Numerical Approximation of Non-Hyperbolic Heteroclinic Orbits

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Abstract

In this paper we consider heteroclinic orbits in discrete time dynamical systems that connect a hyperbolic fixed point to a non-hyperbolic fixed point with a one-dimensional center direction. A numerical method for approximating the heteroclinic orbit by a finite orbit sequence is introduced and a detailed error analysis is presented. The loss of hyperbolicity requires special tools for proving the error estimate – the polynomial dichotomy of linear difference equations and a (partial) normal form transformation near the non-hyperbolic fixed point. This situation appears, for example, when one fixed point undergoes a flip bifurcation. For this case, the approximation method and the validity of the error estimate is illustrated by an example.

Keywords: Discrete time dynamical system, non-hyperbolic transversal heteroclinic orbit, numerical approximation, approximation error, polynomial dichotomy, normal form transformation.

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Contents

1 Introduction

Discrete time dynamical systems where first studied around 1890, by Poincaré in his famous essay on the stability of the solar system. He was especially interested in analyzing orbits, converging towards a periodic orbit. To simplify the analysis of the three body problem, described by ordinary differential equations, he introduced the so called Poincaré map. By this approach, the study of continuous orbits was reduced to the analysis of the associated discrete orbits. He noticed the appearance of complex structures, close to homoclinic orbits. This discovery initialized further analysis e.g. by Birkhoff, Smale and Shilnikov leading to the famous result that the dynamics in a neighborhood of a homoclinic orbit is chaotic.

The calculation of homoclinic or heteroclinic orbits is an important task for the numerical analysis of dynamical systems. In this paper we consider discrete time dynamical systems of the form

$$x_{n+1} = f(x_n, \lambda), \quad n \in \mathbb{Z}, \tag{1}$$

where $f : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$ is sufficiently smooth and a diffeomorphism w.r.t. the x-variable. Assume that a heteroclinic orbit $(\bar{x}_n)_{n\in\mathbb{Z}}$ exists at $\lambda = \bar{\lambda}$, connecting one fixed point ξ_- to another fixed point ξ_+, i.e. $(\bar{x}_n)_{n\in\mathbb{Z}}$ is a solution of (1) and $\lim_{n\to\pm\infty} \bar{x}_n = \xi_\pm$. In case both fixed points are hyperbolic the orbit lies in the intersection of the unstable manifold W^u_- of ξ_- and the stable manifold W^s_+ of ξ_+, see Figure 1. Note that these manifolds generically have a transversal intersection.

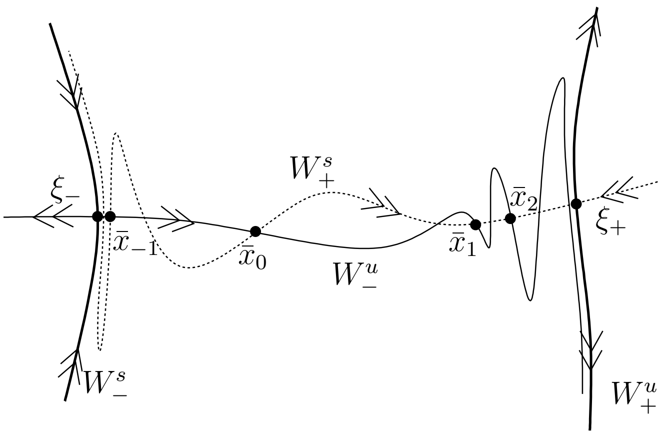

Figure 1: *Schematic picture of a transversal orbit as intersection of the unstable manifold of some hyperbolic fixed point ξ_- and the stable manifold of another hyperbolic fixed point ξ_+.*

The numerical approximation of transversal hyperbolic orbits is considered in [5] and [7]. Here we will consider a specific degenerate case.

There are essentially three ways a transversal hyperbolic orbit can degenerate at the parameter $\lambda = \bar{\lambda}$.

(i) One of the matrices $f_x\big(\xi_\pm, \bar\lambda\big)$ or $f_x\big(\bar x_n, \bar\lambda\big)$, $n \in \mathbb{Z}$ is singular.

(ii) The two manifolds W_-^u and W_+^s do not intersect transversally.

(iii) One of the matrices $f_x\big(\xi_\pm, \bar\lambda\big)$ possesses an eigenvalue of absolute value 1.

The first case appears for example when considering a discretization of a partial differential equation, see Lani-Wayda [15] or Steinlein and Walther [24].

The second case is the main topic of the monograph from Palis and Takens [17]. In [12] Kleinkauf derived a result on this tangent case, that is particularly useful for numerical analysis. He showed that a tangential heteroclinic orbit can computed as the turning point of a parametrized boundary value problem and he provided detailed error estimates.

For the third case Arnold, Afraimovich, Ilyashenko and Shilnikov summarized in [1] results about structural stability for continuous and discrete time dynamical systems, but the question of approximation is not considered. Furthermore, the appearance of an eigenvalue -1 of the matrix $f_x\big(\xi_\pm, \bar\lambda\big)$ is excluded, because according to a lemma of Afraimovich, these systems do not lie on the boundary of the set of Morse-Smale systems.

In [28] Beyn and Zou analyze discretization effects for this case. They have proved under suitable assumptions, that a saddle-node homoclinic orbit of a continuous time system persists under discretization by a one-step method and leads to a closed curve of discrete homoclinic orbits, containing both, transversal and tangential saddle-node orbits.

The paper [3] contains a survey of numerical methods and examples for nondegenerate connecting orbits.

The structure of the paper is as follows. In Section 2 we state our basic assumptions and present the boundary-value problem for approximating a transversal non-hyperbolic heteroclinic orbit. Furthermore, we describe the main results of our approximation theorem, neglecting technical details.

In Section 3 the basic tools used in the approximation theorem are developed. First we analyze the polynomial rate of convergence for the orbit itself and for the solutions of the associated variational equation

$$u_{n+1} = f_x(\bar x_n, \bar\lambda) u_n, \quad n \in \mathbb{Z}.$$

This motivates the definition of a special type of polynomial dichotomy, for which a perturbation result is derived. Combining this with a (partial) normal form transformation near the non-hyperbolic fixed point (without assuming non-resonance conditions) we prove in Section 3.4 that the variational equation possesses a polynomial dichotomy for $n \geq 0$. Finally, we give a characterization of the transversal intersection of the unstable and the center-stable manifold.

In Section 4 these tools are used to prove the approximation theorem.

Section 5 contains an example of a map, where one fixed point $\xi_+(\lambda)$ undergoes a flip bifurcation at $\lambda = \bar{\lambda}$. At this parameter we approximate a heteroclinic orbit and illustrate the validity of the error estimates.

The main results of this paper are taken from the PhD thesis of the second author [8].

2 Basic assumptions and numerical method

In this section, we present the numerical method for approximating a non-hyperbolic heteroclinic orbit. Technical details that are needed for the proof of the corresponding approximation theorem will be provided in later sections.

First we introduce some notation and the basic assumptions.

2.1 Assumptions

Consider a discrete time dynamical system

$$x_{n+1} = f(x_n), \quad n \in \mathbb{Z}. \tag{2}$$

A1 Let $f \in C^\infty(\mathbb{R}^k, \mathbb{R}^k)$ be a diffeomorphism.

A2 The map f possesses two fixed points ξ_+ and ξ_-.

By a translation, one of the fixed points can be shifted to 0, without loss of generality assume $\xi_+ = 0$.

A heteroclinic orbit connecting the fixed points ξ_- and ξ_+ is defined in the following way.

Definition 1 A **heteroclinic orbit** $\bar{x}_\mathbb{Z} := (\bar{x}_n)_{n\in\mathbb{Z}}$ is a solution of the difference equation (2) satisfying $\lim_{n\to\pm\infty} \bar{x}_n = \xi_\pm$. For any $n \in \mathbb{Z}$ the point $y = \bar{x}_n$ is called a **heteroclinic point**.

With $k_{\pm\kappa}$, $\kappa \in \{s, c, u, sc\}$ we denote the dimension of the stable, center, unstable and center-stable subspace of $f'(\xi_\pm)$, respectively. The corresponding subspaces and manifolds are denoted by $X^\kappa_\pm$ and $W^\kappa_\pm$, $\kappa \in \{s, c, u, sc\}$.

A3 Let $k_{-c} = 0$, $k_{+c} = 1$ and $k_{-u} + k_{+sc} = k$.

In **A3** we assume that the fixed point ξ_+ possesses a one-dimensional center manifold and that ξ_- is hyperbolic. Thus an orbit converging towards ξ_+ via the center-stable manifold generically has a component in the slow center direction. This technical assumption is stated in **A4**.

A4 Let $\bar{x}_\mathbb{Z}$ be a heteroclinic orbit such that $\bar{x}_0 \in W^{sc}_+ \setminus W^s_+$.

Finally, we assume that the unstable manifold of ξ_- and the center-stable manifold of ξ_+ intersect transversally.

A5 The invariant manifolds W^u_- and W^{sc}_+ have transversal intersections at $\bar{x}_\mathbb{Z}$.

A schematic picture of a heteroclinic orbit is given in Figure 2. Note that the orbit converges to ξ_- as $n \to -\infty$ with an exponential rate while the convergence towards ξ_+ as $n \to \infty$ has a polynomial rate.

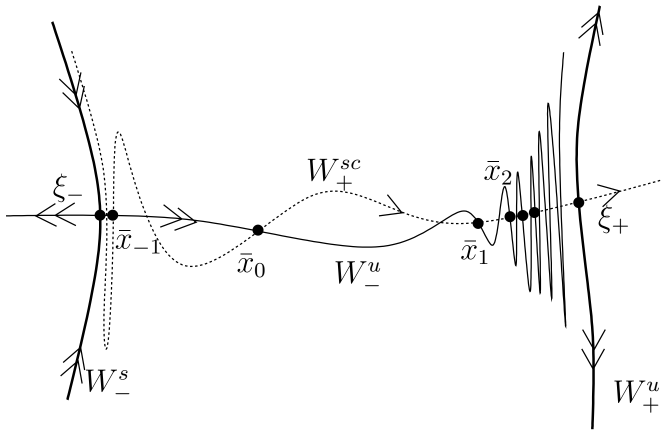

Figure 2: *Schematic picture of a non-hyperbolic transversal orbit as intersection of the unstable manifold of some hyperbolic fixed points ξ_- and the center-stable manifold of another non-hyperbolic fixed point ξ_+.*

2.2 The approximate heteroclinic orbit

In [2] a hyperbolic homoclinic orbit of a differential equation is computed as a solution of a boundary-value problem truncated to a finite interval and supplemented by a phase condition. Schecter modified this approach in [22] for saddle-node homoclinic orbits. For this non-hyperbolic case he showed that the radius of the ball, within which a unique solution of the finite boundary value problem exists, shrinks with the length of the orbit. Furthermore, the boundary condition is assumed to be of sufficiently high order. Note that these two conditions are not required for hyperbolic orbits. By using a sharper version of Banach's fixed point theorem Sandstede [21] reduced the required order of the boundary condition by one.

For discrete time dynamical systems these results cannot be applied directly. Kleinkauf proved an approximation theorem for homoclinic orbits in [11] and extensions can be found in [4] and [5]. For heteroclinic orbits an approximation theorem is stated in [5] and a proof can be found in [7].

Let $J := [n_-, n_+] \cap \mathbb{Z}$ and $\tilde{J} := [n_-, n_+ - 1] \cap \mathbb{Z}$ be discrete intervals, where $n_- = -\infty$, $n_+ = \infty$ is allowed, $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{Z}^+ = \{0, 1, \cdots\}$, $\mathbb{Z}^- = \{\cdots, -1, 0\}$.

Our aim is to approximate the infinite non-hyperbolic f-orbit $\bar{x}_\mathbb{Z}$ by a finite segment $x_J = (x_n)_{n\in J}$. This segment is the solution of the defining system

$$\begin{aligned} x_{n+1} - f(x_n) &= 0, \quad n = n_-, \ldots, n_+ - 1, \\ b(x_{n_-}, x_{n_+}) &= 0, \end{aligned} \tag{3}$$

where $b \in C^1(\mathbb{R}^{2k}, \mathbb{R}^k)$ is an appropriately chosen boundary operator.

In our approximation theorem (cf. [8, Theorem 4.3]) the solution of (3) is unique in some ball

$$\|\bar{x}_{|J} - x_J\|_\infty \leq \frac{\delta}{n_+},$$

where $\bar{x}_{|J}$ is the restriction of the exact orbit to the finite interval J. Furthermore, the approximation error can be estimated as follows:

$$\|\bar{x}_{|J} - x_J\|_\infty \leq C \left\|b(\bar{x}_{n_-}, \bar{x}_{n_+})\right\|.$$

3 Polynomial dichotomy

In this section the basic tool to prove our approximation theorem is introduced; the so called **polynomial dichotomy** for difference equations of the form

$$u_{n+1} = A_n u_n, \quad A_n \in \mathbb{R}^{k,k} \text{ invertible}, \; n \in \mathbb{Z}^+,$$

where A_n converges to a non-hyperbolic matrix A. This dichotomy is motivated by the polynomial rate of convergence on the center manifold that we examine in Section 3.1. As in the theory of exponential dichotomies, see [6] and [18], we derive a perturbation (roughness) theorem and prove estimates for solutions of inhomogeneous equations.

In Section 3.4 we show that the associated variational equation

$$u_{n+1} = f'(\bar{x}_n) u_n, \quad n \in \mathbb{Z}^+ \tag{4}$$

possesses a polynomial dichotomy on $\mathbb{Z}^+$. A major step in this proof is a certain normal form transformation (cf. [8, Theorem 3.4]) that does not need any non-resonance condition. The perturbation theorem then applies to the transformed system. Finally, we give an alternative description of the transversality, stated in assumption **A5**.

3.1 Polynomial rate of convergence in the center direction

Consider the one-dimensional system

$$x_{n+1} = g(x_n), \quad n \in \mathbb{Z}^+, \tag{5}$$

where

$$g(x) = x - \ell x^{q+1} + \mathcal{O}(x^{q+2}), \quad \ell > 0, \; q \in \mathbb{N}, \; x \in \mathbb{R}. \tag{6}$$

Note that the restriction of f to its one-dimensional center manifold at the fixed point ξ_+ is locally of the form (6). To get the precise polynomial rate of convergence for a g-orbit with starting point $0 < x_0 < \bar{x}$, we construct a series of intervals

$$I_n := [a_n^-, a_n^+], \quad a_n^\pm := \frac{1}{(\ell q)^{1/q}(n \mp n^\gamma)^{1/q}}, \quad 0 < \gamma < 1.$$

Lemma 2 *There exist two constants $N \in \mathbb{N}$, $0 < \gamma < 1$ such that*

$$\begin{aligned} I_{n+1} &\supset g(I_n), & (7)\\ I_n \cap I_{n+1} &\neq \emptyset & (8)\end{aligned}$$

hold for all $n \geq N$.

Idea of proof: The function g is monotone increasing in $[0, \bar{x}]$ for $\bar{x}$ sufficiently small. Thus it is sufficient to prove the following assertions:

$$\begin{aligned} g(a_n^+) &\leq a_{n+1}^+, \\ g(a_n^-) &\geq a_{n+1}^-, \\ a_n^- &\leq a_{n+1}^+. \end{aligned}$$

This can be done by direct, but very careful estimates. ■

With this tool at hand, it is easy to prove the following theorem (cf. [8, Theorem 1.4] for the details).

Theorem 3 *There exists a $\bar{x} > 0$, such that for every $0 < x_0 < \bar{x}$ we find three constants $K,\ L \geq 0$ and $0 < \gamma < 1$ such that the orbit $x_{\mathbb{Z}^+} = (g^n(x_0))_{n \in \mathbb{Z}^+}$ satisfies*

$$a_{K+n}^- \leq x_{L+n} \leq a_{K+n}^+ \quad \forall n \in \mathbb{Z}^+ \tag{9}$$

and

$$\lim_{n \to \infty} (\ell q)^{1/q} n^{1/q} x_n = 1. \tag{10}$$

Theorem 3 is an improvement over the previous approach of Hüls, Zou [9] and (10) is also derived by a different method by Szekeres [25].

We will use the rather sharp bounds (9) in order to analyze the associated variational equation

$$u_{n+1} = g'(x_n) u_n, \quad n \in \mathbb{Z}^+. \tag{11}$$

The solution operator Φ of (11) is defined for $n \geq m \geq 0$ by

$$\Phi(n, m) := \prod_{i=m}^{n-1} g'(x_i).$$

A precise estimate for this operator is presented in the following theorem (cf. [8, Theorem 1.6]). The proof is given in Appendix B.1.

Theorem 4 *Assume $\bar{x}$ is given as in Theorem 3, $q \in \mathbb{N}$ and $x_{\mathbb{Z}^+}$ is a g-orbit with starting point $0 < x_0 < \bar{x}$. Under these assumptions there exists a constant $K > 0$ which depends on x_0, such that for all $n \geq m \geq 0$ the following estimate holds*

$$\left|\Phi(n,m)\right| \leq K \left(\tfrac{n+1}{m+1}\right)^{-\nu}, \quad \nu = 1 + \frac{1}{q}. \tag{12}$$

3.2 Polynomial dichotomy

Similar to the well known definition of an exponential dichotomy (cf. [6], [18]), Theorem 4 motivates the definition of a polynomial dichotomy for the difference equation

$$u_{n+1} = A_n u_n, \quad A_n \in \mathbb{R}^{k,k} \text{ invertible}, \ n \in \mathbb{Z}. \tag{13}$$

This definition takes into account the polynomial rate of convergence in the center direction.

Let us first introduce some notations. With k_κ, $\kappa \in \{s, c, u, sc\}$ we denote the dimension of the stable, center, unstable and center-stable subspace of the matrix A, respectively and with X_κ, $\kappa \in \{s, c, u, sc\}$ we denote the corresponding subspaces. Moreover

$$\Phi(n,m) := \begin{cases} A_{n-1} \dots A_m & , n > m, \\ I & , n = m, \\ A_n^{-1} \dots A_{m-1}^{-1} & , n < m \end{cases} \tag{14}$$

defines the solution operator of (13).

Definition 5 The linear difference equation (13) has a **polynomial dichotomy** with data $(K, \alpha, \nu, P_n^{sc}, P_n^u)$ on $J \subset \mathbb{Z}^+$, if there exist two families of projectors P_n^{sc} and $P_n^u = I - P_n^{sc}$ and three constants K, $\alpha > 0$, $\nu > 1$ such that the following assertions hold:

$$P_n^\kappa \Phi(n,m) = \Phi(n,m) P_m^\kappa \quad \forall n, m \in J, \quad \kappa \in \{sc, u\},$$

$$\left.\begin{array}{rcl} \|\Phi(n,m) P_m^{sc}\| & \leq & K \left(\frac{n+1}{m+1}\right)^{-\nu} \\ \|\Phi(m,n) P_n^u\| & \leq & K \mathrm{e}^{-\alpha(n-m)} \end{array}\right. \quad \forall n \geq m, \ n, m \in J.$$

In Figure 3 we show the behavior of a solution of the difference equation (13) that has a polynomial dichotomy.

In [19] and [16] the more general concept of a (h,k)-dichotomy is introduced by López-Fenner and Pinto.

Similar to the results for an exponential dichotomy (cf. [18]) Lemma 6 provides an alternative representation of $\mathcal{R}(P_m^{sc})$.

Lemma 6 *Assume that for some $n_- \geq 0$ the difference equation (13) has a polynomial dichotomy on $J = [n_-, \infty)$ with data $(K, \alpha, \nu, P_n^{sc}, P_n^u)$.*

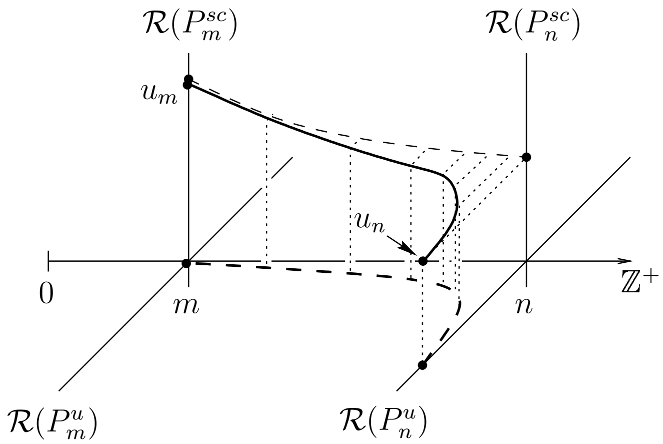

Figure 3: *Schematic picture of the behavior of a solution of a difference equation with polynomial dichotomy.*

Then for every $m \geq n_-$ the representation

$$\mathcal{R}(P_m^{sc}) = \left\{ u \in \mathbb{R}^k : \sup_{n \geq m} \|\Phi(n,m)u\| < \infty \right\} \tag{15}$$

holds. Furthermore, assume $(L, \beta, \mu, Q_n^{sc}, Q_n^u)$ is another set of dichotomy data for (13) on J. Then $\mathcal{R}(Q_n^{sc}) = \mathcal{R}(P_n^{sc})$ for all $n \in J$ and the following estimate holds

$$\|Q_n^{sc} - P_n^{sc}\| \leq KL \left[\mathrm{e}^{-\beta(n-n_-)} \left(\tfrac{n+1}{n_-+1} \right)^{-\nu} \right] \|Q_{n_-}^{sc} - P_{n_-}^{sc}\|. \tag{16}$$

The proof is similar to the one in the hyperbolic situation, cf. [18, Proposition 2.3]. Here the exponential rate from the stable direction is replaced by the polynomial rate from the center-stable direction.

We discuss the inhomogeneous equation

$$u_{n+1} = A_n u_n + r_n, \quad n \in \mathbb{Z}, \tag{17}$$

where r_n is assumed to be bounded in the $\|\cdot\|^*$ norm, defined by

$$\|r_J\|^* := \max \left\{ \sup_{n \leq 0,\ n \in J} \|r_n\|, \sup_{n > 0,\ n \in J} n\|r_n\| \right\}. \tag{18}$$

Lemma 7 *Assume $n_- \geq 0$ and the homogeneous difference equation (13) possesses a polynomial dichotomy on $J = [n_-, \infty)$ with data $(K, \alpha, \nu, P_n^{sc}, P_n^u)$.*

Then the inhomogeneous equation (17) considered for $n \in J$, subject to the boundary condition

$$P_{n_-}^{sc} u_{n_-} = \zeta, \quad \zeta \in \mathcal{R}(P_{n_-}^{sc})$$

has, for every $r_{\tilde{J}}$ with $\|r_{\tilde{J}}\|^* < \infty$, a unique bounded solution u_J.

The proof follows by direct computation from the representation of the solution which uses Green's function, defined in Appendix A.1.

3.2.1 Roughness Theorem

It is well knows that an exponential dichotomy persists under small perturbations of the difference equation (13). The exponents are slightly smaller when small bounded perturbations are allowed. A direct proof is given by Palmer in [18].

For linear parameter dependent differential equations a perturbation result is presented by Sandstede in his masters thesis [20]. This approach is carried over by Kleinkauf to discrete difference equations in [12]. He simplified the proof by introducing an operator whose fixed point is Green's function of the perturbed system

$$u_{n+1} = (A_n + E_n)u_n. \tag{19}$$

With $\|E_J\|_1 := \sum_{n\in J} \|E_n\|$ we denote the L_1-norm in $(\mathbb{R}^{k,k})^J$. The space of sequences bounded in this norm is defined by

$$S^1_J(\mathbb{R}^{k,k}) := \left\{E_J \in (\mathbb{R}^{k,k})^J : \|E_J\|_1 < \infty\right\}.$$

Using this notation, we state our L_1-perturbation theorem for polynomial dichotomies.

Theorem 8 (Roughness Theorem) *Assume the difference equation (13) has a polynomial dichotomy on $J \subset \mathbb{Z}^+$ with data $(K, \alpha, \nu, P^{sc}_n, P^u_n)$. Furthermore, $\|A_n^{-1}\| \le \tau$ holds for all $n \in J$ with a constant $\tau > 0$.*

Then an $\varepsilon > 0$ exists, such that for every sequence $\|E_J\|_1 < \varepsilon$, the perturbed system (19) possesses a polynomial dichotomy with data $(\tilde{K}, \alpha, \nu, Q^{sc}_n, Q^u_n)$, where Q^{sc}_n and Q^u_n are invariant projectors and $\tilde{K} := 2K$.

The projectors Q^{sc}_n and Q^u_n have the same rank as P^{sc}_n and P^u_n, respectively. The difference between these projectors can be estimated by

$$\|Q^{sc}_n - P^{sc}_n\| \le C\|E_J\|_1, \quad n \in J. \tag{20}$$

The proof is indicated in Appendix B.2. Note that the exponents α and ν remain unchanged in this result.

Corollary 9 *Under the assumptions of Theorem 8 we obtain*

$$\lim_{n\to\infty} P^{sc}_n - Q^{sc}_n = 0. \tag{21}$$

3.3 Normal form transformation without non-resonance conditions

To prove an approximation theorem it is essential to show that the variational equation

$$u_{n+1} = f'(\bar{x}_n)u_n, \quad n \in \mathbb{Z} \tag{22}$$

possesses a polynomial dichotomy on $\mathbb{Z}^+$. In order to do this, we introduce a smooth normal form transformation, such that in the new coordinates, the variational equation is of the form

$$v_{n+1} = (A_n + E_n)v_n, \quad n \in \mathbb{Z},$$

where the unperturbed system has a polynomial dichotomy and $E_{\mathbb{Z}^+}$ is a small L_1-perturbation. Then by Theorem 8 we get a polynomial dichotomy of the perturbed system.

The corresponding normal form transformation is presented in this section. Assuming non-resonance conditions it is well known from Takens smooth saddle suspension theorem (cf. [26], [10]) that one can transform the original system locally into the form

$$(s, u, w) \mapsto \big(A(w)s, B(w)u, g(w)\big), \tag{23}$$

where $A : \mathbb{R}^{k_c} \to \mathbb{R}^{k_s,k_s}$, $B : \mathbb{R}^{k_c} \to \mathbb{R}^{k_u,k_u}$ are smooth functions and $A(0) = f'(0)_{|X^s}$, $B(0) = f'(0)_{|X^u}$.

For our purpose it is sufficient to derive a normal form that, compared to (23), allows higher order terms of a specific structure. As we will show this can be achieved without assuming non-resonance conditions (cf. [8, Theorem 3.4]). First we need a technical assumption.

A6 The function, describing the reduced system has a Taylor expansion

$$u \mapsto s_1 u + s_2 d u^{q+1} + \mathcal{O}(u^{q+2}), \quad q \in \mathbb{N}, \quad d > 0, \quad u \in \mathbb{R},$$

where $|s_{1,2}| = 1$ and in addition $s_1 = 1$ if q is odd.

Theorem 10 *Assume **A1** to **A4** and **A6**. Then there exist C^1-functions $A : \mathbb{R} \to \mathbb{R}^{k_s,k_s}$, $B : \mathbb{R} \to \mathbb{R}^{k_u,k_u}$ and $g : \mathbb{R} \to \mathbb{R}$, $A(0) = f'(0)_{|X^s}$, $B(0) = f'(0)_{|X^u}$, $g(0) = 0$, $|g'(0)| = 1$, a small neighborhood of 0 and a C^1-diffeomorphism $\Upsilon : \mathbb{R}^k \to \mathbb{R}^{k_s} \oplus \mathbb{R}^{k_u} \oplus \mathbb{R}$, such that the transformed system $\tilde{f}(s, u, w) = \Upsilon \circ f \circ \Upsilon^{-1}(s, u, w)$ is locally of the form*

$$\tilde{f}\begin{pmatrix} s \\ u \\ w \end{pmatrix} = \begin{pmatrix} A(w)s \\ B(w)u \\ g(w) \end{pmatrix} + \varphi\begin{pmatrix} s \\ u \\ w \end{pmatrix}, \tag{24}$$

where $\varphi = (\varphi_s, \varphi_u, \varphi_c)$,

$$\varphi(s, u, w) = \mathcal{O}\big(|(s, u)|^2\big), \quad \varphi'(s, u, w) = \mathcal{O}\big(|(s, u)|\big) \tag{25}$$

and $\varphi_u(s, 0, w) = 0$. Furthermore, we get

$$g(x) = s_1 x + s_2 b x^{q+1} + \mathcal{O}(x^{q+2}), \quad q \in \mathbb{N},\ b > 0 \tag{26}$$

and

$$(s_1, s_2) \in \begin{cases} \big\{(1, 1), (1, -1)\big\}, & \textit{for } q \textit{ odd}, \\ \big\{(1, -1), (-1, 1)\big\}, & \textit{for } q \textit{ even}. \end{cases}$$

Proof: In the first step a smooth change of coordinates is introduced that transforms the map f locally into the following form

$$\begin{pmatrix} s \\ u \\ w \end{pmatrix} \mapsto \begin{pmatrix} \bar{f}_s(s,u,w) \\ \bar{f}_u(s,u,w) \\ \bar{f}_c(s,u,w) \end{pmatrix}, \tag{27}$$

where

$$\begin{array}{cc} \left\{ \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} : s \in \mathbb{R}^{k_s} \right\}, & \left\{ \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} : u \in \mathbb{R}^{k_u} \right\}, \\ \left\{ \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix} : w \in \mathbb{R} \right\}, & \left\{ \begin{pmatrix} s \\ 0 \\ w \end{pmatrix} : s \in \mathbb{R}^{k_s}, w \in \mathbb{R} \right\} \end{array} \tag{28}$$

are invariant under iteration with the map (27).

To obtain this change of coordinates, we first transform the unstable and the center-stable manifold to the coordinate axis using the local graph representations

$$\begin{array}{rclcrcl} W^u_{\mathrm{loc}} & = & \left\{ x_u + \phi^u(x_u) : x_u \in V^u \right\}, & & \phi^u & : & X^u \to X^{sc}, \\ W^{sc}_{\mathrm{loc}} & = & \left\{ x_{sc} + \phi^{sc}(x_{sc}) : x_{sc} \in V^{sc} \right\}, & & \phi^{sc} & : & X^{sc} \to X^u, \end{array}$$

where $V^u \subset X^u$, $V^{sc} \subset X^{sc}$ are sufficiently small neighborhoods of 0.

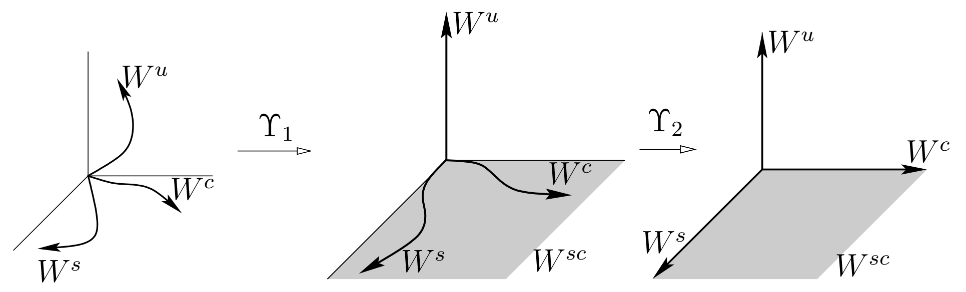

Figure 4: *Order of transformations, rectifying the unstable, center-stable, stable and center manifold.*

We define the first transformation Υ_1 (see Figure 4) by

$$\Upsilon_1 \begin{pmatrix} a \\ b \end{pmatrix} := (E^s, E^c, E^u) \begin{pmatrix} a \\ b \end{pmatrix} + \phi^{sc}\big((E^s, E^c)a\big) + \phi^u(E^u b), \; a \in \mathbb{R}^{k_s+1}, b \in \mathbb{R}^{k_u}.$$

Here E^s, E^c, E^u form a basis of X^s, X^c, X^u, respectively.

By this construction, the bases of the stable, unstable and center subspace for the transformed system

$$\hat{f} = \Upsilon_1^{-1} \circ f \circ \Upsilon_1 \tag{29}$$

are given by

$$\hat{E}_s = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}, \quad \hat{E}_u = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}, \quad \hat{E}_c = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Next, we rectify the stable and a center manifold, lying within the center-stable manifold. The local graph representations for the system (29) have the form

$$\begin{array}{rclcrcl} \hat{W}^s_{\mathrm{loc}} & = & \{x_s + \hat{\phi}^s(x_s) : x_s \in \hat{V}^s\}, & \quad & \hat{\phi}^s & : & \hat{X}^s \to \hat{X}^c, \quad \hat{V}^s \subset \hat{X}^s, \\ \hat{W}^c_{\mathrm{loc}} & = & \{x_c + \hat{\phi}^c(x_c) : x_c \in \hat{V}^c\}, & \quad & \hat{\phi}^c & : & \hat{X}^c \to \hat{X}^s, \quad \hat{V}^c \subset \hat{X}^c. \end{array}$$

We define

$$\Upsilon_2 \begin{pmatrix} s \\ u \\ w \end{pmatrix} := \begin{pmatrix} s \\ w \\ u \end{pmatrix} + \hat{\phi}^s(\hat{E}^s s) + \hat{\phi}^c(\hat{E}^c w), \quad s \in \mathbb{R}^{k_s},\ u \in \mathbb{R}^{k_u},\ w \in \mathbb{R}.$$

The system

$$\bar{f} = \Upsilon_2^{-1} \circ \Upsilon_1^{-1} \circ f \circ \Upsilon_1 \circ \Upsilon_2 \tag{30}$$

is then of the form (27) and the sets (28) are invariant w.r.t. the map $\bar{f}$.

A Taylor expansion shows that (30) has the form

$$\begin{pmatrix} s \\ u \\ w \end{pmatrix} \mapsto \begin{pmatrix} A(w)s + c_1(w)u \\ B(w)u \\ g(w) + c_2(w)s + c_3(w)u \end{pmatrix} + \varphi \begin{pmatrix} s \\ u \\ w \end{pmatrix}, \tag{31}$$

where $A(w) = D_s\bar{f}_s(0,0,w)$, $B(w) = D_u\bar{f}_u(0,0,w)$, $g(w) = \bar{f}_c(0,0,w)$, $c_1(w) = D_u\bar{f}_s(0,0,w)$, $c_2(w) = D_s\bar{f}_c(0,0,w)$, $c_3(w) = D_u\bar{f}_c(0,0,w)$ and φ satisfies (25). Using assumption **A6** it turns out that g has the form (26). Note that for q even, the case $s_1 = s_2$ can be excluded, since the orbit converges towards the fixed point via the center-stable manifold.

Next we define the transformation $t_1(s,u,w) := \big(s + h_1(w)u, u, w\big)$, where h_1 will be chosen, such that the term $c_1(w)u$ vanishes. A direct computation shows that h_1 has to satisfy the functional equation

$$h_1\big(g(w)\big) = A(w)h_1(w)B(w)^{-1} - c_1(w)B(w)^{-1}.$$

Since the operator $L(w)[U] := A(w)UB(w)^{-1}$ is hyperbolic, Appendix A.2 Lemma 19 applies and yields a local solution.

Using a similar approach we eliminate by a transformation of the type

$$t_2(s,u,w) := \big(s, u, w + h_2(w)s + h_3(w)u\big)$$

the terms $c_2(w)s$ and $c_3(w)u$. Therefore, the transformed mapping

$$\tilde{f} := t_2 \circ t_1 \circ \Upsilon_2^{-1} \circ \Upsilon_1^{-1} \circ f \circ \Upsilon_1 \circ \Upsilon_2 \circ t_1^{-1} \circ t_2^{-1}$$

has the form (24). Note that the conditions $\tilde{f}_u(s,0,w) = 0$, $\varphi(s,u,w) = \mathcal{O}(|(s,u)|^2)$ and $\varphi'(s,u,w) = \mathcal{O}(|(s,u)|)$ are preserved under these transformations.

$\blacksquare$

Remark 11 *Without assuming non-resonance conditions we cannot transform (2) into the form (24) such that φ depends only on terms of **cubic and higher order**.*

As a counter example consider the 3-dimensional system

$$(s, u, w) \mapsto \big(\lambda_1 s, \lambda_2 u, w + su\big).$$

In order to eliminate the su-term a transformation

$$t(s, u, w) := \big(s, u, w + h(w)su\big)$$

is needed, where h solves the functional equation $h(w) = \lambda_1\lambda_2 h(w) + 1$. This is impossible for resonant eigenvalues, e.g. $\lambda_1 = 1/2$, $\lambda_2 = 2$.

3.4 Polynomial dichotomy of the variational equation

With the Roughness Theorem and the normal form transformation introduced in Theorem 10 we have all tools at hand to prove that the variational equation (22) possesses a polynomial dichotomy on $\mathbb{Z}^+$. Furthermore, (22) has an exponential dichotomy on $\mathbb{Z}^-$. This is stated in the following theorem (cf. [8, Theorem 3.8]).

Theorem 12 *Assume **A1** to **A4** and **A6**. Then the variational equation (22) possesses an exponential dichotomy with data $(K^-, \alpha^-, P_n^{-s}, P_n^{-u})$ on $\mathbb{Z}^-$ and a polynomial dichotomy with data $(K^+, \alpha^+, \nu, P_n^{+sc}, P_n^{+u})$ on $\mathbb{Z}^+$, where $\nu = 1 + \frac{1}{q}$. The projectors P_n^κ converge to P^κ, $\kappa \in \{-s, -u, +sc, +u\}$.*

Proof: The exponential dichotomy on $\mathbb{Z}^-$ follows from the Roughness Theorem for exponential dichotomies (cf. [18, Proposition 2.10], [12, Lemma 1.1.9]), since $f_n(\bar{x}_n)$ converges towards the hyperbolic matrix $f_x(\xi_-)$ as $n \to -\infty$. This also gives us the convergence of the projectors.

Next we prove the polynomial dichotomy on $\mathbb{Z}^+$. First assume $\xi_+ = 0$ without loss of generality. According to our assumptions, the orbit $\bar{x}_\mathbb{Z}$ converges as $n \to \infty$ towards the fixed point 0. Thus for every small neighborhood $V(0)$, there exists a $N_0 \in \mathbb{N}$, such that $\bar{x}_n \in V(0)$ for all $n \geq N_0$. From Theorem 10 we get the existence of a smooth map $\Upsilon : \mathbb{R}^k \to \mathbb{R}^{k_s} \oplus \mathbb{R}^{k_u} \oplus \mathbb{R}$, transforming

$$x_{n+1} = f(x_n), \quad n \geq N_0$$

into the system

$$\begin{pmatrix} s_{n+1} \\ u_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} A(w_n)s_n \\ B(w_n)u_n \\ g(w_n) \end{pmatrix} + \varphi \begin{pmatrix} s_n \\ u_n \\ w_n \end{pmatrix}, \quad n \geq N_0,$$

where $\varphi(s, u, w) = \mathcal{O}\left(|(s, u)|^2\right)$, $\varphi'(s, u, w) = \mathcal{O}(|(s, u)|)$ and $\varphi_u(s, 0, w) = 0$.

It follows from [23, Theorem III.7] that the orbit $(\bar{x}_n)_{n \geq N_0}$ lies in every center-stable manifold, therefore the u-component is 0. Thus in the transformed system the orbit has the representation $(\bar{s}_n, 0, \bar{w}_n)_{n\geq N_0}$. Since $A(\bar{w}_n)$ converges to the stable matrix $A(0)$ as $n \to \infty$, $\bar{s}_n$ converges exponentially fast towards 0.

The variational equation along this orbit has the form

$$\begin{pmatrix} S_{n+1} \\ U_{n+1} \\ W_{n+1} \end{pmatrix} = \left[\begin{pmatrix} A(\bar{w}_n) & & \\ & B(\bar{w}_n) & \\ & & g'(\bar{w}_n) \end{pmatrix} + E_n \right] \begin{pmatrix} S_n \\ U_n \\ W_n \end{pmatrix}, \tag{32}$$

where

$$E_n = \begin{pmatrix} 0 & 0 & A'(\bar{w}_n)\bar{s}_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varphi' \begin{pmatrix} \bar{s}_n \\ 0 \\ \bar{w}_n \end{pmatrix}.$$

Equation (26) from Theorem 10 together with Theorem 4 guarantee the existence of a constant K, such that

$$\prod_{i=m}^{n-1} g'(\bar{w}_i) \leq K \left(\frac{n+1}{m+1} \right)^{-1-\frac{1}{q}}$$

holds for all $n \geq m \geq N_0$. Therefore the unperturbed system ($E_n = 0$) has a polynomial dichotomy on $[N_0, \infty)$ with data $(\tilde{K}, \alpha^+, 1 + \frac{1}{q}, \tilde{P}_n^{+sc}, \tilde{P}_n^{+u})$.

Set $\varepsilon > 0$ as required in the Roughness Theorem 8. Since $(E_n)_{n\geq N_0}$ converges exponentially fast towards 0, there exists a $N_1 > N_0$ with $\sum_{n\geq N_1} \|E_n\| \leq \varepsilon$. Thus Theorem 8 applies and we obtain a polynomial dichotomy of the perturbed system (32) on $[N_1, \infty)$ with data $(K^+, \alpha^+, 1 + \frac{1}{q}, P_n^{+sc}, P_n^{+u})$.

To finish the proof, we show the convergence of P_n^{+sc} and P_n^{+u} as $n \to \infty$. The unperturbed system has projectors

$$\tilde{P}_n^{+sc} = \tilde{P}^{+sc} = \begin{pmatrix} I & & \\ & 0 & \\ & & 1 \end{pmatrix} \text{ and } \tilde{P}_n^{+u} = \tilde{P}^{+u} = \begin{pmatrix} 0 & & \\ & I & \\ & & 0 \end{pmatrix},$$

and it follows from Corollary 9 that the associated projectors of the perturbed system P_n^{+sc} and P_n^{+u} converge to $\tilde{P}^{+sc}$ and $\tilde{P}^{+u}$, respectively.

Transforming this system back to the original coordinates gives us a polynomial dichotomy of the variational equation (22) on $[N_1, \infty)$ with the convergence of projectors retained. Note that the constants α^+ and $\nu = 1 + \frac{1}{q}$ are preserved, too (cf. [8, Lemma A.8]). By adjusting the constant K^+ we can extend this polynomial dichotomy from $[N_1, \infty)$ to $\mathbb{Z}^+$. ■

3.5 Transversality

In this section we present an equivalent formulation of the transversality in terms of the operator Γ defined by

$$\Gamma : \begin{array}{ccc} S_{\mathbb{Z}} & \to & S_{\mathbb{Z}} \\ x_{\mathbb{Z}} & \mapsto & \big(x_{n+1} - f(x_n)\big)_{n\in\mathbb{Z}} \end{array}. \tag{33}$$

Note that a sequence $u_{\mathbb{Z}}$ solving

$$\Gamma'(\bar{x}_{\mathbb{Z}})u_{\mathbb{Z}} = \big(u_{n+1} - f'(\bar{x}_n)u_n\big)_{n\in\mathbb{Z}} = 0$$

is a solution of the associated variational equation.

The following theorem (cf. [8, Theorem 3.13]) provides us with two different formulations of transversality.

Theorem 13 *Assume **A1** to **A4** and **A6**, let $y \in \mathbb{R}^k$ be a heteroclinic point and let $\bar{x}_{\mathbb{Z}} = (f^n(y))_{n\in\mathbb{Z}}$. Then the following statements are equivalent.*

(i) $\Gamma'(\bar{x}_{\mathbb{Z}})u_{\mathbb{Z}} = 0$ *for* $u_{\mathbb{Z}} \in S_{\mathbb{Z}} \Longrightarrow u_{\mathbb{Z}} = 0$.

(ii) $T_y W_-^u \cap T_y W_+^{sc} = \{0\}$.

In the error estimates below we make use of the analytical condition (i) instead of the geometrical condition (ii). If one accepts condition (i) as definition of transversality then Theorem 13 is not needed. Therefore we omit the proof and refer to [8, Theorem 3.13].

Lemma 14 *Assume **A1** to **A4** and **A6** and let $y \in \mathbb{R}^k$ be a heteroclinic point. The following representation holds for the tangent space of the center-stable manifold*

$$T_y W_+^{sc} = \left\{ z \in \mathbb{R}^k : \sup_{n\geq 0} \|Df^n(y)z\| < \infty \right\}. \tag{34}$$

4 Approximation of non-hyperbolic heteroclinic orbits

Let us approximate a non-hyperbolic heteroclinic orbit by restricting the infinite system (2) to a finite interval $J = [n_-, n_+]$ and replacing the condition $\lim\limits_{n\to\pm\infty} \bar{x}_n = \xi_\pm$ by a boundary condition at n_-, n_+.

The defining system (3) can be written in terms of an operator

$$\Gamma_J : S_J \to S_{\tilde{J}} \times \mathbb{R}^k$$

as

$$\Gamma_J(x_J) = \Big(\big(x_{n+1} - f(x_n)\big)_{n\in\tilde{J}}, b(x_{n_-}, x_{n_+})\Big) = 0, \tag{35}$$

where we assume $b \in C^1(\mathbb{R}^{2k}, \mathbb{R}^k)$.

According to Theorem 10 the restriction to the center manifold of $\xi_+ = 0$ is of the form

$$g(x) = s_1 x + s_2 b x^{q+1} + \mathcal{O}(x^{q+2}), \tag{36}$$

where $q \in \mathbb{N}$, $b > 0$,

$$(s_1, s_2) \in \begin{cases} \{(1,1),(1,-1)\}, & \text{for } q \text{ odd}, \\ \{(1,-1),(-1,1)\}, & \text{for } q \text{ even}. \end{cases}$$

From Theorem 3 it follows, that a g-orbit converges towards the fixed point 0 with a polynomial rate of $-\frac{1}{q}$, i.e. $\|x_n\| \approx Cn^{-1/q}$. Thus the variational equation (22) has a polynomial dichotomy on $\mathbb{Z}^+$ with data $(K^+, \alpha^+, \nu, P_n^{+sc}, P_n^{+u})$, where $\nu = 1 + \frac{1}{q}$, see Theorem 12.

In order to formulate the approximation theorem we have to require further assumptions on the boundary condition.

Definition 15 The boundary condition b is of **order** (p_-, p_+), if there exists a constant C, such that for all sufficiently large $-n_-, n_+$ the following estimate holds

$$\left\| b(\bar{x}_{n_-}, \bar{x}_{n_+}) - b(\xi_-, \xi_+) \right\| \leq C \left(\|\bar{x}_{n_-} - \xi_-\|^{p_-} + \|\bar{x}_{n_+} - \xi_+\|^{p_+} \right).$$

Our assumption on the order is:

A7 The boundary condition possesses the order $p_+ > q$.

We also need the following nondegeneracy condition.

A8 The boundary operator $b \in C^1(\mathbb{R}^{2k}, \mathbb{R}^k)$ fulfills

$$b(\xi_-, \xi_+) = 0$$

and the linear mapping $B \in L(X_s^- \times X_u^+, \mathbb{R}^k)$ defined by

$$B(x_s, x_u) = D_1 b(\xi_-, \xi_+) x_s + D_2 b(\xi_-, \xi_+) x_u, \quad x_s \in X_s^-, \; x_u \in X_u^+$$

is non-singular.

Finally, we denote by S_J^* the space of sequences, bounded in the $\|\cdot\|^*$-Norm, defined in (18) and let

$$B_\varepsilon^*(x_J) := \left\{ y_J \in S_J^* : \|y_J - x_J\|^* \leq \varepsilon \right\}$$

be the closed ball of radius ε with center $x_J \in S_J^*$.

4.1 Approximation Theorem

With the preparations so far we can prove the following approximation theorem (cf. [8, Theorem 4.3]).

Theorem 16 (Approximation Theorem) *Assume **A1** to **A8**. Then there exist constants $N \in \mathbb{N}$ and $\delta,\ C > 0$, such that the system (35) has a unique solution*

$$x_J \in B_{\frac{\delta}{n_+}}(\bar{x}_{|J}) \text{ for all } J = [n_-, n_+] \text{ with } -n_-,\ n_+ \geq N.$$

The approximation error satisfies an estimate

$$\|\bar{x}_{|J} - x_J\|_\infty \leq C \left(\beta^{n-p-} + n_+^{-\frac{p_+}{q}} \right), \tag{37}$$

where β is a strict lower bound for the smallest unstable eigenvalue of $f'(\xi_-)$ in absolute value.

As a first major step in the proof of Theorem 16 we derive an estimate for a linearized finite boundary value problem (cf. [8, Lemma 4.4]).

Lemma 17 *Assume **A1** to **A6**, **A8**. Then there exist constants $N \in \mathbb{N}$ and σ, such that for any $-n_-,\ n_+ \geq N$, the inhomogeneous equation*

$$\begin{aligned} u_{n+1} - f'(\bar{x}_n)u_n &= y_n, \quad n \in \tilde{J} = \{n_-, \ldots, n_+ - 1\}, && (38)\\ D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})u_{n_-} + D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})u_{n_+} &= r && (39) \end{aligned}$$

has a unique solution $u_J \in S_J$, $J = [n_-, n_+]$ for any $(y_{\tilde{J}}, r) \in (S^*_{\tilde{J}}, \mathbb{R}^k)$. Furthermore, we have the estimate

$$\|u_J\|_\infty \leq \sigma^{-1}\big(\|y_{\tilde{J}}\|^* + \|r\|\big), \tag{40}$$

where the constant σ is independent of J and the right hand side.

Proof: First, we consider for $y_{\tilde{J}} \in S^*_{\tilde{J}}$ the inhomogeneous equation

$$u_{n+1} - f'(\bar{x}_n)u_n = y_n, \quad n \in \tilde{J}. \tag{41}$$

Let Φ denote the solution operator of the corresponding homogeneous equation.

According to Theorem 12 this difference equation possesses an exponential dichotomy on $\mathbb{Z}^-$ with data $(K^-, \alpha^-, P_n^{-s}, P_n^{-u})$ and a polynomial dichotomy on $\mathbb{Z}^+$ with data $(K^+, \alpha^+, \nu, P_n^{+sc}, P_n^{+u})$.

A solution of (38) on J will be pieced together by particular solutions on $J \cap \mathbb{Z}^+$ and $J \cap \mathbb{Z}^-$. For the computation we use the Green's function (cf. Appendix A.1):

$$G^\pm(n,m) = \begin{cases} \Phi(n,m)(I - P_m^{\pm u}), & n \geq m, \\ -\Phi(n,m)P_m^{\pm u}, & n < m. \end{cases}$$

We define

$$\begin{aligned} z_n^-(y_{\tilde{J}}) &= \sum_{i=n_-}^{-1} G^-(n, i+1)y_i, \quad n_- \leq n \leq 0, \\ z_n^+(y_{\tilde{J}}) &= \sum_{i=0}^{n_+-1} G^+(n, i+1)y_i, \quad 0 \leq n \leq n_+. \end{aligned}$$

By using the exponential dichotomy on $\mathbb{Z}^-$ and the polynomial dichotomy on $\mathbb{Z}^+$, respectively, we show

$$\left\|z_n^+(y_{\tilde{J}})\right\| \le C\|y_{\tilde{J}}\|^* \text{ for } n_- \le n \le 0 \text{ and } \left\|z_n^-(y_{\tilde{J}})\right\| \le C\|y_{\tilde{J}}\|^* \text{ for } 0 \le n \le n_+. \quad (42)$$

For $n \ge 0$, we get

$$\begin{aligned}\left\|z_n^+(y_{\tilde{J}})\right\| &\le \left\|\sum_{i=0}^{n-1} \Phi(n,i+1)P_{i+1}^{+sc}\, y_i\right\| + \left\|\sum_{i=n}^{n_+-1} -\Phi(n,i+1)P_{i+1}^{+u}\, y_i\right\| \\ &\le K^+\big(T_{sc}(n) + T_u(n)\big)\end{aligned}$$

with

$$T_{sc}(n) = \sum_{i=0}^{n-1} \left(\frac{n+1}{i+2}\right)^{-\nu} \|y_i\|, \quad T_u(n) = \sum_{i=n}^{n_+-1} \mathrm{e}^{-\alpha^+(i+1-n)}\|y_i\|.$$

Obviously $T_u(n) \le C\|y_{\tilde{J}}\|^*$ and for the polynomial term $T_{sc}(n)$ it follows

$$\begin{aligned}T_{sc}(n) &= \sum_{i=2}^{n+1} \left(\frac{n+1}{i}\right)^{-\nu} \frac{1}{i}\cdot i\|y_{i-2}\| \ \le\ C\|y_{\tilde{J}}\|^*\ (n+1)^{-\nu}\sum_{i=2}^{n+1} i^{\nu-1} \\ &\le C\|y_{\tilde{J}}\|^*\ (n+1)^{-\nu}\int_0^{n+2} \tau^{\nu-1}d\tau \ \le\ C\|y_{\tilde{J}}\|^*\ (n+1)^{-\nu}\nu^{-1}(n+2)^{\nu} \\ &\le C\|y_{\tilde{J}}\|^*,\end{aligned}$$

where $C > 0$ is a generic constant. Thus (42) holds.

Arbitrary solutions of the inhomogeneous equation (41) on $J \cap \mathbb{Z}^-$ and $J \cap \mathbb{Z}^+$ are of the form

$$\begin{aligned}u_n^- &= \Phi(n,0)\eta + z_n^-(y_{\tilde{J}}), \quad \text{for } n_- \le n \le 0, \quad \eta \in \mathbb{R}^k, \\ u_n^+ &= \Phi(n,0)\zeta + z_n^+(y_{\tilde{J}}), \quad \text{for } 0 \le n \le n_+, \quad \zeta \in \mathbb{R}^k,\end{aligned}$$

respectively.

To get a solution on $\mathbb{Z}$, η and ζ have to be chosen such that $u_0^+ = u_0^-$ holds, therefore we require

$$\zeta - \eta = z_0^-(y_{\tilde{J}}) - z_0^+(y_{\tilde{J}}) =: Z(y_{\tilde{J}}).$$

Using this notation, the boundary condition (39) has the form

$$\begin{aligned}&D_1b(\bar{x}_{n_-},\bar{x}_{n_+})\Phi(n_-,0)\eta + D_2b(\bar{x}_{n_-},\bar{x}_{n_+})\Phi(n_+,0)\zeta \\ =\ & r - D_1b(\bar{x}_{n_-},\bar{x}_{n_+})z_{n_-}^-(y_{\tilde{J}}) - D_2b(\bar{x}_{n_-},\bar{x}_{n_+})z_{n_+}^+(y_{\tilde{J}}) =: R(y_{\tilde{J}}, r).\end{aligned}$$

Thus $u_n^\pm$ is a solution of (38), (39), iff (η,ζ) solves

$$\begin{pmatrix} -I & I \\ D_1b(\bar{x}_{n_-},\bar{x}_{n_+})\Phi(n_-,0) & D_2b(\bar{x}_{n_-},\bar{x}_{n_+})\Phi(n_+,0)\end{pmatrix}\begin{pmatrix}\eta \\ \zeta\end{pmatrix} = \begin{pmatrix} Z(y_{\tilde{J}}) \\ R(y_{\tilde{J}}, r)\end{pmatrix}. \quad (43)$$

Setting

$$\begin{array}{rcl}\eta_- &=& \Phi(n_-,0)P_0^{-s}\eta = P_{n_-}^{-s}\Phi(n_-,0)\eta \in \mathcal{R}(P_{n_-}^{-s}),\\ \eta_0 &=& P_0^{-u}\eta \in \mathcal{R}(P_0^{-u}),\\ \zeta_+ &=& \Phi(n_+,0)P_0^{+u}\zeta = P_{n_+}^{+u}\Phi(n_+,0)\zeta \in \mathcal{R}(P_{n_+}^{+u}),\\ \zeta_0 &=& P_0^{+sc}\zeta \in \mathcal{R}(P_0^{+sc}),\end{array}$$

η and ζ have the representation

$$\eta = \eta_0 + \Phi(0,n_-)\eta_-, \quad \zeta = \zeta_0 + \Phi(0,n_+)\zeta_+.$$

Let us replace η_- and ζ_+ by variables $\bar\eta_-$ and $\bar\zeta_+$ that lie in spaces independent of $n_\pm$ as follows

$$\bar\eta_- = V_{n_-}\eta_-, \quad \bar\zeta_+ = W_{n_+}\zeta_+,$$

where

$$\begin{array}{rclcrcl}V_{n_-} &:=& I + P^{-s} - P_{n_-}^{-s} &:& \mathcal{R}(P_{n_-}^{-s}) &\to& \mathcal{R}(P^{-s}),\\ W_{n_+} &:=& I + P^{+u} - P_{n_+}^{+u} &:& \mathcal{R}(P_{n_+}^{+u}) &\to& \mathcal{R}(P^{+u}).\end{array}$$

Note that

$$\left\|V_{n_-}^{-1}\right\| \le \frac{1}{1-\left\|P^{-s}-P_{n_-}^{-s}\right\|} \le 2, \quad \left\|W_{n_+}^{-1}\right\| \le \frac{1}{1-\left\|P^{+u}-P_{n_+}^{+u}\right\|} \le 2.$$

The system (43) may now be written as

$$\underbrace{\begin{pmatrix} I^- & I^+ & \Xi_{n_-} & \Xi_{n_+}\\ \Delta^1_{n_\pm} & \Delta^2_{n_\pm} & \Theta^1_{n_\pm} & \Theta^2_{n_\pm}\end{pmatrix}}_{=:A_{n_\pm}} \begin{pmatrix}\eta_0\\ \zeta_0\\ \bar\eta_-\\ \bar\zeta_+\end{pmatrix} = \begin{pmatrix} Z(y_{\tilde J})\\ R(y_{\tilde J}, r)\end{pmatrix}, \tag{44}$$

where we have used the following quantities

$$\begin{array}{ll} I^- := -I_{|\mathcal{R}(P_0^{-u})}, & I^+ := I_{|\mathcal{R}(P_0^{+sc})},\\ \Xi_{n_-} := -\Phi(0,n_-)V_{n_-\,|\mathcal{R}(P^{-s})}^{-1}, & \Xi_{n_+} := \Phi(0,n_+)W_{n_+\,|\mathcal{R}(P^{+u})}^{-1},\\ \Delta^1_{n_\pm} := D_1 b(\bar x_{n_-}, \bar x_{n_+})\Phi(n_-,0)_{|\mathcal{R}(P_0^{-u})}, & \Delta^2_{n_\pm} := D_2 b(\bar x_{n_-}, \bar x_{n_+})\Phi(n_+,0)_{|\mathcal{R}(P_0^{+sc})},\\ \Theta^1_{n_\pm} := D_1 b(\bar x_{n_-}, \bar x_{n_+})V_{n_-\,|\mathcal{R}(P^{-s})}^{-1}, & \Theta^2_{n_\pm} := D_2 b(\bar x_{n_-}, \bar x_{n_+})W_{n_+\,|\mathcal{R}(P^{+u})}^{-1}.\end{array}$$

To prove the existence of a unique solution of (44) we show that the limit matrix $A_{\pm\infty}$, obtained as $n_\pm \to \pm\infty$, is invertible. From this it follows, that the matrices $A_{n_\pm}^{-1}$ exist for $-n_-, n_+$ sufficiently large and are uniformly bounded.

By using the exponential dichotomy on $\mathbb{Z}^-$ and the polynomial dichotomy on $\mathbb{Z}^+$ we find that $\Xi_{n_\pm}$ and $\Delta^{1,2}_{n_\pm}$ converge towards 0 as $n_\pm \to \pm\infty$. It holds

$$\begin{aligned}
\lim_{n_-\to-\infty} \left\|\Xi_{n_-}\right\| &\le \lim_{n_-\to-\infty} \left\|\Phi(0,n_-)P^{-s}_{n_-}\right\| \left\|V^{-1}_{n_-\,|\mathcal{R}(P^{-s})}\right\| \\
&\le 2K^- \lim_{n_-\to-\infty} \mathrm{e}^{-\alpha^-(0-n_-)} = 0, \\
\lim_{n_+\to\infty} \left\|\Xi_{n_+}\right\| &\le \lim_{n_+\to\infty} \left\|\Phi(0,n_+)P^{+u}_{n_+}\right\| \left\|W^{-1}_{n_+\,|\mathcal{R}(P^{+u})}\right\| \\
&\le 2K^+ \lim_{n_+\to\infty} \mathrm{e}^{-\alpha^+ n_+} = 0, \\
\lim_{n_\pm\to\pm\infty} \left\|\Delta^1_{n_\pm}\right\| &\le C \lim_{n_-\to-\infty} \left\|\Phi(n_-,0)P^{-u}_0\right\| \\
&\le CK^- \lim_{n_-\to-\infty} \mathrm{e}^{-\alpha^-(0-n_-)} = 0, \\
\lim_{n_\pm\to\pm\infty} \left\|\Delta^2_{n_\pm}\right\| &\le C \lim_{n_+\to\infty} \left\|\Phi(n_+,0)P^{+sc}_0\right\| \\
&\le CK^+ \lim_{n_+\to\infty} (n_+ + 1)^{-\nu} = 0,
\end{aligned}$$

where $C > 0$ is a generic constant. Furthermore, $\Theta^{1,2}_{n_\pm}$ has the limit

$$\begin{aligned}
\Theta^1 &:= \lim_{n_\pm\to\pm\infty} \Theta^1_{n_\pm} = D_1 b(\xi_-,\xi_+)_{|\mathcal{R}(P^{-s})}, \\
\Theta^2 &:= \lim_{n_\pm\to\pm\infty} \Theta^2_{n_\pm} = D_2 b(\xi_-,\xi_+)_{|\mathcal{R}(P^{+u})}.
\end{aligned}$$

Therefore the matrix $A_{\pm\infty}$ has the form

$$A_{\pm\infty} = \left(\begin{array}{cc|cc} I^- & I^+ & 0 & 0 \\ \hline 0 & 0 & \Theta^1 & \Theta^2 \end{array}\right).$$

We prove that the upper left block is invertible. Suppose $\eta_0 \in \mathcal{R}(P^{-u}_0)$ and $\zeta_0 \in \mathcal{R}(P^{+sc}_0)$ satisfy $\eta_0 = \zeta_0$. A solution of the homogeneous equation

$$\Gamma'(\bar{x}_\mathbb{Z})v_\mathbb{Z} = 0 \tag{45}$$

on $\mathbb{Z}$ is then given by

$$v_n = \begin{cases} \Phi(n,0)P^{-u}_0\eta_0, & \text{for } n \le 0, \\ \Phi(n,0)P^{+sc}_0\zeta_0, & \text{for } n \ge 0. \end{cases}$$

By the transversality assumption **A5** and Theorem 13, (45) has only the trivial solution $v_n = 0$ ($n \in \mathbb{Z}$). Thus the upper left block is invertible. Furthermore, it follows from assumption **A8** that also the lower right block is invertible.

Therefore, the matrix $A^{-1}_{n_\pm}$ exists for sufficiently large $-n_-$, n_+ This proves the existence and uniqueness of vectors η and ζ, such that the two solutions $u^\pm_n$ coincide at 0 and give a solution of the system (38), (39) on J:

$$u_n = \begin{cases} u^-_n, & \text{for } n_- \le n \le 0, \\ u^+_n, & \text{for } 0 \le n \le n_+. \end{cases}$$

Next we prove the estimate (40). For a generic constant $C > 0$ we have for the solution of (44)

$$\|(\eta_0, \zeta_0, \eta_-, \zeta_+)^T\| \le C\big(\|y_{\tilde{J}}\|^* + \|r\|\big).$$

Note that $\|Z(y_{\tilde{J}})\|$, $\|R(y_{\tilde{J}}, r)\| \le C\big(\|y_{\tilde{J}}\|^* + \|r\|\big)$. We get for $n_+ \ge n \ge 0$

$$\begin{aligned}
\|u_n\| &\le \big\|\Phi(n,0)\zeta\big\| + \big\|z_n^+(y_{\tilde{J}})\big\| \\
&\le \big\|\Phi(n,0)\zeta_0\big\| + \big\|\Phi(n,n_+)\zeta_+\big\| + C\|y_{\tilde{J}}\|^* \\
&\le \big\|\Phi(n,0)P_0^{+sc}\big\|\|\zeta_0\| + \big\|\Phi(n,n_+)P_{n_+}^{+u}\big\|\|\zeta_+\| + C\|y_{\tilde{J}}\|^* \\
&\le K^+(n+1)^{-\nu}\|\zeta_0\| + K^+\mathrm{e}^{-\alpha^+(n_+-n)}\|\zeta_+\| + C\|y_{\tilde{J}}\|^* \\
&\le C\big(\|y_{\tilde{J}}\|^* + \|r\|\big).
\end{aligned}$$

Similarly, the estimate

$$\|u_n\| \le C\big(\|y_{\tilde{J}}\|^* + \|r\|\big)$$

holds for $n_- \le n \le 0$. Thus there exists an $N \in \mathbb{N}$ and a constant σ with

$$\|u_J\|_\infty \le \sigma^{-1}\big(\|y_{\tilde{J}}\|^* + \|r\|\big)$$

for all $J = [n_-, n_+]$ with $-n_-,\ n_+ \ge N$. Note that σ is independent of J and the right hand side. ■

Next we prove Theorem 16.

Proof of Theorem 16: We apply Appendix A.3, Lemma 20 to the operator Γ_J with the settings

$$\begin{aligned}
Y &= \big(S_J, \|\cdot\|_\infty\big), & Z &= \big(S_{\tilde{J}}^* \times \mathbb{R}^k, \|\cdot\|^* + \|\cdot\|\big), \\
F &= \Gamma_J, & y_0 &= \bar{x}_{|J}.
\end{aligned}$$

First the assumption (50) is verified. According to Lemma 17 the inhomogeneous equation

$$\Gamma_J'(\bar{x}_{|J})u_J = (y_{\tilde{J}}, r)$$

possesses for all $(y_{\tilde{J}}, r) \in S_{\tilde{J}}^* \times \mathbb{R}^k$ and sufficiently large intervals J a unique solution u_J, fulfilling the estimate

$$\|u_J\|_\infty \le \sigma^{-1}\big(\|y_{\tilde{J}}\|^* + \|r\|\big).$$

Note that σ is independent of $n_\pm$ and the right hand side. Thus $\Gamma_J'(\bar{x}_{|J})^{-1}$ exists and is uniformly bounded. For all $(y_{\tilde{J}}, r) \in S_{\tilde{J}}^* \times \mathbb{R}^k$ and $-n_-$, n_+ sufficiently large we get

$$\big\|\Gamma_J'(\bar{x}_{|J})^{-1}(y_{\tilde{J}}, r)\big\|_\infty = \|u_J\|_\infty \le \sigma^{-1}\big(\|y_{\tilde{J}}\|^* + \|r\|\big),$$

and it follows

$$\big\|\Gamma_J'(\bar{x}_{|J})^{-1}\big\|_{\infty\leftarrow *} \le \sigma^{-1}.$$

Define

$$\begin{aligned}
\Lambda_{n_\pm} &:= \big\|D_1 b(x_{n_-}, x_{n_+}) - D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})\big\| \\
&\quad + \big\|D_2 b(x_{n_-}, x_{n_+}) - D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})\big\|, \\
M &:= \big\{x_J \in S_J : \|x_J\|_\infty = 1\big\}.
\end{aligned}$$

It holds for some constant $C > 0$:

$$\begin{aligned}
& \|\Gamma'_J(x_J) - \Gamma'_J(\bar{x}_{|J})\|_{*\leftarrow\infty} \\
\leq\ & \sup_{z_J \in M} \Big\| \big(z_{n+1} - f'(x_n) z_n - z_{n+1} + f'(\bar{x}_n) z_n\big)_{n \in \tilde{J}} \Big\|^* + \Lambda_{n_\pm} \\
\leq\ & \sup_{z_J \in M} \sup_{n \in \tilde{J}} \max\{1, n\} \, \big\|\big(f'(x_n) - f'(\bar{x}_n)\big) z_n\big\| + \Lambda_{n_\pm} \\
\leq\ & \sup_{n \in \tilde{J}} \max\{1, n\} \big\|f'(x_n) - f'(\bar{x}_n)\big\| + \Lambda_{n_\pm} \\
\leq\ & C \sup_{n \in \tilde{J}} \max\{1, n\} \, \|x_n - \bar{x}_n\| + \Lambda_{n_\pm}.
\end{aligned}$$

Setting $\psi = \delta n_+^{-1}$ with a constant $\delta > 0$, we get for $x_n \in B_\psi(\bar{x}_{|J})$ and $n \in \tilde{J}$

$$\max\{1, n\} \, \|x_n - \bar{x}_n\| \leq n_+ \psi = \delta.$$

Thus for sufficiently small $\delta > 0$ and $-n_-$, n_+ sufficiently large, the following estimate holds

$$\big\|\Gamma'_J(x_J) - \Gamma'_J(\bar{x}_{|J})\big\|_{*\leftarrow\infty} \leq \frac{\sigma}{2}.$$

This implies assumption (50) from Appendix A.3, Lemma 20:

$$\big\|\Gamma'_J(x_J) - \Gamma'_J(\bar{x}_{|J})\big\|_{*\leftarrow\infty} \leq \kappa := \frac{\sigma}{2} < \sigma \leq \frac{1}{\big\|\Gamma'_J(\bar{x}_{|J})^{-1}\big\|_{\infty\leftarrow*}}$$

for all $x_J \in B_\psi(\bar{x}_{|J})$.

The second assumption (51) from Lemma 20, is a consequence of **A7** and **A8**. Here we use that the orbits converges towards the fixed point $\xi_\pm$ as $n \to \pm\infty$ with an exponential or a polynomial rate, respectively (cf. Theorem 3). Let β be a strict lower bound for the smallest unstable eigenvalue of $f'(\xi_-)$ in absolute value. For sufficiently large $-n_-$, n_+ we get the estimates

$$\begin{aligned}
\big\|\Gamma_J(\bar{x}_{|J})\big\|^* &= \| \underbrace{\big(\bar{x}_{n+1} - f(\bar{x}_n)\big)_{n \in \tilde{J}}}_{=0} \|^* + \big\|b(\bar{x}_{n_-}, \bar{x}_{n_+})\big\| \\
&\leq C \left(\|\bar{x}_{n_-} - \xi_-\|^{p_-} + \|\bar{x}_{n_+} - \xi_+\|^{p_+}\right) \\
&\leq C \left[\beta^{n_- p_-} + n_+^{-\frac{p_+}{q}}\right] \leq 2C n_+^{-\frac{p_+}{q}} \leq \frac{\sigma}{2} \delta n_+^{-\frac{q}{q}} \\
&= (\sigma - \kappa)\psi.
\end{aligned}$$

Finally, Appendix A.3, Lemma 20 gives us the existence and uniqueness of the zero $x_J \in B_\psi(\bar{x}_{|J})$ of Γ_J. The error estimate follows from (53) by setting $y_1 = \bar{x}_{|J}$, $y_2 = x_J$:

$$\begin{aligned}\|\bar{x}_{|J} - x_J\|_\infty &\leq \frac{1}{\sigma-\kappa}\big\|\Gamma_J(\bar{x}_{|J}) - \underbrace{\Gamma_J(x_J)}_{=0}\big\|^* = \frac{2}{\sigma}\big\|b(\bar{x}_{n_-}, \bar{x}_{n_+})\big\| \\ &\leq C\left(\beta^{n-p_-} + n_+^{-\frac{p_\pm}{q}}\right).\end{aligned}$$

■

4.2 Boundary conditions

In this section we introduce the boundary condition used for the numerical calculations.

Definition 18 The map $b_{\text{proj}} \in C^1(\mathbb{R}^{2k}, \mathbb{R}^k)$ defined by

$$b_{\text{proj}}(x, y) := P^{-s}(x - \xi_-) + P^{+u}(y - \xi_+), \quad x, y \in \mathbb{R}^k,$$

where P^{-s} and P^{+u} are projectors with $\mathcal{R}(P^{-s}) = X_-^s$, $\mathcal{R}(P^{+u}) = X_+^u$, is called **projection boundary map**.

The boundary condition $b_{\text{proj}}(x_{n_-}, x_{n_+}) = 0$ is of order $(p_-, p_+) \geq (2, 2)$, cf. [11, Proposition 3.1.5], [8, Lemma 4.6]. In case the tangent space X_+^u approximates W_+^u up to order n, this boundary condition is of order $p_+ = n+1$. An example for which the projection boundary condition is of order $p_+ = 3$ is considered in Section 5.

5 Example

In this section we present an example that exhibits a non-hyperbolic heteroclinic orbit. This orbit can be approximated by a solution of the defining system (35). Furthermore, we show that the estimate for the approximation error (37) is confirmed by the numerical results.

Consider the following Hénon-like map

$$f: \begin{array}{rcl} \mathbb{R}^2 \times \mathbb{R} & \to & \mathbb{R}^2 \\ (x, \lambda) & \mapsto & \begin{pmatrix} \left(\frac{1}{2} - \lambda\right) x_1 + x_1^3 + \frac{2}{5}x_1^4 + x_2 \\ \frac{3}{2}x_1 \end{pmatrix}. \end{array}$$

This map possesses two branches of fixed points $\xi_\pm(\lambda)$ for all $\lambda \in \mathbb{R}$, where $\xi_+(\lambda) = 0$. The matrix $f_x\big(\xi_+(\lambda), \lambda\big)$ has the eigenvalues

$$\mu^\pm(\lambda) = \frac{1}{4} - \frac{\lambda}{2} \pm \frac{1}{4}\sqrt{4\lambda^2 - 4\lambda + 25},$$

thus at $\lambda = 0$ the center eigenvalue -1 occurs. The second fixed point has one stable and one unstable eigenvalue for all $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$. The bifurcation diagram, obtained with CONTENT, see [14], is shown in Figure 5.

At $\lambda = 0$ the fixed point $\xi_+(\lambda)$ undergoes a flip-bifurcation, thus the reduced system at $\lambda = 0$ has the normal form (cf. [13])

$$g(x) = x - \ell x^{q+1} + \mathcal{O}(x^{q+2}), \quad \ell > 0,\ q = 2.$$

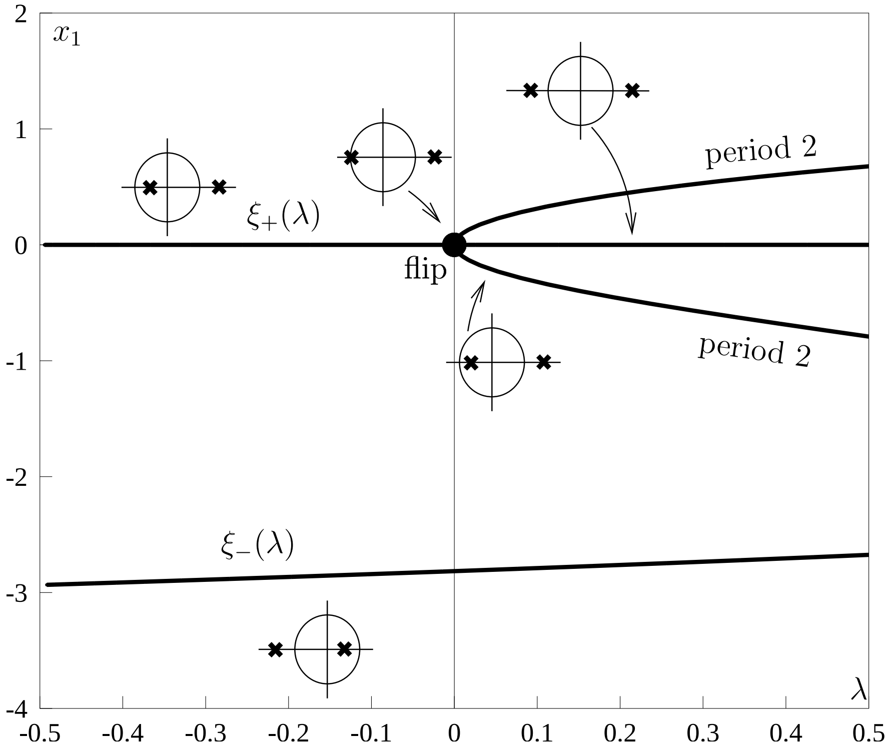

Figure 5: *Bifurcation diagram of fixed points of f, projected to the* (λ, x_1)-*space.*

A non-hyperbolic orbit of length $n_- = -10$, $n_+ = 30$ is plotted in Figure 6. To illustrate the transversality, approximations of W_-^u and W_+^c are also displayed.

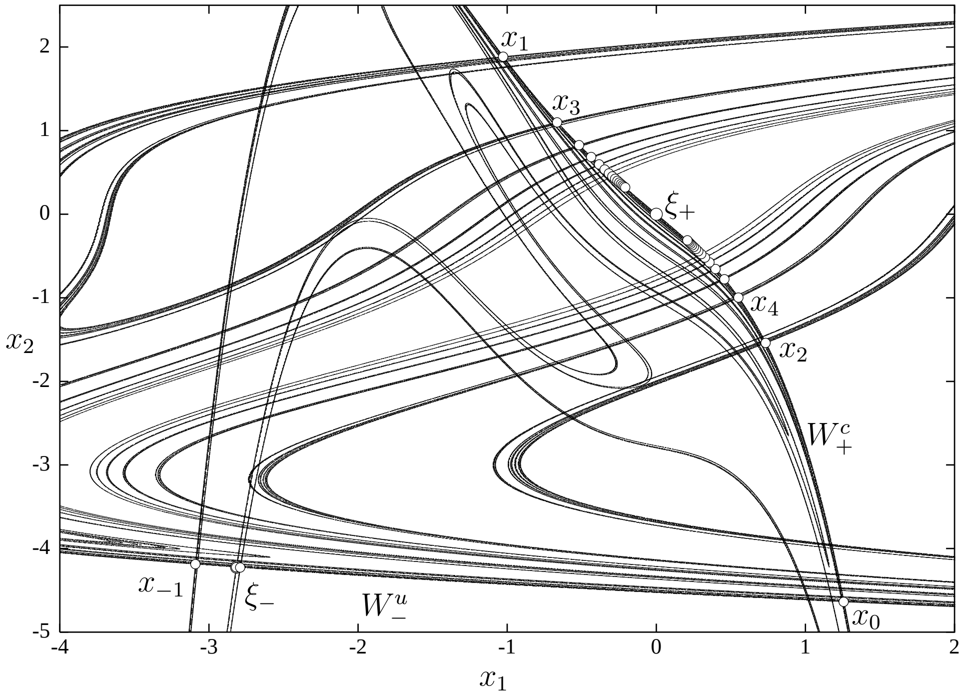

Figure 6: *An orbit of length $n_- = -10$, $n_+ = 30$, plotted together with an approximation of W_-^u and W_+^c.*

For the calculation of the approximation error we compute an orbit of length $n_- = -10^3$, $n_+ = 10^6$ as reference orbit, because an exact orbit is not known. This orbit is compared with 'short' orbits of length $n_- = -10^3$, $n_+ \in [10, 10^5]$. According to Theorem 16 and Theorem 3 we get

$$\|\bar{x}_{|J} - x_J\|_\infty \leq C \left(\beta^{n - p_-} + n_+^{-\frac{p_+}{q}} \right).$$

When $n_+^{-\frac{p_+}{q}}$ is the slowest term in the sum we can estimate $\frac{p_+}{q}$ from

$$\frac{p_+}{q} \approx \frac{-\log_{10}\left(\|\bar{x}_{|J} - x_J\|_\infty\right)}{\log_{10}(n_+)}.$$

Note that the projection boundary condition in this example is of order $p_+ = 3$ (see Section 4.2) and $q = 2$.

Table 1 shows the agreement of the numerical calculation with the theoretical error estimate (37).

n_+	$\log_{10}\left(\lVert\bar{x}_{\vert J} - x_J\rVert_\infty\right)$	$\dfrac{-\log_{10}\left(\lVert\bar{x}_{\vert J} - x_J\rVert_\infty\right)}{\log_{10}(n_+)}$
10^1	-1.768373	1.768373
10^2	-3.382771	1.691385
10^3	-4.943709	1.647903
10^4	-6.464676	1.616169
10^5	-7.971346	1.594269

Table 1: *Numerical computation of the polynomial rate of the approximation error.*

A Appendix

A.1 Green's function

Consider the inhomogeneous difference equation

$$u_{n+1} - A_n u_n = r_n, \quad u_n, r_n \in \mathbb{R}^k,\ n \in \tilde{J} \subset \mathbb{Z}^+ \tag{46}$$

with nonsingular matrices $A_n \in \mathbb{R}^{k,k}$. Denote the solution operator of (46) by Φ and assume that (46) has a polynomial dichotomy on J with data $(K, \alpha, \nu, P_n^{sc}, P_n^u)$. We extend the system (46) by introducing a boundary condition

$$P_{n_-}^{sc} u_{n_-} = \zeta, \quad \zeta \in \mathcal{R}(P_{n_-}^{sc}). \tag{47}$$

Green's function (cf. [18]) is defined by

$$G(n,m) := \begin{cases} \Phi(n,m)P_m^{sc}, & n \geq m, \\ -\Phi(n,m)P_m^u, & n < m. \end{cases}$$

The polynomial dichotomy immediately provides us with estimates for Green's function:

$$\lVert G(n,m)\rVert \leq \begin{cases} K\left(\frac{n+1}{m+1}\right)^{-\nu}, & n \geq m, \\ K\mathrm{e}^{-\alpha(m-n)}, & n < m. \end{cases}$$

Furthermore, Green's function is a solution of

$$G(n+1,m) = A_n G(n,m) + \delta_{n,m-1} I.$$

We obtain a solution of (46), (47) by

$$u_n = \Phi(n, n_-)\zeta + \sum_{m \in \tilde{J}} G(n, m+1) r_m,$$

provided the convergence of $\sum_{m\in\tilde{J}} G(n,m+1)r_m$ can be assured.

A.2 Transformation

The following lemma (cf. [10, Chapter 10, Proposition 3.1], [8, Lemma 3.5]) is useful when solving a functional equation.

Lemma 19 *Let $a, b, c \in \mathbb{N}$. Consider for $x \in \mathbb{R}^c$, $U \in \mathbb{R}^{a,b}$ the system*

$$\begin{pmatrix} x \\ U \end{pmatrix} \mapsto \begin{pmatrix} g(x) \\ L(x)[U] + b(x) \end{pmatrix}, \tag{48}$$

where $L(x)[U] = A(x)UB(x)$, $A \in C^2(\mathbb{R}^c, \mathbb{R}^{a,a})$, $B(x) \in C^2(\mathbb{R}^c, \mathbb{R}^{b,b})$. Assume that $L(0)$ is hyperbolic and $b \in C^2(\mathbb{R}^c, \mathbb{R}^{a,b})$ with $b(0) = 0$. Finally assume $g \in C^2(\mathbb{R}^c, \mathbb{R}^c)$, $g(0) = 0$ and $g'(0)$ has only eigenvalues of absolute value one.

Then there exists a C^2-map $h : \mathbb{R}^c \to \mathbb{R}^{a,b}$ which solves the functional equation

$$h\big(g(x)\big) = L(x)[h(x)] + b(x) \tag{49}$$

for sufficiently small $x \in \mathbb{R}^c$.

A.3 A Lipschitz inverse mapping theorem

The following lemma (cf. Vainikko's Lemma [27]) is used to prove our approximation theorem.

Lemma 20 *Assume Y and Z are Banach spaces, $F \in C^1(Y, Z)$ and $F'(y_0)$ is for $y_0 \in Y$ a homeomorphism. Let κ, σ, $\psi > 0$ be three constants, such that the following estimates hold:*

$$\begin{aligned} \big\|F'(y) - F'(y_0)\big\| &\le \kappa < \sigma \le \frac{1}{\big\|F'(y_0)^{-1}\big\|} \quad \forall y \in B_\psi(y_0), & (50)\\ \big\|F(y_0)\big\| &\le (\sigma - \kappa)\psi. & (51) \end{aligned}$$

Then F has a unique zero $\bar{y} \in B_\psi(y_0)$ and the estimates

$$\begin{aligned} \big\|F'(y)^{-1}\big\| &\le \frac{1}{\sigma - \kappa} \quad \forall y \in B_\psi(y_0), & (52)\\ \|y_1 - y_2\| &\le \frac{1}{\sigma - \kappa}\big\|F(y_1) - F(y_2)\big\| \quad \forall y_1,\ y_2 \in B_\psi(y_0) & (53) \end{aligned}$$

are fulfilled.

B Appendix

B.1 Proof of Theorem 4

Proof of Theorem 4: The variational equation (11) is of the form

$$\begin{aligned} u_{n+1} &= g'(x_n)u_n = \big(1 - \ell(q+1)x_n^q + \varphi(x_n)\big)u_n \\ &= \left(\prod_{i=m}^{n} \big(1 - \ell(q+1)x_i^q + \varphi(x_i)\big)\right) u_m = \Phi(n+1, m)u_m, \end{aligned}$$

where $\varphi(x) = \mathcal{O}(x^{q+1})$. This leads to the representation

$$\begin{aligned}
&\log\left(\left(\tfrac{n+1}{m+1}\right)^{1+\frac{1}{q}}\Phi(n,m)\right) = \left(1+\tfrac{1}{q}\right)\log\left(\tfrac{n+1}{m+1}\right) + \log\left(\Phi(n,m)\right) \\
= \;& \left(1+\tfrac{1}{q}\right)\log\left(\tfrac{n+1}{m+1}\right) - \left(1+\tfrac{1}{q}\right)\sum_{i=m}^{n-1}\ell q x_i^q + \sum_{i=m}^{n-1}\ell(q+1)x_i^q \\
&+\sum_{i=m}^{n-1}\log\left(1-\ell(q+1)x_i^q+\varphi(x_i)\right) \\
= \;& \left(1+\tfrac{1}{q}\right)M(n,m) + \Pi(n,m)
\end{aligned}$$

with

$$\begin{aligned}
M(n,m) &= \log\left(\tfrac{n+1}{m+1}\right) - \sum_{i=m}^{n-1}\ell q x_i^q, \\
\Pi(n,m) &= \sum_{i=m}^{n-1}\ell(q+1)x_i^q + \log\left(1-\ell(q+1)x_i^q+\varphi(x_i)\right), \quad \varphi(x) = \mathcal{O}(x^{q+1}).
\end{aligned}$$

Let $c_n := \sum_{i=1}^{n}\frac{1}{i} - \log(n)$, then c_n converges to Euler's constant $c \approx 0.5772$. Thus $M(n,m)$ can be rewritten in the form

$$M(n,m) = c_{m+1} - c_{n+1} + N(n,m), \text{ where } N(n,m) := \sum_{i=m}^{n-1}\left(\frac{1}{i+2} - \ell q x_i^q\right).$$

By using the estimates for x_n, given in Theorem 3, Equation (9), it follows that $N(n,1)$ is absolutely convergent. This finishes the proof of Theorem 4 since $\Pi(n,1)$ is also absolutely convergent. ■

B.2 Idea of proof of the Roughness Theorem

To prove the Roughness Theorem, we consider for $X \in (\mathbb{R}^{k,k})^{J\times J}$ the weighted norm

$$\|X\|_\diamond := \max\left\{\sup_{n\geq m}\left(\tfrac{n+1}{m+1}\right)^\nu\|X(n,m)\|,\ \sup_{n<m}\mathrm{e}^{\alpha(m-n)}\|X(n,m)\|\right\}.$$

The Banach space $Z_J^{\alpha,\nu}$ of bounded sequences in this norm is given by

$$Z_J^{\alpha,\nu} := \left\{X \in (\mathbb{R}^{k,k})^{J\times J} : \|X\|_\diamond < \infty\right\}. \tag{54}$$

We further introduce the bilinear operators T_1 and T by

$$\begin{aligned}
T_1, T \quad &: \quad Z_J^{\alpha,\nu} \times S_J^1(\mathbb{R}^{k,k}) \to (\mathbb{R}^{k,k})^{J\times J}, \\
T_1(X, E_J) \quad &:= \quad \left(\sum_{l\in\tilde{J}} G(n,l+1)E_l X(l,m)\right)_{(n,m)\in J\times J}, \\
T(X, E_J) \quad &:= \quad G + T_1(X, E_J),
\end{aligned}$$

where G denotes Green's function of the unperturbed system (13) (cf. Appendix A.1). For these operators, the following lemmas hold, cf. [8, Lemma 2.8 and Lemma 2.9].

Lemma 21 *With the assumptions of Theorem 8 the inclusion $\mathcal{R}(T_1) \subset Z_J^{\alpha,\nu}$ holds for every sequence $E_J \in S_J^1(\mathbb{R}^{k,k})$. Furthermore, we have the estimate*

$$\|T_1(\cdot, E_J)\|_\diamond \leq \varrho \|E_J\|_1,$$

where the constant $\varrho > 0$ is independent of E_J.

Lemma 22 *With the assumptions of Theorem 8 we get*

$$T \in C^\infty\big(Z_J^{\alpha,\nu} \times S_J^1(\mathbb{R}^{k,k}), Z_J^{\alpha,\nu}\big). \tag{55}$$

For $X \in Z_J^{\alpha,\nu}$, $E_J \in S_J^1(\mathbb{R}^{k,k})$ the following statements (i) and (ii) are equivalent.

(i) X is a fixed point of $T(\cdot, E_J)$.

(ii) For all $(n, m) \in \tilde{J} \times J$ we have the equations

$$\begin{aligned} X(n+1, m) &= (A_n + E_n)X(n, m) + \delta_{n,m-1}I, & (56)\\ P_{n_-}^{sc} X(n_-, m) &= P_{n_-}^{sc} G(n_-, m), & (57) \end{aligned}$$

where G is Green's function of the unperturbed systems (13).

From Lemma 22, it follows that a fixed point $\bar{X}$ of the operator $T(\cdot, E_J)$ is a Green's function of the perturbed system (19). Then Lemma 21 is used to prove uniform contraction of $T(\cdot, E_J)$ for $\varrho\|E_J\|_1$ sufficiently small.

Define for $n \in J$ the projectors Q_n^{sc}, Q_n^u by

$$Q_n^{sc} := \bar{X}(E_J, n, n), \quad Q_n^u = I - Q_n^{sc}.$$

By using the estimate from Lemma 21 we get the polynomial dichotomy of the perturbed system (19) on J with data $(\tilde{K}, \alpha, \nu, Q_n^{sc}, Q_n^u)$.

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