ & \leq \sup_{x\in\mathbb{R}^+} ||(\mathrm{id} - Q_\rho) D_u f(h(x), 0)||_{L[X^\alpha, X]} + C \left(\sup_{x\in\mathbb{R}} ||h_\rho(x) - h(x)||_{X^\alpha} + |\mu_\rho| \right) \\ & \leq \eta \end{aligned}$$

for some $\eta < \eta_0$ if ρ is sufficiently small. Consider that C can be chosen independent of x since $\lim_{|x|\to\infty} h(x)$ and $\lim_{|x|\to\infty} h_\rho(x)$ exist.

Let $\Phi^s_{+,\rho}(x, x_0)$ and $\Phi^u_{+,\rho}(x_0, x)$ for $x \geq \tau \geq 0$, and $\Phi^s_{-,\rho}(x_0, x)$ and $\Phi^u_{-,\rho}(x, x_0)$ for $x \leq x_0 \leq 0$ be the corresponding solutions operators for (2.33). Using Hypothesis **(K)** and the results of Lemma 2.1.1 and Theorem 1.4.4 shows that $\Phi^u_{+,\rho}(0,0)$ and $\Phi^u_{-,\rho}(0,0)$ are close to $\Phi^s_+(0,0)$ and $\Phi^u_-(0,0)$, respectively. This proves that the solutions h_ρ are nondegenerate. ∎

Corollary 2.3.20

$$\frac{\partial}{\partial x} h_\rho \in C^0(\mathbb{R}, X^\alpha).$$

Proof $\frac{\partial}{\partial x} h_\rho(0) \in R(\Phi^s_{+,\rho}(0,0))$ leads to $\frac{\partial}{\partial x} h_\rho(0) \in X$. Moreover, it is a consequence of Theorem 1.4.6 that $\frac{\partial}{\partial x} h_\rho(x) = \Phi_{+,\rho}(x,0)\frac{\partial}{\partial x} h_\rho(0)$, $x > 0$, is a continuous function into X^α. As the choice of $x = 0$ is arbitrary we have $\frac{\partial}{\partial x} h_\rho \in C^0(\mathbb{R}, X^\alpha)$. ∎

Proof of Theorem 2.1.6 (iii)

The uniqueness statement of Theorem 2.1.6(iii) is a consequence of Theorem 2.2.1.

2.4 The Truncated Boundary Value Problem

In order to analyse the numerical computation of the homoclinic orbits h_ρ of the Galerkin approximation

$$\frac{\partial}{\partial x}u = Au + Q_\rho f(u,\mu), \quad (u,\mu) \in X^\alpha \times \mathbb{R},$$

one truncates the axis $\mathbb{R}$ to a finite interval $[T_-, T_+]$ for some $T_- < 0 < T_+$ and imposes boundary conditions at the end points $x = T_-$ and $x = T_+$. This procedure is the most commonly used one. We follow [15].

In this section we consider truncated boundary value problems of the form

$$\begin{pmatrix} \frac{\partial}{\partial x}u - Au - Q_\rho f(u,\mu) \\ R_\rho(u(T_+), u(T_-), \mu) \\ J_{T,\rho}(u,\mu) \end{pmatrix} = 0, \tag{2.34}$$

where $x \in T := (T_-, T_+)$. $J_{T,\rho}$ describes a phase condition and R_ρ the boundary conditions.

We remark that the translate $h(\cdot + x_0)$ of $h(\cdot)$ is still a homoclinic orbit. In order to choose a particular translate we impose the phase condition $J_{T,\rho}(u,\mu) = 0$. As a consequence, the solution becomes unique. We now add the Hypothesis **(T1)**. Note some important differences to [15].

Hypothesis (T1)

(i) The map $J_{T,\rho} \in C^2(C^0(T, X^\alpha) \times \mathbb{R}, \mathbb{R})$ satisfies $J_{T,\rho}(h_\rho, \mu_\rho) \to 0$ as $|T_\pm| \to \infty$. Moreover, there is a $d_0 > 0$ independent of T_-, T_+ and ρ so that $D_u J_{T,\rho}(h_\rho, \mu_\rho)\frac{\partial}{\partial x}h_\rho \geq d_0$ for all $|T_\pm|$ sufficiently large. $D_u J_{T,\rho}(u,\mu)$ and $D_u^2 J_{T,\rho}(u,\mu)$ are bounded in a ball $B((h_\rho, \mu_\rho), r_1) \subset C^0(T, X^\alpha) \times \mathbb{R}$ of a fixed radius r_1 uniformly in T_-, T_+ and ρ.

(ii) The boundary condition is given by $R_\rho \in C^2(X^\alpha \times X^\alpha \times \mathbb{R}, X^\alpha)$ so that DR_ρ and $D^2 R_\rho$ are bounded in a small ball $B((p_\rho(\mu_\rho), p_\rho(\mu_\rho), \mu_\rho), r_2) \subset X^\alpha \times X^\alpha \times \mathbb{R}$ with radius r_2 uniformly in ρ. R_ρ satisfies $R_\rho(p_0, p_0, 0) = 0$ where p_0 is the hyperbolic equilibrium of **(H6)**. Finally[17],

$$D_{(u_+,u_-)} R_\rho(p_\rho(\mu_\rho), p_\rho(\mu_\rho), \mu_\rho)|_{R(P_{+,\rho}(\mu_\rho)) \times R(P_{-,\rho}(\mu_\rho))}$$

is invertible and the inverse is bounded uniformly in ρ.

Remark 2.4.1 *Hypothesis **(T1)(i)** is well-defined because of $\frac{\partial}{\partial x}h, \frac{\partial}{\partial x}h_\rho \in C^0(T, X^\alpha)$. Consider Corollary 2.3.20.*
In many cases the boundary conditions are separated

$$R_\rho(u_+, u_-, \mu) = (R_{+,\rho}(u_+,\mu), R_{-,\rho}(u_-,\mu)) \in R(P_{+,\rho}(\mu_\rho)) \times R(P_{-,\rho}(\mu_\rho)) = X^\alpha.$$

*In these cases the invertibility condition of **(T1)(ii)** is satisfied if $D_u R_{\pm,\rho}(p_\rho(\mu_\rho), \mu_\rho))|_{R(P_{\pm,\rho}(\mu_\rho))}$ are invertible and their inverses are bounded uniformly in ρ.*

[17] Confer Lemma 2.2.2 for $P_{+,\rho}(\mu_\rho)$ and $P_{-,\rho}(\mu_\rho)$.

In the following C denotes various different constants that are all independent of T_- and T_+.

Theorem 2.4.2 *If the assumptions **(H1), (H3), (H6)-(H9), (K), (Q)** and **(T1)** are satisfied, then there are constants ρ_0, η, $C > 0$ so that on all sufficiently large intervals T the boundary value problem (2.34) has a unique solution $(\bar{h}_\rho, \bar{u}_\rho)$ for any $\rho \in [0, \rho_0)$ in the tube*

$$\left\{(u, \mu) \in C^0([T_-, T_+], X^\alpha) \times \mathbb{R} : |\mu| + \sup_{x \in [T_-, T_+]} ||u(x) - h(x)||_{X^\alpha} \leq \eta\right\}$$

and the estimate

$$|\bar{\mu}_\rho - \mu_\rho| + \sup_{x \in [T_-, T_+]} ||\bar{h}_\rho(x) - h_\rho(x + \gamma_{T,\rho})||_{X^\alpha} \leq C||R_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho)||_{X^\alpha}$$

holds for an appropriate small constant $\gamma_{T,\rho}$.

Corollary 2.4.3

$$|\bar{\mu}_\rho| + \sup_{x \in [T_-, T_+]} ||\bar{h}_\rho(x) - h(x)||_{X^\alpha} \leq C\left(||R_\rho(h(T_+), h(T_-), 0)||_{X^\alpha} + \sup_{x \in \mathbb{R}} ||(id - Q_\rho)h(x)||_{X^\alpha}\right)$$

under the hypotheses of Theorem 2.4.2.

Remark 2.4.4 *Provided that the assumptions of Theorem 2.4.2 are satisfied, we even have the estimate*

$$|\bar{\mu}_\rho| + \sup_{x \in [T_-, T_+]} ||\bar{h}_\rho(x) - h(x)||_{X^\alpha} \leq C\left(e^{\lambda^s T_+} + e^{\lambda^u T_-} + \sup_{x \in \mathbb{R}} ||(id - Q_\rho)h(x)||_{X^\alpha}\right),$$

where $\lambda^s < 0$ and $\lambda^u > 0$ are chosen so that $\{\lambda \in \mathbb{C} | \lambda^s \leq \Re(\lambda) \leq \lambda^u\} \cap \sigma(A + D_u f(p_0, 0)) = \varnothing$. This statement can be proven by showing[18]

$$||h(T_+) - p_0||_{X^\alpha} \leq Ce^{-\beta T_+}$$

for some positive constants C and β.

The results of this section contain also the case of truncating the evolution equation (2.7) directly without imposing a finite-dimensional approximation. In this case we set $Q_\rho = \mathrm{id}$ for all $\rho > 0$.

Proof of the results

Definition 2.4.5 *For $\rho > 0$ we define the maps F_ρ, $\hat{F}_\rho : \mathbb{R} \times X^\alpha \times \mathbb{R} \to X$ by*

$$\begin{aligned} F_\rho(x, v, \nu) &:= D_\mu f(h_\rho(x), \mu_\rho)\nu + \hat{F}_\rho(x, v, \nu), \\ \hat{F}_\rho(x, v, \nu) &:= Q_\rho(f(h_\rho(x) + v, \mu_\rho + \nu) - f(h_\rho(x), \mu_\rho) - D_u f(h(x), 0)v) \\ &\quad - (id - Q_\rho)D_u f(h(x), 0)v - D_\mu f(h_\rho(x), \mu_\rho)\nu. \end{aligned} \tag{2.35}$$

[18]Use the properties of exponential dichotomies.

Conducting the transformation

$$u(x) = h_\rho(x) + v(x),\ x \in \mathbb{R}, \quad \mu = \mu_\rho + \nu \tag{2.36}$$

we obtain from $\frac{\partial}{\partial x} u = Au + Q_\rho f(u, \mu)$ and $\frac{\partial}{\partial x} h_\rho = Ah_\rho + Q_\rho f(h_\rho, \mu_\rho)$ the expression

$$\begin{aligned} \frac{\partial}{\partial x} v &= (A + D_u f(h(x), 0))v + F_\rho(x, v, \nu) \\ &= (A + D_u f(h(x), 0))v + D_\mu f(h_\rho(x), \mu_\rho)\nu + \hat{F}_\rho(x, v, \nu). \end{aligned} \tag{2.37}$$

In the following we search for a strong solution v of this differential equation. Here, strong is defined by

$$v \in C^1((T_-, T_+), X) \cap C^0((T_-, T_+), X^1). \tag{2.38}$$

Definition 2.4.6

$a = (a_+, a_-) \in X_a := R(P_+) \times R(P_-), \quad b = (b_+, b_-) \in X_b := R(\Phi^s_+(0,0)) \times R(\Phi^u_-(0,0)),$
$I_{+,T,\rho} :\ X_a \times X_b \times C^0([0, T_+], X^\alpha) \times \mathbb{R} \to C^0([0, T_+], X^\alpha)$ *with*

$$\begin{aligned} I_{+,T,\rho}(a, b, v_+, \nu)(x) := {}& \Phi^u_+(x, T_+)a_+ + \Phi^s_+(x, 0)b_+ + \int_{T_+}^x \Phi^u_+(x, x_0)F_\rho(x_0, v_+(x_0), \nu)dx_0 \\ &+ \int_0^x \Phi^s_+(x, x_0)F_\rho(x_0, v_+(x_0), \nu)dx_0, \end{aligned}$$

$I_{-,T,\rho} :\ X_a \times X_b \times C^0([T_-, 0], X^\alpha) \times \mathbb{R} \to C^0([T_-, 0], X^\alpha)$ *with*

$$\begin{aligned} I_{-,T,\rho}(a, b, v_-, \nu)(x) := {}& \Phi^s_-(x, T_-)a_- + \Phi^u_-(x, 0)b_- + \int_{T_-}^x \Phi^s_-(x, x_0)F_\rho(x_0, v_-(x_0), \nu)dx_0 \\ &+ \int_0^x \Phi^u_-(x, x_0)F_\rho(x_0, v_-(x_0), \nu)dx_0. \end{aligned}$$

The maps are well-defined and even smooth which can be shown as in the proofs of the previous sections.

Theorem 2.4.7

(i) *If v is a strong solution of (2.37) it satisfies*

$$\begin{aligned} 0 &= v_+(x) - I_{+,T,\rho}(a, b, v_+, \nu)(x), \quad x \in [0, T_+], \\ 0 &= v_-(x) - I_{-,T,\rho}(a, b, v_-, \nu)(x), \quad x \in [T_-, 0], \\ v_+(0) &= v_-(0) \end{aligned} \tag{2.39}$$

for some $a, b \in X_a \times X_b$, where $v_+ := v|_{[0,T_+]}$ and $v_- := v|_{[T_-,0]}$.

(ii) *If $(v_+, v_-) \in C^0([0, T_+], X^\alpha) \times C^0([T_-, 0], X^\alpha)$ are solutions of (2.39) for some $a, b \in X_a \times X_b$, i.e. mild solutions of (2.37), then the function v defined by*

$$v(x) := \begin{cases} v_-(x), & x \in [T_-, 0] \\ v_+(x), & x \in [0, T_+] \end{cases}$$

is a strong solution of (2.37).

Proof Here, we refer to the proof of Theorem 2.3.8 and to Lemma 2.3.7.

Definition 2.4.8

$$
\begin{aligned}
V:\, & C^0([0,T_+],X^\alpha)\times C^0([T_-,0],X^\alpha)\to C^0([T_-,T_+],X^\alpha),\\
& V(v_+,v_-)(x)=\begin{cases} v_+(x)+v_-(0)-v_+(0), & x>0\\ v_-(x), & x\le 0.\end{cases}
\end{aligned}
\tag{2.40}
$$

V is a linear and bounded operator. In the following we have to solve the phase and boundary conditions

$$
\begin{aligned}
R_\rho(h_\rho(T_+)+v_+(T_+),h_\rho(T_-)+v_-(T_-),\mu_\rho+\nu)&=0,\\
J_{T,\rho}(h_\rho+V(v_+,v_-),\mu_\rho+\nu)&=0.
\end{aligned}
\tag{2.41}
$$

Lemma 2.4.9

(i) *R_ρ satisfies*

$$
\begin{aligned}
&R_\rho(h_\rho(T_+)+v_+,h_\rho(T_-)+v_-,\mu_\rho+\nu)=R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\\
&+D_{(u_+,u_-,\nu)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)(v_+,v_-,\nu)+\hat R_\rho(h_\rho(T_+),h_\rho(T_-),v_+,v_-,\nu)\quad \textit{with}\\
&||D_{(v_+,v_-,\nu)}\hat R_\rho(h_\rho(T_+),h_\rho(T_-),v_+,v_-,\nu)||_{L[X^\alpha\times X^\alpha\times\mathbb{R},X^\alpha]}\le C(||v_+||_{X^\alpha}+||v_-||_{X^\alpha}+|\nu|)
\end{aligned}
$$

for (v_+,v_-,ν) in a sufficiently small ball in $X^\alpha\times X^\alpha\times\mathbb{R}$ centered at the origin. The constant C is independent of ρ.

(ii) *$J_{T,\rho}$ satisfies*

$$
\begin{aligned}
&J_{T,\rho}(h_\rho+v,\mu_\rho+\nu)=J_{T,\rho}(h_\rho,\mu_\rho)+D_vJ_{T,\rho}(h_\rho,\mu_\rho)v+D_\mu J_{T,\rho}(h_\rho,\mu_\rho)\nu+\hat J_{T,\rho}(h_\rho,v,\nu)\\
&\textit{with}\quad ||D_{(v,\nu)}\hat J_{T,\rho}(h_\rho,v,\nu)||_{L[C^0(T,X^\alpha)\times\mathbb{R},\mathbb{R}]}\le C(||v||_{C^0(T,X^\alpha)}+|\nu|)
\end{aligned}
$$

for (v,ν) in a sufficiently small ball in $C^0(T,X^\alpha)\times\mathbb{R}$ centered at the origin. The constant C is independent of ρ.

Proof (i) The Taylor expansion of R_ρ leads to the asserted equation. This results in the expression

$$
\begin{aligned}
&D_{(v_+,v_-,\nu)}\hat R_\rho(h_\rho(T_+),h_\rho(T_-),v_+,v_-,\nu)\\
&=D_{(v_+,v_-,\nu)}(R_\rho(h_\rho(T_+)+v_+,h_\rho(T_-)+v_-,\mu_\rho+\nu)-R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\\
&\quad -D_{(u_+,u_-,\nu)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)[v_+,v_-,\nu])\\
&=D_{(u_+,u_-,\nu)}R_\rho(h_\rho(T_+)+v_+,h_\rho(T_-)+v_-,\mu_\rho+\nu)-D_{(u_+,u_-,\nu)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\\
&=D^2_{(u_+,u_-,\nu)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)[v_+,v_-,\nu]\\
&\quad +R_1(D_{(u_+,u_-,\nu)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho),(v_+,v_-,\nu)).
\end{aligned}
$$

Due to Theorem A.1.2

$$
\begin{aligned}
&||R_1(D_{(u_+,u_-,\nu)}R_\rho,(h_\rho(T_+),h_\rho(T_-),\mu_\rho),(v_+,v_-,\nu))||_{L[X^\alpha\times X^\alpha\times\mathbb{R},X^\alpha]}\\
&\le \max_{0\le t\le 1}||D^2_{(u_+,u_-,\nu)}R_\rho(h_\rho(T_+)+tv_+,h_\rho(T_-)+tv_-,\mu_\rho+t\nu)\\
&-D^2_{(u_+,u_-,\nu)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)||_{L[X^\alpha\times X^\alpha\times\mathbb{R},L[X^\alpha\times X^\alpha\times\mathbb{R},X^\alpha]]}\cdot||(v_+,v_-,\nu)||_{X^\alpha\times X^\alpha\times\mathbb{R}}\\
&\le C(||v_+||_{X^\alpha}+||v_-||_{X^\alpha}+|\nu|)
\end{aligned}
$$

for (v_+,v_-,ν) in a sufficiently small ball around the origin. In the last step we used that $D^2_{(u_+,u_-,\nu)}R_\rho(\cdot,\cdot,\cdot)$ is uniformly continuous on an open neighbourhood of the compact set $\{h_\rho(x):x\in\mathbb{R}\}\times\{h_\rho(x):x\in\mathbb{R}\}\times\{\mu_\rho\}$, see **(T1)(ii)**.

Using this argument once more we obtain from the first equation the asserted estimate

$$||D_{(v_+,v_-,\nu)}\hat{R}_\rho(h_\rho(T_+),h_\rho(T_-),v_+,v_-,\nu)||_{L[X^\alpha\times X^\alpha\times\mathbb{R},X^\alpha]}\le C(||v_+||_{X^\alpha}+||v_-||_{X^\alpha}+|\nu|).$$

(ii) is similar to (i). ∎

Definition 2.4.10

$$
\begin{aligned}
&G_{T,\rho}:Y\to\hat{Y},\\
&Y:=X_a\times X_b\times C^0([0,T_+],X^\alpha)\times C^0([T_-,0],X^\alpha)\times\mathbb{R},\\
&\hat{Y}:=C^0([0,T_+],X^\alpha)\times C^0([T_-,0],X^\alpha)\times X^\alpha\times X^\alpha\times\mathbb{R},\\
&G_{T,\rho}(a,b,v_+,v_-,\nu):=\begin{pmatrix} v_+-I_{+,T,\rho}(a,b,v_+,\nu)\\ v_--I_{-,T,\rho}(a,b,v_-,\nu)\\ I_{+,T,\rho}(a,b,v_+,\nu)(0)-I_{-,T,\rho}(a,b,v_-,\nu)(0)\\ R_\rho(h_\rho(T_+)+v_+(T_+),h_\rho(T_-)+v_-(T_-),\mu_\rho+\nu)\\ J_{T,\rho}(h_\rho+V(v_+,v_-),\mu_\rho+\nu)\end{pmatrix}.
\end{aligned}
\tag{2.42}
$$

We note that $G_{T,\rho}$ is well-defined and smooth which can be proven as in Section 2.3. Again, we intend to apply Theorem 2.2.1. Thus, we need to ensure that its preconditions are met. Therefore, we expose the following definitions and statements.

Definition 2.4.11

$$
\begin{aligned}
\hat{I}_{T_+,\rho}(a,b,\nu)(x):=&\,\Phi^u_+(x,T_+)a_++\Phi^s_+(x,0)b_++\nu\left(\int_{T_+}^x\Phi^u_+(x,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\right.\\
&\left.+\int_0^x\Phi^s_+(x,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\right)\quad x\in[0,T_+],\\
\hat{I}_{T_-,\rho}(a,b,\nu)(x):=&\,\Phi^s_-(x,T_-)a_-+\Phi^u_-(x,0)b_-+\nu\left(\int_{T_-}^x\Phi^s_-(x,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\right.\\
&\left.+\int_0^x\Phi^u_-(x,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\right)\quad x\in[T_-,0].
\end{aligned}
\tag{2.43}
$$

Lemma 2.4.12

$$\hat{I}_{T_+,\rho} \in L[X_a \times X_b \times \mathbb{R}, C^0([0,T_+], X^\alpha)], \quad \sup_{T_+ \in \mathbb{R}^+} ||\hat{I}_{T_+,\rho}(a,b,\nu)||_{L[X_a \times X_b \times \mathbb{R}, C^0([0,T_+],X^\alpha)]} < \infty$$
$$\hat{I}_{T_-,\rho} \in L[X_a \times X_b \times \mathbb{R}, C^0([T_-,0], X^\alpha)], \quad \sup_{T_- \in \mathbb{R}^-} ||\hat{I}_{T_-,\rho}(a,b,\nu)||_{L[X_a \times X_b \times \mathbb{R}, C^0([T_-,0],X^\alpha)]} < \infty$$

Proof Using Theorem 1.4.6 several times yields

$$\begin{aligned}
&||\hat{I}_{T_+,\rho}(a,b,\nu)||_{C^0} \\
&\leq \sup_{x\in[0,T_+]} \Bigg|\Bigg| \Phi_+^u(x,T_+)a_+ + \Phi_+^s(x,0)b_+ + \nu \left(\int_{T_+}^x \Phi_+^u(x,x_0) D_\mu f(h_\rho(x_0),\mu_\rho) dx_0 \right. \\
&\qquad\qquad \left. + \int_0^x \Phi_+^s(x,x_0) D_\mu f(h_\rho(x_0),\mu_\rho) dx_0 \right) \Bigg|\Bigg|_{X^\alpha} \\
&\leq C(||a||_{X_+} + ||b_+||_{X^\alpha}) + |\nu| \left(\sup_{x\in[0,\infty)} \left\{ \int_x^\infty ||\Phi_+^u(x,x_0)||_{L[X,X^\alpha]} ||D_\mu f(h_\rho(x_0),\mu_\rho)||_X dx_0 \right\} \right. \\
&\quad \left. + \sup_{x\in[0,\infty)} \left\{ \int_0^x ||\Phi_+^s(x,x_0)||_{L[X,X^\alpha]} ||D_\mu f(h_\rho(x_0),\mu_\rho)||_X dx_0 \right\} \right). \\
&\leq C(||a||_{X_+} + ||b_+||_{X^\alpha} + |\nu|) \\
&\leq C||(a,b,\nu)||_{X_a \times X_b \times \mathbb{R}}.
\end{aligned}$$

In the same way we can show $||\hat{I}_{T_-,\rho}(a,b,\nu)||_{C^0} \leq C||(a,b,\nu)||_{X_a \times X_b \times \mathbb{R}}$. Note that the constants C are independent of T_+ and T_-, respectively. ∎

We decompose the map $G_{T,\rho}$ into a linear and nonlinear part:

$$G_{T,\rho}(a,b,v_+,v_-,\nu) = L_{T,\rho}(a,b,v_+,v_-,\nu) + \hat{G}_{T,\rho}(a,b,v_+,v_-,\nu), \tag{2.44}$$

$$\begin{aligned}
&L_{T,\rho}(a,b,v_+,v_-,\nu) \\
&= \begin{pmatrix}
v_+ - \hat{I}_{T_+,\rho}(a,b,\nu) \\
v_- - \hat{I}_{T_-,\rho}(a,b,\nu) \\
\hat{I}_{T_+,\rho}(a,b,\nu)(0) - \hat{I}_{T_-,\rho}(a,b,\nu)(0) \\
DR_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)(\hat{I}_{T_+,\rho}(a,b,\nu)(T_+), \hat{I}_{T_-,\rho}(a,b,\nu)(T_-),\nu) \\
D_v J_{T,\rho}(h_\rho,\mu_\rho) V(\hat{I}_{T_+,\rho}(a,b,\nu), \hat{I}_{T_-,\rho}(a,b,\nu)) + D_\mu J_{T,\rho}(h_\rho,\mu_\rho)\nu
\end{pmatrix}, \\
&\hat{G}_{T,\rho}(a,b,v_+,v_-,\nu) \\
&= \begin{pmatrix}
-\int_{T_+}^{\cdot} \Phi_+^u(\cdot,x_0)\hat{F}_\rho(x_0,v_+(x_0),\nu)dx_0 - \int_0^{\cdot} \Phi_+^s(\cdot,x_0)\hat{F}_\rho(x_0,v_+(x_0),\nu)dx_0 \\
-\int_{T_-}^{\cdot} \Phi_-^s(\cdot,x_0)\hat{F}_\rho(x_0,v_-(x_0),\nu)dx_0 - \int_0^{\cdot} \Phi_-^u(\cdot,x_0)\hat{F}_\rho(x_0,v_-(x_0),\nu)dx_0 \\
\int_{T_+}^0 \Phi_+^u(0,x_0)\hat{F}_\rho(x_0,v_+(x_0),\nu)dx_0 - \int_{T_-}^0 \Phi_-^s(0,x_0)\hat{F}_\rho(x_0,v_-(x_0),\nu)dx_0 \\
R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho) + \hat{R}_\rho(h_\rho(T_+),h_\rho(T_-),\hat{I}_{T_+,\rho}(a,b,\nu)(T_+),\hat{I}_{T_-,\rho}(a,b,\nu)(T_-),\nu) \\
J_{T,\rho}(h_\rho,\mu_\rho) + \hat{J}_{T,\rho}(h_\rho, V(\hat{I}_{T_+,\rho}(a,b,\nu),\hat{I}_{T_-,\rho}(a,b,\nu)),\nu)
\end{pmatrix}.
\end{aligned}$$

Lemma 2.4.13 *$L_{T,\rho} : Y \to \hat{Y}$ is continuously invertible and there is a constant $C > 0$ independent of ρ and T so that $||L_{T,\rho}^{-1}||_{L[\hat{Y},Y]} \leq C$.*

Proof In the following we prove that for any $(g_+, g_-, c, r, j) \in \hat{Y}$ the linear system

$$L_{T,\rho}(a, b, v_+, v_-, \nu) = (g_+, g_-, c, r, j) \tag{2.45}$$

has a unique solution $(a, b, v_+, v_-, \nu) \in Y$ and that

$$||L_{T,\rho}^{-1}(g_+, g_-, c, r, j)||_Y = ||(a, b, v_+, v_-, \nu)||_Y \leq C||(g_+, g_-, c, r, j)||_{\hat{Y}}, \tag{2.46}$$

where C is a positive constant independent of ρ and T.

The first two equations of (2.45) are solved by

$$(v_+, v_-) = W_1(a, b, \nu, g_+, g_-) := (g_+ + \hat{I}_{T_+,\rho}(a, b, \nu), g_- + \hat{I}_{T_-,\rho}(a, b, \nu)).$$

It follows from Lemma 2.4.12:

$$\begin{aligned}
&||W_1(a, b, \nu, g_+, g_-)||_{C^0([0,T_+],X^\alpha)\times C^0([T_-,0],X^\alpha)} \\
&\leq ||g_+||_{C^0([0,T_+],X^\alpha)} + ||\hat{I}_{T_+,\rho}(a, b, \nu)||_{C^0([0,T_+],X^\alpha)} + ||g_-||_{C^0([T_-,0],X^\alpha)} \\
&\quad + ||\hat{I}_{T_-,\rho}(a, b, \nu)||_{C^0([T_-,0],X^\alpha)} \\
&\leq ||g_+||_{C^0([0,T_+],X^\alpha)} + C||(a, b, \nu)||_{X_a\times X_b\times\mathbb{R}} + ||g_-||_{C^0([T_-,0],X^\alpha)} + C||(a, b, \nu)||_{X_a\times X_b\times\mathbb{R}} \\
&\leq C||(g_+, g_-, a, b, \nu)||_{X_a\times X_b\times\mathbb{R}},
\end{aligned}$$

where C is independent of ρ and T.

The forth equation of (2.45) can be written in the form

$$\begin{aligned}
r &= DR_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho)(w_+, w_-, \nu) \quad \text{with} \\
w_+ &= \Phi_+^u(T_+, T_+)a_+ + \Phi_+^s(T_+, 0)b_+ + \nu \int_0^{T_+} \Phi_+^s(T_+, x_0)D_\mu f(h_\rho(x_0), \mu_\rho)dx_0, \\
w_- &= \Phi_-^s(T_-, T_-)a_- + \Phi_-^u(T_-, 0)b_- + \nu \int_0^{T_-} \Phi_-^u(T_-, x_0)D_\mu f(h_\rho(x_0), \mu_\rho)dx_0.
\end{aligned} \tag{2.47}$$

Because of **(K)**, Lemma 2.1.3 and Theorem 1.4.4 the operators[19] $\Phi_+^u(T_+, T_+)$ and P_+ as well as $\Phi_-^s(T_-, T_-)$ and P_- are close to each other for all sufficiently large Intervals T and sufficiently small ρ.

This statement justifies that

$$D_{(u_+,u_-)}R_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho)(\Phi_+^u(T_+, T_+)|_{R(P_+)}, \Phi_-^s(T_-, T_-)|_{R(P_-)})$$

[19] See Definition 2.3.3.

is a linear and invertible map from $R(P_+) \times R(P_-)$ to $X^\alpha = X^\alpha_+ \oplus X^\alpha_-$ uniformly in ρ because of **(T1)(ii)** and

$$\begin{aligned}
&D_{(u_+,u_-)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\left(\Phi^u_+(T_+,T_+)\left|_{R(P_+)},\Phi^s_-(T_-,T_-)\right|_{R(P_-)}\right)\\
=\quad &D_{(u_+,u_-)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\left(\Phi^u_+(T_+,T_+)\left|_{R(P_+)},\Phi^s_-(T_-,T_-)\right|_{R(P_-)}\right)\\
&-D_{(u_+,u_-)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\left(P_+\left|_{R(P_+)},P_-\right|_{R(P_-)}\right)\\
&+D_{(u_+,u_-)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\left(P_+\left|_{R(P_+)},P_-\right|_{R(P_-)}\right)\\
&-D_{(u_+,u_-)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\left(P_{+,\rho}(\mu_\rho)\left|_{R(P_+)},P_{-,\rho}(\mu_\rho)\right|_{R(P_-)}\right)\\
&+D_{(u_+,u_-)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\left(P_{+,\rho}(\mu_\rho)\left|_{R(P_+)},P_{-,\rho}(\mu_\rho)\right|_{R(P_-)}\right)\\
&-D_{(u_+,u_-)}R_\rho(p_\rho(\mu_\rho),p_\rho(\mu_\rho),\mu_\rho)\left(P_{+,\rho}(\mu_\rho)\left|_{R(P_+)},P_{-,\rho}(\mu_\rho)\right|_{R(P_-)}\right)\\
&+D_{(u_+,u_-)}R_\rho(p_\rho(\mu_\rho),p_\rho(\mu_\rho),\mu_\rho)\left(P_{+,\rho}(\mu_\rho)\left|_{R(P_+)},P_{-,\rho}(\mu_\rho)\right|_{R(P_-)}\right).
\end{aligned} \tag{2.48}$$

Let E be the first six summands and $\tilde{A}$ be the last summand of the right hand side of (2.48). For sufficiently small ρ and large T we obtain $||\tilde{A}^{-1}E||_{L[R(P_+)\times R(P_-)]} < 1$. Note that $\tilde{A}$ is invertible and its inverse is bounded uniformly by **(T1)(ii)**. Therefore, considering the Neumann series we obtain that

$$D_{(u_+,u_-)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\left(\Phi^u_+(T_+,T_+)\left|_{R(P_+)},\Phi^s_-(T_-,T_-)\right|_{R(P_-)}\right) = \tilde{A}(\tilde{A}^{-1}E + \mathrm{id})$$

is invertible uniformly in ρ.

Now we can solve (2.47) for $a = (a_+, a_-)$:

$$\begin{aligned}
&\begin{pmatrix} a_+ \\ a_- \\ 0 \end{pmatrix} = \\
&\left(\begin{matrix} \left(D_{(u_+,u_-)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\left(\Phi^u_+(T_+,T_+)\left|_{R(P_+)},\Phi^s_-(T_-,T_-)\right|_{R(P_-)}\right)\right)^{-1} & 0 \\ 0 & \mathrm{id} \end{matrix}\right) r \\
&-\left(\begin{matrix} \left(D_{(u_+,u_-)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\left(\Phi^u_+(T_+,T_+)\left|_{R(P_+)},\Phi^s_-(T_-,T_-)\right|_{R(P_-)}\right)\right)^{-1} & 0 \\ 0 & \mathrm{id} \end{matrix}\right) \\
&\quad\begin{pmatrix} D_{(u_+,u_-)}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho) & 0 \\ 0 & D_\mu R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho) \end{pmatrix} \\
&\quad\begin{pmatrix} \Phi^s_+(T_+,0)b_+ + \nu\int_0^{T_+}\Phi^s_+(T_+,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0 \\ \Phi^u_-(T_-,0)b_- + \nu\int_0^{T_-}\Phi^u_-(T_-,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0 \\ \nu \end{pmatrix}.
\end{aligned}$$

We can write $a = W_2(b,\nu,r)$ with

$$||W_2(b,\nu,r)||_{X^\alpha} \le C\left(e^{-\kappa T_+}||b_+||_{X^\alpha} + e^{\kappa T_-}||b_-||_{X^\alpha} + |\nu| + ||r||_{X^\alpha}\right)$$

due to **(T1)** and to the estimates

$$||\Phi_+^s(T_+,0)b_+||_{X^\alpha} \leq Ce^{-\kappa T_+}||b_+||_{X^\alpha}, \quad ||\Phi_+^u(T_-,0)b_-||_{X^\alpha} \leq Ce^{\kappa T_-}||b_-||_{X^\alpha}. \tag{2.49}$$

In the following we regard the equation of (2.45) which contains the phase condition. Using the estimates (2.49), the definition (2.43) and the estimates for a results in

$$\begin{aligned}
&V(\hat{I}_{T_+,\rho},\hat{I}_{T_-,\rho})(a,b,\nu)(x) = \hat{I}_{T_+,\rho}(a,b,\nu)(x) + \hat{I}_{T_-,\rho}(a,b,\nu)(0) - \hat{I}_{T_+,\rho}(a,b,\nu)(0)\\
&= \Phi_+^u(x,T_+)a_+ + \Phi_+^s(x,0)b_+\\
&\quad + \nu\left(\int_{T_+}^x \Phi_+^u(x,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0 + \int_0^x \Phi_+^s(x,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\right)\\
&\quad + \Phi_-^s(0,T_-)a_- + \Phi_-^u(0,0)b_- + \nu\int_{T_-}^0 \Phi_-^s(0,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\\
&\quad - \Phi_+^u(0,T_+)a_+ - \Phi_+^s(0,0)b_+ - \nu\int_{T_+}^0 \Phi_+^u(0,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\\
&= \Phi_+^s(x,0)b_+ + b_- - b_+ - \Phi_-^s(0,0)b_- + \Phi_+^u(0,0)b_+ + (\Phi_+^u(x,T_+) - \Phi_+^u(0,T_+))a_+\\
&\quad + \Phi_-^s(0,T_-)a_- + \nu\left(\int_{T_+}^x \Phi_+^u(x,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0 + \int_0^x \Phi_+^s(x,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\right)\\
&\quad + \nu\int_{T_-}^0 \Phi_-^s(0,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0 - \nu\int_{T_+}^0 \Phi_+^u(0,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\\
&= \Phi_+^s(x,0)b_+ + b_- - b_+ + W_3(b,\nu,r)(x), \quad x > 0,
\end{aligned}$$

$$\begin{aligned}
&V(\hat{I}_{T_+,\rho},\hat{I}_{T_-,\rho})(a,b,\nu)(x) = \hat{I}_{T_-,\rho}(a,b,\nu)(x)\\
&= \Phi_-^s(x,T_-)a_- + \Phi_-^u(x,0)b_-\\
&\quad + \nu\left(\int_{T_+}^x \Phi_+^u(x,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0 + \int_0^x \Phi_+^s(x,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\right)\\
&= \Phi_-^u(x,0)b_- + W_4(b,\nu,r)(x), \quad x \leq 0.
\end{aligned}$$

Here, W_3 and W_4 satisfy the estimates

$$||W_{3,4}(b,\nu,r)(x)||_{X^\alpha} \leq C\left(e^{-\kappa T_+}||b_+||_{X^\alpha} + e^{\kappa T_-}||b_-||_{X^\alpha} + |\nu| + ||r||_{X^\alpha}\right).$$

As we have seen in Section 2.3 we can write

$$(b_+,b_-) = (\hat{b}_+,\hat{b}_-) + \gamma\left(\frac{\partial}{\partial x}h(0), \frac{\partial}{\partial x}h(0)\right), \quad (\hat{b}_+,\hat{b}_-) \in \hat{X}_b,$$

where $\hat{X}_b \oplus \operatorname{span}\left\{\left(\frac{\partial}{\partial x}h(0), \frac{\partial}{\partial x}h(0)\right)\right\} = X_b = R(\Phi_+^s(0,0)) \times R(\Phi_-^u(0,0))$. This leads to

$$\begin{aligned}
V(\hat{I}_{T_+,\rho},\hat{I}_{-,t,\rho})(a,b,\nu)(x) &= \gamma\frac{\partial}{\partial x}h(x) + W_5(b,\nu,r)(x) \quad \text{with}\\
||W_5(b,\nu,r)(x)||_{X^\alpha} &\leq C\left(e^{-\kappa|T|}|\gamma| + ||\hat{b}_+||_{X^\alpha} + ||\hat{b}_-||_{X^\alpha} + |\nu| + ||r||_{X^\alpha}\right),
\end{aligned}$$

where $|T| := \min\{|T_-|, T_+\}$.

Now we can rewrite the phase condition and the continuity equation of (2.45):

$$\begin{aligned} j &= \gamma D_v J_{T,\rho}(h_\rho,\mu_\rho)\frac{d}{dx}h + D_v J_{T,\rho}(h_\rho,\mu_\rho)W_5(b,\nu,r) \\ &= \gamma D_v J_{T,\rho}(h_\rho,\mu_\rho)\frac{d}{dx}h + W_6(b,\nu,r), \\ c &= \Phi_+^u(0,T_+)W_{2,+}(b_+,\nu,r_+) - \Phi_-^s(0,T_-)W_{2,-}(b_-,\nu,r_-) + \hat{b}_+ - \hat{b}_- \\ &\quad - \nu\left(\int_0^{T_+}\Phi_+^u(0,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0 + \int_{T_-}^0 \Phi_-^s(0,x_0)D_\mu f(h_\rho(x_0),\mu_\rho)dx_0\right). \end{aligned} \tag{2.50}$$

We obtain the estimate

$$|W_6(b,\nu,r)| \leq C\left(e^{-\kappa T_+}||b_+||_{X^\alpha} + e^{\kappa T_-}||b_-||_{X^\alpha} + |\nu| + ||r||_{X^\alpha}\right).$$

Moreover, we have

$$\begin{aligned} &||\Phi_+^u(0,T_+)W_{2,+}(b_+,\nu,r_+) - \Phi_-^s(0,T_-)W_{2,-}(b_-,\nu,r_-)||_{X^\alpha} \\ &\leq C\left(e^{-\kappa T_+} + e^{\kappa T_-}\right)(||b_+||_{X^\alpha} + ||b_-||_{X^\alpha} + |\nu| + ||r||_{X^\alpha}). \end{aligned}$$

The integral

$$\int_{T_-}^{T_+}\langle\psi(x), D_\mu f(h_\rho(x),\mu_\rho)\rangle dx$$

is bounded away from 0 because of **(H9)**, persistence Theorem 2.1.6 and because of the exponential decay of $\psi(x)$.

Finally, one can solve (2.50) for $(\hat{b},\gamma,\nu)$ employing Theorem 2.1.6, **T1(i)** and the arguments of Section 2.3 (Lemma 2.3.13 and Theorem 2.3.14). ∎

The following lemma relates to the maps of (2.35) and (2.42).

Lemma 2.4.14
(a)

$$||D_{(v,\nu)}\hat{F}_\rho(x,v,\nu)||_{L[X^\alpha\times\mathbb{R},X]} \leq C(||v||_{X^\alpha} + |\nu|) + g(\rho)$$

for sufficiently small ρ, v and ν. The function $g(\rho)$ satisfies $g(\rho)\to 0$ as $\rho\to 0$.

(b)

$$||G_{T,\rho}(0,0,0,0,0)||_{\hat{Y}} \leq C||R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)||_{X^\alpha}.$$

Proof (a) Consider

$$\begin{aligned} &||D_{(v,\nu)}\hat{F}_\rho(x,v,\nu)||_{L[X^\alpha\times\mathbb{R},X]} = \\ &\left|\left|\begin{pmatrix} Q_\rho(D_u f(h_\rho(x)+v,\mu_\rho+\nu) - D_u f(h(x),0)) - (\mathrm{id}-Q_\rho)D_u f(h(x),0) \\ Q_\rho(D_\mu f(h_\rho(x)+v,\mu_\rho+\nu) - D_\mu f(h_\rho(x),\mu_\rho)) - (\mathrm{id}-Q_\rho)D_\mu f(h_\rho(x),\mu_\rho))] \end{pmatrix}\right|\right|_{L[X^\alpha\times\mathbb{R},X]} \\ &\leq C||D_u f(h_\rho(x)+v,\mu_\rho+\nu) - D_u f(h(x),0)||_{L[X^\alpha,X]} + ||(\mathrm{id}-Q_\rho)D_u f(h(x),0))||_{L[X^\alpha,X]} \\ &\quad + C||D_\mu f(h_\rho(x)+v,\mu_\rho+\nu) - D_\mu f(h_\rho(x),\mu_\rho)||_X + ||(\mathrm{id}-Q_\rho)D_\mu f(h_\rho(x),\mu_\rho))||_X. \end{aligned}$$

Since $D_\mu f(\cdot,\cdot)$ and $D_u f(\cdot,\cdot)$ are continuous and $K := \{h_\rho(x) : x\in\mathbb{R}\}\times\{\mu_\rho\}$ is compact there exists an open neighbourhood $\tilde{O}$ of K so that[20] $D_\mu f(\cdot,\cdot)$ and $D_u f(\cdot,\cdot)$ is uniformly continuous

[20] Confer proof of Lemma 2.3.11.

on $\tilde{O}$. Because of Theorem 2.1.6 (ii) we have the estimate

$$||D_u f(h_\rho(x)+v,\mu_\rho+\nu) - D_u f(h(x),0)||_{L[X^\alpha,X]} + ||D_\mu f(h_\rho(x)+v,\mu_\rho+\nu) - D_\mu f(h_\rho(x),\mu_\rho)||_X$$
$$\leq C(||v||_{X^\alpha} + |\nu|)$$

for sufficiently small ρ, v and ν.

Furthermore,

$$\sup_{x\in\mathbb{R}} ||(\mathrm{id} - Q_\rho) D_u f(h(x),0)||_{L[X^\alpha,X]} \to 0 \quad \text{as} \quad \rho \to 0,$$
$$\sup_{x\in\mathbb{R}} ||(\mathrm{id} - Q_\rho) D_\mu f(h_\rho(x),\mu_\rho)||_X \to 0 \quad \text{as} \quad \rho \to 0$$

are consequences of Lemma 2.1.3.

(b) We obtain $G_{T,\rho}(0,0,0,0,0) = (0,0,0,R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho),J_{T,\rho}(h_\rho,\mu_\rho))$. There is a small shift $\gamma_{T,\rho}$ so that replacing $h_\rho(\cdot)$ by $h_\rho(\cdot+\gamma_{T,\rho})$ yields $J_{T,\rho}(h_\rho,\mu_\rho)=0$. This leads to the estimate (b). ∎

Lemma 2.4.15 *$\hat{G}_{T,\rho}$ is smooth in every $(a_0,b_0,v_{0,+},v_{0,-},\nu_0) \in Y$ and its Frechet derivative converges to zero as $(a_0,b_0,v_{0,+},v_{0,-},\nu_0) \to 0$.*

Proof To prove the statement regarding the first three components of $\hat{G}_{T,\rho}$ consider the Lemma 2.4.14 (a) and proceed as in Section 2.3.

The last two components of $\hat{G}_{T,\rho}$ are given by the maps

$$\begin{aligned}(a,b,\nu) \mapsto\ & R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)\\ & + \hat{R}_\rho(h_\rho(T_+),h_\rho(T_-),\hat{I}_{T_+,\rho}(a,b,\nu)(T_+),\hat{I}_{T_-,\rho}(a,b,\nu)(T_-),\nu),\\ (a,b,\nu) \mapsto\ & J_{T,\rho}(h_\rho,\mu_\rho) + \hat{J}_{T,\rho}(h_\rho, V(\hat{I}_{T_+,\rho}(a,b,\nu),\hat{I}_{T_-,\rho}(a,b,\nu)),\nu).\end{aligned} \tag{2.51}$$

The chain rule yields for the Frechet derivative of the first map of (2.51)

$$\begin{aligned}(a_0,b_0,\nu_0) \mapsto & D_{(u_+,u_-,\nu)}\hat{R}_\rho(h_\rho(T_+),h_\rho(T_-),\hat{I}_{T_+,\rho}(a_0,b_0,\nu_0)(T_+),\hat{I}_{T_-,\rho}(a_0,b_0,\nu_0)(T_-),\nu_0)\\ & (\hat{I}_{T_+,\rho}(a_0,b_0,\nu_0)(T_+),\hat{I}_{T_-,\rho}(a_0,b_0,\nu_0)(T_-),1).\end{aligned}$$

Consider **(T1)** and Lemma 2.4.12. It follows from Lemma 2.4.12 and 2.4.9 the continuity of the Frechet derivative and the statement

$$\begin{aligned}&||D_{(u_+,u_-,\nu)}\hat{R}_\rho(h_\rho(T_+),h_\rho(T_-),\hat{I}_{T_+,\rho}(a_0,b_0,\nu_0)(T_+),\hat{I}_{T_-,\rho}(a_0,b_0,\nu_0)(T_-),\nu_0)\\ &\quad (\hat{I}_{T_+,\rho}(a_0,b_0,\nu_0)(T_+),\hat{I}_{T_-,\rho}(a_0,b_0,\nu_0)(T_-),1)||_{L[X_a\times X_b\times\mathbb{R},X^\alpha]}\\ &\leq ||D_{(u_+,u_-,\nu)}\hat{R}_\rho(h_\rho(T_+),h_\rho(T_-),\hat{I}_{T_+,\rho}(a_0,b_0,\nu_0)(T_+),\hat{I}_{T_-,\rho}(a_0,b_0,\nu_0)(T_-),\nu_0)||_{L[X^\alpha\times X^\alpha\times\mathbb{R},X^\alpha]}\\ &\quad ||(\hat{I}_{T_+,\rho}(a_0,b_0,\nu_0)(T_+),\hat{I}_{T_-,\rho}(a_0,b_0,\nu_0)(T_-),1)||_{L[X_a\times X_b\times\mathbb{R},X^\alpha]}\\ &\leq C(||\hat{I}_{T_+,\rho}(a_0,b_0,\nu_0)(T_+)||_{X^\alpha}) + ||\hat{I}_{T_-,\rho}(a_0,b_0,\nu_0)(T_-)||_{X^\alpha} + |\nu|)\\ &\leq C(||\hat{I}_{T_+,\rho}(a_0,b_0,\nu_0)||_{C^0([0,T_+],X^\alpha)} + ||\hat{I}_{T_-,\rho}(a_0,b_0,\nu_0)||_{C^0([T_-,0],X^\alpha)} + |\nu|)\\ &\quad \to 0 \quad \text{as } (a_0,b_0,\nu_0) \to 0.\end{aligned}$$

In the same way we can prove the statement regarding $J_{T,\rho}$. ∎

Now we can apply Theorem 2.2.1 to the map $G_{T,\rho}$, see (2.42).

Proof of Theorem 2.4.2 The spaces Y and $\hat{Y}$ of (2.42) are Banach spaces. Lemma 2.4.15 yields the smoothness of $G_{T,\rho}$ and Lemma 2.4.13 the continuous invertibility of $L_{T,\rho}$. Due to Lemma 2.4.15 there are numbers $0 < r$ and $0 < \kappa < 1$ so that

$$||\mathrm{id} - L_{T,\rho}^{-1} DG_{T,\rho}(a, b, v_+, v_-, \nu)||_{L[Y]} \leq C||D\hat{G}_{T,\rho}(a, b, v_+, v_-, \nu)||_{L[Y,\hat{Y}]} \leq \kappa$$

for all $(a, b, v_+, v_-, \nu) \in B(0, r)$. Due to Lemma 2.4.14 (b), **(T1)(ii)** and Theorem 2.1.6 the estimate

$$||L_{T,\rho}^{-1} G_{T,\rho}(0, 0, 0, 0, 0)||_Y \leq C||R_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho)||_{X^\alpha}(1 - q) \leq r(1 - q)$$

holds for some $\kappa < q < 1$, for every sufficiently small ρ and sufficiently large interval T. Therefore, Theorem 2.2.1 yields for these values of ρ and $T_\pm$ a unique solution $(\bar{a}, \bar{b}, \bar{v}_+, \bar{v}_-, \bar{\nu})$ of $G_{T,\rho}(a, b, v_+, v_-, \nu) = 0$ in a ball centered at the origin.

Because of transformation (2.36) and phase and boundary condition (2.41)

$$\bar{h}_\rho(x) := h_\rho(x + \gamma_{T,\rho}) + \bar{v}(x), \quad \bar{\mu}_\rho := \mu_\rho + \bar{\nu}, \quad \rho \in [0, \rho_0)$$

is the unique solution of the boundary value problem (2.34) in the tube

$$\left\{ (u, \mu) \in C^0([T_-, T_+], X^\alpha) \times \mathbb{R} \, : \, |\mu| + \sup_{x \in [T_-, T_+]} ||u(x) - h(x)||_{X^\alpha} \leq \bar{\eta} \right\}$$

for some positive numbers $\bar{\eta}, \rho_0$. We can choose the tube this way due to Theorem 2.1.6 and due to the estimates

$$\begin{aligned} ||\bar{h}_\rho - h_\rho(\cdot + \gamma_{T,\rho})||_{C^0} &\leq ||\bar{h}_\rho - h||_{C^0} + ||h - h_\rho(\cdot + \gamma_{T,\rho})||_{C^0} \leq \bar{\eta} + g_1(\rho), \\ |\bar{\mu}_\rho - \mu_\rho| &\leq |\bar{\mu}_\rho| + |\mu_\rho| \leq \bar{\eta} + g_2(\rho) \end{aligned}$$

for some functions $g_{1,2}$ with $g_{1,2}(\rho) \to 0$ as $\rho \to 0$. Moreover, we obtain from (2.11) the estimate

$$\begin{aligned} &||(\bar{a}, \bar{b}, \bar{v}_+, \bar{v}_-, \bar{\nu})||_Y \\ &= ||\bar{a}||_{X_a} + ||\bar{b}||_{X_b} + ||\bar{h}_\rho - h_\rho(\cdot + \gamma_{T,\rho})||_{C^0} + ||\bar{h}_\rho - h_\rho(\cdot + \gamma_{T,\rho})||_{C^0} + |\bar{\mu}_\rho - \mu_\rho| \\ &\leq (1 - q)^{-1} ||L_{T,\rho}^{-1} G_{T,\rho}(0, 0, 0, 0, 0)||_Y \leq C||R_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho)||_{X^\alpha} \end{aligned}$$

for some positive number C. This results in

$$|\bar{\mu}_\rho - \mu_\rho| + \sup_{x \in [T_-, T_+]} ||\bar{h}_\rho(x) - h_\rho(x + \gamma_{T,\rho})||_{X^\alpha} \leq C||R_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho)||_{X^\alpha}.$$

∎

Proof of Corollary 2.4.3 Combining Theorems 2.4.2 and 2.1.6 results in

$$
\begin{aligned}
&|\bar{\mu}_\rho| + \sup_{x\in[T_-,T_+]} ||\bar{h}_\rho(x) - h(x)||_{X^\alpha} \\
&\le |\bar{\mu}_\rho - \mu_\rho| + |\mu_\rho| + \sup_{x\in[T_-,T_+]} ||\bar{h}_\rho(x) - h_\rho(x+\gamma_{T,\rho})||_{X^\alpha} + \sup_{x\in[T_-,T_+]} ||h_\rho(x+\gamma_{T,\rho}) - h(x)||_{X^\alpha} \\
&\le C||R_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho)||_{X^\alpha} + |\mu_\rho| + \sup_{x\in[T_-,T_+]} ||h_\rho(x+\gamma_{T,\rho}) - h(x)||_{X^\alpha} \\
&\le C||R_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho) - R_\rho(h(T_+), h(T_-), 0)||_{X^\alpha} + ||R_\rho(h(T_+), h(T_-), 0)||_{X^\alpha} \\
&\quad + |\mu_\rho| + \sup_{x\in[T_-,T_+]} ||h_\rho(x+\gamma_{T,\rho}) - h(x)||_{X^\alpha} \\
&\le C\left(|\mu_\rho| + \sup_{x\in\mathbb{R}} ||h_\rho(x) - h(x)||_{X^\alpha}\right) + ||R_\rho(h(T_+), h(T_-), 0)||_{X^\alpha} \\
&\quad + |\mu_\rho| + \sup_{x\in[T_-,T_+]} ||h_\rho(x+\gamma_{T,\rho}) - h(x)||_{X^\alpha} \\
&\le C\left(||R_\rho(h(T_+), h(T_-), 0)||_{X^\alpha} + \sup_{x\in\mathbb{R}} ||(\mathrm{id} - Q_\rho)h(x)||_{X^\alpha}\right) + \sup_{x\in[T_-,T_+]} ||h_\rho(x+\gamma_{T,\rho}) - h(x)||_{X^\alpha}.
\end{aligned}
$$

∎

2.5 Projection Boundary Conditions

In this section we present the algorithm in practice, where the Galerkin approximation is considered on the space $R(Q_\rho)$ and where we use projection boundary conditions. Confer also [15]. In the third chapter we will consider a numerical example with such boundary conditions.

Definition 2.5.1 ***(Boundary-value problem on*** $R(Q_\rho)$***)***
*Let X be a Hilbert space and $\{Q_\rho\}_{\rho>0}$ a Galerkin approximation which satisfies **(Q)**. Moreover, let $Q_{+,\rho}(\mu)$ and $Q_{-,\rho}(\mu)$ be the stable and unstable spectral projections in $R(Q_\rho)$ of the operator $(A + Q_\rho D_u f(p_\rho(\mu),\mu))|_{R(Q_\rho)}$. Then we define the boundary value problem on $R(Q_\rho)$ by*

$$
\begin{aligned}
\frac{\partial}{\partial x} q &= Aq + Q_\rho f(q,\mu), \quad (q,\mu) \in R(Q_\rho) \times \mathbb{R}, \\
J_{T,\rho}(q,\mu) &= \int_{T_-}^{T_+} \left\langle \frac{\partial}{\partial x} h_\rho(x), q(x) - h_\rho(x) \right\rangle_X dx = 0, \\
R_{+,\rho}(q(T_+),\mu) &= Q_{+,\rho}(\mu)(q(T_+) - p_\rho(\mu)) = 0, \\
R_{-,\rho}(q(T_-),\mu) &= Q_{-,\rho}(\mu)(q(T_-) - p_\rho(\mu)) = 0.
\end{aligned}
\tag{2.52}
$$

Theorem 2.5.2 *Provided that **(H1)**, **(H3)**, **(H6)-(H9)**, **(Q)**, and **(K)** are satisfied and that X is a Hilbert space, there exist constants $\rho_0, \eta, C > 0$ so that for all sufficiently large* *intervals T and for any $\rho \in [0, \rho_0)$ the boundary value problem on $R(Q_\rho)$ has a unique solution $(\bar{h}_\rho, \bar{\mu}_\rho)$ in*

$$\left\{ (q, \mu) \in C^0([T_-, T_+], R(Q_\rho) \times \mathbb{R}) \, : \, |\mu| + \sup_{x \in [T_-, T_+]} ||q(x) - h(x)||_{X^\alpha} \leq \eta \right\}.$$

Moreover,

$$|\bar{\mu}_\rho| + \sup_{x \in [T_-, T_+]} ||\bar{h}_\rho(x) - h(x)||_{X^\alpha} \leq C \left(e^{2\lambda^s T_+} + e^{2\lambda^u T_-} + \sup_{x \in \mathbb{R}} ||(id - Q_\rho)h(x)||_{X^\alpha} \right)$$

where $\lambda^s < 0$ and $\lambda^u > 0$ are chosen so that $\{\lambda \in \mathbb{C} | \lambda^s \leq \Re(\lambda) \leq \lambda^u\} \cap \sigma(A + D_u f(p_0, 0)) = \varnothing$.

Proof Due to **(Q)** and Lemma 2.1.1 the operators Q_ρ are projections in $L[X^\alpha]$. Hence, we can use the decomposition

$$u = q + w =: \begin{pmatrix} q \\ w \end{pmatrix}, \quad q \in R(Q_\rho), \, w \in N(Q_\rho). \tag{2.53}$$

This results in

$$\begin{aligned}
&(A + Q_\rho D_u f(p_\rho(\mu), \mu))u \\
&= (A + Q_\rho D_u f(p_\rho(\mu), \mu))q + (A + Q_\rho D_u f(p_\rho(\mu), \mu))w \\
&= (A + Q_\rho D_u f(p_\rho(\mu), \mu))|_{R(Q_\rho)} q + Q_\rho D_u f(p_\rho(\mu), \mu)|_{N(Q_\rho)} w + A|_{N(Q_\rho)} w \\
&\overset{(*)}{=} \begin{pmatrix} (A + Q_\rho D_u f(p_\rho(\mu), \mu))|_{R(Q_\rho)} q + Q_\rho D_u f(p_\rho(\mu), \mu)|_{N(Q_\rho)} w \\ A|_{N(Q_\rho)} w \end{pmatrix} \\
&= \begin{pmatrix} (A + Q_\rho D_u f(p_\rho(\mu), \mu))|_{R(Q_\rho)} & Q_\rho D_u f(p_\rho(\mu), \mu)|_{N(Q_\rho)} \\ 0 & A|_{N(Q_\rho)} \end{pmatrix} \begin{pmatrix} q \\ w \end{pmatrix}.
\end{aligned}$$

The notation in $(*)$ is also used in the Chapter V.5 of [25], which will also corroborate the following considerations.
Let $\hat{P}_\pm$ and $P_{\pm,\rho}(\mu)$ be the spectral projections of the operators A and $A + Q_\rho D_u f(p_\rho(\mu), \mu)$, respectively. Defining also $Q_{\pm,\rho}(\mu)$ as the spectral projections of $(A + Q_\rho D_u f(p_\rho(\mu), \mu))|_{R(Q_\rho)}$ one obtains some bounded operators $D_{\pm,\rho}(\mu)$ so that

$$P_{\pm,\rho}(\mu) = \begin{pmatrix} Q_{\pm,\rho}(\mu) & D_{\pm,\rho}(\mu) \\ 0 & (\mathrm{id} - Q_\rho)\hat{P}_\pm \end{pmatrix}.$$

Implementing the substitution subject to (2.53) the equation $\frac{\partial}{\partial x} u = Au + Q_\rho f(u, \mu)$ is equivalent to

$$\frac{\partial}{\partial x} q = Aq + Q_\rho f(q + w, \mu), \quad \frac{\partial}{\partial x} w = Aw. \tag{2.54}$$

The phase and boundary conditions are given by

$$\begin{aligned}
\tilde{J}((q, w), \mu) &= \int_{T_-}^{T_+} \left\langle \frac{d}{dx} h_\rho(x), q(x) + w(x) - h_\rho(x) \right\rangle_X dx, \\
\tilde{R}_+((q, w)(T_+), \mu) &= P_{+,\rho}(\mu_\rho)(q(T_+) + w(T_+) - p_\rho(\mu)), \\
\tilde{R}_-((q, w)(T_-), \mu) &= P_{-,\rho}(\mu_\rho)(q(T_-) + w(T_-) - p_\rho(\mu)).
\end{aligned} \tag{2.55}$$

In order to show that (2.55) possesses a unique solution we refer to the Remark 2.4.1. We obtain exemplary for $\mu = \mu_\rho$

$$D_u \tilde{R}_+(p_\rho(\mu_\rho), \mu_\rho)\Big|_{R(P_{\pm,\rho}(\mu_\rho)))} = P_{\pm,\rho}(\mu_\rho)|_{P_{\pm,\rho}(\mu_\rho)}$$

which is invertible as an operator into $R(P_{\pm,\rho}(\mu_\rho))$. For μ close to μ_ρ it is still invertible with uniform inverse. Using Theorem 2.4.2 finally proves that (2.55) has a unique solution.

Because $A|_{N(Q_\rho)}$ is hyperbolic $w = 0$ is the only solution of

$$\frac{\partial}{\partial x} w = Aw, \quad P_+ w(T_+) = 0, \quad P_- w(T_-) = 0.$$

This results in the consilience of (2.54)-(2.55) and (2.52). That is why we can conclude that $(q, w) = (q, 0)$ meets (2.54)-(2.55) if, and only if, q is a solution of the boundary value problem (2.52).

Finally, we have

$$||R_{+,\rho}(h(T_+), 0)||_{X^\alpha} \leq C||h(T_+)||^2_{X^\alpha} \leq Ce^{2\lambda^s T_+}.$$

Since an analogous estimate holds for $R_{-,\rho}(h(T_-), 0)$ the proof is completed. ∎

Remark 2.5.3 *One can also prove the superconvergence property*

$$|\bar{\mu}_\rho| \leq C \left(e^{(2\lambda^s - \lambda^u)T_+} + e^{(2\lambda^u - \lambda^s)T_-} + \sup_{x \in \mathbb{R}} ||(id - Q_\rho)h(x)||_{X^\alpha} \right),$$

see [21].

3 A Numerical Example with Projection Boundary Conditions

At the beginning of this chapter we discuss the more general case of semilinear elliptic equations, where the reflexive Banach space X is chosen to be a Hilbert space. The densely defined and closed operator A is defined on a product space. In many applications it consists of the Laplacian. Hereupon we consider a concrete example and compare the numerical computations with the theoretical predictions of the last chapter. We choose projection boundary conditions when truncating the boundary value problem on the unbounded domain to a bounded domain. For the discussion and the numerical example confer [15].

Abstract Elliptic Equations

Let Y be a Hilbert space and consider a densely defined, self-adjoint and positive definite operator

$$L : D(L) \subset Y \to Y. \tag{3.1}$$

We also assume that L has a compact resolvent. Recall the interpolation spaces given in Definition 1.1.2. For $u \in Y^\alpha$ we analyse the abstract elliptic equation

$$u_{xx} - Lu = g(u, u_x), \tag{3.2}$$

where $x \in \mathbb{R}$ and $g \in C^k(Y^{(1+\alpha)/2} \oplus Y^{\alpha/2}, Y)$ for some $\alpha \in [0, 1)$.

Formulating (3.2) as first order system yields

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & id \\ L & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ g(u,v) \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + G(u, v). \tag{3.3}$$

Here, $(u, v) = (u, u_x)$, $G(u, v) = (0, g(u, v))$ and

$$A = \begin{pmatrix} 0 & id \\ L & 0 \end{pmatrix} : Y^1 \oplus Y^{1/2} \to Y^{1/2} \oplus Y.$$

In the following we discuss the required hypotheses in order to apply the theoretical statements of the last chapter. We explain the setting in some detail but refer to [15] for a more comprehensive verification of the basic hypotheses.

Assumption **(H1)** is satisfied and the associated projections are given by

$$P_\pm = \frac{1}{2}\begin{pmatrix} \mathrm{id} & \pm L^{-1/2} \\ \pm L^{1/2} & \mathrm{id} \end{pmatrix} : Y^{1/2} \oplus Y \to Y^{1/2} \oplus Y.$$

Moreover, the interpolation spaces are $X^\alpha = Y^{\frac{1+\alpha}{2}} \oplus Y^{\frac{\alpha}{2}}$ and $G : X^\alpha \subset X^{\alpha-\varepsilon} \to X$ is two times continuously differentiable due to the smoothness properties of g. The operator L

having compact resolvent leads to compact resolvent of A. In [8] conditions can be found which enable to verify **(H8)**. The hyperbolicity of equilibria according to **(H6)** and the transverse unfolding **(H9)** are generic properties if there is a particular solitary wave solution. At least nonlinearities of the form $g\left(y, u, u_x, \nabla_y u, \mu\right)$ guarantee that these parts of the hypotheses are satisfied. The remaining assumptions not regarding the Galerkin approximation can also be verified, see [15].

To examine numerical examples we use Galerkin approximation which leads to a discretization of the cross-section. Regarding elliptic equations (3.2) it is useful to choose the projections Q_ρ, $\rho \in \left\{\frac{1}{k} | k \in \mathbb{N}\right\}$, as the orthogonal Galerkin projections onto the first m eigenfunctions of the operator L. The completeness of the orthogonal system of eigenfunctions results in the hypothesis **(Q)**.

Finally, we have to discuss the boundary conditions at $x = T_-$ and $x = T_+$ if we regard concrete applications. It can be difficult to determine them because the projections $P_{\pm,\rho}$ might not be easily given. Periodic boundary conditions or the actual computation of P_+ appear to be the generic possibilities. Dirichlet boundary conditions $v(T_\pm) = p$ and Neumann boundary conditions $v(T_\pm) = 0$ are usually choices that will not succeed. But if the numerical problems are reversible systems or equations of variational type, confer [15], there are interesting cases where Dirichlet and Neumann conditions can be applied.

Numerical example with projection boundary conditions

From now on we will consider the following elliptic equation with Neumann boundary conditions:

$$\begin{cases} u_{xx} + u_{yy} + cu_x = u(1 + 2p - u) + p_{yy} - p(1 + p), & (x, y) \in \mathbb{R} \times (-1, 1) \\ u_y(x, \pm 1) = 0, \quad x \in \mathbb{R}. \end{cases} \tag{3.4}$$

Here, c is some real constant and p is given by the polynomial $p(y) := (1+y^2)(1-y^2) = (1-y^2)^2$. The equations $p_y(\pm 1) = 0$ and (3.4) with $u = p$ hold for any constant c. For $c = 0$ there is the solitary wave solution

$$h(x, y) = p(y) + \frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right) \tag{3.5}$$

which is plotted on the next page in Figure 3.1. We choose $(x, y) \in [-15, 15] \times [-1, 1]$.

Setting $g(y, u, u_x) := p(1 + p) - p_{yy} - u(1 + 2p - u) + cu_x$ the differential equation of (3.4) becomes

$$u_{xx} + \Delta_y u + g(y, u, u_x) = 0, \qquad (x, y) \in \mathbb{R} \times (-1, 1).$$

Now we reformulate this equation by defining

$$A := \begin{pmatrix} 0 & id \\ -\Delta_y & 0 \end{pmatrix}, \quad f(u, v, c) := \begin{pmatrix} 0 \\ \hat{g}(u, v) \end{pmatrix},$$

where $\hat{g}(u, v)(y) := -g(y, u(y), v(y))$. This leads to the first order system

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + f(u, v, c)$$

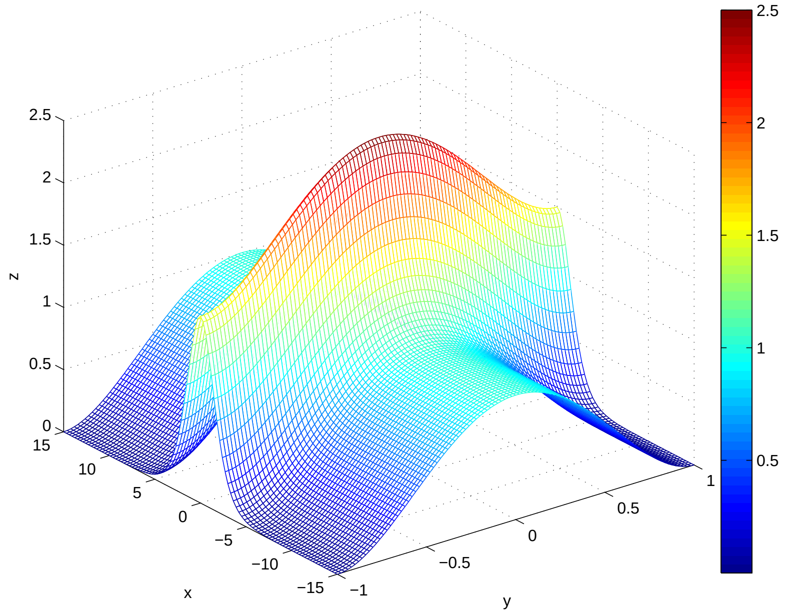

Figure 3.1: Solitary wave h.

where the related reflexive Banach space is $L^2(-1,1)\times L^2(-1,1)$. The operator A with $D(A) = H^1(-1,1)\times L^2(-1,1)$ is densely defined and closed and the assumptions **(H1)**, **(H3)**, **(H6)**-**(H9)** and **(K)** are satisfied, confer [15]. The hyperbolic equilibrium and the homoclinic solution are given by $(p,0)$ and (h,h_x), respectively. Note that $(h,h_x)\to(p,0)$ as $|x|\to\infty$.

Now we implement the projection boundary conditions. Thus we consider the linearization at the equilibrium $(p,0)$:

$$A + D_{(u,v)}f(p,0,c) = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta_y & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathrm{id}+2p-2p & -c \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta_y+\mathrm{id} & -c \end{pmatrix}.$$

For $k\in\mathbb{Z}\setminus\{0\}$ and $c=0$ the even eigenfunctions and the corresponding eigenvalues are given by

$$q_k(y) = \frac{1}{\sqrt{2}}\begin{pmatrix} (1+\pi^2k^2)^{-\frac{1}{2}} \\ \pm 1 \end{pmatrix}\cos(k\pi y), \quad \lambda_k = \pm\sqrt{1+\pi^2k^2},$$

respectively. Here, replace ± 1 by $+1$ for positive k and by -1 for negative k. Moreover, for $c = 0$ there are the even eigenfunctions and eigenvalues indexed by $k = \pm 0$ and given by

$$q_{+0} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_{+0} = 1, \quad q_{-0} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_{-0} = -1,$$

respectively. Defining

$$\mathbb{Z}_{\pm 0} := (\mathbb{Z} \setminus \{0\}) \cup \{+0, -0\}, \quad \mathbb{Z}_{+0} := \big(\mathbb{Z}^+ \setminus \{0\}\big) \cup \{+0\}, \quad \mathbb{Z}_{-0} := \big(\mathbb{Z}^- \setminus \{0\}\big) \cup \{-0\}$$

the set $\{q_k\}_{k \in \mathbb{Z}_{\pm 0}}$ is an orthonormal system in $H^1(-1,1) \times L^2(-1,1)$. Hence

$$\langle q_k, q_l \rangle_{H^1 \times L^2} = \delta_{k,l}, \quad k, l \in \mathbb{Z}_{\pm 0}.$$

In the following we consider the Galerkin approximation

$$Q_n = \sum_{\substack{k=-n \\ \pm 0}}^{n} \langle q_k, \cdot \rangle_{H^1 \times L^2} \; q_k, \quad n \in \mathbb{N}.$$

The sum indicates that k takes the values $\{-n, ..., -0, +0, ..., n\}$. Note that Q_n satisfies hypothesis **(Q)**. In order to compare the theoretical predictions with numerical computations we solve the following differential equation with projection boundary conditions

$$\begin{aligned}
\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} &= A\begin{pmatrix} u \\ v \end{pmatrix} + Q_n f(u, v, c) \\
&= Q_n \begin{pmatrix} v \\ -u_{yy} - cv + u(1 + 2p - u) + p_{yy} - p(1+p) \end{pmatrix} \\
0 &= \int_{-T}^{T} \langle Q_n(h_x, h_{xx})(x), (u, v)(x) - Q_n(h, h_x)(x) \rangle_{L^2 \times L^2} dx \\
0 &= Q_{+,n}(c)((u, v)(T) - (p_n(c), 0)), \\
0 &= Q_{-,n}(c)((u, v)(T) - (p_n(c), 0))
\end{aligned} \tag{3.6}$$

on $(-T, T)$ with $(u, v) \in R(Q_n)$. Since **(T1)** is satisfied we can apply Theorem 2.5.2 and compare its theoretical statements with a concrete computation of (3.6) which reduces to a system of ordinary differential equations (ODEs).

We refer to appendix B for the detailed computations which lead to the system with the corresponding boundary conditions. Equations (B.1) and (B.2) yield

$$
\begin{aligned}
\frac{\partial}{\partial x}a_k &= \frac{1}{2}\left(2b(k)^{-1} \mp c + \frac{-45+16k^4\pi^4}{15k^4\pi^4}b(k)\right)(\pm a_k) \\
&\quad + \frac{1}{2}\left(\pm c + \frac{-45+16k^4\pi^4}{15k^4\pi^4}b(k)\right)\{\pm a_{-k}\} \\
&\quad + \sum_{\substack{l=-n \\ |l|\neq|k|, l\neq 0}}^{n} \frac{b(l)}{2}\frac{48(-1)^{1+k+l}(2k^4+12k^2l^2+2l^4)}{\pi^4(l^8+k^8-4k^6l^2+6k^4l^4-4k^2l^6)}\; a_l \\
&\quad \pm \frac{1}{2\sqrt{2}}\frac{96(-1)^{k+1}}{k^4\pi^4} + \frac{2b(k)}{4}(-a_{+0}-a_{-0})((\pm a_k)+\{a_{-k}\}) \\
&\quad \pm \frac{1}{\sqrt{2}}\frac{48(-1)^k(k^4\pi^4+1680-160k^2\pi^2+k^6\pi^6)}{k^8\pi^8}, \qquad k=-n,...,-1,1,...,n, \\
\frac{\partial}{\partial x}a_{-0} &= \left(-\frac{4}{4}-\frac{2c}{4}-\frac{1}{4}\frac{32}{15}\right)a_{-0} + \left(\frac{2c}{4}-\frac{1}{4}\frac{32}{15}\right)a_{+0} - \sum_{\substack{l=-n \\ l\neq 0}}^{n}\frac{b(l)}{2\sqrt{2}}\frac{96(-1)^{1+l}}{\pi^4 l^4}a_l \\
&\quad + \sum_{\substack{l=-n \\ l\neq 0}}^{n}\frac{b(l)^2}{4}a_l(a_l+a_{-l}) + \frac{2}{8}(a_{+0}+a_{-0})^2 + \frac{1}{2}\frac{592}{315}, \\
\frac{\partial}{\partial x}a_{+0} &= \left(\frac{4}{4}-\frac{2c}{4}+\frac{1}{4}\frac{32}{15}\right)a_{+0} + \left(\frac{2c}{4}+\frac{1}{4}\frac{32}{15}\right)a_{-0} + \sum_{\substack{l=-n \\ l\neq 0}}^{n}\frac{b(l)}{2\sqrt{2}}\frac{96(-1)^{1+l}}{\pi^4 l^4}a_l \\
&\quad - \sum_{\substack{l=-n \\ l\neq 0}}^{n}\frac{b(l)^2}{4}a_l(a_l+a_{-l}) - \frac{2}{8}(a_{+0}+a_{-0})^2 - \frac{1}{2}\frac{592}{315}, \\
\frac{\partial}{\partial x}a_{n+1} &= (h_x(x,y)+h_{xx}(x,y))\left\{a_{+0} - \left(\frac{8}{15}+\frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)+h_x(x,y)\right)\right\} \\
&\quad + (h_x(x,y)-h_{xx}(x,y))\left\{a_{-0} - \left(\frac{8}{15}+\frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)-h_x(x,y)\right)\right\}.
\end{aligned}
\tag{3.7}
$$

The corresponding boundary conditions are given by (B.2) and (B.5):

$$
\begin{aligned}
a_k(T) &= d_k, \qquad k=1,...,n, \\
a_{-k}(-T) &= d_{-k}, \quad k=-1,...,-n, \\
a_{+0}(T) &= d_0, \\
a_{-0}(-T) &= d_0, \\
a_{n+1}(T) &= 0, \\
a_{n+1}(-T) &= 0.
\end{aligned}
\tag{3.8}
$$

Here, the first boundary values are the Galerkin modes of the equilibrium (B.4). They are computed by applying Newton's method to the function (B.3). An initial guess for the boundary value problem (BVP) is given by $Q_n(p(y),0)$. Equations (B.6) yield the corresponding

Galerkin modes for the guess:

$$
\begin{aligned}
a_k &= \frac{48(-1)^{1+k}}{k^4\pi^4}\,\frac{b(k)^{-1}}{\sqrt{2}}, \quad k=-n,...,-1,1,...,n,\\
a_{-0} &= \frac{1}{2}\left(\frac{16}{15}+3\operatorname{sech}^2\left(\frac{x}{2}\right)+3\tanh\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)\right),\\
a_{+0} &= \frac{1}{2}\left(\frac{16}{15}+3\operatorname{sech}^2\left(\frac{x}{2}\right)-3\tanh\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)\right).
\end{aligned}
\tag{3.9}
$$

Solving the boundary value problem with Matlab's routine bvp4c

To solve the BVP (3.6), (3.7) we use Matlab and its solver called bvp4c. It is able to solve a large class of BVPs for ODEs in the Matlab problem solving environment (PSE). For the following short discussion of the routine and of its theoretical backgrounds we refer to [14].

The solver bvp4c is capable of solving ODEs with two-point boundary conditions of the form

$$
\begin{aligned}
\frac{d}{dx}u &= f(x,u,p), \quad x\in[a,b],\\
0 &= g(u(a),u(b),p).
\end{aligned}
\tag{3.10}
$$

Here, p is a vector of unknown parameters.
One of the advantages of bvp4c is that it does not require analytical partial derivatives of the nonlinearities. However, if the user can determine the partial derivatives bvp4c can use them and can thus be more efficient. Moreover, bvp4c is very capable of handling poor guesses for the mesh and for the solution compared to other routines.
The solver bvp4c is based on a collocation method with a C^1-piecewise cubic polynomial S. If $a = x_0 < ... < x_N = b$ is the related mesh the polynomial collocates at the ends of each subinterval $[x_i, x_{i+1}]$ and at the midpoint. The choice of the mesh and the error estimation is determined by the residual of S. This collocation method is equivalent to the 3-stage Lobatto IIIa implicit Runge-Kutta formula. It is also called Simpson method as it becomes the Simpson formula when a quadrature problem is treated. The routine bvp4c neglects some accuracy in favor of a simple behaviour of the residual. Therewith a more inexpensive and asymptotically correct estimate of the residual is possible. The Simpson method leads to algebraic equations which are solved by using a simplified Newton (chord) method. [13] proves that with modest assumptions the piecewise cubic polynomial S and a corresponding isolated solution y satisfy

$$||y(x) - S(x)|| \leq Ch^4$$

for $h = \max_i\{x_{i+1} - x_i\}$ and for all $x \in [a,b]$. bvp4c needs the following three input arguments: A function handle[1] for the right hand side of the differential equations (3.10), a function handle for the residual in the boundary conditions and a structure which contains an initial guess for the solution and an initial mesh. The latter can be created by using Matlab's routine bvpinit. Generally, BVP solvers require an initial guess for the solution since BVPs can have more than one solution.
Moreover, there is an optional integration argument. Matlab's bvpset function creates this

[1] In Matlab a function handle is a value that provides a means of calling a function indirectly. Confer Matlab's product help.

argument which is a structure. Confering Matlab's product help we summarise shortly the categories of optional properties that can be added:
Error tolerance properties, vectorization, analytical partial derivatives, singular BVPs, mesh size property, solution statistic property.
The error tolerance is divided into absolute and relative error tolerance. We use only the optional property of relative error tolerance which applies to all components of the residual vector. It is a measure of the residual relative to the size of the right hand side of the differential equation (3.10). We will mainly use the value 0.001 that corresponds to 0.1% accuracy. As mentioned above, the user can provide analytical expressions of the partial derivatives to make bvp4c be more efficient.

Now we solve the BVP (3.7), (3.8) on the x-y-plane $[-T, T] \times [-1, 1]$. We create an initial guess for the solution and an initial mesh using (3.9) and Matlab's routine bvpinit.

As first example, we analyse $T = 15$ and $n = 14$. We choose $P = 100$ equidistant points beginning at $T = -15$ and ending at $T = 15$ as initial mesh. Moreover, we choose as relative error tolerance the value 0.001. Then we obtain numerically a solitary wave which is plotted in Figure 3.2.

To compare quantitatively the numerical solution of (3.7) and (3.8), denoted by

$$\bar{h}_n = \sum_{\substack{k=-n \\ \pm 0}}^{n} a_k q_k,$$

with the exact solitary wave solution (3.5),

$$h(x,y) = p(y) + \frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right),$$

we refer to Theorem 2.5.2 and compute the following difference:

$$\begin{aligned}
\Delta(T,n) &:= \sup_{x\in[-T,T]} \left\{ \left|\left| \bar{h}_n(x,\cdot) - \begin{pmatrix} h(x,\cdot) \\ h_x(x,\cdot) \end{pmatrix} \right|\right|_{L^2\times L^2} \right\} \\
&= \sup_{x\in[-T,T]} \left\{ \sqrt{\left(\left|\left| \sum_{\substack{k=-n \\ \pm 0}}^{n} a_k (q_k)_u - h(x,\cdot) \right|\right|^2_{L^2\times L^2} + \left|\left| \sum_{\substack{k=-n \\ \pm 0}}^{n} a_k (q_k)_v - h_x(x,\cdot) \right|\right|^2_{L^2\times L^2} \right)} \right\}.
\end{aligned}$$

According to Theorem 2.5.2 we expect to obtain the estimate

$$\Delta(T,n) \leq C\left(e^{-2T} + ||(\mathrm{id} - Q_n)(p,0)||_{L^2\times L^2}\right) \approx C\left(e^{-2T} + n^{-\frac{9}{2}}(an+b)\right)$$

for some constants a and b. That is why we first plot the scaled error $\ln(\Delta(T,n))$ versus the length T, see Figure 3.3. We choose $n = 20$. For values of T smaller than 30 the scaled error $\ln(\Delta(T,n))$ decreases and we have a linear behaviour. But for larger values the error because of truncating the Galerkin modes prevails. If the number n of Galerkin modes increases the latter error becomes smaller. One can also verify that the constant C of the predicted estimate is independent of n, see Figure 8.2 in [15]. Secondly, one analyses the scaled error $n^{9/2}\Delta(T,n)$

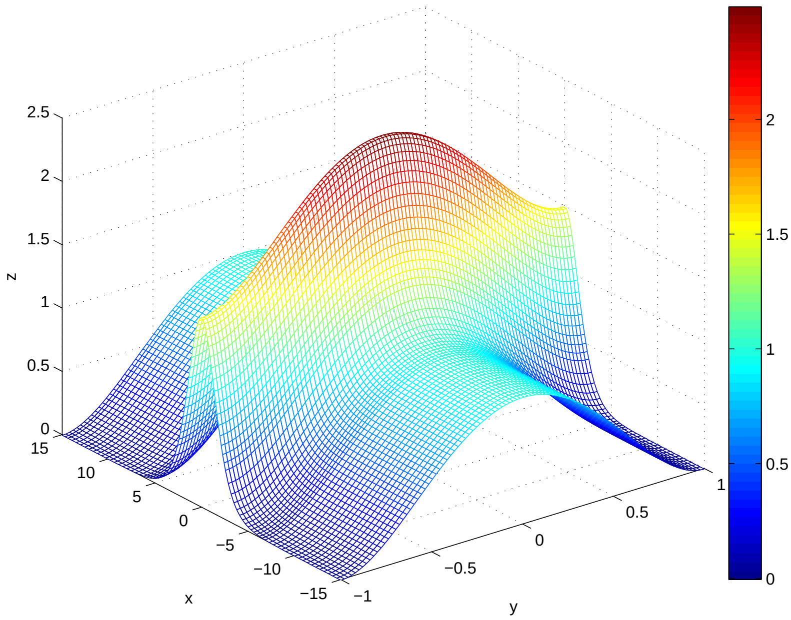

Figure 3.2: Solitary wave solution numerically computed with bvp4c.

for some fixed length T and for different numbers n of the Galerkin modes. We expect a linear behaviour which is confirmed by Figure 8.1 in [15].

Note that we could not exactly recover the diagrams shown in [15]. While the behaviour of T is reproduced correctly the error levels for large T differ from those in [15]. We were not able to finally trace the reasons for this difference. However, note that the authors of [15] used another solver for the BVP and possibly also other norms for their computations.

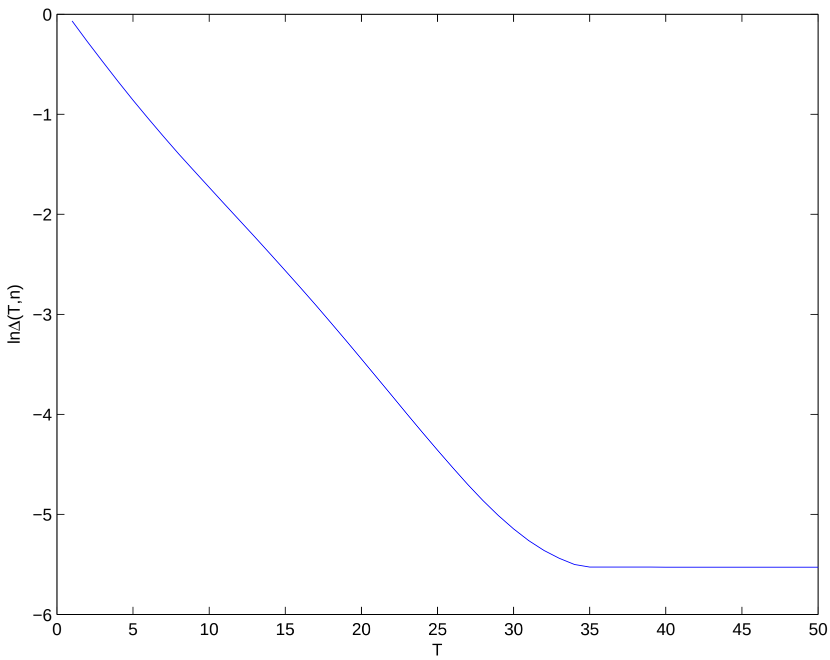

Figure 3.3: Scaled error $\ln(\Delta(T, 20))$ versus the length T of the interval $[-T, T]$.

A Appendix - Background Theory

A.1 Classical Analysis

The following two theorems are from [2], Chapter VII.

Theorem A.1.1 ***(Frechet differentiability)*** *[2]*
Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces over $\mathbb{K}$ and let $U \subset X$ be open. Moreover, let $f : U \to Y$ and $x_0 \in U$. Then the following statements are equivalent:

1. *f is differentiable in x_0.*
2. *There is $A_{x_0} \in L[X, Y]$ and $r_{x_0} : U \to Y$ continuous in x_0 with $r_{x_0}(x_0) = 0$ so that*
$$f(x) = f(x_0) + A_{x_0}(x - x_0) + r_{x_0}(x)||x - x_0||_X, \quad x \in X.$$
3. *There exists a $A_{x_0} \in L[X, Y]$ with*
$$f(x) = f(x_0) + A_{x_0}(x - x_0) + o(||x - x_0||_X) \quad (x \to x_0).$$

The operator A_{x_0} is uniquely determined and is denoted by $Df(x_0)$.

Theorem A.1.2 ***(Taylor's theorem)*** *[2]*
Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces over $\mathbb{K}$, let $U \subset X$ be open and $q \in \mathbb{N} \setminus \{0\}$. If $f \in C^q(U, Y)$, $x \in X$, $h \in U$ and if the line from x to $x + h$ is in U, then

$$f(x+h) = \sum_{k=0}^{q} \frac{1}{k!} D^k f(x)[h]^k + R_q(f, x; h)$$
$$\text{with } R_q(f, x; h) := \int_0^1 \frac{(1-t)^{q-1}}{(q-1)!} [D^q f(x+th) - D^q f(x)][h]^q dt \in Y.$$

Moreover,

$$R_q(f, x; h) = o(||h||^q) \quad \text{with} \quad ||R_q(f, x; h)||_Y \le \frac{1}{q!} \max_{0 \le t \le 1} ||D^q f(x+th) - D^q f(x)||_{L[X,Y]} ||h||^q.$$

The following theorem is taken from [5] and is very important for the proof of the main results of Sections 2.1 and 2.4.

Theorem A.1.3 ***(Contraction mapping theorem with parameters)*** *[5]*
Let $(X, ||\cdot||)$ be a Banach space and

$$F : U \subset X \times \mathbb{R}^l \to X, \quad (u, \mu) \mapsto F(u, \mu)$$

be continuous, where U is open and $l \in \mathbb{N}$. Moreover, let $g_0 : V \subset \mathbb{R}^l \to X$ be continuous with V open and let $r > 0$ so that

$$S = \{(u, \mu) \in X \times \mathbb{R}^l : ||u - g_0(\mu)|| \leq r, \mu \in V\} \subset U.$$

Let $q \in [0, 1)$ so that

$$||F(u, \mu) - F(w, \mu)|| \leq q||u - w|| \quad \forall (u, \mu), (w, \mu) \in S,$$
$$||F(g_0(\mu), \mu) - g_0(\mu)|| \leq r(1 - q) \quad \forall \mu \in V.$$

Then, for every $\mu \in V$ the fixed point problem $F(u, \mu) = u$ has a unique solution $\bar{y} = g(\mu)$ in $\{u \in X : ||u - g_0(\mu)|| \leq r\}$ and $g : V \to X$ is continuous. Furthermore,

$$||u - w|| \leq \frac{1}{1 - q}||u - F(u, \mu) - (w - F(w, \mu))|| \quad \forall (u, \mu), (w, \mu) \in S. \tag{A.1}$$

Finally, $g \in C^p(V, X)$ if $F \in C^p(U, X)$ and $g_0 \in C^p(V, X)$.

The next definition and theorem are from [1], Chapter 1.

Definition A.1.4 ***(Hölder constant, Hölder spaces, Hölder continuous)*** *[1]*
Let $S \subset \mathbb{R}^n$, where $\mathbb{R}^n$ is equipped with an arbitrary norm $||\cdot||$. Let $(X, ||\cdot||_X)$ be a Banach space, $\vartheta > 0$ and let $f : S \to X$. Then we call

$$Höl_\vartheta(f, S) := \sup\left\{ \frac{||f(x) - f(y)||_X}{||x - y||^\vartheta} \ : \ x, y \in S, x \neq y \right\} \tag{A.2}$$

Hölder constant of f on S for the exponent ϑ. Moreover, we define the Hölder spaces by

$$C^{m,\vartheta}(S, X) := \{f \in C^m(S, X) \ : \ Höl_\vartheta(\partial^s f, S) < \infty \text{ for } |s| = m\}, \tag{A.3}$$

where $m \in \mathbb{N}$. We call a function $f \in C^{0,\vartheta}(S, X)$ Hölder continuous. The case $\vartheta = 1$ yields Lipschitz continuous functions. Finally, we set

$$||f||_{C^{m,\vartheta}(S)} := \sum_{|s| \leq m} ||\partial^s f||_{C^0(S)} + \sum_{|s|=m} Höl_\vartheta(\partial^s f, S). \tag{A.4}$$

Theorem A.1.5 *[1]*
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, let $m \in \mathbb{N}$, $0 < \vartheta \leq 1$ and let $(X, ||\cdot||_X)$ be a Banach space. Then $(C^{m,\vartheta}(\bar{\Omega}, X), ||\cdot||_{C^{m,\vartheta}(\bar{\Omega})})$ is a Banach space.

A.2 Functional Analysis

Definition A.2.1 *Let U_1 and U_2 be subspaces of a normed vector spaces $(X, ||\cdot||_X)$ with $U_1 \cap U_2 = \{0\}$. Let U_1 be equipped with a norm $||\cdot||_{U_1}$ and U_2 with $||\cdot||_{U_2}$. Then $U_1 \oplus U_2$ notes the direct sum of U_1 and U_2 with the norm $||w||_{\oplus} = ||w||_{U_1} + ||w||_{U_2}$, where every $w \in U_1 \oplus U_2$ can be uniquely written as $w = w_1 + w_2$ with $w_1 \in U_1$ and $w_2 \in U_2$. We define also[1]*

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} := w \in U_1 \oplus U_2. \tag{A.5}$$

Let $(Y, ||\cdot||_Y)$ be another normed vector space. Then $X \oplus Y$ denotes the direct sum of X and Y with the norm $||(w_X, w_Y)||_{\oplus} = ||w_X||_X + ||w_Y||_Y$, where $w_X \in X$ and $w_Y \in Y$.

The next definition and lemma are from [25].

Definition A.2.2 ***(Resolvent set, spectrum)*** *[25]*
Let $A : D(A) \subset X \to X$ be an operator on a normed vector space X. Then

$$\rho(A) := \{\lambda \in \mathbb{C} | R(\lambda - A) \text{ is dense and } (\lambda - A) \text{ has a continuous inverse}\}$$

is called the resolvent set of the operator and $R_\lambda(A) := (\lambda - A)^{-1}$ the resolvent of A at the point $\lambda \in \rho(A)$. Moreover, $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is defined as the spectrum of A.

Remark A.2.3 *If X is a Banach space and A is closed, $\lambda \in \rho(A)$ results in $R_\lambda(A) \in L[X]$.*

Lemma A.2.4 *[25]*
Let $A : D(A) \subset X \to X$ be an operator on a normed vector space X. Suppose A is such that $R(\lambda - A) = X$ if $\lambda \in \rho(A)$. Then, if $\lambda, \mu \in \rho(A)$ one obtains

$$\begin{aligned} R_\lambda(A) - R_\mu(A) &= (\mu - \lambda) R_\lambda R_\mu, \quad \textit{(resolvent identity)} \\ R_\lambda(A) R_\mu(A) &= R_\mu(A) R_\lambda(A). \end{aligned} \tag{A.6}$$

The next definition of compact operators and maps is taken from [26].

Definition A.2.5 ***(Compact operator, compact map)*** *[26]*

- *A compact operator between normed vector spaces X and Y maps bounded sets on relatively compact sets. One denotes the set of all compact operators by $K[X, Y]$. Moreover, one sets $K[X] := K[X, X]$.*
- *A compact map is a continuous map between Banach spaces that maps bounded sets on relatively compact sets.*

[1] This notation is also used in [25], Chapter V.5.

Theorem A.2.6 *[26]*
An operator $A : D(A) \subset X \to X$ on a normed vector space X with $R(\lambda - A) = X$ for $\lambda \in \rho(A)$ has a compact resolvent, if $A^{-1} \in K[X]$.

The following theorem is taken from [12], Chapter 7.

Theorem A.2.7 *[12]*
Let X be a compact metric space, Y be a Banach space and $F \subset C_b(X, Y)$. Then F is relative compact if and only if F is equicontinuous and $\{f(x)|f \in F\}$ is relative compact in Y for all $x \in X$.

The next definition of the dual space and the next theorem is from [26], Chapter II.

Definition A.2.8 ***(Dual space)*** *[26]*
Let $(X, ||\cdot||_X)$ be a normed vector space. Then let $(X', ||\cdot||_{X'})$ denote the Banach space $L[X, \mathbb{K}]$ equipped with the norm $||x'||_{X'} := ||x||_{L[X,\mathbb{K}]}$. We call X' the dual space of X and its elements $x' \in X'$ functionals. Moreover, we define

$$\langle x', x\rangle := x'(x), \quad x' \in X', \, x \in X.$$

Theorem A.2.9 *[26]*
Let $A : D(A) \subset X \to Y$ be a bounded operator, where $D(A)$ is a dense subspace of a normed vector space $(X, ||\cdot||_X)$ and where $(Y, ||\cdot||_Y)$ is a Banach space. Then there exists a unique continuous extension $\hat{A} \in L[X, Y]$, i.e. there is a bounded operator with $\hat{A}|_{D(A)} = A$. Furthermore, $||\hat{A}||_{L[X,Y]} = ||A||_{L[D(A),Y]}$.

Now we define the conjugate of a densely defined linear operator by citing [25]:

Definition A.2.10 ***(Conjugate of a densely defined linear operator)*** *[25]*
Let $A : D(A) \subset X \to Y$ be a densely defined operator, where $(X, ||\cdot||_X)$, $(Y, ||\cdot||_Y)$ are normed vector spaces and $D(A)$ is a subspace of X. Let

$$D(A') = \{y' \in Y' \, : \, y' \circ A \in L[D(A), \mathbb{K}]\}. \tag{A.7}$$

Because of Theorem A.2.9 the operator $y' \circ A$ has a unique extension to a continuous linear functional on $\overline{D(A)} = X$. We denote this element of X' by $A'y'$. Let the conjugate of A be the linear operator $A' : D(A') \subset Y' \to X'$ defined by

$$(A'y')(x) = y'(Ax), \quad x \in D(A), y' \in D(A'). \tag{A.8}$$

Theorem A.2.11 *[25]*
Let A satisfy the assumptions of the above definition. Then A' is closed.

The next theorem is a continuation version of the Hahn-Banach theorem. It is taken from [26], Chapter III.

Theorem A.2.12 ***(Hahn-Banach theorem, continuation version)*** *[26]*
Let $(X, ||\cdot||_X)$ be a normed vector space and U be a subspace of X. Then every $u' \in L[U, \mathbb{K}]$ has a unique continuation $x' \in X'$ with

$$x'|_U = u', \quad ||x'||_{X'} = ||u||_{L[U,\mathbb{K}]}.$$

Hereupon we define annihilators. Confer again [26], Chapter III.

Definition A.2.13 ***(Annihilator)*** *[26]*
Let $(X, ||\cdot||_X)$ be a normed vector space and $U \subset X$, $V \subset X'$. Then

$$\begin{aligned} U^\perp &:= \{x' \in X' \,|\, x'(x) = 0 \quad \forall x \in U\}, \\ V_\perp &:= \{x \in X \,|\, x'(x) = 0 \quad \forall x' \in V\} \end{aligned} \tag{A.9}$$

are closed subspaces of X' and X, respectively. $U^\perp$ is called annihilator of U in X' and $V_\perp$ annihilator of V in X.

Theorem A.2.14 *[26]*
Let $(X, ||\cdot||_X)$ be a normed vector space and U a closed subspace of X. Then there exist canonical isometric isomorphism

$$(X/U)' \cong U^\perp, \quad U' \cong X'/U^\perp.$$

The following definition and theorem deal with Fredholm operators. We cite [23].

Definition A.2.15 ***(Fredholm operator)*** *[23]*
Let X and Y be Banach spaces. $A \in L[X, Y]$ is called a Fredholm operator if

- $\dim N(A) < \infty$,
- *$R(A)$ is closed,*
- $\dim N(A') < \infty$.

The Fredholm index of A is defined as

$$\operatorname{ind}(A) = \dim(N(A)) - \dim(N(A')).$$

We note that $\dim N(A') < \infty$ can be replaced by $\operatorname{codim}_Y(R(A)) = \dim(Y/R(A)) < \infty$.

Theorem A.2.16 ***(Compact perturbation of a Fredholm operator)*** *[23]*
If $F \in L[X, Y]$ is a Fredholm operator and $K \in L[X, Y]$ is a compact operator from Banach spaces X to Y, then $F + K \in L[X, Y]$ is also a Fredholm operator and satisfies ind$(F + K) =$ ind(F).

The last theorem of this section is an analogon to the open mapping theorem for closed operators, confer [26], Chapter IV.

Theorem A.2.17 ***(Analogon to the open mapping theorem for closed operators)*** *[26]*
Let X and Y be Banach spaces and let $A : D(A) \subset X \to Y$ be a closed and surjective operator, where $D(A)$ is a subspace of X. Then A is open, i.e. A maps open sets onto open sets. If A is also injective A^{-1} is continuous.

A.3 Sectorial Operators, Analytic Semigroups and Fractional Powers of Operators

In this section all definitions and theorems are taken from [11], Chapter 1.

Definition A.3.1 ***(Sectorial operator)***
Let $(X, ||\cdot||)$ be a Banach space and let $A : D(A) \subset X \to X$ be densely defined and closed. A is called a sectorial operator if for some $\phi \in (0, \frac{\pi}{2})$, some $M \geq 1$ and real a

$$S_{a,\phi} := \{\lambda \in \mathbb{C} \,|\, \phi \leq |\arg(\lambda - a)| \leq \pi, \, \lambda \neq a\} \subset \rho(A),$$
$$||(\lambda - A)^{-1}|| \leq \frac{M}{|\lambda - a|} \quad \forall \lambda \in S_{a,\phi}.$$

Definition A.3.2 ***(Analytic semigroup)***
Let $(X, ||\cdot||)$ be a Banach space and $\{T(t)\}_{t\geq 0}$ a family of continuous linear operators on X which satisfy

1. *$T(0) = id, \quad T(t)T(s) = T(t+s) \quad$ for $t \geq 0, \, s \geq 0$,*
2. *$T(t)x \to x$ as $t \to 0^+$, for each $x \in X$,*
3. *$t \mapsto T(t)x$ is real analytic on $0 < t < \infty$ for each $x \in X$.*

Then $\{T(t)\}_{t\geq 0}$ is called an analytic semigroup on X. Furthermore, we define

$$D(L) := \left\{ x \in X \,:\, \exists \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x) \right\}, \quad Lx := \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x) \quad \forall x \in D(L)$$

and call L the infinitesimal generator of the analytic semigroup. We usually write $T(t) = e^{Lt}$.

Theorem A.3.3 *Let A be a sectorial operator and define*

$$e^{-tA} := \frac{1}{2\pi i} \int_\Gamma (\lambda + A)^{-1} e^{\lambda t} d\lambda,$$

where Γ is a contour in $\rho(-A)$ with $\arg(\lambda) \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in \left(\frac{\pi}{2}, \pi\right)$. Then $\left\{e^{-tA}\right\}_{t\geq 0}$ forms an analytic semigroup and $-A$ is its infinitesimal generator. Moreover, e^{-At} can be continued analytically into $\{t \neq 0 \,:\, |arg(t)| < \varepsilon\}$ which contains the positive real axis. We obtain for $t > 0$

$$\frac{d}{dt} e^{-At} = -Ae^{-At}.$$

If $\Re(\sigma(A)) > a$ it follows for $t > 0$

$$||e^{-At}||_{L[X]} \leq Ce^{-at}, \quad ||Ae^{-At}||_{L[X]} \leq \frac{C}{t}e^{-at}$$

for some constant C. Conversely, if $-A$ generates an analytic semigroup, then A is sectorial.

Corollary A.3.4

- *If A is sectorial and $m \in \mathbb{N} \setminus \{0\}$, then $R(e^{-At}) \subset D(A^m)$ for $t > 0$. Consequently, $D(A^m)$ is dense in X for every $m \geq 1$.*
- *If $\{e^{-At}, t \geq 0\}$ is a strongly continuous semigroup ((i) and (ii) of the definition of an analytical semigroup are met for $t > 0$) and $||e^{-At}|| \leq C$, $||Ae^{-At}|| \leq Ct^{-1}$ for $0 < t \leq 1$, then $\{e^{-At}, t \geq 0\}$ is an analytic semigroup.*
- *If $A \in L[X]$ with X Banach space, then e^{-At} as defined above extends to a group of linear operators*

$$e^{-At}e^{-As} = e^{-A(t+s)} \quad \textit{for } s, t \in \mathbb{R} \tag{A.10}$$

and $e^{-At} = \sum_{n=0}^{\infty} \frac{(-At)^n}{n!}$.

Definition A.3.5 ***(Fractional powers of sectorial operators)***
Let A be a sectorial operator with $\Re(\sigma(A)) > 0$. We define for any $\alpha > 0$

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-At} dt, \tag{A.11}$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt$ is the Gamma function.

Theorem A.3.6 *If A is a sectorial operator on a Banach space X with $\Re(\sigma(A)) > 0$, then $A^{-\alpha} \in L[X]$ for any $\alpha > 0$. Moreover, for any $\alpha > 0$, $\beta > 0$ the operator $A^{-\alpha}$ is injective and satisfies $A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$.*

If $0 < \alpha < 1$ we obtain

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^{\infty} \lambda^{-\alpha}(\lambda + A)^{-1} d\lambda.$$

Definition A.3.7 *If A is a sectorial operator on a Banach space X with $\Re(\sigma(A)) > 0$ we define A^{α} as the inverse of $A^{-\alpha}$ with $D(A^{\alpha}) = R(A^{-\alpha})$ for $\alpha > 0$ and A^0 as the identity on the space X.*

Lemma A.3.8

- *A^{α} is closed and densely defined for $\alpha > 0$.*
- *$\alpha \geq \beta \Rightarrow D(A^{\alpha}) \subset D(A^{\beta})$.*
- *$A^{\alpha}A^{\beta} = A^{\beta}A^{\alpha} = A^{\alpha+\beta}$ on $D(A^{\gamma})$ with $\gamma = \max(\alpha, \beta, \alpha + \beta)$.*
- *$[A^{\alpha}, e^{-At}] = 0$ on $D(A^{\alpha})$ for $t > 0$.*

Theorem A.3.9 *Let A be sectorial on a Banach space $(X, ||\cdot||)$ with $\Re(\sigma(A)) > \delta > 0$. It follows that for each $\alpha \geq 0$ there is a constant $C_\alpha < \infty$ such that*

$$||A^\alpha e^{-At}||_{L[X]} \leq C_\alpha t^{-\alpha} e^{-\delta t} \quad \textit{for } t > 0$$

and if $0 < \alpha \leq 1$, $x \in D(A^\alpha)$,

$$||(e^{-At} - id)x|| \leq \frac{1}{\alpha} C_{1-\alpha} t^\alpha ||A^\alpha x|| \quad \textit{for } t > 0.$$

Moreover, C_α is bounded for α in any compact interval which is contained in $(0, \infty)$.

Theorem A.3.10

- *If A is self adjoint and positive definite, then so is A^α for all $\alpha > 0$.*
- *Let A be a sectorial operator with $\Re(\sigma(A)) > 0$, then:*
  A^{-1} is compact $\Leftrightarrow$ $A^{-\alpha}$ is compact for all $\alpha > 0$ $\Leftrightarrow$ e^{-At} is compact for $t > 0$.
- *For each $x \in X$, $t \mapsto tAe^{-At}x$ is continuous from $[0, \infty)$ to X and $||tAe^{-At}|| \to 0$ as $t \to 0^+$.*
- *If $x \in X$ and A is sectorial on X with $\Re(\sigma(A)) > 0$, then $t^\alpha ||A^\alpha e^{-At}x|| \to 0$ as $t \to 0^+$ for $0 < \alpha \leq 1$.*

Definition A.3.11 ***(Interpolation space)***
Let A be a sectorial operator on a Banach space X. We define $A_1 := A + a\, id$ with a chosen so that $\Re(\sigma(A_1)) > 0$. Furthermore, we define for each $a \geq 0$ the interpolation space

$$\begin{aligned} X^\alpha &= D(A_1^\alpha) \quad \textit{with the graph norm} \\ ||x||_{X^\alpha} &= ||A_1^\alpha x||, \quad x \in X^\alpha. \end{aligned} \tag{A.12}$$

Remark A.3.12 *One can prove that different choices of a give equivalent norms on X^α. Therefore, we suppress the dependence on a.*

Theorem A.3.13 *If A is a sectorial operator on a Banach space X, then X^α is a Banach space in the norm $||\cdot||_{X^\alpha}$ for $\alpha \geq 0$, $X^0 = X$, and for $\alpha \geq \beta \geq 0$, X^α is a dense subspace of X^β with continuous inclusion. If A has compact resolvent, the inclusion $X^\alpha \subset X^\beta$ is compact when $\alpha > \beta \geq 0$.*
If A_1, A_2 are sectorial operators in X with the same domain and $\Re(\sigma(A_j)) > 0$ for $j = 1, 2$, and if $(A_1 - A_2)A_1^{-\alpha}$ is bounded for some $\alpha < 1$, then with $X_j^\beta = D(A_j^\beta)\,(j = 1, 2)$, $X_1^\beta = X_2^\beta$ with equivalent norms for $0 \leq \beta \leq 1$.

B Appendix - Details of the Numerical Example from Chapter 3

Here, we will present the computations which lead to a system of ordinary differential equations. First, we give some useful definitions and simple computations:

$$
\begin{aligned}
b(k) &:= (1+\pi^2k^2)^{-\frac{1}{2}} \quad k \in \mathbb{Z}, \\
h(x,y) &= p(y) + \frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right) \\
\Rightarrow h_x(x,y) &= -\frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right) \\
\Rightarrow h_{xx}(x,y) &= \frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)\tanh^2\left(\frac{x}{2}\right) - \frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)\left(\frac{1}{2}-\frac{1}{2}\tanh^2\left(\frac{x}{2}\right)\right).
\end{aligned}
$$

If $(u,v) \in R(Q_n)$ then there are coefficients $\{a_k\}_{k\in\{-n,\dots,-0,+0,\dots,n\}}$ so that

$$
\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{\substack{k=-n \\ \pm 0}}^{n} a_k q_k = \sum_{\substack{k=-n \\ \pm 0}}^{n} a_k \begin{pmatrix} (q_k)_u \\ (q_k)_v \end{pmatrix}.
$$

The coefficients a_k are called Galerkin modes and they depend on x but not on y. Moreover, note that $(q_k)_u = \frac{b(k)}{\sqrt{2}}\cos(k\pi y)$ and $(q_k)_v = \pm\frac{1}{\sqrt{2}}\cos(k\pi y)$ for $k \in \mathbb{Z}\setminus\{0\}$ and $(q_{\pm 0})_u = \frac{1}{2}$, $(q_{\pm 0})_v = \pm\frac{1}{2}$. We define also

$$
(\pm a_k) := \begin{cases} a_k, & k \in \mathbb{Z}_{+0} \\ -a_k, & k \in \mathbb{Z}_{-0} \end{cases}, \quad [\pm a_{-k}] := \begin{cases} -a_{-k}, & k \in \mathbb{Z}_{+0} \\ a_{-k}, & k \in \mathbb{Z}_{-0} \end{cases},
$$

$$
\{\pm a_k\} := \begin{cases} a_{-k}, & k \in \mathbb{Z}_{+0} \\ -a_{-k}, & k \in \mathbb{Z}_{-0} \end{cases},
$$

$$
(\sqrt{2},2)_k := \begin{cases} \sqrt{2}, & k \in \mathbb{Z}\setminus\{0\} \\ 2, & k \in \{-0,+0\} \end{cases}.
$$

If the symbol $\pm$ appears but not standing in front of a Galerkin mode, it corresponds to the sign of k which belongs to q_k.

Differential equation of (3.6)

$$
\begin{aligned}
&Q_n \begin{pmatrix} v \\ -u_{yy} - cv + u(1+2p-u) + p_{yy} - p(1+p) \end{pmatrix} \\
&= \sum_{\substack{k=-n \\ \pm 0}}^{n} \left\langle q_k, \begin{pmatrix} v \\ -u_{yy} - cv + u(1+2p-u) + p_{yy} - p(1+p) \end{pmatrix} \right\rangle_{H^1 \times L^2} q_k \\
&= \sum_{\substack{k=-n \\ \pm 0}}^{n} \left\langle q_k, \begin{pmatrix} \sum_{\substack{l=-n \\ \pm 0}}^{n} a_l (q_l)_v \\ + \sum_{\substack{l=-n \\ \pm 0}}^{n} a_l (q_l)_u (1+2p-u) + p_{yy} - p(1+p) \end{pmatrix} \right\rangle_{H^1 \times L^2} q_k \\
&= \sum_{\substack{k=-n \\ \pm 0}}^{n} \left\langle (q_k)_u, \sum_{\substack{k=-n \\ \pm 0}}^{n} a_l (q_l)_v \right\rangle_{H^1} q_k + \sum_{\substack{k=-n \\ \pm 0}}^{n} \left\langle (q_k)_v, - \sum_{\substack{l=-n \\ \pm 0}}^{n} a_l \frac{\partial^2}{\partial y^2} (q_l)_u - c \sum_{\substack{l=-n \\ \pm 0}}^{n} a_l (q_l)_v \right\rangle_{L^2} q_k \\
&\quad + \sum_{\substack{k=-n \\ \pm 0}}^{n} \left\langle (q_k)_v, \sum_{\substack{l=-n \\ \pm 0}}^{n} a_l (q_l)_u \left(1 + 2p - \sum_{\substack{j=-n \\ \pm 0}}^{n} a_j (q_j)_u \right) + p_{yy} - p(1+p) \right\rangle_{L^2} q_k \\
&= \sum_{\substack{k=-n \\ \pm 0}}^{n} \left\langle \frac{b(k)}{(\sqrt{2},2)_k} \cos(k\pi y), \sum_{\substack{l=-n \\ l \neq 0}}^{n} \frac{\pm a_l}{\sqrt{2}} \cos(k\pi y) + \frac{a_{+0}}{2} - \frac{a_{-0}}{2} \right\rangle_{L^2} q_k \\
&\quad + \sum_{\substack{k=-n \\ k \neq 0}}^{n} \left\langle -\frac{b(k)}{\sqrt{2}} k\pi \sin(k\pi y), \sum_{\substack{l=-n \\ l \neq 0}}^{n} \pm a_l \frac{b(k)}{\sqrt{2}} (-l\pi) \sin(k\pi y) \right\rangle_{L^2} q_k \\
&\quad + \sum_{\substack{k=-n \\ \pm 0}}^{n} \left\langle \frac{\pm 1}{(\sqrt{2},2)_k} \cos(k\pi y), \sum_{\substack{l=-n \\ l \neq 0}}^{n} \frac{a_l}{\sqrt{2}} b(l) l^2 \pi^2 \cos(l\pi y) - c \sum_{\substack{l=-n \\ l \neq 0}}^{n} \frac{\pm a_l}{\sqrt{2}} \cos(k\pi y) - c\frac{a_{+0}}{2} + c\frac{a_{-0}}{2} \right\rangle_{L^2} q_k \\
&\quad + \sum_{\substack{k=-n \\ \pm 0}}^{n} \left\langle \frac{\pm 1}{(\sqrt{2},2)_k} \cos(k\pi y), \sum_{\substack{l=-n \\ l \neq 0}}^{n} \frac{a_l}{\sqrt{2}} b(l) \cos(l\pi y) \left(1 + 2p - \sum_{\substack{j=-n \\ j \neq 0}}^{n} \frac{a_j}{\sqrt{2}} b(j) \cos(j\pi y) - \frac{a_{+0}}{2} - \frac{a_{-0}}{2} \right) \right. \\
&\qquad\qquad \left. + \frac{1}{2}(a_{+0} + a_{-0}) \left(1 + 2p - \sum_{\substack{j=-n \\ j \neq 0}}^{n} \frac{a_j}{\sqrt{2}} b(j) \cos(j\pi y) - \frac{a_{+0}}{2} - \frac{a_{-0}}{2} \right) \right\rangle_{L^2} q_k \\
&\quad + \sum_{\substack{k=-n \\ k \neq 0}}^{n} \left\langle \frac{\pm 1}{\sqrt{2}} \cos(k\pi y), 12y^2 - 4 - p(1+p) \right\rangle_{L^2} q_k + \left\langle \frac{1}{2}, 12y^2 - 4 - p(1+p) \right\rangle_{L^2} q_{+0} \\
&\quad + \left\langle -\frac{1}{2}, 12y^2 - 4 - p(1+p) \right\rangle_{L^2} q_{-0}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{2}(\pm a_k)q_k + \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{2}[\pm a_{-k}]q_k + \frac{2}{4}(a_{+0}-a_{-0})q_{+0} + \frac{2}{4}(a_{+0}-a_{-0})q_{-0} \\
&+ \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{2}k^2\pi^2(\pm a_k)q_k - \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{2}(-k^2\pi^2)[\pm a_{-k}]q_k \\
&+ \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{2}k^2\pi^2(\pm a_k)q_k + \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{2}k^2\pi^2\{\pm a_{-k}\}q_k \\
&- \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{c}{2}(\pm a_k)(\pm q_k) - \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{c}{2}[\pm a_{-k}](\pm q_k) + \frac{2}{4}(-ca_{+0}+ca_{-0})q_{+0} - \frac{2}{4}(-ca_{+0}+ca_{-0})q_{-0} \\
&+ \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{2}(\pm a_k)q_k + \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{2}\{\pm a_{-k}\}q_k \\
&+ \sum_{\substack{k=-n\\k\neq 0}}^{n} \sum_{\substack{l=-n\\|l|\neq|k|,l\neq 0}}^{n} \frac{b(l)}{2}\frac{48(-1)^{1+k+l}(2k^4+12k^2l^2+2l^4)}{\pi^4(l^8+k^8-4k^6l^2+6k^4l^4-4k^2l^6)}a_l(\pm q_k) \\
&+ \sum_{\substack{l=-n\\l\neq 0}}^{n} \frac{b(l)}{2\sqrt{2}}\frac{96(-1)^{1+l}}{\pi^4l^4}a_lq_{+0} - \sum_{\substack{l=-n\\l\neq 0}}^{n} \frac{b(l)}{2\sqrt{2}}\frac{96(-1)^{1+l}}{\pi^4l^4}a_lq_{-0} + \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{2}\frac{-45+16k^4\pi^4}{15k^4\pi^4}(\pm a_k)q_k \\
&+ \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{2}\frac{-45+16k^4\pi^4}{15k^4\pi^4}\{\pm a_{-k}\}q_k - \sum_{\substack{l=-n\\l\neq 0}}^{n} \frac{b(l)^2}{4}a_l^2q_{+0} - \sum_{\substack{l=-n\\l\neq 0}}^{n} \frac{b(l)^2}{4}a_la_{-l}q_{+0} + \sum_{\substack{l=-n\\l\neq 0}}^{n} \frac{b(l)^2}{4}a_l^2q_{-0} \\
&+ \sum_{\substack{l=-n\\l\neq 0}}^{n} \frac{b(l)^2}{4}a_la_{-l}q_{-0} + \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{4}(-a_{+0}-a_{-0})(\pm a_k)q_k + \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{4}(-a_{+0}-a_{-0})\{\pm a_{-k}\}q_k \\
&+ \frac{2}{4}(a_{+0}+a_{-0})q_{+0} - \frac{2}{4}(a_{+0}+a_{-0})q_{-0} + \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{1}{2\sqrt{2}}\frac{96(-1)^{1+k}}{k^4\pi^4}(a_{+0}+a_{-0})(\pm q_k) \\
&+ \frac{1}{4}\;\frac{32}{15}(a_{+0}+a_{-0})q_{+0} - \frac{1}{4}\;\frac{32}{15}(a_{+0}+a_{-0})q_{-0} - \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{4}(a_{+0}+a_{-0})(\pm a_k)q_k \\
&- \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{b(k)}{4}(a_{+0}+a_{-0})\{\pm a_{-k}\}q_k - \frac{2}{8}(a_{+0}+a_{-0})^2q_{+0} + \frac{2}{8}(a_{+0}+a_{-0})^2q_{-0} \\
&+ \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{1}{\sqrt{2}}\;\frac{48(-1)^k(k^4\pi^4+1680-160k^2\pi^2+k^6\pi^6)}{k^8\pi^8}(\pm q_k) - \frac{1}{2}\;\frac{592}{315}q_{+0} + \frac{1}{2}\;\frac{592}{315}q_{-0}.
\end{aligned}
$$

Considering the left hand side of the differential equation (3.6),

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\partial}{\partial x}\sum_{\substack{k=-n \\ \pm 0}}^{n} a_k q_k = \sum_{\substack{k=-n \\ \pm 0}}^{n} \frac{\partial}{\partial x} a_k \; q_k = \sum_{\substack{k=-n \\ k\neq 0}}^{n} \frac{\partial}{\partial x} a_k \; q_k + \frac{\partial}{\partial x} a_{+0} \; q_{+0} + \frac{\partial}{\partial x} a_{-0} \; q_{-0},$$

and equating coefficients yield the following system of differential equations:

$$\begin{aligned}
\frac{\partial}{\partial x} a_k = & \frac{1}{2}\left(2b(k)^{-1} \mp c + \frac{-45+16k^4\pi^4}{15k^4\pi^4} b(k)\right)(\pm a_k) \\
& + \frac{1}{2}\left(\pm c + \frac{-45+16k^4\pi^4}{15k^4\pi^4} b(k)\right)\{\pm a_{-k}\} \\
& + \sum_{\substack{l=-n \\ |l|\neq|k|, l\neq 0}}^{n} \frac{b(l)}{2} \frac{48(-1)^{1+k+l}(2k^4+12k^2l^2+2l^4)}{\pi^4(l^8+k^8-4k^6l^2+6k^4l^4-4k^2l^6)} \; a_l \pm \frac{1}{2\sqrt{2}} \frac{96(-1)^{k+1}}{k^4\pi^4} \\
& + \frac{2b(k)}{4}(-a_{+0}-a_{-0})((\pm a_k)+\{a_{-k}\}) \\
& \pm \frac{1}{\sqrt{2}} \frac{48(-1)^k(k^4\pi^4+1680-160k^2\pi^2+k^6\pi^6)}{k^8\pi^8} \\
& \text{for } k=-n,...,-1,1,...,n, \\
\frac{\partial}{\partial x} a_{-0} = & \left(-\frac{4}{4} - \frac{2c}{4} - \frac{1}{4}\frac{32}{15}\right) a_{-0} + \left(\frac{2c}{4} - \frac{1}{4}\frac{32}{15}\right) a_{+0} - \sum_{\substack{l=-n \\ l\neq 0}}^{n} \frac{b(l)}{2\sqrt{2}} \frac{96(-1)^{1+l}}{\pi^4 l^4} a_l \\
& + \sum_{\substack{l=-n \\ l\neq 0}}^{n} \frac{b(l)^2}{4} a_l(a_l+a_{-l}) + \frac{2}{8}(a_{+0}+a_{-0})^2 + \frac{1}{2}\frac{592}{315}, \\
\frac{\partial}{\partial x} a_{+0} = & \left(\frac{4}{4} - \frac{2c}{4} + \frac{1}{4}\frac{32}{15}\right) a_{+0} + \left(\frac{2c}{4} + \frac{1}{4}\frac{32}{15}\right) a_{-0} + \sum_{\substack{l=-n \\ l\neq 0}}^{n} \frac{b(l)}{2\sqrt{2}} \frac{96(-1)^{1+l}}{\pi^4 l^4} a_l \\
& - \sum_{\substack{l=-n \\ l\neq 0}}^{n} \frac{b(l)^2}{4} a_l(a_l+a_{-l}) - \frac{2}{8}(a_{+0}+a_{-0})^2 - \frac{1}{2}\frac{592}{315}.
\end{aligned} \tag{B.1}$$

Integral condition of (3.6)

At first some auxiliary computations:

$$\left\langle q_{\pm 0}, \begin{pmatrix} h \\ h_x \end{pmatrix} \right\rangle_{H^1\times L^2} = \left\langle \begin{pmatrix} \frac{1}{2} \\ \pm\frac{1}{2} \end{pmatrix}, \begin{pmatrix} h \\ h_x \end{pmatrix} \right\rangle_{H^1\times L^2} = \int_{-1}^{1} \frac{1}{2} h(x,y)dy + 0 \pm \int_{-1}^{1} \frac{1}{2} h_x(x,y)dy$$

$$= \frac{8}{15} + \frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right) \pm h_x(x,y)$$

$$\left\langle q_l, \begin{pmatrix} h_x \\ h_{xx} \end{pmatrix} \right\rangle_{H^1\times L^2} = \left\langle \begin{pmatrix} \frac{b(l)}{(\sqrt{2},2)_l}\cos(l\pi y) \\ \frac{\pm 1}{(\sqrt{2},2)_l}\cos(l\pi y) \end{pmatrix}, \begin{pmatrix} h_x \\ h_{xx} \end{pmatrix} \right\rangle_{H^1\times L^2}$$

$$= \begin{cases} 0, \quad l \neq 0 \\ \int_{-1}^{1} \frac{1}{2} h_x(x,y)dy \pm \int_{-1}^{1} \frac{1}{2} h_{xx}(x,y)dy = h_x(x,y) \pm h_{xx}(x,y), \ l = \pm 0, \end{cases}$$

$$\langle q_l, q_k \rangle_{L^2\times L^2} = \int_{-1}^{1} \frac{b(l)}{(\sqrt{2},2)_l}\cos(l\pi y)\frac{b(k)}{(\sqrt{2},2)_k}\cos(k\pi y)dy$$
$$+ \int_{-1}^{1} \frac{\pm 1}{(\sqrt{2},2)_l}\cos(l\pi y)\frac{\pm 1}{(\sqrt{2},2)_k}\cos(k\pi y)dy$$
$$= \begin{cases} \frac{b(l)^2}{(\sqrt{2},2)_l(\sqrt{2},2)_k}\delta_{|l|,|k|} \pm \frac{1}{(\sqrt{2},2)_l(\sqrt{2},2)_k}\delta_{|l|,|k|} \quad \text{für } (k \neq 0 \vee l \neq 0) \\ \frac{1}{2} \pm \frac{1}{2} \quad \text{für } k,l \in \{-0,+0\} \end{cases}$$

with $\pm$ becoming positive for same signs of k and l and becoming negative for different signs.

Now we can express the integral condition in Galerkin modes:

$$\int_{-T}^{T} \langle Q_n(h_x,h_{xx})(x), (u,v)(x) - Q_n(h,h_x)(x) \rangle_{L^2\times L^2} dx$$
$$= \int_{-T}^{T} \left\langle \sum_{\substack{l=-n \\ \pm 0}}^{n} \left\langle q_l, \begin{pmatrix} h_x \\ h_{xx} \end{pmatrix} \right\rangle_{H^1\times L^2} q_l, \sum_{\substack{k=-n \\ \pm 0}}^{n} a_k q_k - \sum_{\substack{j=-n \\ \pm 0}}^{n} \left\langle q_j, \begin{pmatrix} h \\ h_x \end{pmatrix} \right\rangle_{H^1\times L^2} q_j \right\rangle dx$$
$$= \int_{-T}^{T} \left(\sum_{\substack{l=-n \\ \pm 0}}^{n} \sum_{\substack{k=-n \\ \pm 0}}^{n} \left\langle q_l, \begin{pmatrix} h_x \\ h_{xx} \end{pmatrix} \right\rangle_{H^1\times L^2} a_k \langle q_l, q_k \rangle_{L^2\times L^2} \right.$$
$$\left. - \sum_{\substack{l=-n \\ \pm 0}}^{n} \sum_{\substack{j=-n \\ \pm 0}}^{n} \left\langle q_l, \begin{pmatrix} h_x \\ h_{xx} \end{pmatrix} \right\rangle_{H^1\times L^2} \langle q_l, q_j \rangle_{L^2\times L^2} \left\langle q_j, \begin{pmatrix} h \\ h_x \end{pmatrix} \right\rangle_{H^1\times L^2} \right) dx$$

$$
\begin{aligned}
&+\int_{-T}^{T}\Bigg[\sum_{\substack{k=-n\\ \pm 0}}^{n}\{(h_x(x,y)+h_{xx}(x,y))a_k\langle q_{+0},q_k\rangle_{L^2\times L^2}+(h_x(x,y)-h_{xx}(x,y))a_k\langle q_{-0},q_k\rangle_{L^2\times L^2}\}\\
&\qquad -\sum_{\substack{j=-n\\ \pm 0}}^{n}\Bigg\{(h_x(x,y)+h_{xx}(x,y))\langle q_{+0},q_j\rangle_{L^2\times L^2}\left\langle q_j,\begin{pmatrix} h\\ h_x\end{pmatrix}\right\rangle_{H^1\times L^2}\\
&\qquad\qquad +(h_x(x,y)-h_{xx}(x,y))\langle q_{-0},q_j\rangle_{L^2\times L^2}\left\langle q_j,\begin{pmatrix} h\\ h_x\end{pmatrix}\right\rangle_{H^1\times L^2}\Bigg\}\Bigg]\,dx\\
&=\int_{-T}^{T}\Bigg[(h_x(x,y)+h_{xx}(x,y))\left\{a_{+0}-\left(\frac{8}{15}+\frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)+h_x(x,y)\right)\right\}\\
&\qquad +(h_x(x,y)-h_{xx}(x,y))\left\{a_{-0}-\left(\frac{8}{15}+\frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)-h_x(x,y)\right)\right\}\Bigg]\,dx.
\end{aligned}
$$

We will consider this integral condition by adding the following differential equation and boundary conditions:

$$
\begin{aligned}
\frac{\partial}{\partial x}a_{n+1} &= (h_x(x,y)+h_{xx}(x,y))\left\{a_{+0}-\left(\frac{8}{15}+\frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)+h_x(x,y)\right)\right\}\\
&\quad +(h_x(x,y)-h_{xx}(x,y))\left\{a_{-0}-\left(\frac{8}{15}+\frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)-h_x(x,y)\right)\right\}, \qquad \text{(B.2)}\\
a_{n+1}(T) &= 0\\
a_{n+1}(-T) &= 0
\end{aligned}
$$

Computation of the equilibrium of (3.6)

We search an element

$$
\begin{pmatrix} u\\ v\end{pmatrix}=\sum_{\substack{k=-n\\ \pm 0}}^{n}a_k q_k=\sum_{\substack{k=-n\\ \pm 0}}^{n}a_k\begin{pmatrix}(q_k)_u\\ (q_k)_v\end{pmatrix}\in R(Q_n)
$$

that satisfies

$$
Q_n\begin{pmatrix} v\\ -u_{yy}-cv+u(1+2p-u)+p_{yy}-p(1+p)\end{pmatrix}=0.
$$

This is equivalent to

$$
\begin{aligned}
0&=\langle (q_j)_u,v\rangle_{H^1}+\langle (q_j)_v,-u_{yy}-cv+u(1+2p-u)+p_{yy}-p(1+p)\rangle_{L^2},\\
&\quad j=-n,...,-0,+0,...,n.
\end{aligned}
$$

Because of $(q_{-j})_u=(q_j)_u$ and $(q_{-j})_v=-(q_j)_v$ for $j=+0,1,...,n$ we choose the ansatz

$a_j = a_{-j}$ for $j = +0, 1, ..., n$ which results in

$$v = \sum_{\substack{k=-n \\ \pm 0}}^{n} a_k (q_k)_v = -\sum_{k=+0}^{n} a_k (q_k)_v + \sum_{k=+0}^{n} a_k (q_k)_v = 0,$$

$$u = \sum_{\substack{k=-n \\ \pm 0}}^{n} a_k (q_k)_u = \sum_{k=+0}^{n} a_k (q_k)_u + \sum_{k=+0}^{n} a_k (q_k)_u = 2\sum_{k=+0}^{n} a_k (q_k)_u.$$

Due to $v = 0$ and

$$\begin{aligned} &\langle (q_j)_v, -u_{yy} - cv + u(1 + 2p - u) + p_{yy} - p(1 + p)\rangle_{L^2} \\ &= -\langle (q_{-j})_v, -u_{yy} - cv + u(1 + 2p - u) + p_{yy} - p(1 + p)\rangle_{L^2}, \quad j = +0, 1, ..., n \end{aligned}$$

it suffices to solve

$$\begin{aligned} &\langle (q_j)_v, -u_{yy} - cv + u(1 + 2p - u) + p_{yy} - p(1 + p)\rangle_{L^2} = 0, \quad j = +0, 1, ..., n \\ &\text{with } u = 2\sum_{k=+0}^{n} a_k (q_k)_u. \end{aligned}$$

We divide

$$\begin{aligned} &\langle (q_j)_v, -u_{yy} - cv + u(1 + 2p - u) + p_{yy} - p(1 + p)\rangle_{L^2} \\ &= \Bigg\langle \frac{1}{(\sqrt{2}, 2)_j} \cos(j\pi y), -2\sum_{k=+0}^{n} \frac{b(k)}{(\sqrt{2}, 2)_k} a_k \frac{\partial^2}{\partial y^2} \cos(k\pi y) \\ &\quad +2\sum_{k=+0}^{n} \frac{b(k)}{(\sqrt{2}, 2)_k} a_k \cos(k\pi y) \left(1 + 2p(y) - 2\sum_{l=+0}^{n} \frac{b(l)}{(\sqrt{2}, 2)_l} a_l \cos(l\pi y)\right) + p_{yy}(y) - p(y)(1 + p(y)) \Bigg\rangle_{L^2} \\ &= \left\langle \frac{1}{(\sqrt{2}, 2)_j} \cos(j\pi y), 2\sum_{k=+0}^{n} \frac{b(k)}{(\sqrt{2}, 2)_k} a_k k^2 \pi^2 \cos(k\pi y) \right\rangle_{L^2} \\ &\quad + \left\langle \frac{1}{(\sqrt{2}, 2)_j} \cos(j\pi y), 2\sum_{k=+0}^{n} \frac{b(k)}{(\sqrt{2}, 2)_k} a_k \cos(k\pi y) \right\rangle_{L^2} \\ &\quad + \left\langle \frac{1}{(\sqrt{2}, 2)_j} \cos(j\pi y), 4\sum_{k=+0}^{n} \frac{b(k)}{(\sqrt{2}, 2)_k} a_k \cos(k\pi y)(1 - y^2)^2 \right\rangle_{L^2} \\ &\quad + \left\langle \frac{1}{(\sqrt{2}, 2)_j} \cos(j\pi y), -4\sum_{k=+0}^{n} \frac{b(k)}{(\sqrt{2}, 2)_k} a_k \cos(k\pi y) \sum_{l=+0}^{n} \frac{b(l)}{(\sqrt{2}, 2)_l} a_l \cos(l\pi y) \right\rangle_{L^2} \\ &\quad + \left\langle \frac{1}{(\sqrt{2}, 2)_j} \cos(j\pi y), 12y^2 - (1 - y^2)^2 \left(1 + (1 - y^2)^2\right) \right\rangle_{L^2} \end{aligned}$$

into the cases $j = 1, ..., n$ and $j = +0$:

$$\begin{aligned}
&\langle (q_j)_v, -u_{yy} - cv + u(1+2p-u) + p_{yy} - p(1+p)\rangle_{L^2} \\
&= \left(b(j)^{-1} + \frac{-45 + 16j^4\pi^4}{15j^4\pi^4} b(j) \right) a_j + \frac{1}{\sqrt{2}2} \frac{192(-1)^{1+j}}{j^4\pi^4} a_{+0} \\
&\quad + \sum_{\substack{k=1 \\ k\neq j}}^{n} \frac{96(-1)^{1+j+k}(2j^4 + 12j^2k^2 + 2k^4)}{\pi^4(-4j^6k^2 + 6j^4k^4 + k^8 - 4j^2k^6 + j^8)} \frac{b(k)}{2} a_k - 2b(j)a_j a_{+0} \\
&\quad + \frac{1}{\sqrt{2}} 48(-1)^j \frac{(j^4\pi^4 + 1680 + j^6\pi^6 - 160j^2\pi^2)}{j^8\pi^8}, \quad j = 1, ..., n, \\
&\langle (q_{+0})_v, -u_{yy} - cv + u(1+2p-u) + p_{yy} - p(1+p)\rangle_{L^2} \\
&= \frac{1}{4}\left(4 + \frac{64}{15}\right) a_{+0} + \sum_{k=1}^{n} \frac{192(-1)^{1+k}}{k^4\pi^4} \frac{b(k)}{2\sqrt{2}} a_k - \sum_{k=1}^{n} a_k^2 b(k)^2 - a_{+0}^2 - \frac{1}{2}\frac{592}{315}.
\end{aligned}$$

In the following we apply Newton's method to the function

$$\begin{aligned}
F(a_1, ..., a_n, a_{+0}) = &\begin{pmatrix} A_1(1) & & A_2(j,k) & A_3(1) \\ & \ddots & & \\ A_2(j,k) & & A_1(n) & A_3(n) \\ A_4(1) & \cdots & A_4(n) & A_5(n+1) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a_{+0} \end{pmatrix} \\
&+ \begin{pmatrix} f_1(a_1, ..., a_n, a_{+0}) \\ \vdots \\ f_n(a_1, ..., a_n, a_{+0}) \\ \tilde{f}(a_1, ..., a_n, a_{+0}) \end{pmatrix},
\end{aligned} \tag{B.3}$$

where

$$\begin{aligned}
A_1(j) &= \left(b(j)^{-1} + \frac{-45 + 16j^4\pi^4}{15j^4\pi^4} b(j) \right), \quad j \in \{1, ..., n\}, \\
A_2(j,k) &= \frac{96(-1)^{1+j+k}(2j^4 + 12j^2k^2 + 2k^4)}{\pi^4(-4j^6k^2 + 6j^4k^4 + k^8 - 4j^2k^6 + j^8)} \frac{b(k)}{2}, \quad j,k \in \{1, ..., n\},\ k \neq j, \\
A_3(j) &= \frac{1}{\sqrt{2}\,2} \frac{192(-1)^{1+j}}{j^4\pi^4} a_{+0}, \quad j \in \{1, ..., n\}, \\
A_4(j) &= \frac{192(-1)^{1+j}}{j^4\pi^4} \frac{b(j)}{2\sqrt{2}}, \quad j \in \{1, ..., n\}, \\
A_5(n+1) &= \frac{31}{15}, \\
f_j(a_1, ..., a_n, a_{+0}) &= -2b(j)a_j a_{+0} + \frac{1}{\sqrt{2}} 48(-1)^j \frac{(j^4\pi^4 + 1680 + j^6\pi^6 - 160j^2\pi^2)}{j^8\pi^8}, \quad j \in \{1, ..., n\}, \\
\tilde{f}_j(a_1, ..., a_n, a_{+0}) &= -\sum_{k=1}^{n} a_k^2 b(k)^2 - a_{+0}^2 - \frac{1}{2}\frac{592}{315}.
\end{aligned}$$

Because the Fourier series of p is given by

$$\frac{8}{15} + \frac{48}{\pi^4} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \cos(k\pi y)$$

we use the first guess

$$(a_{+0})_0 = \frac{8}{15}, \quad (a_k)_0 = \frac{48(-1)^{k+1}}{k^4\pi^4}\frac{1}{\sqrt{2}}b(k)^{-1} \quad k = 1, ..., n.$$

for Newton's method. In the following we will call the solution coefficients of the equilibrium d_{+0} and d_k so that

$$p_n(c) = 2\sum_{k=+0}^{n} d_k(q_k)_u. \tag{B.4}$$

Boundary value conditions of (3.6)

For the following computations consider (B.4).

$$\begin{aligned}
&Q_{+,n}(c)((u,v)(T) - (p_n(c),0))\\
&= \sum_{k=+0}^{n} \{\langle (q_k)_u, u(T) - p_n(c)\rangle_{H^1} + \langle (q_k)_v, v(T)\rangle_{L^2}\}\, q_k\\
&= \sum_{k=+0}^{n} \left\{ \int_{-1}^{1} \frac{b(k)}{(\sqrt{2},2)_k} \cos(k\pi y) \left(\sum_{\substack{l=-n\\ l\neq 0}}^{n} \frac{b(l)}{\sqrt{2}} a_l(T)\cos(l\pi y) + \frac{1}{2}a_{+0}(T) + \frac{1}{2}a_{-0}(T) - p_n(c) \right) dy \right.\\
&\quad + \int_{-1}^{1} \frac{b(k)}{(\sqrt{2},2)_k}(-k\pi)\sin(k\pi y) \left(\sum_{\substack{l=-n\\ l\neq 0}}^{n} \frac{b(l)}{\sqrt{2}} a_l(T)(-l\pi)\sin(l\pi y) + 0 + 0 - \frac{\partial}{\partial y}p_n(c) \right) dy\\
&\quad \left. + \int_{-1}^{1} \frac{1}{(\sqrt{2},2)_k} \cos(k\pi y) \left(\sum_{\substack{l=-n\\ l\neq 0}}^{n} \frac{1}{\sqrt{2}}(\pm a_l(T))\cos(l\pi y) + \frac{1}{2}a_{+0}(T) + \frac{1}{2}a_{-0}(T) \right) dy + \right\} q_k\\
&= \sum_{k=1}^{n} \frac{b(k)^2}{2} a_k(T) q_k + \sum_{k=1}^{n} \frac{b(k)^2}{2} a_{-k}(T) q_k + (\frac{1}{2}a_{+0}(T) + \frac{1}{2}a_{-0}(T))q_{+0}\\
&\quad + \sum_{k=+0}^{n} \left(\int_{-1}^{1} \frac{b(k)}{(\sqrt{2},2)_k} \cos(k\pi y)\, 2\sum_{l=+0}^{n} d_l(q_l)_u \right) q_k + \sum_{k=+0}^{n} \frac{b(k)^2}{2} k^2\pi^2 a_k(T) q_k\\
&\quad + \sum_{k=1}^{n} (-k\pi)b(k)a_{-k}(T)b(k)(-1)(- - k\pi)q_k + \sum_{k=1}^{n} \int_{-1}^{1} \frac{b(k)}{\sqrt{2}} k\pi \sin(k\pi y) \frac{\partial}{\partial y}\left(2\sum_{l=+0}^{n} d_l(q_l)_u \right) dy\, q_k\\
&\quad + \sum_{k=1}^{n} \frac{1}{2} a_k(T) q_k + \sum_{k=1}^{n} \frac{1}{2}(-a_{-k}(T))q_k + \left(\frac{1}{2}a_{+0}(T) - \frac{1}{2}a_{-0}(T) \right) q_{+0}\\
&= 0
\end{aligned}$$

$$\iff$$
$$\begin{aligned}
0 &= \frac{1}{2}(b(k)^2 + k^2\pi^2 b(k)^2 + 1)a_k(T) + \frac{1}{2}(b(k)^2 + k^2\pi^2 b(k)^2 - 1)a_{-k}(T) - 2d_k, \quad k = 1, ..., n,\\
0 &= a_{+0}(T) - d_0
\end{aligned}$$
$$\iff$$
$$\begin{aligned}
a_k(T) &= d_k, \quad k = 1, ..., n,\\
a_{+0}(T) &= d_0.
\end{aligned}$$

Similar computations lead to

$$Q_{-,n}(c)((u,v)(-T) - (p_n(c), 0)) = 0 \quad \iff \quad \begin{cases} a_k(-T) = d_{-k}, & k = -1, ..., -n\\ a_{-0}(-T) = d_0. \end{cases}$$

Finally, we summarise these boundary conditions:

$$\begin{array}{rlr} a_k(T) = d_k, & k = 1, ..., n, & a_{+0}(T) = d_0,\\ a_{-k}(-T) = d_{-k}, & k = -1, ..., -n, & a_{-0}(-T) = d_0. \end{array} \tag{B.5}$$

Initial guess for the boundary value problem (3.6)

$$\begin{aligned}
Q_n\begin{pmatrix} h\\ h_x \end{pmatrix} &= \sum_{\substack{k=-n\\ \pm 0}}^{n} \langle (q_k)_u, h\rangle_{H^1}\, q_k + \sum_{\substack{k=-n\\ \pm 0}}^{n} \langle (q_k)_v, h_x\rangle_{L^2}\, q_k\\
&= \sum_{\substack{k=-n\\ k\neq 0}}^{n} \left\langle \frac{1}{\sqrt{2}}\cos(k\pi y), p(y) + \frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)\right\rangle_{L^2} q_k + \left\langle \frac{1}{2}, p(y) + \frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)\right\rangle_{L^2} q_{-0}\\
&\quad + \left\langle \frac{1}{2}, p(y) + \frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right)\right\rangle_{L^2} q_{+0} + \sum_{\substack{k=-n\\ k\neq 0}}^{n} \left\langle -\frac{b(k)}{\sqrt{2}} k\pi \sin(k\pi y), \frac{\partial}{\partial y} p(y)\right\rangle_{L^2} q_k\\
&\quad + \sum_{\substack{k=-n\\ k\neq 0}}^{n} \left\langle \frac{\pm 1}{\sqrt{2}}\cos(k\pi y), -\frac{3}{2}\tanh\left(\frac{x}{2}\right)\mathrm{sech}^2\left(\frac{x}{2}\right)\right\rangle_{L^2} q_k + \left\langle -\frac{1}{2}, -\frac{3}{2}\tanh\left(\frac{x}{2}\right)\mathrm{sech}^2\left(\frac{x}{2}\right)\right\rangle_{L^2} q_{-0}\\
&\quad + \left\langle \frac{1}{2}, -\frac{3}{2}\tanh\left(\frac{x}{2}\right)\mathrm{sech}^2\left(\frac{x}{2}\right)\right\rangle_{L^2} q_{+0}\\
&= \sum_{\substack{k=-n\\ k\neq 0}}^{n} \frac{48(-1)^{1+k}}{k^4\pi^4}\frac{b(k)^{-1}}{\sqrt{2}} q_k + \frac{1}{2}\left(\frac{16}{15} + 3\,\mathrm{sech}^2\left(\frac{x}{2}\right) + 3\tanh\left(\frac{x}{2}\right)\mathrm{sech}^2\left(\frac{x}{2}\right)\right) q_{-0}\\
&\quad + \frac{1}{2}\left(\frac{16}{15} + 3\,\mathrm{sech}^2\left(\frac{x}{2}\right) - 3\tanh\left(\frac{x}{2}\right)\mathrm{sech}^2\left(\frac{x}{2}\right)\right) q_{+0}.
\end{aligned}$$

This leads to the following initial guess for the Galerkin modes:

$$
\begin{aligned}
a_k &= \frac{48(-1)^{1+k}}{k^4\pi^4}\,\frac{b(k)^{-1}}{\sqrt{2}}, \quad k=-n,...,-1,1,...,n,\\
a_{-0} &= \frac{1}{2}\left(\frac{16}{15}+3\operatorname{sech}^2\left(\frac{x}{2}\right)+3\tanh\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)\right),\\
a_{+0} &= \frac{1}{2}\left(\frac{16}{15}+3\operatorname{sech}^2\left(\frac{x}{2}\right)-3\tanh\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)\right).
\end{aligned}
\tag{B.6}
$$

Error estimate determined by the approximation of the true solution using Galerkin modes

Let

$$\bar{h}_n = \sum_{\substack{k=-n\\ \pm 0}}^{n} a_k q_k$$

be a solution of the BVP that is given by (B.1) (B.2) and (B.5). Now we consider the error

$$
\begin{aligned}
\Delta(T,n) &= \sup_{x\in[-T,T]}\left\{\left\|\bar{h}_n(x,\cdot)-\begin{pmatrix} h(x,\cdot)\\ h_x(x,\cdot)\end{pmatrix}\right\|_{L^2\times L^2}\right\}\\
&= \sup_{x\in[-T,T]}\left\{\sqrt{\left(\left\|\sum_{\substack{k=-n\\ \pm 0}}^{n} a_k (q_k)_u - h(x,\cdot)\right\|^2_{L^2\times L^2}+\left\|\sum_{\substack{k=-n\\ \pm 0}}^{n} a_k (q_k)_v - h_x(x,\cdot)\right\|^2_{L^2\times L^2}\right)}\right\}.
\end{aligned}
$$

First summand:

$$
\begin{aligned}
&\left\|\sum_{\substack{k=-n\\ \pm 0}}^{n} a_k (q_k)_u - h(x,\cdot)\right\|^2_{L^2\times L^2}\\
&= \int_{-1}^{1}\left(\sum_{\substack{k=-n\\ \pm 0}}^{n} a_k \frac{b(k)}{(\sqrt{2},2)_k}\cos(k\pi y)-(1-y^2)^2-\frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)\right)^2 dy\\
&= \int_{-1}^{1}\left(\sum_{\substack{k=-n\\ k\neq 0}}^{n} a_k \frac{b(k)}{\sqrt{2}}\cos(k\pi y)+\frac{1}{2}a_{-0}+\frac{1}{2}a_{+0}\right)\left(\sum_{\substack{l=-n\\ l\neq 0}}^{n} a_l \frac{b(l)}{\sqrt{2}}\cos(l\pi y)+\frac{1}{2}a_{-0}+\frac{1}{2}a_{+0}\right) dy\\
&\quad + 2\int_{-1}^{1}\left(\sum_{\substack{k=-n\\ k\neq 0}}^{n} a_k \frac{b(k)}{\sqrt{2}}\cos(k\pi y)+\frac{1}{2}a_{-0}+\frac{1}{2}a_{+0}\right)\left(-(1-y^2)^2-\frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)\right) dy\\
&\quad + \int_{-1}^{1}\left((1-y^2)^2+\frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)\right)^2 dy
\end{aligned}
$$

$$= \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{1}{2} b(k)^2 a_k^2 + \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{1}{2} b(k)^2 a_k a_{-k} + \frac{1}{2}(a_{-0} + a_{+0})^2 - 2 \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{48(-1)^{1+k}}{k^4\pi^4} \frac{b(k)}{\sqrt{2}} a_k - \frac{16}{15}(a_{-0} + a_{+0})$$
$$- 3\mathrm{sech}^2\left(\frac{x}{2}\right)(a_{-0} + a_{+0}) + \int_{-1}^{1} (1 - y^2)^4 dy + \int_{-1}^{1} 2(1 - y^2)^2 \frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right) dy + \int_{-1}^{1} \frac{9}{4}\mathrm{sech}^4\left(\frac{x}{2}\right) dy$$
$$= \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{1}{2} b(k)^2 a_k^2 + \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{1}{2} b(k)^2 a_k a_{-k} - 2 \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{48(-1)^{1+k}}{k^4\pi^4} \frac{b(k)}{\sqrt{2}} a_k - \frac{16}{15}(a_{-0} + a_{+0})$$
$$- 3\mathrm{sech}^2\left(\frac{x}{2}\right)(a_{-0} + a_{+0}) + \frac{1}{2}(a_{-0} + a_{+0})^2 + \frac{256}{315} + \frac{48}{15}\mathrm{sech}^2\left(\frac{x}{2}\right) + \frac{9}{2}\mathrm{sech}^4\left(\frac{x}{2}\right).$$

Second summand:

$$\left\| \sum_{\substack{k=-n\\ \pm 0}}^{n} a_k (q_k)_v - h_x(x,\cdot) \right\|_{L^2\times L^2}^2$$
$$= \int_{-1}^{1} \left(\sum_{\substack{k=-n\\ \pm 0}}^{n} a_k \frac{\pm 1}{(\sqrt{2}, 2)_k} \cos(k\pi y) - (1 - y^2)^2 - -\frac{3}{2}\tanh\left(\frac{x}{2}\right)\mathrm{sech}^2\left(\frac{x}{2}\right) \right)^2 dy$$
$$= \int_{-1}^{1} \left(\sum_{\substack{k=-n\\k\neq 0}}^{n} a_k \frac{\pm 1}{\sqrt{2}} \cos(k\pi y) - \frac{1}{2}a_{-0} + \frac{1}{2}a_{+0} \right) \left(\sum_{\substack{l=-n\\l\neq 0}}^{n} a_l \frac{\pm 1}{\sqrt{2}} \cos(l\pi y) - \frac{1}{2}a_{-0} + \frac{1}{2}a_{+0} \right) dy$$
$$+ 2\int_{-1}^{1} \left(\sum_{\substack{k=-n\\k\neq 0}}^{n} a_k \frac{\pm 1}{\sqrt{2}} \cos(k\pi y) - \frac{1}{2}a_{-0} + \frac{1}{2}a_{+0} \right) \frac{3}{2}\tanh\left(\frac{x}{2}\right)\mathrm{sech}^2\left(\frac{x}{2}\right) dy$$
$$+ \int_{-1}^{1} \left(\frac{3}{2}\tanh\left(\frac{x}{2}\right)\mathrm{sech}^2\left(\frac{x}{2}\right) \right)^2 dy$$
$$= \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{1}{2} a_k^2 - \sum_{\substack{k=-n\\k\neq 0}}^{n} \frac{1}{2} a_k a_{-k} + \frac{1}{2}(-a_{-0} + a_{+0})^2 + (-a_{-0} + a_{+0})\ 3\ \tanh\left(\frac{x}{2}\right)\mathrm{sech}^2\left(\frac{x}{2}\right)$$
$$+ \frac{9}{2}\tanh^2\left(\frac{x}{2}\right)\mathrm{sech}^4\left(\frac{x}{2}\right).$$

Note that the coefficients a_k depend on x.

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Erklärung

Hiermit versichere ich, die vorliegende Arbeit selbstständig angefertigt und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben.

Bielefeld, den 06.08.2010

Symbol Dictionary

$\hookrightarrow$	embedding
$C^0(X,Y)$	set of all continuous maps from a metric space X to a metric space Y
$C^m(U,Y)$	set of all m-times continuously differentiable maps from an open subspace U of X to Y, where X and Y are Banach spaces
$C^{m,\vartheta}(S,X)$	Hölder space, where $S \subset \mathbb{R}^n$ and X is a Banach space
$L[X,Y]$	set of all bounded operators from normed vector spaces X to Y
$L[X]$	set of all bounded operators on a normed vector space X
$\mathring{J}$	interior of a set J which is contained in a metric space
$R(A), N(A)$	range and kernel of a linear operator A, respectively
$R_\lambda(A), \rho(A), \sigma(A)$	reslovent, resolvent set and spectrum of an operator A, respectively
$[A,B] = AB - BA$	commutator
$\oplus$	direct sum of normed vector spaces
$K[X]$	set of all compact operators on a normed vector space X
X'	dual space of a normed vector space X
$\langle x', x \rangle$	dual pairing of $x' \in X'$ and $x \in X$
A'	conjugate of a densely defined operator A
$U^\perp, V_\perp$	annihilators of U in X' and of V in X, respectively, where X is a normed vector space
$\Re(z), \Im(z)$	real and imaginary part of a complex number z
$f_x = \frac{\partial f}{\partial x}$	partial derivative of f with respect to x
$\Delta_y = \sum_{k=1}^n \frac{\partial^2}{\partial y_k^2}$	Laplace operator with respect to y
$\nabla_y = \left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right)$	del operator with respect to y

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