Construction of a conjugacy and closeness estimates in the discretized fold bifurcation

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Dedicated to the memory of Gyula Farkas (1972–2002)

Abstract

Our present work is a case study of numerical structural stability of flows under discretization in the vicinity of a non-hyperbolic equilibrium. One-dimensional ordinary differential equations with the origin undergoing *fold bifurcation* at bifurcation parameter value $\alpha = 0$ are considered together with their discretizations. In a neighbourhood of this equilibrium *a conjugacy is constructed* between the time-h-map of the solution flow of the ODE and its stepsize-h discretization of order $p \geq 1$ *in the limiting case* $h \to 0^+$.

As we have shown in [4], the conjugacy problem between the original ODE and its discretization can be reduced to the construction of a conjugacy between the corresponding *normal forms*, which in turn amounts to solving a one-dimensional functional equation depending on the two parameters h and α. A solution—being the required conjugacy map—is now obtained by applying the technique of fundamental domains.

The main emphasis in this work is put on *estimating the distance* between the conjugacy map $J(h, \cdot, \alpha)$ and the identity on $[-\varepsilon_0, \varepsilon_0]$ for $0 < h \leq h_0$ and $-\alpha_0 \leq \alpha \leq \alpha_0$. Since the origin is a fold bifurcation point, we can assume that both normal forms possess two fixed points for $\alpha < 0$ which merge at $\alpha = 0$ then disappear for $\alpha > 0$. For $\alpha \leq 0$, we show that $|\,x - J(h, x, \alpha)|$ is $\mathcal{O}(h^p)$ small, uniformly in x and $\alpha \leq 0$, further, that this closeness result is optimal. For $\alpha > 0$ however, we are currently unable to establish uniform $\mathcal{O}(h^p)$-closeness: only a weakly singular estimate $\mathcal{O}(h^p \cdot \ln \frac{1}{\alpha})$ is proved. Nevertheless, numerical experiments suggest that this estimate is sharp for our particular construction of J. Uniform $\mathcal{O}(h^p)$-closeness in the $\alpha > 0$ case is proved under an additional assumption on the normal forms.

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1 Introduction

Numerical structural stability of flows under discretization assuming some kind of hyperbolicity has been thoroughly investigated in the past years, see, e.g., [1] and [2].

Numerical counterparts of classical results on ordinary differential equations have been established—such as the numerical flow box theorem (around a non-equilibrium point) or the numerical Grobman-Hartman Lemma (around a hyperbolic equilibrium).

On one hand, the question of numerical structural stability of flows is captured by the existence of a conjugacy mapping between the original flow and its discretization. Such a conjugacy preserves the topological structure of the phase portrait, thus the corresponding dynamical systems are topologically equivalent. Consequently, it is often convenient to think of conjugacies as coordinate-transformations and discretizations as special, small perturbations of the original flow.

On the other hand, numerical structural stability can quantitatively be expressed by measuring the distance between the conjugacy and the identity map on the center manifold. Under various hyperbolicity assumptions, this quantity can be shown to be $\mathcal{O}(h^p)$-small, where h is the stepsize and p is the order of the discretization method applied.

It is thus natural to compare exact and discretized dynamics in the simplest non-hyperbolic case, that is, in one-parameter families of ODE's with hyperbolicity violated at a single value of the parameter.

A paper [3] by Gyula Farkas has been a step in this direction. He has constructed a conjugacy between the time-1-map $\Phi(1,\cdot,\alpha)$ of an ordinary differential equation and the N^{th} iterate ($N \in \mathbb{N}^+$) of its stepsize $h = 1/N$ discretization $\varphi^{[N]}(h,\cdot,\alpha)$ in the vicinity of a fold bifurcation point. Here $\alpha \in \mathbb{R}$ is the bifurcation parameter and the bifurcation point is chosen to be the origin. He also "showed" that the constructed conjugacy is $\mathcal{O}(h^p)$-close to the identity on the center manifold, where p is the order of the one-step discretization method φ. Quotation marks have been used in the previous sentence because the proof of the main estimate in [3] contains some gaps in the $\alpha \leq 0$ case. However, the gaps are much larger in the $\alpha > 0$ case, which can not be considered as proven: the main technical difficulty remained unnoticed and unresolved here.

Our primary aim is to generalize and correct the above result in the limiting case $h \to 0^+$, that is to construct a conjugacy $J(h,\cdot,\alpha)$ between the time-h-map $\Phi(h,\cdot,\alpha)$ of the ODE and its corresponding discretization $\varphi(h,\cdot,\alpha)$, further, to prove a suitable closeness between the conjugacy map and the identity. Although $\Phi(h,\cdot,\alpha)$ in the present work is only an $\mathbb{R} \to \mathbb{R}$ map, some preliminary computations suggest that the same type of conjugacy result will hold near fold bifurcation points of ODE's and their discretizations in higher space dimensions. In addition, the closeness estimate will remain true for the corresponding maps restricted to their center manifolds—with the modification that all h^p in the final estimates will be replaced by h^{p-1}. (A discretization of order p means $\mathcal{O}(h^{p+1})$-closeness between the maps $\Phi(h,\cdot,\alpha)$ and $\varphi(h,\cdot,\alpha)$, but only $\mathcal{O}(h^p)$-closeness of the center manifolds, so the restricted one-dimensional maps on the respective center manifolds have also $\mathcal{O}(h^p)$-closeness.)

In our previous work [4] we have shown that the conjugacy problem can be reduced to the conjugacy of two normal forms depending on the stepsize h and the bifurcation parameter α. Let us denote by

$$\mathcal{N}_\Phi(h,x,\alpha) := h\alpha + x + hx^2 + hx^3 \cdot \widehat{\eta}_3(h,x,\alpha)$$

the normal form of the map $\Phi(h,\cdot,\alpha)$ computed in Lemma 2.2 in [4], and let

$$\mathcal{N}_\varphi(h,x,\alpha) := h\alpha + x + hx^2 + hx^3 \cdot \widetilde{\eta}_3(h,x,\alpha)$$

denote the normal form of the discretization map $\varphi(h,\cdot,\alpha)$ in Theorem 2.5 in [4]. These normal forms correspond to the case when there are two equilibria for $\alpha < 0$ and there is no equilibrium for $\alpha > 0$. The converse case can, of course, be treated symmetrically.

Notice that [3] worked with slightly different normal forms having also an additional factor $a > 0$ in the coefficient of the quadratic term. We have shown in [4] that this coefficient can be chosen as $a = 1$, which will be vital for our later arguments in Section 5.

Theorem 2.5 in [4] implies the following important inequality. There exists a constant $c > 0$ such that for any $h \in (0, h_0]$, $x \in [-\varepsilon_0, \varepsilon_0]$ and $\alpha \in [-\alpha_0, \alpha_0]$ (with $h_0 > 0$, $\varepsilon_0 > 0$ and $\alpha_0 > 0$ being sufficiently small) we have that

$$|\mathcal{N}_\Phi(h,x,\alpha) - \mathcal{N}_\varphi(h,x,\alpha)| \leq c \cdot h^{p+1}|x|^3. \tag{1}$$

In what follows, the original ODE and its discretization will not be present explicitly. All later closeness estimates of $|J(h,x,\alpha) - x|$ will be derived as consequences of (1).

Therefore, our task is to construct a homeomorphism $J(h,\cdot,\alpha)$ in a small neighbourhood of the origin with $h \in (0, h_0]$ and $\alpha \in [-\alpha_0, \alpha_0]$ as parameters such that $J(h,\cdot,\alpha)$ solves the conjugacy equation

$$\mathcal{N}_\Phi(h, J(h,x,\alpha), \alpha) = J(h, \mathcal{N}_\varphi(h,x,\alpha), \alpha), \tag{2}$$

and also to estimate the quantity

$$\sup_{h\in(0,h_0]} \sup_{|x|\leq\varepsilon_0} \sup_{|\alpha|\leq\alpha_0} \frac{|\, x - J(h,x,\alpha)|}{h^p}.$$

This clearly shows that our investigations belong to the quantitative theory of functional equations as well. (They can be considered as generalizations of the one-dimensional Grobman-Hartman Lemma established by [5]. For a general treatment of functional equations, we refer, e.g., to [6]. For general conjugacy results, including the Grobman-Hartman Lemma and structural stability, see, e.g., [7].)

Let us now briefly summarize the content of each section. In Section 2, the conjugacy is defined for $\alpha \leq 0$, $h > 0$, $x \in [-\varepsilon_0, \varepsilon_0]$, using forward iteration of the normal forms for $x \leq 0$ (where the branch of attractive fixed points is located) and inverse iteration of the normal forms for $x \geq 0$ (containing the repelling fixed points). Section 3 develops the uniform $\mathcal{O}(h^p)$-estimates for $\alpha \leq 0$, moreover, it shows that even a better closeness estimate holds in the region between the fixed points, and that the given estimates at the fixed points are optimal.

Section 4 analyzes the growth rate of iterates of the normal forms for $\alpha > 0$, while in Section 5 we construct a conjugacy J, then prove a singular $\mathcal{O}(h^p \cdot \ln \frac{1}{\alpha})$ logarithmic estimate of the distance between the map J and the identity. A uniform $\mathcal{O}(h^p)$-estimate under an additional assumption on the distance of the normal forms is also proved: if $|x|^3$ in (1) is replaced by $|x|^4$, then uniform closeness holds. Another uniform $\mathcal{O}(h^p)$-closeness estimate without any additional assumption is proved on a shrinking parabola-shaped domain in the (α, x)-plane. In conjecturing sharper versions of some inequalities, and even in proving Lemma 4.8, the computer program *Mathematica* has extensively been used.

In the last section, some comments and open questions are collected concerning the fixed-point-free $\alpha > 0$ case, where estimates turned out to be surprisingly harder than their $\alpha \leq 0$ counterparts. We do believe that the current logarithmic estimate is sharp—at least for our natural construction of J, as demonstrated in Sections 6.2–6.3 by some convincing numerical tests arranged in tabular forms and plots.

Sections 2–3 and Sections 4–5 are basically independent of each other—however their basic theme is the same: after the conjugacy J has been defined recursively, the (convergence) speed of sequences generated by the iterates of normal forms is analyzed to prove the closeness estimates.

A notable feature of our estimates is that they are fairly *explicit*, meaning that h_0, ε_0, α_0 and the closeness estimates themselves are all expressed in terms of p, c and K, where $p \geq 1$ is the order of the discretization method, $c > 0$ is the constant in (1), and $K > 0$ is a common uniform bound of the moduli of the functions $x \mapsto \widehat{\eta}_3(h, x, \alpha)$ and $x \mapsto \widetilde{\eta}_3(h, x, \alpha)$ together with their first and second derivatives on $[0, h_0] \times [-\varepsilon_0, \varepsilon_0] \times [-\alpha_0, \alpha_0]$. In other words, we specify how large the domain of definition of J and the coefficients of the h^p terms in the closeness estimates are.

As for future work, there are at least four directions planned to be examined or elaborated further. Most importantly, instead of a one-dimensional setting, the fold bifurcation problem can be considered in N dimension and center manifold reduction applied afterwards. Secondly, it is yet to be decided whether the logarithmic estimate in the $\alpha > 0$ case can be improved by possibly modifying the construction of J. Thirdly, in the current construction, continuity of the mapping $\alpha \mapsto J(h, x, \alpha)$ at $\alpha = 0$ and $x \geq 0$ is still an unresolved question of interest. Finally, other simple types of bifurcations are also intended to be treated.

Concluding this introduction, some more notation is introduced. The superscript ^{E} will denote *function evaluation at h and α* for functions from $\mathbb{R}^3$ to $\mathbb{R}$, that is, for example, J^E stands for the function $J(h, \cdot, \alpha)$. The range of parameters h and α will be clear from the context. The symbol $f^{[-1]}$ means the *inverse* of a real function f. Similarly, $f^{[k]}$ is the k^{th} *iterate* ($k \in \mathbb{Z}$) of $f : \mathbb{R} \to \mathbb{R}$. The symbol *id* denotes the identity function on $\mathbb{R}$. Symbols $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$, as usual, denote the *floor* and *ceiling* functions, that is the greatest integer and least integer functions, respectively (for a reference and origin of their usage, see, e.g., the online encyclopedia at `http://mathworld.wolfram.com/IntegerPart.html`). The set of nonnegative integers is denoted by $\mathbb{N}$. $\#A$ will denote the number of elements of the (finite) set A. Finally, for any $a, b \in \mathbb{R}$, the symbol $[\{a, b\}]$ represents the closed *interval between* the elements of the set $\{a, b\}$, that is $[\{a, b\}] := [\min(a, b), \max(a, b)]$.

2 Construction of the conjugacy in the $\alpha \le 0$ case

Consider first the case $\alpha < 0$. Let us denote the negative fixed point of $\mathcal{N}^E_\varphi$ and $\mathcal{N}^E_\Phi$ near the origin by $\omega_{\varphi,-} \equiv \omega_{\varphi,-}(h,\alpha)$ and $\omega_{\Phi,-} \equiv \omega_{\Phi,-}(h,\alpha)$, respectively.

Lemma 2.1 *For every $0 < h \le h_0$ and $-\alpha_0 \le \alpha < 0$ we have that*

$$-\sqrt{2}\sqrt{|\alpha|} \le \omega_{\varphi,-} \le -\sqrt{\frac{2}{3}}\sqrt{|\alpha|}$$

and

$$-\sqrt{2}\sqrt{|\alpha|} \le \omega_{\Phi,-} \le -\sqrt{\frac{2}{3}}\sqrt{|\alpha|},$$

provided that $\alpha_0 \le \frac{1}{8K^2}$.

Proof. By definition, $\omega_{\varphi,-} < 0$ solves $\alpha + x^2 + x^3 \cdot \widetilde{\eta}_3(h,x,\alpha) = 0$. Since if $|x| \le \frac{1}{2K}$, then $|x^3 \widetilde{\eta}_3| \le \frac{1}{2}x^2$ and hence

$$\alpha + \frac{x^2}{2} \le \alpha + x^2 + x^3 \cdot \widetilde{\eta}_3(h,x,\alpha) \le \alpha + \frac{3x^2}{2}$$

holds, we get the desired estimates provided that $\sqrt{2}\sqrt{|\alpha|} \le \frac{1}{2K}$, which is true if $|\alpha| \le \frac{1}{8K^2}$. The proof for $\omega_{\Phi,-}$ is similar. ■

By iterating one of the normal forms, let us define two sequences x_k and y_k. Let $x_k \equiv x_k(h,\alpha)$ be defined as

$$x_{k+1} := \mathcal{N}_\varphi(h, x_k, \alpha), \quad k = 0, 1, 2, \ldots$$

with $x_0 := 0$, further let $y_k \equiv y_k(h,\alpha)$ be defined as

$$y_{k+1} := \mathcal{N}_\varphi(h, y_k, \alpha), \quad k = 0, 1, 2, \ldots \tag{3}$$

with $y_0 < \omega_{\varphi,-}$, being independent of both h and α, and $|y_0|$ being chosen appropriately, see below. Note that y_0 is a negative number.

Since, by Lemma 2.1, if h and $|\alpha|$ are sufficiently small, $0 < (\mathcal{N}^E_\varphi)'(\omega_{\varphi,-}) < 1$ holds, the fixed point $\omega_{\varphi,-}$ is attracting, hence $\lim_{k\to\infty} x_k(h,\alpha) = \lim_{k\to\infty} y_k(h,\alpha) = \omega_{\varphi,-}$. Moreover, a simple calculation shows that $y_0 < y_1(h,\alpha)$ can also be achieved, for example, $-\frac{1}{4K} \le y_0 \le -2\sqrt{\alpha_0}$ suffices, hence it follows by induction that the sequence y_k is monotone increasing. Similarly, it can be assumed that the sequence x_k is monotone decreasing.

We remark that suitable values of h_0, α_0 and y_0 have been built into the conditions of the following lemmas and theorems corresponding to the $\alpha \le 0$ case.

The following figure shows the branch of stable and unstable fixed points of $\mathcal{N}^E_\varphi$ in the (α, x)-plane together with the first few terms of the inner sequence $x_k(h,\alpha)$ and the outer sequence $y_k(h,\alpha)$ with some $h > 0$ and $\alpha < 0$ fixed. The arrows point toward terms of the sequences with larger k indices.

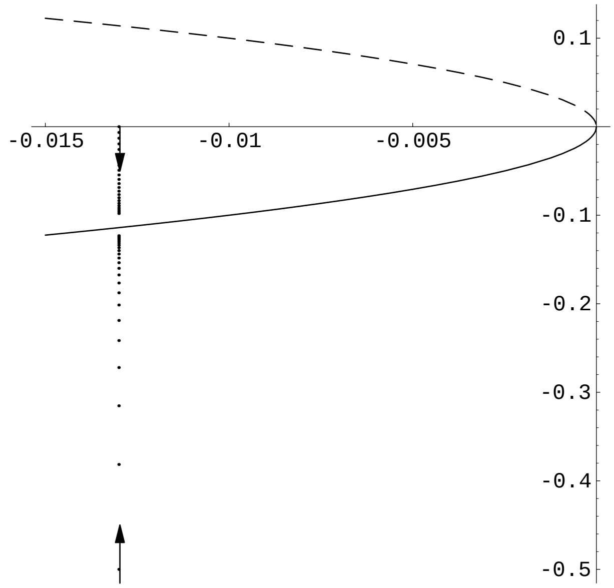

Figure 2.1

The intervals $[x_{k+1}, x_k]$ and $[y_k, y_{k+1}]$ $(k \in \mathbb{N})$ constitute the so-called *fundamental domains* on which the homeomorphism J^E is now piecewise defined.

Fix $0 < h \leq h_0$ and $-\alpha_0 \leq \alpha < 0$ arbitrarily.

Let $J^E(x) := x$ for $x \in [x_1, x_0] \equiv [h\alpha, 0]$. For $n > 1$, set

$$J^E(x_n) := \left(\mathcal{N}_\Phi^E\right)^{[n]}(x_0),$$

and recursively, for $n > 1$ and for $x \in (x_n, x_{n-1})$, let

$$J^E(x) := \left(\mathcal{N}_\Phi^E \circ J^E \circ \left(\mathcal{N}_\varphi^E\right)^{[-1]}\right)(x). \tag{4}$$

Here the right hand side has already been defined by the recursion. Finally, set

$$J^E(\omega_{\varphi,-}) := \omega_{\Phi,-}.$$

Then J^E is continuous, strictly monotone increasing on $[\omega_{\varphi,-}, 0]$, as it is a composition of three such functions, and satisfies (2).

Fix $-\frac{1}{4K} \leq y_0 \leq -2\sqrt{\alpha_0}$ as well. Let $J^E(y_0) := y_0$, and for $n > 1$, set

$$J^E(y_n) := \left(\mathcal{N}_\Phi^E\right)^{[n]}(y_0).$$

On the interval $[y_0, y_1]$, extend J^E linearly. Recursively, for $n > 1$ and for $y \in (y_{n-1}, y_n)$, set

$$J^E(y) := \left(\mathcal{N}_\Phi^E \circ J^E \circ \left(\mathcal{N}_\varphi^E\right)^{[-1]}\right)(y).$$

Then J^E is continuous, strictly monotone increasing on $[y_0, \omega_{\varphi,-}]$ and satisfies (2).

The same construction is carried out for $\alpha = 0$. This time, however, only the sequence y_k is needed, since the two fixed points merge then disappear as α passes through 0^-. Of course, $J(h, 0, 0) := 0$.

Currently, the construction is halfway ready—the function J has been defined on $(0, h_0] \times [-|y_0|, 0] \times [-\alpha_0, 0]$ so far.

On $(0, h_0] \times [0, |y_0|] \times [-\alpha_0, 0]$, that is in the region of *repelling* fixed points, the *inverses* of the normal forms are iterated. For any $0 < h \le h_0$ and $-\alpha_0 \le \alpha < 0$, set $\widetilde{x}_0 := x_0 = 0$ and for $k = 1, 2, \ldots$

$$\widetilde{x}_k \equiv \widetilde{x}_k(h, \alpha) := \left(\mathcal{N}_\varphi^E\right)^{[-k]}(\widetilde{x}_0),$$

further for any $0 < h \le h_0$ and $-\alpha_0 \le \alpha \le 0$, let $\widetilde{y}_0 := |y_0|$ and for $k = 1, 2, \ldots$

$$\widetilde{y}_k \equiv \widetilde{y}_k(h, \alpha) := \left(\mathcal{N}_\varphi^E\right)^{[-k]}(\widetilde{y}_0).$$

Then for $\alpha < 0$, the monotone increasing sequence $\widetilde{x}_k$ tends to $\omega_{\varphi,+}$, while the monotone decreasing sequence $\widetilde{y}_k$ converges to $\omega_{\Phi,+}$, where $\omega_{\varphi,+}$ and $\omega_{\Phi,+}$ denote the *positive* fixed points of $\mathcal{N}_\varphi^E$ and $\mathcal{N}_\Phi^E$, respectively.

The construction for J^E is analogous: for example, we set

$$J^E(\widetilde{x}_n) := \left(\mathcal{N}_\Phi^E\right)^{[-n]}(\widetilde{x}_0) \quad \text{and} \quad J^E(\widetilde{y}_n) := \left(\mathcal{N}_\Phi^E\right)^{[-n]}(\widetilde{y}_0),$$

but now the relation $J^E = \left(\mathcal{N}_\Phi^E\right)^{[-1]} \circ J^E \circ \mathcal{N}_\varphi^E$ is used in the recursive extensions.

Remark 2.1 Notice that our construction is more direct than the one in [3], since the intermediate pure quadratic function $g(x, \alpha)$ as well as the two auxiliary homeomorphisms H and G in [3] are eliminated.

3 The closeness estimate for the conjugacy in the $\alpha \le 0$ case

3.1 The inner region

We now prove that the constructed conjugacy J^E is $\mathcal{O}(h^p)$-close to the identity on the interval $[\omega_{\varphi,-}, 0]$ uniformly for any $h \in (0, h_0]$ and $\alpha \in [-\alpha_0, 0)$.

We mention that $\mathcal{O}(h^{p-1})$-closeness could be proved easily by arguing as [3] with estimates formulated in terms of the sequence x_k itself. Nevertheless, it turns out that restoring this lost order is possible by a different subdivision of $[\omega_{\varphi,-}, 0]$, established by the following preparatory lemma. The sequence s_n defined below successfully bridges the gap between two different orders of magnitude: it connects the "micro" level $\mathcal{O}(h\alpha)$ with the "meso" level $\mathcal{O}(\sqrt{|\alpha|})$. The meso-level and the "macro" level $\mathcal{O}(1)$ will be connected by the sequence y_k in Section 3.2.

Lemma 3.1 *Let $h_0 \le \frac{1}{16}$ and $\sqrt{\alpha_0} \le \min\left(\frac{1}{2}, \frac{1}{\sqrt{8K}}\right)$. For every $h \in (0, h_0]$ and $\alpha \in [-\alpha_0, 0)$, define*

$$m \equiv m(h) := \lfloor \log_2 \log_2 \frac{1}{h} \rfloor$$

and for $1 \le n \le m$

$$s_n \equiv s_n(h, \alpha) := -\sqrt[2^n]{h}\sqrt{|\alpha|},$$

further let $s_0 := h\alpha \equiv x_1$. *Then* $m \geq 2$,

$$\omega_{\varphi,-} < -\frac{\sqrt{|\alpha|}}{2} \leq s_m \leq -\frac{\sqrt{|\alpha|}}{4}$$

and for any $1 \leq n < m$ *we have that*

$$\omega_{\varphi,-} < \mathcal{N}_\varphi^E(s_{n+1}) < s_{n+1} < \mathcal{N}_\varphi^E(s_n) < s_n < \ldots < \mathcal{N}_\varphi^E(s_1) < s_1 < \mathcal{N}_\varphi^E(s_0) < s_0 < 0.$$

Proof. It is seen that $\frac{1}{2} \geq h^{2^{-m}} \geq \frac{1}{4}$ is equivalent to $0 \leq -m + \log_2 \log_2 \frac{1}{h} \leq 1$, which is always satisfied due to the definition of $m(h)$. $\sqrt{\alpha_0}$ has been chosen so small that Lemma 2.1 can be applied, hence $\omega_{\varphi,-} \leq -\sqrt{\frac{2}{3}}\sqrt{|\alpha|} < -\frac{1}{2}\sqrt{|\alpha|}$. Also notice that (due to the definition of $\omega_{\varphi,-}$, $\mathcal{N}_\varphi^E(0) < 0$ and continuity) $\omega_{\varphi,-} < \mathcal{N}_\varphi^E(x) < x$ holds for $x \in (\omega_{\varphi,-}, 0]$. It is easy to see that $h\alpha + x \leq \mathcal{N}_\varphi^E(x) < 0$, if $|x| \leq \frac{1}{K}$. The already shown inequality $-\frac{\sqrt{|\alpha|}}{2} \leq s_n$ $(1 \leq n \leq m)$ and condition $\sqrt{|\alpha|} \leq \frac{1}{\sqrt{8K}}$ imply $|s_n| \leq \frac{1}{K}$, hence it is sufficient to prove that $s_{n+1} < h\alpha + s_n$ holds for $1 \leq n < m$. (The case $n = 0$ can be verified directly.) But this is equivalent to $h^{2^{-n-1}} + h \cdot h^{-(2^{-n-1})}\sqrt{|\alpha|} < 1$. The second term is strictly less than $\frac{1}{2}$, hence $h^{2^{-n-1}} \leq \frac{1}{2}$ remains to be shown. However, this reduces to $\log_2 \log_2 \frac{1}{h} \geq n + 1$, which is true, since $m > n$. ■

Now the desired closeness is shown to hold on each of the subintervals defined above. However, since the number of these subintervals tends to infinity as $h \to 0^+$, the constants on the right hand sides of the estimates should be controlled carefully. Thus, instead of a generic positive constant *const*, the symbol $c > 0$ being the same as in (1) with *fixed value* is used throughout the proof.

Lemma 3.2 *Suppose that* $h_0 \leq \min\left(\frac{1}{16}, \sqrt[p]{\frac{1}{8c}}\right)$ *and* $\sqrt{\alpha_0} \leq \min\left(\frac{1}{2}, \frac{1}{19K}\right)$. *Then using the notations of the previous lemma, for every* $h \in (0, h_0]$, $\alpha \in [-\alpha_0, 0)$ *and* $0 \leq n < m$ *we have the following estimates:*

$$\sup_{[s_0,0]} |\, id - J^E| = 0, \tag{5}$$

$$\sup_{[\mathcal{N}_\varphi^E(s_0), s_0]} |\, id - J^E| \leq c \cdot h^{p+4}|\alpha|^3, \tag{6}$$

$$\sup_{[\mathcal{N}_\varphi^E(s_{n+1}), \mathcal{N}_\varphi^E(s_n)]} |\, id - J^E| \leq c \cdot h^{p+2^{-n-1}}\sqrt{|\alpha|}, \tag{7}$$

$$\sup_{[\omega_{\varphi,-}, \mathcal{N}_\varphi^E(s_m)]} |\, id - J^E| \leq 12c \cdot h^p \sqrt{|\alpha|}. \tag{8}$$

Proof. Step 1. h_0 and α_0 have been chosen such that $\max(|\,\omega_{\varphi,-}|, |\,\omega_{\Phi,-}|) \leq \min\left(1, \frac{1}{13K}\right)$ (see Lemma 2.1), which implies that $0 < (\mathcal{N}_\Phi^E)' \leq 1 + h \cdot id$ and $(\mathcal{N}_\Phi^E)'$ is monotone increasing (due to $0 < (\mathcal{N}_\Phi^E)''$) on $[\omega_{\varphi,-}, 0] \cup [\omega_{\Phi,-}, 0]$. So the above estimates can be evaluated at any $x \in [\omega_{\varphi,-}, 0]$ and therefore at any $J^E(x) \in [\omega_{\Phi,-}, 0]$. (We remind that, by construction, J^E maps the interval $[\omega_{\varphi,-}, 0]$ onto $[\omega_{\Phi,-}, 0]$.) Hence for any $x \in [\omega_{\varphi,-}, 0]$

$$\sup_{[\{x, J^E(x)\}]} (\mathcal{N}_\Phi^E)' \leq 1 + h \cdot \max(x, J^E(x)) \tag{9}$$

holds. (With the $[\{\cdot,\cdot\}]$ notation introduced earlier, both cases $x \le J^E(x)$ and $J^E(x) < x$ can be treated simultaneously.) Taking also into account that J^E is strictly monotone increasing by construction, further inequality (1) and definition (4), we get for any $\omega_{\varphi,-} \le a < b \le 0$ that

$$\sup_{[\mathcal{N}_\varphi^E(a),\mathcal{N}_\varphi^E(b)]} |\, id - J^E| = \sup_{[\mathcal{N}_\varphi^E(a),\mathcal{N}_\varphi^E(b)]} \left| \mathcal{N}_\varphi^E \circ (\mathcal{N}_\varphi^E)^{[-1]} - \mathcal{N}_\Phi^E \circ J^E \circ (\mathcal{N}_\varphi^E)^{[-1]} \right| \le$$

$$\le \sup_{[a,b]} \left| \mathcal{N}_\varphi^E - \mathcal{N}_\Phi^E \right| + \sup_{[a,b]} \left| \mathcal{N}_\Phi^E - \mathcal{N}_\Phi^E \circ J^E \right| \le$$

$$\le c \cdot h^{p+1} |a|^3 + \sup_{x\in[a,b]} \left(\left(\sup_{[\{x,J^E(x)\}]} (\mathcal{N}_\Phi^E)' \right) |x - J^E(x)| \right) \le$$

$$\le c \cdot h^{p+1} |a|^3 + \left(1 + h \cdot \max\left(b, J^E(b)\right)\right) \sup_{[a,b]} |\, id - J^E|. \tag{10}$$

Step 2. $\sup_{[s_0,0]} |\, id - J^E| = \sup_{[x_1,x_0]} |\, id - J^E| = 0$, since $J^E = id$ on $[x_1, x_0]$ by construction.

Step 3. By (10) and the previous step,

$$\sup_{[\mathcal{N}_\varphi^E(s_0),s_0]} |\, id - J^E| = \sup_{[\mathcal{N}_\varphi^E(x_1),\mathcal{N}_\varphi^E(x_0)]} |\, id - J^E| \le$$

$$\le c \cdot h^{p+1} |x_1|^3 + \left(1 + h \cdot \max\left(x_0, J^E(x_0)\right)\right) \sup_{[x_1,x_0]} |\, id - J^E| = c \cdot h^{p+4} |\alpha|^3.$$

Step 4. By (10), we have that

$$\sup_{[\mathcal{N}_\varphi^E(s_1),\mathcal{N}_\varphi^E(s_0)]} |\, id - J^E| \le c \cdot h^{p+1} |s_1|^3 + \left(1 + h \cdot \max\left(s_0, J^E(s_0)\right)\right) \sup_{[s_1,s_0]} |\, id - J^E|. \tag{11}$$

Here $J^E(s_0) = J^E(x_1) = -h|\alpha| = s_0$, $|s_1|^3 = h^{\frac{3}{2}} |\alpha|^{\frac{3}{2}}$ and narrowing the interval in the supremum on the left hand side yields that

$$\sup_{[s_1,\mathcal{N}_\varphi^E(s_0)]} |\, id - J^E| \le$$

$$\le c \cdot h^{p+1+\frac{3}{2}} |\alpha|^{\frac{3}{2}} + \left(1 - h^2|\alpha|\right) \max \left(\sup_{[s_1,\mathcal{N}_\varphi^E(s_0)]} |\, id - J^E|, \sup_{[\mathcal{N}_\varphi^E(s_0),s_0]} |\, id - J^E| \right).$$

If the maximum is attained on the second term, the estimate from Step 3 is used (together with $h \le 1$ and $\sqrt{|\alpha|} \le \frac{1}{2}$), while if the maximum is attained on the first term, the resulting inequality is solved. In any case, we can establish that

$$\sup_{[s_1,\mathcal{N}_\varphi^E(s_0)]} |\, id - J^E| \le c \cdot h^{p+\frac{1}{2}} \sqrt{|\alpha|}, \tag{12}$$

with the same c as before. Now, turning to (11) again, but this time also using Step 3 and (12), we get that

$$\sup_{[\mathcal{N}_\varphi^E(s_1),\mathcal{N}_\varphi^E(s_0)]} |\, id - J^E| \le$$

$$\le c \cdot h^{p+2+\frac{1}{2}} |\alpha|^{\frac{3}{2}} + \left(1 - h^2|\alpha|\right) \max \left(c \cdot h^{p+\frac{1}{2}} \sqrt{|\alpha|}, c \cdot h^{p+4} |\alpha|^3 \right).$$

Again, it is easy to see that in any case the right hand side can not be greater than $c \cdot h^{p+\frac{1}{2}} \sqrt{|\alpha|}$.

Step 5. Repeating inductively, we get for $1 \le n < m$ that

$$\sup_{[\mathcal{N}_\varphi^E(s_n),\mathcal{N}_\varphi^E(s_{n-1})]} |\, id - J^E| \le c \cdot h^{p+2^{-n}} \sqrt{|\alpha|}.$$

By (10),

$$\sup_{[\mathcal{N}_\varphi^E(s_{n+1}),\mathcal{N}_\varphi^E(s_n)]} |\, id - J^E| \le$$

$$\le c \cdot h^{p+1} |s_{n+1}|^3 + \big(1 + h \cdot \max\big(s_n, J^E(s_n)\big)\big) \sup_{[s_{n+1},s_n]} |\, id - J^E|. \tag{13}$$

Here $|s_{n+1}|^3 = h^{3 \cdot 2^{-n-1}} |\alpha|^{\frac{3}{2}}$. Further, since $s_n \in [\mathcal{N}_\varphi^E(s_n), \mathcal{N}_\varphi^E(s_{n-1})]$, by the induction hypothesis we have that

$$J^E(s_n) - s_n \le |J^E(s_n) - s_n| \le c \cdot h^{p+2^{-n}} \sqrt{|\alpha|},$$

from which it is easy to deduce that

$$J^E(s_n) \le -\frac{\sqrt{|\alpha|}}{2} h^{2^{-n}}$$

using that $h^p \le \frac{1}{8c} \le \frac{1}{2c}$ by assumption. Obviously, $s_n \le -\frac{\sqrt{|\alpha|}}{2} h^{2^{-n}}$ holds as well. So (13) yields that

$$\sup_{[\mathcal{N}_\varphi(s_{n+1}),\mathcal{N}_\varphi^E(s_n)]} |\, id - J^E| \le c \cdot h^{p+1+3 \cdot 2^{-n-1}} |\alpha|^{\frac{3}{2}} +$$

$$+ \left(1 - \frac{\sqrt{|\alpha|}}{2} h^{1+2^{-n}}\right) \max\left(\sup_{[s_{n+1},\mathcal{N}_\varphi^E(s_n)]} |\, id - J^E|, \sup_{[\mathcal{N}_\varphi^E(s_n),s_n]} |\, id - J^E| \right). \tag{14}$$

Clearly, the supremum on the left hand side is not increased if it is taken only on $[s_{n+1}, \mathcal{N}_\varphi^E(s_n)]$. Evaluating the first case in the maximum we have that

$$\sup_{[s_{n+1},\mathcal{N}_\varphi^E(s_n)]} |\, id - J^E| \le \frac{c \cdot h^{p+1+3 \cdot 2^{-n-1}} |\alpha|^{\frac{3}{2}}}{\frac{\sqrt{|\alpha|}}{2} h^{1+2^{-n}}} = 2c \cdot h^{p+2^{-n-1}} |\alpha| \le c \cdot h^{p+2^{-n-1}} \sqrt{|\alpha|},$$

since $\sqrt{|\alpha|} \le \frac{1}{2}$, and similarly, evaluating the second case in the maximum on the right hand side of (14) (and using the induction hypothesis also) yields the same, since for $1 \le n < m$

$$c \cdot h^{p+1+3 \cdot 2^{-n-1}} |\alpha|^{\frac{3}{2}} + c \cdot h^{p+2^{-n}} \sqrt{|\alpha|} \le c \cdot h^{p+2^{-n-1}} \sqrt{|\alpha|}. \tag{15}$$

Therefore, we have shown that

$$\sup_{[s_{n+1},\mathcal{N}_\varphi^E(s_n)]} |\, id - J^E| \le c \cdot h^{p+2^{-n-1}} \sqrt{|\alpha|}.$$

Now with this additional information substituted back into the right hand side of (14) together with the induction hypothesis, we see as in (15) that

$$\sup_{[\mathcal{N}_\varphi^E(s_{n+1}),\mathcal{N}_\varphi^E(s_n)]} |\, id - J^E| \le c \cdot h^{p+2^{-n-1}} \sqrt{|\alpha|}.$$

The induction is complete.

Step 6. Finally, by using (10) we get that

$$\sup_{[\omega_{\varphi,-},\mathcal{N}_\varphi^E(s_m)]}|\,id-J^E|\leq c\cdot h^{p+1}|\omega_{\varphi,-}|^3+\left(1+h\cdot\max\left(s_m,J^E\left(s_m\right)\right)\right)\sup_{[\omega_{\varphi,-},s_m]}|\,id-J^E|,$$

since $\omega_{\varphi,-}$ is a fixed point of $\mathcal{N}_\varphi^E$. Now we use inequality $|\,\omega_{\varphi,-}|\leq\sqrt{2}\sqrt{|\alpha|}$ from Lemma 2.1, further inequality $\mathcal{N}_\varphi^E(s_m)\leq s_m\leq-\frac{\sqrt{|\alpha|}}{4}$ from Lemma 3.1 and (7) with $n=m-1$ together with the assumption $h^p\leq\frac{1}{8c}$ to obtain $J^E\left(s_m\right)\leq-\frac{\sqrt{|\alpha|}}{4}+c\cdot h^p\sqrt{|\alpha|}\leq-\frac{\sqrt{|\alpha|}}{8}$ and $s_m\leq-\frac{\sqrt{|\alpha|}}{8}$, finally the decomposition $[\omega_{\varphi,-},s_m]=[\omega_{\varphi,-},\mathcal{N}_\varphi^E(s_m)]\cup[\mathcal{N}_\varphi^E(s_m),s_m]$ in the supremum on the right hand side to point out in the first case that

$$\sup_{[\omega_{\varphi,-},\mathcal{N}_\varphi^E(s_m)]}|\,id-J^E|\leq\frac{\sqrt{8}c\cdot h^{p+1}|\alpha|^{\frac{3}{2}}}{h\frac{\sqrt{|\alpha|}}{8}}=16\sqrt{2}c\cdot h^p|\alpha|\leq 8\sqrt{2}c\cdot h^p\sqrt{|\alpha|},$$

while in the second case—using (7) again—that

$$\sup_{[\omega_{\varphi,-},\mathcal{N}_\varphi^E(s_m)]}|\,id-J^E|\leq 2c\cdot h^p\sqrt{|\alpha|}.$$

Now the proof of the lemma is complete. ∎

Remark 3.1 At $\omega_{\varphi,-}$, we can obtain a slightly better estimate in terms of α. Namely, we have

$$|\,id-J^E|(\omega_{\varphi,-})=|\,\omega_{\varphi,-}-\omega_{\Phi,-}|\leq c\cdot h^{p+1}|\omega_{\varphi,-}|^3+\left(\sup_{[\{\omega_{\varphi,-},\omega_{\Phi,-}\}]}(\mathcal{N}_\Phi^E)'\right)|\,\omega_{\varphi,-}-\omega_{\Phi,-}|,$$

which—since the positive supremum is at most $1-\frac{h}{2}\sqrt{|\alpha|}$ together with Lemma 2.1—implies that

$$|\,\omega_{\varphi,-}-\omega_{\Phi,-}|\leq 2c\cdot h^p\frac{|\omega_{\varphi,-}|^3}{\sqrt{|\alpha|}}\leq 4\sqrt{2}c\cdot h^p|\alpha|.$$

Remark 3.2 on optimality. The following explicit example illustrates that the distance of fixed points of functions satisfying (1) may be bounded from *below* by $\mathcal{O}(h^p)$ $(h\to 0)$. Hence the fact that fixed points must be mapped into nearby fixed points by the conjugacy J^E implies that better estimates than $\mathcal{O}(h^p)$ of $|\,id-J^E|$ generally can not be expected.

Indeed, set $\widehat{\eta}_3(h,x,\alpha):=0$ and $\widetilde{\eta}_3(h,x,\alpha):=h^p\cdot x$. Then $\mathcal{N}_\Phi(h,x,\alpha)=h\alpha+x+hx^2$ and $\mathcal{N}_\varphi(h,x,\alpha)=h\alpha+x+hx^2+h^{p+1}x^4$ satisfy (1) in a neighbourhood of the origin, further, $\omega_{\Phi,-}=-\sqrt{|\alpha|}$ and $\omega_{\varphi,-}=-\sqrt{\frac{1}{2h^p}\left(\sqrt{1+4h^p|\alpha|}-1\right)}$. Using inequality $1+\frac{t}{2}-\frac{t^2}{8}\leq\sqrt{1+t}\leq 1+\frac{t}{2}-\frac{t^2}{8}+\frac{t^3}{16}$ for $0\leq t\leq 1$, one can show that

$$|\,\omega_{\varphi,-}-\omega_{\Phi,-}|\geq h^p\left(\frac{1}{2}|\alpha|^{\frac{3}{2}}-h^p|\alpha|^{\frac{5}{2}}\right)$$

holds, if, for example, $h\leq 1$ and $\sqrt{|\alpha|}\leq\frac{1}{2}$.

3.2 The outer region

In the following lemma—also interesting in itself—we first estimate the growth of iterates of the normal form $\mathcal{N}_{\varphi}^{E}$, *i.e.* the convergence speed of $y_k(h,\alpha)$.

Lemma 3.3 *Suppose that the positive numbers h_0 and α_0 are small enough, further $|y_0|$ has been chosen appropriately. Then for $-\alpha_0 \le \alpha \le 0$, $0 < h \le h_0$ and $k \ge 0$ we have that*

$$-|y_0| \le y_k(h,\alpha) \le 0,$$

while for $-\alpha_0 \le \alpha \le 0$, $0 < h \le h_0$ and $k \ge \lfloor \frac{1}{h} \rfloor + 1$ we have that

$$-\sqrt{2|\alpha|} - \frac{2}{kh} \le y_k(h,\alpha) < 0. \tag{16}$$

Proof. The first estimate follows from the fact that the sequence y_k is monotone increasing, as we have seen (for example, the condition $-\frac{1}{4K} \le y_0 \le -2\sqrt{\alpha_0}$ guarantees this).

As for the second estimate, we prove by induction on k. For $k = \lfloor \frac{1}{h} \rfloor + 1$, $kh \le 2$ holds if h is small enough, hence $-\sqrt{2|\alpha|} - \frac{2}{kh} \le -1 \le y_0 \le y_k < 0$, if $|y_0|$ is small enough.

For $k > \lfloor \frac{1}{h} \rfloor + 1$, we have that $y_{k+1} = \mathcal{N}_{\varphi}(h, y_k, \alpha) \ge h\alpha + y_k + \frac{h}{2} y_k^2$, if $|y_k|$ is small enough (for example, if $|y_k| \le |y_0| \le \frac{1}{2K}$). Hence it is sufficient to prove

$$h\alpha + y_k + \frac{h}{2} y_k^2 \ge -\sqrt{2|\alpha|} - \frac{2}{(k+1)h}. \tag{17}$$

To this end, notice that the function $x \mapsto h\alpha + x + \frac{h}{2}x^2$ is monotone increasing provided that $x > -\frac{1}{h}$. It is easy to see that $-\sqrt{2|\alpha|} - \frac{2}{kh} > -\frac{1}{h}$, if $k > \lfloor \frac{1}{h} \rfloor + 1$, further h and $|\alpha|$ are small enough. Then, by the induction hypothesis, we see that

$$h\alpha + \left(-\sqrt{2|\alpha|} - \frac{2}{kh}\right) + \frac{h}{2}\left(-\sqrt{2|\alpha|} - \frac{2}{kh}\right)^2 \ge -\sqrt{2|\alpha|} - \frac{2}{(k+1)h} \tag{18}$$

implies (17). However, since now $|\alpha| = -\alpha$, (18) is equivalent to $hk\sqrt{2|\alpha|} \ge -\frac{1}{k+1}$. Therefore, the induction is complete. ■

Remark 3.3 The precise conditions for h_0, α_0 and $|y_0|$ are collected in the next lemma.

Remark 3.4 Estimate (16) has been devised by superimposing the following pieces of information: on one hand, the convergence speed of the sequence $y_k(h,0) = \mathcal{O}(\frac{1}{hk})$ $(k \to \infty)$ can be inferred from [8], while, on the other hand, we know from Lemma 2.1 that $\lim_{k\to\infty} y_k(h,\alpha) = \mathcal{O}(\sqrt{|\alpha|})$, if $-\alpha_0 \le \alpha < 0$ is small.

Our estimate of y_k is simpler and more explicit than the corresponding one in [3], in which a majorizing sequence z_k containing a fractional power of k is used. We will only need fractional powers in the finer analyses in Sections 4 and 5 for $\alpha > 0$.

Now it is proved that the conjugacy J^E is $\mathcal{O}(h^p)$-close to the identity on the interval $[-|y_0|, \omega_{\varphi,-})$ for any $h \in (0, h_0]$ and $\alpha \in [-\alpha_0, 0)$, as well as on the interval $[-|y_0|, 0]$ for any $h \in (0, h_0]$ when $\alpha = 0$.

Lemma 3.4 *Suppose that* $h_0 \le \frac{1}{5}$, $\sqrt{\alpha_0} \le \min\left(\frac{1}{2}, \frac{1}{26K}\right)$ *and* $\max\left(-1, -\frac{1}{13K}\right) \le y_0 \le -2\sqrt{\alpha_0}$. *Then for each* $h \in (0, h_0]$ *and* $\alpha \in [-\alpha_0, 0)$ *we have that*

$$\sup_{[y_0,\omega_{\varphi,-})} |\,id - J^E| \le c\left(3y_0^2 + 4\sqrt{\alpha_0} + \frac{4}{1-h_0} + 12\right)h^p, \tag{19}$$

and similarly, for $h \in (0, h_0]$ *and* $\alpha = 0$ *the estimate*

$$\sup_{[y_0,0]} |\,id - J^E| \le c\left(3y_0^2 + \frac{4}{1-h_0}\right)h^p \tag{20}$$

holds.

Proof. Step 1. The assumptions have been set up such that Lemma 2.1 and Lemma 3.3 are both applicable (hence $\omega_{\varphi,-} < -\frac{\sqrt{|\alpha|}}{2}$ holds for example, when $\alpha < 0$), further $0 < (\mathcal{N}_\Phi^E)' \le 1 + h \cdot id$ and $(\mathcal{N}_\Phi^E)'$ is monotone increasing on $[-|y_0|, 0]$.

Step 2a. Consider the case $\alpha \in [-\alpha_0, 0)$ first. It is clear that

$$\sup_{[y_0,\omega_{\varphi,-})} |\,id - J^E| = \sup_{n\in\mathbb{N}} \sup_{[y_n,y_{n+1}]} |\,id - J^E|. \tag{21}$$

Step 2b. Now since $J^E(y_0) = y_0$ and J^E is linear on $[y_0, y_1]$, we get that

$$\sup_{[y_0,y_1]} |\,id - J^E| = |\,y_1 - J^E(y_1)| = |\mathcal{N}_\varphi^E(y_0) - \mathcal{N}_\Phi^E(y_0)| \le c \cdot h^{p+1} y_0^2,$$

by a weaker form of (1).

Step 2c. For $n \ge 1$, similarly to (10), we obtain that

$$\sup_{[y_n,y_{n+1}]} |\,id - J^E| \le \sup_{[y_n,y_{n+1}]} \left|\mathcal{N}_\varphi^E \circ (\mathcal{N}_\varphi^E)^{[-1]} - \mathcal{N}_\Phi^E \circ (\mathcal{N}_\varphi^E)^{[-1]}\right| +$$

$$+ \sup_{[y_n,y_{n+1}]} \left|\mathcal{N}_\Phi^E \circ (\mathcal{N}_\varphi^E)^{[-1]} - \mathcal{N}_\Phi^E \circ J^E \circ (\mathcal{N}_\varphi^E)^{[-1]}\right| =$$

$$= \sup_{[y_{n-1},y_n]} \left|\mathcal{N}_\varphi^E - \mathcal{N}_\Phi^E\right| + \sup_{[y_{n-1},y_n]} \left|\mathcal{N}_\Phi^E - \mathcal{N}_\Phi^E \circ J^E\right| \le$$

$$\le \sup_{[y_{n-1},y_n]} \left|\mathcal{N}_\varphi^E - \mathcal{N}_\Phi^E\right| + \sup_{y\in[y_{n-1},y_n]} \left(\left(\sup_{[\{y,J^E(y)\}]} (\mathcal{N}_\Phi^E)'\right) |\,y - J^E(y)|\right) \le$$

$$\le c \cdot h^{p+1} y_{n-1}^2 + \left(1 - \frac{h}{2}\sqrt{|\alpha|}\right) \sup_{[y_{n-1},y_n]} |\,id - J^E|,$$

using the fact that for $y \in [y_0, \omega_{\varphi,-}]$ the inclusion $[\{y, J^E(y)\}] \subset [y_0, \omega_{\varphi,-}] \cup [y_0, \omega_{\Phi,-}]$ holds, further, $\sup_{[y_0,\max(\omega_{\varphi,-},\omega_{\Phi,-})]} (\mathcal{N}_\Phi^E)' \le 1 + h \cdot \max(\omega_{\varphi,-}, \omega_{\Phi,-}) \le 1 - \frac{h}{2}\sqrt{|\alpha|}$ by Step 1.

Step 2d. Repeating inductively, for any $n \ge 1$ we have that

$$\sup_{[y_n,y_{n+1}]} |\,id - J^E| \le$$

$$\le 1 \cdot \sup_{[y_0,y_1]} |\,id - J^E| + c \cdot h^{p+1} \sum_{i=0}^{n-1} \left(1 - \frac{h}{2}\sqrt{|\alpha|}\right)^{n-1-i} y_i^2$$

with c being the same constant as in (1). Hence in order to show (19), it is sufficient to verify—by virtue of Step 2a and 2b—that

$$\sup_{h\in(0,h_0]}\sup_{\alpha\in[-\alpha_0,0)}\sup_{k\in\mathbb{N}}\left(h\sum_{i=0}^{k}\left(1-\frac{h}{2}\sqrt{|\alpha|}\right)^{k-i}y_i^2(h,\alpha)\right)\le const \tag{22}$$

holds with a suitable $const > 0$.

Step 2e. We first estimate (22) for $0 \le k \le \lfloor\frac{1}{h}\rfloor$, using the first estimate of Lemma 3.3.

$$h\sum_{i=0}^{k}\left(1-\frac{h}{2}\sqrt{|\alpha|}\right)^{k-i}y_i^2 \le h\sum_{i=0}^{k}y_i^2 \le h\sum_{i=0}^{\lfloor\frac{1}{h}\rfloor}y_0^2 \le$$
$$\le h\left(\frac{1}{h}+1\right)y_0^2 \le 2y_0^2.$$

Step 2f. We can now estimate (22) for $k \ge \lfloor\frac{1}{h}\rfloor+1$ by making use of the second estimate of Lemma 3.3 and Step 2e.

$$h\left(\sum_{i=0}^{\lfloor\frac{1}{h}\rfloor}+\sum_{i=\lfloor\frac{1}{h}\rfloor+1}^{k}\right)\left(1-\frac{h}{2}\sqrt{|\alpha|}\right)^{k-i}y_i^2 \le$$

$$\le 2y_0^2+h\sum_{i=\lfloor\frac{1}{h}\rfloor+1}^{k}\left(1-\frac{h}{2}\sqrt{|\alpha|}\right)^{k-i}\left(2|\alpha|+\frac{4\sqrt{2|\alpha|}}{ih}+\frac{4}{i^2h^2}\right)\le\dots$$

Now we use $ih \ge \left(\lfloor\frac{1}{h}\rfloor+1\right)h > \frac{1}{h}\cdot h = 1$, and $\sum_{i=\lfloor\frac{1}{h}\rfloor+1}^{k}\left(1-\frac{h}{2}\sqrt{|\alpha|}\right)^{k-i} \le \frac{1}{1-\left(1-\frac{h}{2}\sqrt{|\alpha|}\right)}$ to proceed.

$$\dots\le 2y_0^2+h\frac{1}{1-\left(1-\frac{h}{2}\sqrt{|\alpha|}\right)}\left(2|\alpha|+4\sqrt{2|\alpha|}\right)+h\sum_{i=\lfloor\frac{1}{h}\rfloor+1}^{k}1^{k-i}\frac{4}{i^2h^2}\le$$

$$\le 2y_0^2+2\left(2\sqrt{|\alpha|}+4\sqrt{2}\right)+\frac{4}{h}\int_{\lfloor\frac{1}{h}\rfloor}^{\infty}\frac{1}{i^2}\mathrm{d}i\le$$
$$\le 2y_0^2+4(\sqrt{\alpha_0}+2\sqrt{2})+\frac{4}{h}\frac{1}{\frac{1}{h}-1}\le$$
$$\le 2y_0^2+4\sqrt{\alpha_0}+\frac{4}{1-h_0}+12,$$

which is a suitable choice for $const$ in (22). This estimate, substituted back into (21), yields (19). The proof of the lemma in the case $\alpha\in[-\alpha_0,0)$ is complete.

Step 3. Consider now the case $\alpha = 0$. Then the estimate $0 < (\mathcal{N}_\Phi^E)' \le 1$ can be used on $[y_0, 0]$. As in Step 2d, we arrive at the following condition

$$h\sum_{i=0}^{k}y_i^2 \le const \tag{23}$$

to be proved, uniformly in h and α for all values of $k\in\mathbb{N}$. But (23) can be proved along the lines of Step 2e and 2f, being now much simpler, thanks to the last two estimates of Lemma 3.3 at $\alpha = 0$. ■

3.3 Conclusion and further remarks

Taking into account Lemma 3.2 and Lemma 3.4, we have thus proved the following theorem.

Theorem 3.5 *Suppose that* $h_0 \le \min\left(\frac{1}{16}, \sqrt[p]{\frac{1}{8c}}\right)$ *and* $\sqrt{\alpha_0} \le \min\left(\frac{1}{2}, \frac{1}{26K}\right)$*, further* $\max\left(-1, -\frac{1}{13K}\right) \le y_0 \le -2\sqrt{\alpha_0}$*. Then, for every* $h \in (0, h_0]$ *and* $\alpha \in [-\alpha_0, 0]$*, the conjugacy defined in Section 2 satisfies*

$$\sup_{[y_0,0]} |\, id - J^E| \le 22c \cdot h^p,$$

where $c > 0$ *is the same as in (1).*

Now a similar task has to be carried out to acquire the appropriate estimates on $[0, |y_0|]$ as well, for any $h \in (0, h_0]$ and $-\alpha_0 \le \alpha \le 0$. These proofs are however a bit more technical due to the ubiquitous *inverses* of the normal forms. We only illustrate how *some* of the estimates can be derived in this case by showing two fragments of the proof. The case $-\alpha_0 \le \alpha < 0$ is considered now and attention is focused only near the boundary of the interval $[0, \omega_{\varphi,+}]$.

1. We begin proving the counterpart of Lemma 3.2. Let us formulate two basic inequalities first.

$$|\, id - J^E|(x) = \left| \left(\mathcal{N}_\Phi^E\right)^{[-1]} \circ \mathcal{N}_\Phi^E - \left(\mathcal{N}_\Phi^E\right)^{[-1]} \circ J^E \circ \mathcal{N}_\varphi^E \right|(x) \le$$

$$\le \left(\sup_{[\{\mathcal{N}_\Phi^E(x), J^E \circ \mathcal{N}_\varphi^E(x)\}]} \left((\mathcal{N}_\Phi^E)^{[-1]} \right)' \right) \left(\left|\mathcal{N}_\Phi^E - \mathcal{N}_\varphi^E\right|(x) + \left|\mathcal{N}_\varphi^E - J^E \circ \mathcal{N}_\varphi^E\right|(x) \right) \le$$

$$\le \left(\sup_{[\{\mathcal{N}_\Phi^E(x), J^E \circ \mathcal{N}_\varphi^E(x)\}]} \left((\mathcal{N}_\Phi^E)^{[-1]} \right)' \right) \left(c \cdot h^{p+1} |x|^3 + |\, id - J^E| \left(\mathcal{N}_\varphi^E(x)\right) \right). \tag{24}$$

On the other hand, by using that $\left(\left(\mathcal{N}_\Phi^E\right)^{[-1]} \right)' \le 2$ is valid on a small neighbourhood of the origin, inequality

$$\left| \left(\mathcal{N}_\Phi^E\right)^{[-1]}(x) - \left(\mathcal{N}_\varphi^E\right)^{[-1]}(x) \right| = \left| \left(\mathcal{N}_\Phi^E\right)^{[-1]}(x) - \left(\mathcal{N}_\Phi^E\right)^{[-1]} \circ \mathcal{N}_\Phi^E \circ \left(\mathcal{N}_\varphi^E\right)^{[-1]}(x) \right| \le$$

$$\le \left(\sup_{[\{x, \mathcal{N}_\Phi^E \circ \left(\mathcal{N}_\varphi^E\right)^{[-1]}(x)\}]} \left((\mathcal{N}_\Phi^E)^{[-1]} \right)' \right) \left| \mathcal{N}_\varphi^E \circ \left(\mathcal{N}_\varphi^E\right)^{[-1]}(x) - \mathcal{N}_\Phi^E \circ \left(\mathcal{N}_\varphi^E\right)^{[-1]}(x) \right| \le$$

$$\le 2c \cdot h^{p+1} \left| \left(\mathcal{N}_\varphi^E\right)^{[-1]}(x) \right|^3 \tag{25}$$

is also at our disposal.

Now using (24) and the definitions $\widetilde{x}_1 \equiv \left(\mathcal{N}_\varphi^E\right)^{[-1]}(0)$, further $J^E(0) = 0$, we establish that

$$\sup_{[0,\widetilde{x}_1]} |\, id - J^E| \le \left(\sup_{[\{\mathcal{N}_\Phi^E(\widetilde{x}_1), 0\}]} \left((\mathcal{N}_\Phi^E)^{[-1]} \right)' \right) \cdot c \cdot h^{p+1} |\widetilde{x}_1|^3 \le 2c \cdot h^{p+1} |\widetilde{x}_1|^3.$$

Now it is verified that $\widetilde{x}_1$ has the correct order of magnitude in terms of h and α. To this end, use the fact that, say, $\frac{1}{2} \leq \left((\mathcal{N}_\varphi^E)^{[-1]}\right)' \leq 2$ holds on a small neighbourhood of the origin to get

$$\frac{1}{2}h|\alpha| \leq \left(\inf_{[h\alpha,0]} \left((\mathcal{N}_\varphi^E)^{[-1]}\right)'\right)|h\alpha| = \left(\inf_{[\{0,\mathcal{N}_\varphi^E(0)\}]} \left((\mathcal{N}_\varphi^E)^{[-1]}\right)'\right)|0-\mathcal{N}_\varphi^E(0)| \leq$$

$$\leq \left|\left(\mathcal{N}_\varphi^E\right)^{[-1]}(0) - \left(\mathcal{N}_\varphi^E\right)^{[-1]}\left(\mathcal{N}_\varphi^E(0)\right)\right| \equiv |\widetilde{x}_1| \leq$$

$$\leq \left(\sup_{[\{0,\mathcal{N}_\varphi^E(0)\}]} \left((\mathcal{N}_\varphi^E)^{[-1]}\right)'\right)|0-\mathcal{N}_\varphi^E(0)| = \left(\sup_{[h\alpha,0]} \left((\mathcal{N}_\varphi^E)^{[-1]}\right)'\right)|h\alpha| \leq 2h|\alpha|.$$

Thus, $\sup_{[0,\widetilde{x}_1]} |\, id - J^E| \leq 16c \cdot h^{p+4}|\alpha|^3$. (However, with a little analysis, similar to (26) below, one can show that $\sup_{[h\alpha,0]} \left((\mathcal{N}_\varphi^E)^{[-1]}\right)'=1$, hence estimate $\sup_{[0,\widetilde{x}_1]} |\, id - J^E| \leq 2c \cdot h^{p+4}|\alpha|^3$ is closer to the truth.) The rest of the proof can be carried over similarly.

2. Secondly, it is shown that the repelling fixed points are sufficiently close to each other, *i.e.* the conjugacy J^E is $\mathcal{O}(h^p)$-close to the identity also at $\omega_{\varphi,+}$. By (24) at $x = \omega_{\varphi,+}$ we have that

$$|\, \omega_{\varphi,+} - \omega_{\Phi,+}| = |\, id - J^E|(\omega_{\varphi,+}) \leq$$

$$\leq \left(\sup_{[\{\mathcal{N}_\Phi^E(\omega_{\varphi,+}),\omega_{\Phi,+}\}]} \left((\mathcal{N}_\Phi^E)^{[-1]}\right)'\right)\left(c \cdot h^{p+1} \cdot \omega_{\varphi,+}^3 + |\, \omega_{\varphi,+} - \omega_{\Phi,+}|\right).$$

Now let us examine this supremum. We observe that

$$\sup_{[\{\mathcal{N}_\Phi^E(\omega_{\varphi,+}),\omega_{\Phi,+}\}]} \left((\mathcal{N}_\Phi^E)^{[-1]}\right)' = \sup_{[\{\mathcal{N}_\Phi^E(\omega_{\varphi,+}),\omega_{\Phi,+}\}]} \frac{1}{(\mathcal{N}_\Phi^E)' \circ (\mathcal{N}_\Phi^E)^{[-1]}} =$$

$$\sup_{[\{\omega_{\varphi,+},\omega_{\Phi,+}\}]} \frac{1}{(\mathcal{N}_\Phi^E)'} \leq \frac{1}{(\mathcal{N}_\Phi^E)'(\min(\omega_{\varphi,+},\omega_{\Phi,+}))} \leq$$

$$\frac{1}{(\mathcal{N}_\Phi^E)'\left(\frac{\sqrt{|\alpha|}}{2}\right)} \leq \frac{1}{1+\frac{h}{2}\sqrt{|\alpha|}} \leq 1 - \frac{h}{4}\sqrt{|\alpha|}, \tag{26}$$

by taking into account that the positive function $(\mathcal{N}_\Phi^E)'$ is monotone increasing, the corresponding estimates (cf. Lemma 2.1) $\frac{\sqrt{|\alpha|}}{2} \leq \sqrt{\frac{2}{3}}\sqrt{|\alpha|} \leq \omega_{\varphi,+}, \omega_{\Phi,+} \leq \sqrt{2}\sqrt{|\alpha|}$ for both positive fixed points of the normal forms, further the fact that $(\mathcal{N}_\Phi^E)'(x) \geq 1 + hx$, if $0 \leq x$ is sufficiently small, finally the inequality $\frac{1}{1+x} \leq 1 - \frac{x}{2}$, if $0 \leq x \leq 1$. From these we express the desired quantity to get

$$|\, \omega_{\varphi,+} - \omega_{\Phi,+}| \cdot \frac{h}{4}\sqrt{|\alpha|} \leq \left(1 - \frac{h}{4}\sqrt{|\alpha|}\right) c \cdot h^{p+1} \cdot \omega_{\varphi,+}^3$$

which in turn results in the inequality

$$|\, \omega_{\varphi,+} - \omega_{\Phi,+}| \leq 8\sqrt{2}\, c \cdot h^p \cdot |\alpha|.$$

4 Preparation of the closeness estimates: speed of the orbits in the $\alpha > 0$ case

For the construction of a conjugacy and the corresponding closeness estimates in the $\alpha > 0$ case, the current preparatory section analyzes some properties of the orbit of 0 under mappings of the form $x \mapsto h\alpha + x + hx^2 + hx^3 \cdot \eta(h, x, \alpha)$ with η from a suitable function class. (Then, of course, η will be replaced either by $\widehat{\eta}_3$ or $\widetilde{\eta}_3$.)

Let $p_n \equiv p_n(h, \alpha)$ denote any sequence satisfying $p_0 = 0$ and

$$p_{n+1} = \mathcal{N}_\eta(h, p_n, \alpha) \tag{27}$$

for $n \in \mathbb{N}$, where $\mathcal{N}_\eta(h, x, \alpha) := h\alpha + x + hx^2 + hx^3 \cdot \eta(h, x, \alpha)$ and η is any smooth function with $|\eta|$, $|\frac{\mathrm{d}}{\mathrm{d}x}\eta|$ and $|\frac{\mathrm{d}}{\mathrm{d}x^2}\eta|$ bounded, again, by some $K > 0$ uniformly for all $h \in (0, h_0]$, $x \in [-\varepsilon_0, \varepsilon_0]$ and $\alpha \in (0, \alpha_0]$, when $h_0 > 0$, $\varepsilon_0 > 0$ and $\alpha_0 > 0$ are sufficiently small. In what follows, we fix parameters $h \in (0, h_0]$ and $\alpha \in (0, \alpha_0]$ arbitrarily.

First notice that the asymptotic behaviour of p_n can be qualitatively different for different choices of η—for example, p_n can be unbounded, but can tend to a finite limit as well. In order to make its behaviour uniform, we will cut p_n at some suitable value $\kappa > 0$ and consider only the terms of the sequence below this *cutting-level*.

Lemma 4.1 *Let $\kappa := \min\left(\frac{3}{8}, (13\widetilde{K})^{-1}\right)$ with $\widetilde{K} := K + 3h_0$. Then the sequence p_n reaches level κ at some n, further, for $p_n \le \kappa$ the sequence is strictly monotone increasing, and for $0 < x \le \kappa$ both $\left(\mathcal{N}_\eta^E\right)'(x)$ and $\left(\mathcal{N}_\eta^E\right)''(x)$ are positive.*

Proof. Since $0 < x \le 1$, we have $\left(\mathcal{N}_\eta^E\right)'(x) \ge 1 + hx(2 - 3Kx - Kx)$ being positive due to $x \le (4K)^{-1}$. Similarly, $\left(\mathcal{N}_\eta^E\right)''(x) \ge h(2 - 6Kx - 6Kx - Kx) > 0$ because of $x \le (13K)^{-1}$. Strict monotonicity of p_n follows easily from $p_{n+1} - p_n > hp_n^2(1 - Kp_n) > 0$. Finally, since $(p_{n+2} - p_{n+1}) - (p_{n+1} - p_n) = h \cdot (t(p_{n+1}) - t(p_n))$ with $t(x) := x^2 + x^3 \cdot \eta^E(x)$, and $t(p_{n+1}) - t(p_n) = t'(\xi) \cdot (p_{n+1} - p_n)$ with some $\xi \in (p_n, p_{n+1})$, further $t'(\xi) \ge \xi(2 - 3\xi K - \xi K) > 0$, the proof is complete. ■

Remark 4.1 The role of $\widetilde{K}$ will be explained by Lemma 4.2, while that of $\frac{3}{8}$ by Lemma 4.8.

Having assured the strict monotonicity of p_n, the rest of this section will be devoted to devising suitable upper estimates for the sequence.

The behaviour of p_n under the level κ is the juxtaposition of two, qualitatively different phases.

In the interval $[0, \sqrt{\alpha}]$, the sequence is mainly determined by the term $h\alpha$ in the recursive definition (27), hence here $p_n \approx nh\alpha$, see (33) in the proof of Lemma 4.6.

However, after the level $\mathcal{O}(\sqrt{\alpha})$ has been passed, higher order terms begin to dominate and the linear growth suddenly turns into a steep increase.

Therefore, splitting our investigations into two is natural: the "trivial" linear part, and the tail part of the sequence will be treated separately. In this latter region—due to the fact that higher order terms are only "weakly" α-dependent, as α is present only in η—it is reasonable to expect some similarities between the

$\alpha > 0$ ($\alpha \to 0^+$) and the $\alpha = 0$ cases—and indeed, we will explicitly exploit this phenomenon. Hence, an essential part of the proofs in this section will not contain α. It seems hard, however, to control the growth rate of p_n effectively as *n increases*, that is, to devise suitable *global* estimates of p_n in terms of n *and* to say something meaningful about the index where the sequence reaches level κ, see Proposition 6.1 and the subsequent remarks. Nevertheless, a "backward" approach will work, that is, properties of the *inverse-iteration* can be grasped better by considering p_{N-k} as k increases, where N is chosen such that $p_N \approx \kappa$.

In exploring quantitative properties of this sequence described by the current Section, the program *Mathematica* has been heavily relied upon.

We first obtain an *a priori* inverse estimate for one term in the sequence in terms of its successor.

Lemma 4.2 *Suppose $h_0 \le \frac{1}{3}$, $h\alpha \le 1$, further κ and $\widetilde{K}$ are as above. Then for all $n \ge 1$ satisfying $p_n \le \kappa$ we have that*

$$p_{n-1} \le p_n - h\alpha + h^2\alpha - hp_n^2 + h\widetilde{K}p_n^3. \tag{28}$$

Proof. Substituting $n-1$ (instead of n) into (27), rearranging, and using the upper and lower bounds of η together with the fact that p_{n-1} is nonnegative, we see that

$$p_n - h\alpha - hp_{n-1}^2 - hKp_{n-1}^3 \le p_{n-1} \le p_n - h\alpha - hp_{n-1}^2 + hKp_{n-1}^3. \tag{29}$$

From the left hand side inequality—since p_n is monotone increasing and positive—we get

$$p_n - h\alpha - hp_n^2 - hKp_n^3 \le p_{n-1}. \tag{30}$$

Now we first show that the left hand side of (30) is nonnegative for $n \ge 2$. Using $p_n \le \frac{1}{2}$, $Kp_n \le \frac{1}{2}$ and $h \le \frac{1}{3}$, we have $h(\alpha + p_n^2 + Kp_n^3) \le h(\alpha + \frac{1}{2}p_n + \frac{1}{4}p_n) \le h\alpha + \frac{1}{4}p_n$. From this we get that the left hand side of (30) is nonnegative if $\frac{4}{3}h\alpha \le p_n$. But since $p_1 = h\alpha$ and $p_2 > 2h\alpha$, condition $\frac{4}{3}h\alpha \le p_n$ is implied by $n \ge 2$. So we temporarily assume $n \ge 2$.

Rearrangement of (30) thus yields

$$-(p_n - h\alpha - hp_n^2 - hKp_n^3)^2 \ge -p_{n-1}^2$$

for $n \ge 2$. Now let us combine this with the right hand side inequality of (29), showing for $n \ge 2$ that

$$p_{n-1} \le p_n - h\alpha - h(p_n - h\alpha - hp_n^2 - hKp_n^3)^2 + hKp_{n-1}^3.$$

Now we will simplify the right hand side here to arrive at the desired result. To this end, first replace the term hKp_{n-1}^3 with hKp_n^3 by monotonicity, then expand the square to get

$$p_{n-1} \le p_n - h\alpha - hp_n^2 + hKp_n^3 + 2h^2p_n(\alpha + p_n^2 + Kp_n^3) - h^3(\alpha + p_n^2 + Kp_n^3)^2. \tag{31}$$

Let us examine the last two terms above. The last negative term can safely be omitted, so we are left with estimating $2h^2p_n(\alpha + p_n^2 + Kp_n^3)$ from above. But using again $p_n \le \frac{1}{2}$ and $Kp_n \le \frac{1}{2}$, we get $2h^2p_n(\alpha + p_n^2 + Kp_n^3) = 2p_nh^2\alpha + 2h^2p_n^3(1 + Kp_n) \le h^2\alpha + hp_n^3 3h$.

Hence, by suitable upper estimates, (31) has been transformed into (28), with $\widetilde{K} := K + 3h_0$ for $n \ge 2$.

Finally, we directly verify (28) for $n = 1$. A direct substitution $p_0 = 0$ and $p_1 = h\alpha$ yields that it is sufficient to have $h^2\alpha - hp_1^2 + h\widetilde{K}p_1^3 \geq 0$, which is, however, implied by $h\alpha \leq 1$. ■

Remark 4.2 Inequalities (28) and (30) quantitatively express the natural fact that the inverse mapping of (identity + higher order terms), *i.e.* of (27), has the form (identity − perturbed higher order terms). Of course, it is not apparent at the first sight, how big this perturbation can be in terms of the parameters h and α.

Let us denote by $N \equiv N(h, \alpha)$ the unique index where the sequence p_n *passes level* κ, that is determine $N \in \mathbb{N}$ in such a way that $p_N \leq \kappa$ but $p_{N+1} > \kappa$.

Although $p_N \leq \kappa$, it will be important later to exclude the possibility of p_N being too small as $h, \alpha \to 0^+$. The next Lemma shows that, under appropriate conditions, p_N is uniformly separated from 0.

Lemma 4.3 *Suppose that the conditions of Lemma 4.2 hold and* $\alpha_0 \leq \kappa$. *Then* $p_N \geq \frac{\kappa}{2}$ *for all* $h \in (0, h_0]$ *and* $\alpha \in (0, \alpha_0]$.

Proof. Suppose, to the contrary, that $p_N < \frac{\kappa}{2}$ holds. Then by $K\kappa \leq 1$, $3h_0 \leq 1$ and $\kappa \leq 1$, one would get

$$\kappa < p_{N+1} = h\alpha + p_N + hp_N^2 + hp_N^3 \cdot \eta^E(p_N) \leq h\alpha + \frac{\kappa}{2} + h\frac{\kappa^2}{4} + h\frac{\kappa^3}{8}K \leq$$

$$h\alpha + \frac{\kappa}{2} + h\frac{\kappa^2}{4} + h\frac{\kappa^2}{8} = \frac{\kappa}{2} + h\alpha + \frac{3h\kappa^2}{8} \leq \frac{\kappa}{2} + \frac{\alpha_0}{3} + \frac{\kappa^2}{8} \leq \frac{\kappa}{2} + \kappa\left(\frac{1}{3} + \frac{1}{8}\right) < \kappa,$$

a contradiction. ■

In a similar fashion, we can replace the level κ to be passed by $\sqrt{\alpha}$. This type of result will also be needed later.

Lemma 4.4 *Suppose that the conditions of Lemma 4.2 hold and* $0 < \alpha_0 \leq \kappa^2$. *If* m *is the index such that* $p_m \leq \sqrt{\alpha}$, *but* $p_{m+1} > \sqrt{\alpha}$, *then* $p_m \geq \frac{\sqrt{\alpha}}{2}$ *for any* $h \in (0, h_0]$ *and* $\alpha \in (0, \alpha_0]$.

Proof. Suppose, to the contrary, that $p_m < \frac{\sqrt{\alpha}}{2}$. Then by $p_m K \leq \kappa K \leq 1$, $3h_0 \leq 1$ and $\alpha \leq 1$, we would get

$$\sqrt{\alpha} < p_{m+1} = h\alpha + p_m + hp_m^2 + hp_m^3 \cdot \eta^E(p_m) \leq h\alpha + p_m + 2hp_m^2 \leq$$

$$h\alpha + \frac{\sqrt{\alpha}}{2} + h\frac{\alpha}{2} \leq \frac{\sqrt{\alpha}}{2} + \frac{3}{2}h\alpha \leq \frac{\sqrt{\alpha}}{2} + \frac{\sqrt{\alpha}}{2} = \sqrt{\alpha},$$

a contradiction. ■

For $k \in \mathbb{N}$ sufficiently large, we now deduce a rough, but direct auxiliary estimate for p_{N-k} based on Lemma 4.2. However, it will be required later to have estimates not only for p_{N-k}, but also for p_{N^*-k}, where $0 < N^* \leq N$, so we have to prove a bit more general statement.

Lemma 4.5 *Suppose that the conditions of Lemma 4.2 hold. Set* $k_1 := \frac{1}{h\kappa}$ *and let* $N^* \in \mathbb{N}^+$ *be arbitrary with* $N^* \leq N$. *Then for* $k_1 \leq k \leq N^*$

$$p_{N^*-k} \leq \frac{2}{hk}. \tag{32}$$

Proof. Due to the monotone increasing property of p_n and the fact that we are dealing with upper estimates, it is sufficient to prove everything for N instead of $N^* \le N$.

The proof is by induction on k. Since $\kappa h \le 1$, it is clear for $k = \lceil k_1 \rceil$ that $p_{N-k} \le p_N \le \kappa \le \frac{2}{h(\frac{1}{h\kappa}+1)} \le \frac{2}{hk}$.

So assume (32) is true for some $k \ge k_1$. Then by (28)—omitting the nonpositive $-h\alpha + h^2\alpha$ due to $h \le 1$—we see that

$$p_{N-(k+1)} \le p_{N-k} - hp_{N-k}^2 + h\widetilde{K}p_{N-k}^3.$$

Since the function $p \mapsto p - hp^2 + h\widetilde{K}p^3$ is monotone increasing for $0 < p \le \kappa < \frac{1}{2h}$, it is enough to show

$$\frac{2}{hk} - h\frac{4}{h^2k^2} + h\widetilde{K}\frac{8}{h^3k^3} \le \frac{2}{h(k+1)}$$

to finish the induction. But the above is equivalent to

$$\frac{8\widetilde{K}}{h^2k^3} \le \frac{4}{hk^2} - \frac{2}{hk(k+1)},$$

which is a bit more strengthened if its right hand side is decreased by writing $\frac{4}{hk^2} - \frac{2}{hk^2}$. So it is sufficient to establish

$$\frac{8\widetilde{K}}{h^2k^3} \le \frac{2}{hk^2},$$

being equivalent to $4\widetilde{K} \le hk$, which latter is however implied by $4\widetilde{K} \le \frac{1}{\kappa} = hk_1 \le hk$ due to the definition of κ and k_1. ■

Remark 4.3 The lemma above can be considered as a counterpart of Lemma 3.3.

Remark 4.4 In this elementary argument one could replace the 2 in the numerator in (32) by $1+\delta$, with δ being an arbitrarily small positive number. However, the limiting process $\delta \to 0^+$ is not allowed, because this would shift k_1 to infinity (or the cutting-level κ to zero).

On the other hand, by scanning the proofs of (46) and (52), it can be seen that a constant strictly greater than 1 in the numerator in (32) would destroy the order of magnitude of the upper estimate (52): instead of the logarithmic singularity $\ln\frac{1}{\alpha}$, one would get only $\left(\frac{1}{\alpha}\right)^{\varepsilon}$, with a suitable $\varepsilon > 0$ as $\alpha \to 0^+$. So, for the sake of a sharper result, we are going to analyze deeper the growth rate of p_n.

First, we prove that the maximal index $N \equiv N(h,\alpha)$ is $\mathcal{O}(\frac{1}{h\sqrt{\alpha}})$, as $h, \alpha \to 0^+$.

Lemma 4.6 *Suppose that the conditions of Lemma 4.2 hold and $0 < \alpha_0 \le \kappa^2$. Then we have*

$$\begin{aligned}
\#\{p_n | p_n \in [0, \sqrt{\alpha}]\} &\le \frac{1}{h\sqrt{\alpha}}, \\
\#\{p_n | p_n \in [\sqrt{\alpha}, \kappa]\} &\le \frac{2}{h\sqrt{\alpha}} + \frac{1}{h\kappa},
\end{aligned}$$

and hence

$$N \le \frac{3}{h\sqrt{\alpha}} + \frac{1}{h\kappa} \le \frac{4}{h\sqrt{\alpha}}.$$

Proof. Since $0 \le p_n \le \kappa$, again $hp_n^2(1 + p_n \cdot \eta^E(p_n)) \ge 0$, so for $0 \le n \le N$ we get the *trivial lower estimate* from (27)

$$p_n \ge nh\alpha, \tag{33}$$

which yields the upper estimate to the number of elements of the first set in the Lemma. To get the second estimate, apply (32) but noticing that at most $\frac{1}{h\kappa}$ elements should be counted separately due to the restriction on the starting index k in (32). ■

Remark 4.5 It is possible to prove $N \le \frac{2+\delta}{h\sqrt{\alpha}}$, with $\delta > 0$ being arbitrary small. But, as remarked previously, such an improvement would be of no help for the following estimates.

Returning to our "backward approach", we develop two lemmas—the refined counterparts of Lemma 4.2 and Lemma 4.5.

The first subtle step is to conceal cubic terms in the inverse iteration by introducing a new sequence s_k.

Lemma 4.7 *Suppose that the conditions of Lemma 4.2 hold, and again,* $\frac{1}{h\kappa} =: k_1 \le k \le N^*$ *with some* $N^* \le N$. *Then*

$$p_{N^*-(k+1)} \le p_{N^*-k} - hs_k p_{N^*-k}^2, \tag{34}$$

where $s_k \equiv s_k(h, \kappa) := 1 - \frac{1}{hk\kappa}$ *for* $k \ge k_1$.

Proof. Inequality (28) implies that $p_{N^*-(k+1)} \le p_{N^*-k} - hp_{N^*-k}^2(1 - \widetilde{K}p_{N^*-k})$. Now use (32) and the definition of κ to obtain

$$1 - \widetilde{K}p_{N^*-k} \ge 1 - \widetilde{K}\frac{2}{hk} = \frac{hk - 2\widetilde{K}}{hk} \ge \frac{hk - \frac{1}{\kappa}}{hk} = s_k \ge 0.$$

These yield (34). ■

Now we can state and prove our main tool in the $\alpha > 0$ case. The Lemma below yields additional information on the convergence speed of the backward iteration, being fundamental to the final closeness estimates.

Lemma 4.8 (The $\frac{3}{2}$-Lemma) *Suppose that the conditions of Lemma 4.2 hold, but now* $\frac{1}{h\kappa^2} \le k \le N^*$ *with some* $N^* \le N$. *Then*

$$p_{N^*-k} \le \frac{1}{hk} + \frac{1/\kappa}{(hk)^{3/2}}. \tag{35}$$

Proof. Again, it is enough to prove everything for N instead of N^*.

We prove by induction on k. The induction can be started because for $k = \frac{1}{h\kappa^2} \ge \frac{1}{h\kappa}$, (32) yields that $p_{N-\lceil k \rceil} \le 2(h\lceil \frac{1}{h\kappa^2} \rceil)^{-1} \le 2\kappa^2 = \frac{1}{hk} + \frac{1/\kappa}{(hk)^{3/2}}$.

Let us now introduce the abbreviation $P(h, k, \kappa) := \frac{1}{hk} + \frac{1/\kappa}{(hk)^{3/2}}$ and suppose that $p_{N-k} \le P(h, k, \kappa)$ holds for some $k \ge \frac{1}{h\kappa^2}$. Then using (34) together with the monotonicity of the function $p \mapsto p - hs_k p^2$ (being true since $p_{N-k} \le \kappa < \frac{1}{2hs_k}$), we get

$$p_{N-(k+1)} \le p_{N-k} - hs_k p_{N-k}^2 \le P(h, k, \kappa) - hs_k P^2(h, k, \kappa).$$

Hence, clearly, in order to finish the induction, it is sufficient to establish that

$$Q(h,k,\kappa) := P(h,k+1,\kappa) - P(h,k,\kappa) + hs_k P^2(h,k,\kappa) \geq 0,$$

for all $h \in (0,1)$, $\kappa \in (0, \frac{3}{8}]$ and $k \geq \frac{1}{h\kappa^2}$.

To this end, we first shift the second argument in Q by setting $\ell := k - \frac{1}{h\kappa^2}$, then introduce a new variable $A := 1 + h\kappa^2\ell$ to get

$$Q(h,k,\kappa) \equiv Q\left(h, \frac{1}{h\kappa^2} + \ell, \kappa\right) \equiv Q\left(h, \frac{A}{h\kappa^2}, \kappa\right).$$

(So $\ell \geq 0$ is arbitrary, thus $A \geq 1$ is also arbitrary.) Albeit the expressions above are mathematically equivalent, yet, from a structural point of view, they are substantially different: the last form can be simplified to

$$Q\left(h, \frac{A}{h\kappa^2}, \kappa\right) \equiv$$

$$\kappa^2\left(\frac{1}{A+h\kappa^2} - \frac{1}{A} + \frac{1}{(A+h\kappa^2)^{3/2}} - \frac{1}{A^{3/2}} + \frac{(1+\sqrt{A})^2(A-\kappa)h\kappa^2}{A^4}\right),$$

where a new parameter $\nu := h\kappa^2$ is immediately introduced. Also dropping the positive factor κ^2 outside, and noticing that the whole expression is not increased if the only explicitly remaining κ is replaced by its maximal value $\frac{3}{8}$ in $(A-\kappa)$, we arrive at the following inequality in *two* variables

$$0 \leq \frac{1}{A+\nu} - \frac{1}{A} + \frac{1}{(A+\nu)^{3/2}} - \frac{1}{A^{3/2}} + \frac{(1+\sqrt{A})^2(A-\frac{3}{8})\nu}{A^4} \tag{36}$$

to be shown for all $A \geq 1$ and (even for all) $\nu \geq 0$.

Let us abbreviate the right hand side of (36) by $R(A,\nu)$ and notice that $R(A,0) = 0$ for all $A \geq 1$. Furthermore, notice that for $\nu > 0$, the partial derivative $\partial_\nu R(A,\nu)$ satisfies

$$\partial_\nu R(A,\nu) = \frac{(1+\sqrt{A})^2(A-\frac{3}{8})}{A^4} - \frac{3}{2(A+\nu)^{5/2}} - \frac{1}{(A+\nu)^2} >$$

$$\frac{(1+\sqrt{A})^2(A-\frac{3}{8})}{A^4} - \frac{3}{2A^{5/2}} - \frac{1}{A^2} = \frac{(\sqrt{A}-1)(4A+9\sqrt{A}+3)}{8A^4} \geq 0.$$

The proof is complete. ■

Remark 4.6 The exponent in (35) has been postulated to be $\frac{3}{2}$, because it is the "simplest" number between 1 and 2. Numerical tests suggest that this fractional order is necessary, since if the exponent $\frac{3}{2}$ was replaced by 2, then—according to the tests—nonnegativity of the counterpart of $Q\left(h, \frac{A}{h\kappa^2}, \kappa\right)$ would not hold uniformly, *i.e.* for any small $\kappa > 0$ in the factor $(A-\kappa)$, it is possible to choose $A \gg 1$ and $0 < \nu \ll 1$ such that the corresponding Q-expression is negative.

The very same fact is indicated in [8] as well, when studying the recursion

$$u_{k+1} = g(u_k),$$

with $g(x) \equiv x - bx^{q+1} + \mathcal{O}(x^{q+2})$ and $b > 0$, $q \in \mathbb{N}$ being fixed parameters. If $u_0 > 0$ is *sufficiently small*, then the sequence u_k tends to 0 as $k \to \infty$. The convergence speed of this iteration in terms of b and q together with strict lower and upper bounds for u_k are given in [8], provided that k is *large enough*. We remark that the same limiting relation in the $q = 1$ case—though with a different proof—also appears in [5]. It is seen that the iteration u_k with $q = 1$ is of the same type as our backward iteration p_{N-k} when $\alpha = 0$. The upper bounds given in [8] have a similar fractional order structure as (35). However, it will be important for us to have *explicit* estimates from a starting index of the form $\frac{\text{const}}{h}$; estimates only from a sufficiently large and unspecified starting index k would be insufficient.

We add that the sharpest estimate concerning this class of iteration we know about is contained in [9]. Define, similarly as above,

$$u_{k+1} = u_k - u_k^2 + \mathcal{O}(u_k^3)$$

and suppose that $u_0 > 0$ is chosen so small such that $u_k \to 0$. Then [9] sketches the proof of

$$u_k = \frac{1}{k} + \mathcal{O}\left(\frac{\log k}{k^2}\right).$$

However, the above sharper convergence rate is not yet an explicit estimate, and even if it was, it would not make our later closeness estimates better.

Remark 4.7 After the form of inequality (35) to be proved was set, the maximal value $\frac{3}{8}$ of κ became sharp—it originates from the Taylor series expansion of $Q\left(h, \frac{1}{h\kappa^2}, \kappa\right)$ about the origin:

$$\lim_{h\to 0} \frac{\mathrm{d}}{\mathrm{d}h}\left(Q\left(h, \frac{1}{h\kappa^2}, \kappa\right)\right) = \frac{\kappa^4}{2}(3 - 8\kappa).$$

This necessary condition $\kappa \leq \frac{3}{8}$ for nonnegativity of Q turned out to be sufficient as well, further $\kappa = \frac{3}{8}$ allows the nice factorization in the lower estimate of $\partial_\nu R(A, \nu)$.

During the search for the proof of $Q \geq 0$, the combined symbolic, numeric and graphical capabilities of *Mathematica* proved to be indispensable. The main source of problems has been the fact that the function Q with two parameters fixed often exhibits unimodality. The simple structural manipulations described in the Lemma above successfully eliminate unimodality as well as reduce the number of parameters by suitably grouping them together.

5 Construction of a conjugacy and closeness estimates in the $\alpha > 0$ case

Let us consider our mappings

$$x \mapsto \mathcal{N}_\Phi(h, x, \alpha) \equiv h\alpha + x + hx^2 + hx^3 \cdot \widehat{\eta}_3(h, x, \alpha) \tag{37}$$

and

$$x \mapsto \mathcal{N}_\varphi(h, x, \alpha) \equiv h\alpha + x + hx^2 + hx^3 \cdot \widetilde{\eta}_3(h, x, \alpha) \tag{38}$$

for any fixed $h \in (0, h_0]$ and $\alpha \in (0, \alpha_0]$. Both $\widehat{\eta}_3$ and $\widetilde{\eta}_3$ satisfy the assumptions at the beginning of Section 4, that is they are smooth functions with a *common* uniform bound $K > 0$. Suppose, that they are sufficiently close, that is there exists a positive constant $c > 0$ such that

$$|\mathcal{N}_\Phi(h, x, \alpha) - \mathcal{N}_\varphi(h, x, \alpha)| \leq c \cdot h^{p+1}|x|^\omega \tag{39}$$

holds for all $h \in (0, h_0]$, $x \in [-\varepsilon_0, \varepsilon_0]$ and $\alpha \in (0, \alpha_0]$, where the order is assumed to be $\omega = 3$ (what we have proved in [4]) or $\omega = 4$ (an additional assumption).

For every fixed $h \in (0, h_0]$ and $\alpha \in (0, \alpha_0]$ we construct a conjugacy between (37) and (38), that is a strictly monotone increasing map $x \mapsto J(h, x, \alpha)$ in a neighbourhood $[-\varepsilon_0, \varepsilon_0]$ of the origin such that

$$\mathcal{N}_\Phi^E \circ J^E = J^E \circ \mathcal{N}_\varphi. \tag{40}$$

We will deal only with the case $x \in [0, \varepsilon_0]$, the negative part $x \in [-\varepsilon_0, 0]$ (using the appropriate inverse mappings) is similar.

To this end, suppose—again as in Section 4—that the sequence p_n is the orbit of 0 under (37), while the sequence q_n is the orbit of 0 under (38), with $p_0 = q_0 \equiv 0$. Hence, all the results of Section 4 can be applied to both p_n and q_n—quantities κ, K and $\widetilde{K}$ are the same in both cases, however, of course, a clear distinction should be made between the cutting indices: let us denote by N_p the index where $p_{N_p} \le \kappa$ but $p_{N_p+1} > \kappa$, and similarly, by N_q the index where $q_{N_q} \le \kappa$ but $q_{N_q+1} > \kappa$. Since we are going to work with p_n and q_n simultaneously, they both should be kept below κ for the results of Section 4 to work, so a common cutting index N^* is now defined as

$$N^* := \min(N_p, N_q).$$

The following figure shows the first few (but same number of) terms of the sequences $q_n(h, \alpha_1)$ and $q_n(h, \alpha_2)$ in the (α, x)-plane with some $\alpha_1 > 0$, $\alpha_2 > 0$ and $h > 0$ fixed. Condensation of the sequences near the horizontal axis is clearly visible, however, for any $\alpha > 0$, due to the absence of the fixed points, they will eventually pass this axis, then begin increasing rapidly. (Note that for the sake of a better comparison, the value of q_0 has been redefined on this plot as $q_0 := -\frac{1}{2}$. The branch of stable and unstable fixed points of $\mathcal{N}_\varphi^E$ are also displayed. Again, the arrows point toward terms of the sequences with larger n indices.)

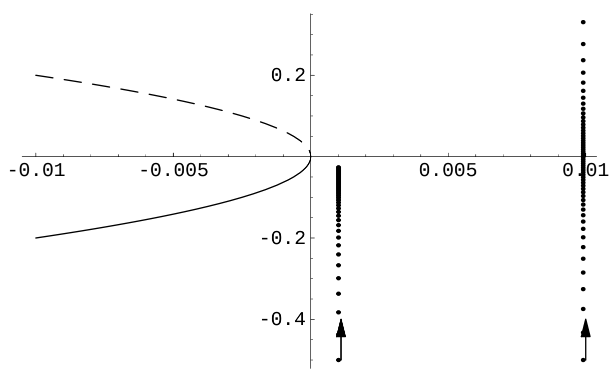

Figure 5.1

The figure below depicts part of the global dynamics of the map $\mathcal{N}_\varphi^E$ near the bifurcation point in the (α, x)-plane. The same n_0 $(0 \le n \le n_0)$ number of terms of the sequences $y_n(h, \alpha)$, $y_0 := -\frac{1}{2}$ and $q_n(h, \alpha)$, $q_0 := -\frac{1}{2}$ are displayed together (cf. Figure 2.1 and Figure 5.1), with $h > 0$ fixed and α running from -0.01 to 0.01 on an equidistant grid.

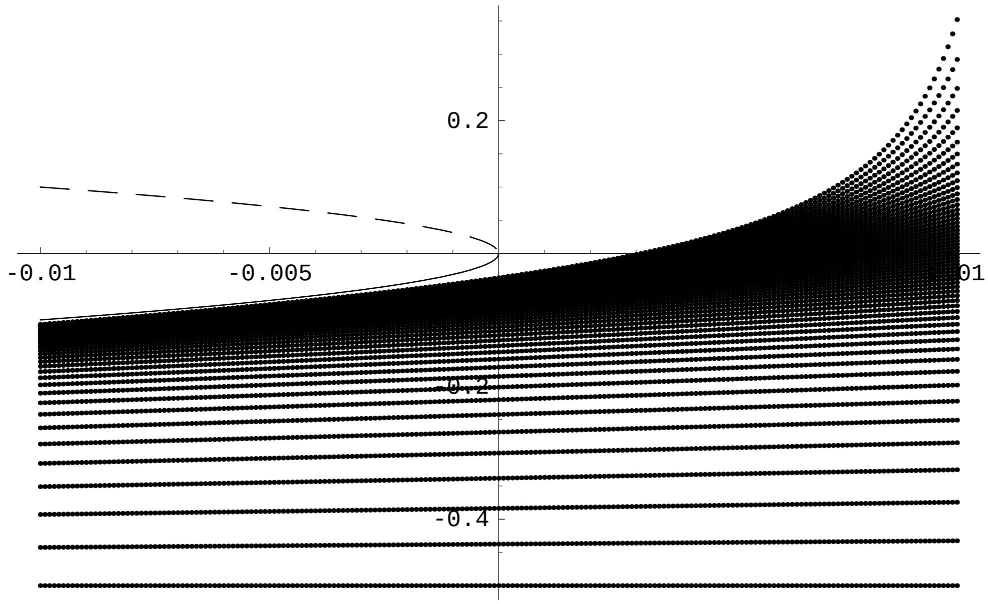

Figure 5.2

Now fix $h \in (0, h_0]$ and $\alpha \in (0, \alpha_0]$. Set

$$J^E(0) := 0. \tag{41}$$

Then (40) recursively forces the definition of J^E at q_n to be

$$J^E(q_n) := \left(\mathcal{N}_\Phi^E\right)^{[n]}(J^E(0)) \equiv \left(\mathcal{N}_\Phi^E\right)^{[n]}(0) \equiv p_n.$$

Define further $J^E(x) := x$ for $x \in [0, q_1]$. This will be a compatible extension, since $J^E(q_1) = p_1$ by definition, but $q_1 = p_1 \equiv h\alpha$. Then using these, together with (40) recursively, we can extend J^E homeomorphically in an (upper semi-)neighbourhood of the origin. We have thus proved the following theorem.

Theorem 5.1 *For every fixed $h \in (0, h_0]$ and $\alpha \in (0, \alpha_0]$, there exists a conjugacy $J(h, \cdot, \alpha)$ between (37) and (38) defined in a uniform neighbourhood $[-\varepsilon_0, \varepsilon_0]$ of the origin with $\varepsilon_0 := \kappa > 0$.*

Our aim now will be to measure the distance of J^E from the identity.

Remark 5.1 Due to the monotonicity of the mapping J^E, its growth rate and its distance from the identity can not be affected by the chosen extension on $[0, q_1]$, hence the only degree of freedom in the construction is prescribing the value of $J^E(0)$.

First we present an auxiliary estimate, similar to (10) before. Suppose $0 < a < b$. Using (40), Lemma 4.1 for the monotonicity and (39), we get

$$\sup_{[\mathcal{N}_\varphi^E(a), \mathcal{N}_\varphi^E(b)]} |\, id - J^E| = \sup_{[\mathcal{N}_\varphi^E(a), \mathcal{N}_\varphi^E(b)]} \left| \mathcal{N}_\varphi^E \circ (\mathcal{N}_\varphi^E)^{[-1]} - \mathcal{N}_\Phi^E \circ J^E \circ (\mathcal{N}_\varphi^E)^{[-1]} \right| \leq$$

$$\leq \sup_{[a,b]} \left| \mathcal{N}_\varphi^E - \mathcal{N}_\Phi^E \right| + \sup_{[a,b]} \left| \mathcal{N}_\Phi^E - \mathcal{N}_\Phi^E \circ J^E \right| \leq$$

$$\leq c \cdot h^{p+1} b^\omega + \sup_{x \in [a,b]} \left(\left(\sup_{[\{x, J^E(x)\}]} (\mathcal{N}_\Phi^E)' \right) |x - J^E(x)| \right) \leq$$

$$\leq c \cdot h^{p+1} b^\omega + (\mathcal{N}_\Phi^E)' \left(\max \left(b, J^E(b) \right) \right) \cdot \sup_{[a,b]} |\, id - J^E|. \tag{42}$$

For $n \in \mathbb{N}^+$, let us abbreviate the supremum by

$$S_n \equiv S_n(h, \alpha) := \sup_{[q_{n-1}, q_n]} |\, id - J^E|$$

and the derivative by

$$D_n \equiv D_n(h, \alpha) := (\mathcal{N}_\Phi^E)' \left(\max \left(q_n, J^E(q_n) \right) \right).$$

With $n \leq N^*$, $a = q_{n-1}$ and $b = q_n$, (42) therefore becomes

$$S_{n+1} \leq D_n S_n + c \cdot h^{p+1} q_n^\omega.$$

Applying this recursively, we construct the upper estimate for $n \leq N^*$

$$S_{n+1} \leq \left(\prod_{i=1}^{n} D_i \right) S_1 + c \cdot h^{p+1} \left(\sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} D_j \right) q_i^\omega \right), \tag{43}$$

where, of course, the product $\prod_{j=n+1}^{n} D_j$ is understood to be 1.

The first term on the right hand side vanishes, since $S_1 \equiv 0$ by construction. The second term, however, is monotone increasing in $n \leq N^*$, so we get

$$\sup_{[0, q_{N^*+1}]} |\, id - J^E| \leq c \cdot h^p \left(h \sum_{i=1}^{N^*} \left(\prod_{j=i+1}^{N^*} D_j \right) q_i^\omega \right). \tag{44}$$

In order to be able to estimate the right hand side of (44), we prove an important estimate concerning the sum of powers of μ_{N^*-k}, where

$$\mu_n \equiv \mu_n(h, \alpha) := \max(q_n, p_n),$$

for $0 \leq n \leq N^*$.

Lemma 5.2 *Suppose that the conditions of Lemma 4.2 hold and* $0 < \alpha_0 \leq \kappa^2$. *Then there exist positive constants* $\mathrm{const}_1(\kappa) > 0$ *and* $\mathrm{const}_2(\kappa) > 0$, *depending only on* κ, *such that for any index* $i \in \{0, 1, \ldots, N^*\}$ *and for any* $h \in (0, h_0]$ *and* $\alpha \in (0, \alpha_0]$ *we have that*

$$2h \sum_{k=0}^{i} \mu_{N^*-k} \leq \mathrm{const}_1(\kappa), \tag{45}$$

provided that $0 \leq i \leq \lfloor \frac{1}{h\kappa^2} \rfloor$, *and*

$$2h \sum_{k=0}^{i} \mu_{N^*-k} \leq \mathrm{const}_2(\kappa) + 2 \ln(h\, i), \tag{46}$$

provided that $\lfloor \frac{1}{h\kappa^2} \rfloor + 1 \le i \le N^*$.

Further, for any $\delta > 0$ *there exists a positive constant* $\mathrm{const}_3(\delta,\kappa) > 0$*, depending on* δ *and* κ*, such that for any* $h \in (0, h_0]$ *and* $\alpha \in (0, \alpha_0]$ *we have that*

$$h \sum_{k=0}^{N^*} (\mu_{N^*-k})^{1+\delta} \le \mathrm{const}_3(\delta,\kappa). \tag{47}$$

Proof. By the definition of N^* and μ_n, $0 \le \mu_n \le \kappa$ for any $n \in \{0, 1, \dots, N^*\}$. If $i \le \lfloor \frac{1}{h\kappa^2} \rfloor$, then

$$2h \sum_{k=0}^{i} \mu_{N^*-k} \le 2h \sum_{k=0}^{\lfloor \frac{1}{h\kappa^2} \rfloor} \kappa \le 2h\kappa \left(\frac{1}{h\kappa^2} + 1 \right) \le \frac{2}{\kappa} + 2h_0\kappa < \frac{2}{\kappa} + \kappa,$$

which shows (45).

Assume now that $N^* > \lfloor \frac{1}{h\kappa^2} \rfloor + 1$ and $\lfloor \frac{1}{h\kappa^2} \rfloor + 1 \le i \le N^*$. (Case $N^* = \lfloor \frac{1}{h\kappa^2} \rfloor + 1$ is just like (45).) From (45) and Lemma 4.8 we deduce that

$$2h \sum_{k=0}^{i} \mu_{N^*-k} \le \left(\frac{2}{\kappa} + \kappa \right) + \kappa + 2h \sum_{k=\lfloor \frac{1}{h\kappa^2} \rfloor + 2}^{i} \frac{1}{hk} + \frac{1/\kappa}{(hk)^{3/2}} \le$$

$$\left(\frac{2}{\kappa} + 2\kappa \right) + 2h \int_{\lfloor \frac{1}{h\kappa^2} \rfloor + 1}^{i} \left(\frac{1}{hx} + \frac{1/\kappa}{(hx)^{3/2}} \right) \mathrm{d}x \le$$

$$\left(\frac{2}{\kappa} + 2\kappa \right) + 2h \int_{\frac{1}{h\kappa^2}}^{i} \left(\frac{1}{hx} + \frac{1/\kappa}{(hx)^{3/2}} \right) \mathrm{d}x =$$

$$\left(\frac{2}{\kappa} + 2\kappa \right) + 4 - \frac{4}{\kappa\sqrt{h\, i}} + 4\ln\kappa + 2\ln(h\, i) \le \left(\frac{2}{\kappa} + 2\kappa + 4 + 4\ln\kappa \right) + 2\ln(h\, i),$$

proving (46). (We remark that keeping the term $-\frac{4}{\kappa\sqrt{h\, i}}$ would not make (46) any sharper, since $-4 \le -\frac{4}{\kappa\sqrt{h\, i}} \le 0$.)

Finally, for (47) use $N^* \le \frac{4}{h\sqrt{\alpha}}$ from Lemma 4.6. Then it suffices to turn to the weaker estimate (32) to get that

$$h \sum_{k=0}^{N^*} (\mu_{N^*-k})^{1+\delta} \le h \sum_{k=0}^{\lfloor \frac{1}{h\kappa} \rfloor + 1} \kappa^{1+\delta} + h \sum_{k=\lfloor \frac{1}{h\kappa} \rfloor + 2}^{N^*} \left(\frac{2}{hk} \right)^{1+\delta} \le$$

$$\kappa^{1+\delta} h \left(\frac{1}{h\kappa} + 2 \right) + h \int_{\frac{1}{h\kappa}}^{\frac{4}{h\sqrt{\alpha}}} \left(\frac{2}{hx} \right)^{1+\delta} \mathrm{d}x =$$

$$\left(\kappa^\delta + 2h\kappa^{1+\delta} \right) + \frac{2^{1+\delta}\kappa^\delta}{\delta} - \frac{2^{1-\delta}\alpha^{\delta/2}}{\delta} \le \kappa^\delta + \kappa^{1+\delta} + \frac{2^{1+\delta}\kappa^\delta}{\delta},$$

completing the proof.

It is seen that the choices for the constants $\mathrm{const}_1(\kappa) := \frac{2}{\kappa} + \kappa$, $\mathrm{const}_2(\kappa) := \frac{2}{\kappa} + 2\kappa + \frac{1}{10}$ (due to $4 + 4\ln\kappa \le 4 + 4\ln\frac{3}{8} < \frac{1}{10}$) and $\mathrm{const}_3(\delta,\kappa) := \kappa^\delta + \kappa^{1+\delta} + \frac{2^{1+\delta}\kappa^\delta}{\delta}$ are appropriate. ■

Now let us examine the product $\prod_{j=i+1}^{N^*} D_j$ in (44) for $i \in \{1, 2, \ldots, N^* - 1\}$. Computing $D_j \equiv (\mathcal{N}_\Phi^E)'(\mu_j)$ and using $1 + x \le e^x$ ($x \in \mathbb{R}$), we get that

$$\prod_{j=i+1}^{N^*} D_j \le \exp\left(2h \sum_{j=i+1}^{N^*} \mu_j + 3hK \sum_{j=i+1}^{N^*} \mu_j^2 + hK \sum_{j=i+1}^{N^*} \mu_j^3\right),$$

but taking into account (47), the right hand side can be simplified further to get

$$\prod_{j=i+1}^{N^*} D_j \le \mathrm{const}_4 \cdot \exp\left(2h \sum_{j=i+1}^{N^*} \mu_j\right), \tag{48}$$

with a suitable positive constant $\mathrm{const}_4 > 0$, uniformly in h and α.

Using the value of $\mathrm{const}_3(\delta, \kappa)$ set at the very end of the proof of Lemma 5.2, $\kappa \le 1$, $h_0 \le \frac{1}{3}$ and $\kappa K \le \frac{1}{13}$ we see that

$$3hK \sum_{j=i+1}^{N^*} \mu_j^2 + hK \sum_{j=i+1}^{N^*} \mu_j^3 \le 3hK\mathrm{const}_3(1,\kappa) + hK\mathrm{const}_3(2,\kappa) \le$$

$$3hK(5\kappa + \kappa^2) + hK(5\kappa^2 + \kappa^3) \le (5\kappa + \kappa^2)\, 4hK \le 24 h_0 \kappa K \le \frac{8}{13},$$

hence $e^{8/13} < 2 =: \mathrm{const}_4$ is a possible choice.

Substituting this into (44), we arrive at the estimate

$$\sup_{[0, q_{N^*+1}]} |\, id - J^E| \le 2c \cdot h^p \left(h \sum_{i=1}^{N^*} \exp\left(2h \sum_{j=i+1}^{N^*} \mu_j\right) \cdot q_i^\omega\right), \tag{49}$$

where c is the same as in (39).

Remark 5.2 Since $e^{x - x^2/2} \le 1 + x \le e^x$ ($x \in \mathbb{R}^+$), it is seen that

$$\mathrm{const} \cdot \exp\left(2h \sum_{j=i+1}^{N^*} \mu_j\right) \le \prod_{j=i+1}^{N^*} D_j \le 2 \exp\left(2h \sum_{j=i+1}^{N^*} \mu_j\right)$$

also holds with a suitable uniform constant $\mathrm{const} > 0$.

Now we are prepared to prove the following Theorem.

Theorem 5.3 *Suppose that κ has been defined as in Lemma 4.1. Suppose further, that $h_0 \le \min\left(\frac{1}{3}, \sqrt[p]{\frac{\exp(-2/\kappa)}{128c}}\right)$ and $0 < \alpha_0 \le \kappa^2$, with $c > 0$ being the same as in (39). If $\omega = 4$ in (39), then the conjugacy J^E defined between (37) and (38) satisfies*

$$\sup_{[0, \kappa/4]} |\, id - J^E| \le \left(12\, c\, e^{2/\kappa}\right) h^p, \tag{50}$$

uniformly in $h \in (0, h_0]$ and $\alpha \in (0, \alpha_0]$.

Proof. The choice of h_0 and α_0 satisfy the assumptions of all Lemmas listed so far.

First, by using $q_0 \equiv 0$ and reindexing the sums, we show that the complicated part of (49)

$$h\sum_{i=0}^{N^*-1} \exp\left(2h\sum_{j=i+1}^{N^*} \mu_j\right)\cdot q_i^4 \equiv h\sum_{i=1}^{N^*}\exp\left(2h\sum_{j=0}^{i-1}\mu_{N^*-j}\right)\cdot q_{N^*-i}^4 \leq$$

$$h\sum_{i=1}^{N^*}\exp\left(2h\sum_{j=0}^{i}\mu_{N^*-j}\right)\cdot q_{N^*-i}^4$$

is uniformly bounded. Applying (45), the trivial estimate $q_{N^*-i} \leq \kappa$, (46) and (32), further inequalities $\mathrm{const}_2(\kappa) \geq \mathrm{const}_1(\kappa)$ from the end of the proof of Lemma 5.2 and $N^* \leq \frac{4}{h\sqrt{\alpha}}$ from Lemma 4.6, we have that

$$h\sum_{i=1}^{N^*}\exp\left(2h\sum_{j=0}^{i}\mu_{N^*-j}\right)\cdot q_{N^*-i}^4 =$$

$$h\left(\sum_{i=1}^{\lfloor\frac{1}{h\kappa^2}\rfloor} + \sum_{i=\lfloor\frac{1}{h\kappa^2}\rfloor+1}^{N^*}\right)\exp\left(2h\sum_{j=0}^{i}\mu_{N^*-j}\right)\cdot q_{N^*-i}^4 \leq$$

$$h\sum_{i=1}^{\lfloor\frac{1}{h\kappa^2}\rfloor} e^{\mathrm{const}_1(\kappa)}\cdot\kappa^4 + h\sum_{i=\lfloor\frac{1}{h\kappa^2}\rfloor+1}^{N^*} e^{\mathrm{const}_2(\kappa)+2\ln(h\,i)}\cdot\left(\frac{2}{h\,i}\right)^4 \leq$$

$$e^{\mathrm{const}_2(\kappa)}\left(h\kappa^4\frac{1}{h\kappa^2} + h\sum_{i=\lfloor\frac{1}{h\kappa^2}\rfloor+1}^{N^*} h^2 i^2\frac{16}{h^4 i^4}\right) =$$

$$e^{\mathrm{const}_2(\kappa)}\left(\kappa^2 + \frac{16}{h\left(\lfloor\frac{1}{h\kappa^2}\rfloor+1\right)^2} + 16\sum_{i=\lfloor\frac{1}{h\kappa^2}\rfloor+2}^{N^*}\frac{1}{hi^2}\right) \leq$$

$$e^{\mathrm{const}_2(\kappa)}\left(\kappa^2 + 16h\kappa^4 + 16\int_{\frac{1}{h\kappa^2}}^{\frac{4}{h\sqrt{\alpha}}}\frac{1}{hx^2}\mathrm{d}x\right) =$$

$$e^{\mathrm{const}_2(\kappa)}\left(\kappa^2 + 16h\kappa^4 - 4\sqrt{\alpha} + 16\kappa^2\right) \leq e^{\mathrm{const}_2(\kappa)}\left(17\kappa^2 + 16h_0\kappa^4\right) \leq$$

$$\kappa\, e^{2/\kappa+2\kappa+1/10}\left(17\kappa + \frac{16}{3}\kappa^3\right) < \kappa\, e^{2/\kappa}\cdot 16.$$

Hence we have proved so far that

$$\sup_{[0,q_{N^*+1}]} |id - J^E| \leq 32c\kappa\, e^{2/\kappa}\cdot h^p \leq 12c\, e^{2/\kappa}\cdot h^p, \tag{51}$$

uniformly in $h \in (0, h_0]$ and $\alpha \in (0, \alpha_0]$.

Finally we show, that the interval on which the supremum is taken is uniformly large. There are two possibilities: $N^* = N_q$ or $N^* = N_p$. In the first case, Lemma 4.3 applied to the sequence q_n (with its own cutting-index) yields that $[0, q_{N^*+1}] \supset [0, q_{N_q}] \supset [0, \kappa/2]$. In the second case however, when $N^* = N_p$, we can turn to the left inequality in (51) *itself* together with the fact that $J^E(q_{N^*}) = p_{N^*}$ by construction, to establish relation

$$|q_{N_p} - p_{N_p}| = |q_{N_p} - J^E(q_{N_p})| \leq \sup_{[0,q_{N^*+1}]} |\, id - J^E| \leq 32c\kappa\, e^{2/\kappa} \cdot h_0^p \leq \frac{\kappa}{4},$$

if $h_0 \leq \sqrt[p]{\frac{\exp(-2/\kappa)}{128c}}$. But the result of Lemma 4.3 is again that $p_{N_p} \geq \frac{\kappa}{2}$, so $q_{N_p} \geq \frac{\kappa}{4}$ must be true. Therefore $[0, q_{N^*+1}] \supset [0, q_{N_p}] \supset [0, \kappa/4]$ and the Theorem is proved. ■

Remark 5.3 Since, by definition, $\kappa \leq \frac{1}{13K}$, that is $26K \leq \frac{2}{\kappa}$, we see that if the common uniform bound K in the mappings (37) and (38) is increased, then the upper estimate (50) *and* the upper bounds on h_0 and α_0 become worse.

For the case $\omega = 3$ in (39), the situation currently seems to be not so "uniform".

Theorem 5.4 *Suppose that κ has been defined as in Lemma 4.1. Suppose further, that $h_0 \leq \frac{1}{3}$, $0 < \alpha_0 \leq \kappa^2$, and $c > 0$ is the same as in (39). If $\omega = 3$ in (39), then the conjugacy J^E defined between (37) and (38) satisfies*

$$\sup_{[0,q_{N^*+1}]} |\, id - J^E| \leq c\left(\text{const}_5(\kappa) + \text{const}_6(\kappa)\ln\frac{1}{\alpha}\right) \cdot h^p. \tag{52}$$

Proof. Estimate (49) will be used with $\omega = 3$. We apply the same type of manipulations as in the proof of Theorem 5.3 to get

$$h\sum_{i=1}^{N^*} \exp\left(2h\sum_{j=i+1}^{N^*}\mu_j\right) \cdot q_i^3 \leq h\sum_{i=1}^{N^*} \exp\left(2h\sum_{j=0}^{i}\mu_{N^*-j}\right) \cdot q_{N^*-i}^3$$

$$h\sum_{i=1}^{\lfloor\frac{1}{h\kappa^2}\rfloor} e^{\text{const}_1(\kappa)} \cdot \kappa^3 + h\sum_{i=\lfloor\frac{1}{h\kappa^2}\rfloor+1}^{N^*} e^{\text{const}_2(\kappa)+2\ln(h\,i)} \cdot \left(\frac{2}{h\,i}\right)^3 \leq$$

$$e^{\text{const}_2(\kappa)}\left(h\kappa^3\frac{1}{h\kappa^2} + h\sum_{i=\lfloor\frac{1}{h\kappa^2}\rfloor+1}^{N^*} h^2 i^2 \frac{8}{h^3 i^3}\right) =$$

$$e^{\text{const}_2(\kappa)}\left(\kappa + \frac{8}{\lfloor\frac{1}{h\kappa^2}\rfloor+1} + 8\sum_{i=\lfloor\frac{1}{h\kappa^2}\rfloor+2}^{N^*}\frac{1}{i}\right) \leq$$

$$e^{\text{const}_2(\kappa)}\left(\kappa + 8h\kappa^2 + 8\int_{\frac{1}{h\kappa^2}}^{\frac{4}{h\sqrt{\alpha}}}\frac{1}{x}\,\mathrm{d}x\right) =$$

$$e^{\text{const}_2(\kappa)}\left(\kappa + 8h\kappa^2 + 8\ln 4 + 16\ln\kappa + 4\ln\frac{1}{\alpha}\right). \quad ■$$

Remark 5.4 Unfortunately, estimate (52) is singular as $\alpha \to 0^+$. Besides this, we can not control the interval $[0, q_{N^*}]$ in the supremum, so it may shrink too much if $N^* = N_p$ as $\alpha \to 0^+$.

A positive result in the $\omega = 3$ case we have is that on a special shrinking domain, namely on a parabola-shaped domain in the (α, x)-plane, a better closeness result holds.

Theorem 5.5 *Suppose that κ has been defined as in Lemma 4.1. Suppose further, that $h_0 \le \min\left(\frac{1}{3}, \sqrt[p]{\frac{1}{16ce^2\kappa}}\right)$ and $0 < \alpha_0 \le \kappa^2$, with $c > 0$ being the same as in (39). If $\omega = 3$ in (39), then the conjugacy J^E defined between (37) and (38) satisfies*

$$\sup_{[0,\sqrt{\alpha}/4]} |\, id - J^E| \le \left(4ce^2\alpha\right) h^p.$$

Proof. Similarly to the cutting-indices N_q, N_p and N^*, let us define $N_q(\sqrt{\alpha})$ to be the index such that $q_{N_q(\sqrt{\alpha})} \le \sqrt{\alpha}$, but $q_{N_q(\sqrt{\alpha})+1} > \sqrt{\alpha}$. Let us denote by $N_p(\sqrt{\alpha})$ the corresponding index for the sequence p_n. Further, let $N^*(\sqrt{\alpha}) := \min(N_q(\sqrt{\alpha}), N_p(\sqrt{\alpha}))$. Then it is easy to reconsider that all formulae (44)–(49) are still valid if N^* is replaced by this (not greater) $N^*(\sqrt{\alpha})$. So, as a starting point, we have

$$\sup_{[0,q_{N^*(\sqrt{\alpha})+1}]} |\, id - J^E| \le 2c \cdot h^p \left(h \sum_{i=1}^{N^*(\sqrt{\alpha})} \exp\left(2h \sum_{j=i+1}^{N^*(\sqrt{\alpha})} \mu_j\right) \cdot q_i^3 \right).$$

But, by the definition of $N^*(\sqrt{\alpha})$, further using Lemma 4.6 to get $N^*(\sqrt{\alpha}) \le \frac{1}{h\sqrt{\alpha}}$, we see that

$$h \sum_{i=1}^{N^*(\sqrt{\alpha})} \exp\left(2h \sum_{j=i+1}^{N^*(\sqrt{\alpha})} \mu_j\right) \cdot q_i^3 \le h \sum_{i=1}^{N^*(\sqrt{\alpha})} \exp\left(2h \sum_{j=2}^{N^*(\sqrt{\alpha})} \sqrt{\alpha}\right) \cdot \sqrt{\alpha}^3 \le$$

$$h\, e^2 \sum_{i=1}^{N^*(\sqrt{\alpha})} \sqrt{\alpha}^3 \le e^2\alpha \left(\frac{1}{h\sqrt{\alpha}} + 1\right) h\sqrt{\alpha} \le 2e^2\alpha.$$

Hence we know that

$$\sup_{[0,q_{N^*(\sqrt{\alpha})+1}]} |\, id - J^E| \le 4ce^2\alpha \cdot h^p. \tag{53}$$

Now, similarly to the end of the proof of Theorem 5.3, we show that the domain of the supremum contains $[0, \sqrt{\alpha}/4]$, uniformly in $h \in (0, h_0]$. If $N^*(\sqrt{\alpha}) = N_q(\sqrt{\alpha})$, then by Lemma 4.4 with $m = N_q(\sqrt{\alpha})$ we see that $q_m \ge \frac{\sqrt{\alpha}}{2}$, while if $N^*(\sqrt{\alpha}) = N_p(\sqrt{\alpha})$, then by (53)

$$|q_{N_p(\sqrt{\alpha})} - p_{N_p(\sqrt{\alpha})}| = |q_{N_p(\sqrt{\alpha})} - J^E(q_{N_p(\sqrt{\alpha})})| \le \sup_{[0,q_{N^*+1}]} |\, id - J^E| \le 4ce^2\alpha \cdot h_0^p \le \frac{\sqrt{\alpha}}{4},$$

if, for example, $h_0 \le \sqrt[p]{\frac{1}{16ce^2\kappa}}$. But again, by Lemma 4.4 with $m = N_p(\sqrt{\alpha})$, $p_m \ge \frac{\sqrt{\alpha}}{2}$, so $q_m \ge \frac{\sqrt{\alpha}}{4}$. ■

Remark 5.5 We have tacitly assumed (especially in Lemma 5.2) that $N^* \ge \frac{1}{h\kappa^2}$. However, this is not a real restriction, since otherwise every estimate is *ab ovo* trivial—just as the proof of (45) in Lemma 5.2—and we would have uniform boundedness in the Theorems.

6 Further results for the $\alpha > 0$ case

6.1 The tangent estimate

Although the following nice proposition finally has not been used in the closeness estimates, we still include it, because it reveals some information about the behaviour of the direct iteration p_n, and, together with the subsequent remarks, served as a motivation for the "backward" approach. (The number N, of course, is N_p here.)

Proposition 6.1 (The Tan-Estimate) *Suppose that the conditions of Lemma 4.2 hold and $0 < \alpha_0 \leq \kappa$. Then for $0 \leq n < \min\left(N, \frac{\pi}{2h\sqrt{2\alpha}} - 1\right)$ we have*

$$p_n \leq \sqrt{\frac{\alpha}{2}} \tan(\sqrt{2\alpha}\, hn).$$

Proof. We prove by induction on n. The case $n = 0$ is trivial. If $n = 1$, then

$$p_1 \equiv h\alpha \leq \sqrt{\frac{\alpha}{2}} \tan(\sqrt{2\alpha}\, h)$$

is equivalent to $\sqrt{2\alpha}\, h \leq \tan(\sqrt{2\alpha}\, h)$, but this latter is true since $x \leq \tan x$ (if, e.g., $0 \leq x \leq 1$), and $\sqrt{2\alpha}\, h \leq \sqrt{2\alpha_0} h_0 \leq \sqrt{2\kappa} h_0 \leq \sqrt{2 \cdot \frac{3}{8} \cdot \frac{1}{3}} < 1$.

So suppose the induction hypothesis is true for some $n \geq 1$. Then

$$p_{n+1} \leq h\alpha + \sqrt{\frac{\alpha}{2}} \tan(\sqrt{2\alpha}\, hn) + 2h \left(\sqrt{\frac{\alpha}{2}} \tan(\sqrt{2\alpha}\, hn) \right)^2$$

holds, since $p_{n+1} = \mathcal{N}_\eta(h, p_n, \alpha) \leq h\alpha + p_n + 2hp_n^2$, if, e.g., $0 \leq p_n \leq \kappa \leq \frac{1}{K}$ (implied by $n < N$). In order to finish the induction, it is sufficient to establish

$$h\alpha + \sqrt{\frac{\alpha}{2}} \tan(\sqrt{2\alpha}\, hn) + 2h \left(\sqrt{\frac{\alpha}{2}} \tan(\sqrt{2\alpha}\, hn) \right)^2 \leq \sqrt{\frac{\alpha}{2}} \tan(\sqrt{2\alpha}\, h(n+1)).$$

By using the abbreviation $x := \sqrt{2\alpha}\, h$, the inequality above can be rewritten as

$$x + \tan(nx) + x \tan^2(nx) \leq \tan((n+1)x). \tag{54}$$

Since $n < \frac{\pi}{2x} - 1$ by assumption, we know that

$$0 < \tan((n+1)x) = \frac{\tan x + \tan(nx)}{1 - \tan x \cdot \tan(nx)}.$$

But due to $0 < x < 1$ and $n < \frac{\pi}{2x}$, both $\tan x$ and $\tan(nx)$ are positive, so the denominator above is also positive. Hence, instead of (54), it is enough to prove

$$\left(x + \tan(nx) + x \tan^2(nx)\right)(1 - \tan x \cdot \tan(nx)) - \tan x - \tan(nx) \leq 0.$$

However, the left hand side can be factored to get

$$-\left(1 + \tan^2(nx)\right)(\tan x - x + x \tan x \cdot \tan(nx)),$$

so it must be nonpositive, because $\tan x - x \geq 0$ and $x, \tan x, \tan(nx) \geq 0$. ■

Remark 6.1 The tangent estimate, *i.e.* the function $t \mapsto \sqrt{\frac{\alpha}{2}}\tan(\sqrt{2\alpha}\,ht)$, has been obtained as the solution of the initial value problem $\dot{X} = h\alpha + 2hX^2$, $X(0) = 0$. (Of course, the multiplying constant 2 could be replaced by $1+\delta$, for any positive δ, but the limit $\delta \to 0^+$ is not allowed.) This ordinary differential equation has been chosen because its stepsize-1 explicit Euler discretization is just the sequence p_n with a slightly modified definition $p_{n+1} := h\alpha + p_n + 2hp_n^2$. Thus, we have proved in the Proposition above that for this particular equation the explicit Euler discretization is a lower approximation to the true solution—although, of course, from our viewpoint the roles are reversed: the known true solution is an upper estimate for the more implicit sequence p_n. The previous observation can be extended to a general class of ordinary differential equations: it can be shown that under a simple assumption on the sign of the right hand side and its derivative of the ordinary differential equation, the explicit/implicit Euler discretization is a lower/upper approximation to the exact solution, provided that the discretization stepsize is sufficiently small, see in [10]. This more general result however can not be directly applied to prove Proposition 6.1, because here the stepsize is 1. A fundamental and very interesting question would be to determine classes of equations where—or explain, in our case, why—the discretization is such a surprisingly sharp estimate of the true solution even with so large stepsizes.

Remark 6.2 The tangent estimate is "nearly global": it is a very good upper estimate of p_n as long as the tangent function is defined and not "too large". Of course, when the tangent reaches its first singularity, it becomes a useless estimate of p_n. This is exactly the main difficulty with the "direct" approach: estimating p_n as n increases is hard in the region when the tangent estimate is no longer valid but still $p_n < \kappa$ for some time.

Remark 6.3 We mention [11] as a peculiar result concerning the forward and backward iterates of the sequence $w_{n+1} = w_n^2 + \frac{1}{4} + \alpha$, $w_0 = \frac{1}{2}$. These recursions appear several times in the literature in connection with the phenomenon of intermittency, but probably this is the first paper containing a proof of the following observation. If $S(\alpha)$ denotes the number of steps needed for w_n to reach, say, 1, then [11] shows that $\lim_{\alpha\to 0^+} \sqrt{\alpha}\, S(\alpha) = \frac{\pi}{2}$. The calculations in the proof are elementary, but quite involved—the basic idea is to compare the difference equation with the corresponding differential equation similar to the one mentioned in Remark 6.1 above, and prove that the leading coefficients in the series expansion of their solutions satisfy the same type of recursive relations. This asymptotic relation in the case of p_n with $\eta \equiv 0$ in (27)—simply being a shifted version of w_n above—would mean that $\lim_{\alpha\to 0^+} h\sqrt{\alpha}\, N(h,\alpha) = \frac{\pi}{2}$.

6.2 Numerical test results

The optimality of Theorems 5.3–5.5 above—*under assumption* (41)—will now be illustrated by some numerical tests.

The following setting has been chosen: for $n \in \mathbb{N}^+$, let

$$q_{n+1} := h\alpha + q_n + hq_n^2$$

denote the pure quadratic iteration with $q_0 := 0$, while

$$p_{n+1} := h\alpha + p_n + hp_n^2 + \frac{1}{2}h^{p+1}p_n^{\omega}$$

with $p_0 := 0$ is a perturbed sequence. Choice of the cutting level $\kappa := \frac{3}{8}$ conforms to the requirements of Lemma 4.1.

What we measure in every case is the quantity

$$\text{dist} := \frac{|p_{N^*} - q_{N^*}|}{h^p}$$

under different choices of the exponents $p \in \mathbb{N}^+$ and $\omega \in \{3, 4\}$, further, the parameters h and α. The quantity dist $\cdot\, h^p$ is clearly a numerical *lower estimate* of $\sup_{[0,q_{N^*+1}]} |\, id - J^E|$, see, e.g., at the end of the proof of Theorem 5.3.

For the sake of comparison, we will also indicate the value of N^*. Since $p_n \geq q_n \geq 0$, we have $N^* = N_p$.

Due to its simplicity and elegance, we include the actual *Mathematica 5* code devised to perform the computations.

After fixing the values of p and ω, the following definition

```
perturbedsequence=Compile[{h,α},NestWhile[
  {h α+#[[1]]+h #[[1]]^2+1/2 h^(p+1) #[[1]]^ω,Last[#]+1}&,{0.,1},#[[1]]<3/8&]]
```

will yield $\{p_{N_p}, N_p\}$ in a list, while

```
quadraticsequence=Compile[{h,α,iternumber},Nest[
  {h α+#[[1]]+h #[[1]]^2,Last[#]+1}&,{0.,1},iternumber-1]]
```

will determine $\{q_{N_p}, N_p\}$, with `iternumber`$:= N_p$. Now—with h_1 and α_1 representing concrete numerical quantities—evaluate the following three commands

```
perturbedsequence[h1,α1]
quadraticsequence[h1,α1,Last[%]]
Abs[First[%]-First[%%]]/h1^p
```

to obtain finally the value of ”dist”.

Remark 6.4 The code for `perturbedsequence` and `quadraticsequence` given above uses machine precision numbers (see the `Compile` commands and the dots behind the 0's), since this substantially reduces the time needed to obtain ”dist” when α is very small. For large and medium values of α, the moderate computing time made it possible to exploit *Mathematica*'s arbitrary precision arithmetic as well. At $h = 10^{-1}$ and 10^{-2}, we have experienced total agreement between calculations based on machine precision and arbitrary precision—providing a good reliability check. However, for $h = 10^{-5}$, and $p = 3$, $\alpha = 10^{-3}$, for example, machine precision turned out to be insufficient, so arbitrary precision has been applied, since in this case $|p_{N^*} - q_{N^*}| \leq 1.5 \cdot 10^{-16}$.

The actual output—together with a graphical representation some of the data—is listed below. The arrangement of these tables is explained by the fact that in this way it is a bit easier for the eye to compare pairs of α-exponents and recognize the logarithmic law.

6.2.1 $\omega = 3,\ p = 2,\ h = 10^{-1}$

α	10^{-1}	10^{-2}	10^{-4}	10^{-8}
dist	$1.602 \cdot 10^{-2}$	$6.814 \cdot 10^{-2}$	$2.169 \cdot 10^{-1}$	$5.239 \cdot 10^{-1}$
N^*	29	134	1549	$1.5706 \cdot 10^{5}$

α	10^{-3}	10^{-6}	10^{-9}	10^{-12}
dist	$1.397 \cdot 10^{-1}$	$3.650 \cdot 10^{-1}$	$6.066 \cdot 10^{-1}$	$8.292 \cdot 10^{-1}$
N^*	474	15688	$4.9671 \cdot 10^{5}$	$1.5708 \cdot 10^{7}$

α	10^{-5}	10^{-10}
dist	$3.061 \cdot 10^{-1}$	$6.835 \cdot 10^{-1}$
N^*	4947	$1.5708 \cdot 10^{6}$

α	10^{-7}	10^{-14}
dist	$4.748 \cdot 10^{-1}$	1.0096
N^*	49655	$1.5708 \cdot 10^{8}$

The relation between α and "dist" is illustrated graphically by the following logarithmic plot: on the horizontal axis, values of $\log_{10}\left(\frac{1}{\alpha}\right)$ are displayed against the values of "dist" on the vertical axis. For the sake of convenience, linear interpolation has been used between the discrete points.

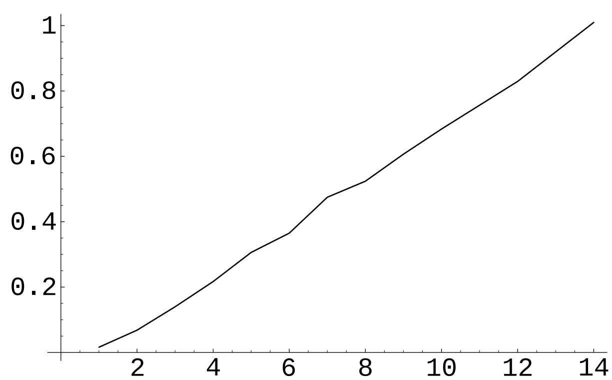

Figure 6.1

6.2.2 $\omega = 3,\ p = 3,\ h = 10^{-1}$

α	10^{-1}	10^{-2}	10^{-4}	10^{-8}
dist	$1.601 \cdot 10^{-2}$	$6.799 \cdot 10^{-2}$	$2.155 \cdot 10^{-1}$	$5.579 \cdot 10^{-1}$
N^*	29	134	1549	$1.5706 \cdot 10^{5}$

α	10^{-3}	10^{-6}	10^{-9}	10^{-12}
dist	$1.392 \cdot 10^{-1}$	$3.909 \cdot 10^{-1}$	$6.444 \cdot 10^{-1}$	$8.745 \cdot 10^{-1}$
N^*	474	15689	$4.9671 \cdot 10^{5}$	$1.5708 \cdot 10^{7}$

α	10^{-5}	10^{-10}
dist	$3.036 \cdot 10^{-1}$	$7.243 \cdot 10^{-1}$
N^*	4947	$1.5708 \cdot 10^6$

α	10^{-7}	10^{-14}
dist	$4.690 \cdot 10^{-1}$	1.060
N^*	49655	$1.5708 \cdot 10^8$

The corresponding graph is quite similar to the one above:

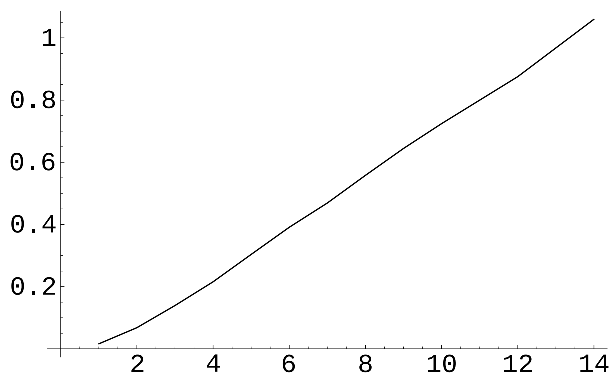

Figure 6.2

6.2.3 $\omega = 3$, $p = 3$, varying h

α	10^{-3}				
h	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
dist	$1.392 \cdot 10^{-1}$	$1.399 \cdot 10^{-1}$	$1.403 \cdot 10^{-1}$	$1.402 \cdot 10^{-1}$	$1.402 \cdot 10^{-1}$
N^*	474	4705	47017	$4.701 \cdot 10^5$	$4.701 \cdot 10^6$

α	10^{-4}			
h	10^{-1}	10^{-2}	10^{-3}	10^{-4}
dist	$2.155 \cdot 10^{-1}$	$2.189 \cdot 10^{-1}$	$2.199 \cdot 10^{-1}$	$2.198 \cdot 10^{-1}$
N^*	1549	15446	154419	$1.5441 \cdot 10^6$

α	10^{-5}			
h	10^{-1}	10^{-2}	10^{-3}	10^{-4}
dist	$3.036 \cdot 10^{-1}$	$3.017 \cdot 10^{-1}$	$3.006 \cdot 10^{-1}$	$3.006 \cdot 10^{-1}$
N^*	4947	49413	$4.9407 \cdot 10^6$	$4.9406 \cdot 10^7$

6.2.4 $\omega = 4$, $p = 2$, $h = 10^{-1}$

α	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
dist	$2.145 \cdot 10^{-2}$	$2.412 \cdot 10^{-2}$	$2.619 \cdot 10^{-2}$	$2.708 \cdot 10^{-2}$	$2.681 \cdot 10^{-2}$
N^*	474	1549	4947	15689	49655

α	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}
dist	$2.731 \cdot 10^{-2}$	$2.753 \cdot 10^{-2}$	$2.734 \cdot 10^{-2}$	$2.746 \cdot 10^{-2}$	$2.662 \cdot 10^{-2}$
N^*	157063	496714	$1.5708 \cdot 10^6$	$4.9672 \cdot 10^6$	$1.5708 \cdot 10^7$

The logarithmic plot this time is completely different:

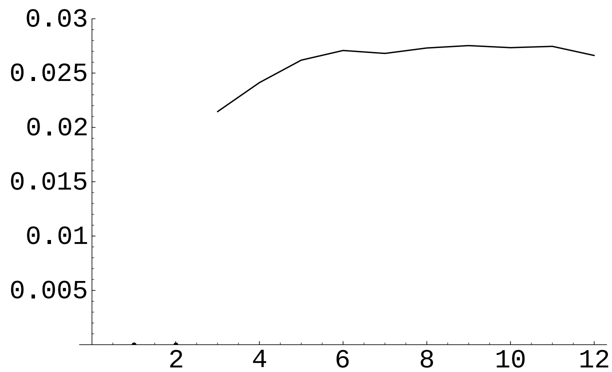

Figure 6.3

6.3 Conclusions of the numerical tests

Case $\omega = 3$. From the *linear* graphs of Figure 6.1 and 6.2, it is seen that the quantity "dist" grows like const $\cdot \ln \frac{1}{\alpha}$, as $\alpha \to 0^+$. Since dist $\cdot\, h^p$ is a numerical lower estimate of $\sup_{[0,q_{N^*+1}]} |\, id - J^E|$, this numerical evidence—together with the right hand side of estimate (52)—gives a convincing argument that *if the crucial (but natural) assumption $J^E(0) := 0$ is made, then the distance of the constructed conjugacy and the identity map indeed shows a logarithmic singularity as $\alpha \to 0^+$.*

Further, "dist" seems to be more or less independent of p, as Figure 6.1 and 6.2 are nearly the same, moreover, values of "dist" show stabilization as $h \to 0^+$ and $\alpha > 0$ is fixed.

Case $\omega = 4$. Numerical results together with Figure 6.3 clearly show uniform boundedness of "dist", which, of course, has been proved in Theorem 5.3.

In all cases, the values of N^* very closely follow the asymptotic formula $N^* \approx \frac{\pi}{2h\sqrt{\alpha}} \approx \frac{1.5708}{h\sqrt{\alpha}}$ $(\alpha \to 0^+)$ stated in Remark 6.3 in Section 6.1.

6.4 Open questions

A conjugacy J^E has been constructed between the mappings (37) and (38) in a uniform neighbourhood of the origin for all, sufficiently small values of the parameters h and α. As for the closeness estimates, further the continuity of the mappings $J(\cdot, x, \alpha)$ and $J(h, x, \cdot)$, we remark the following.

1. Originally, in our earlier work [4], we have proved cubic closeness of the normal forms, *i.e.* $\omega = 3$ in (39). Currently, however, as we have seen, the desired uniform closeness of J^E and the identity map is proved only under the extra assumption $\omega = 4$. We do not know whether it is possible to modify the construction of the conjugacy J in such a way that $\frac{|J - id|}{h^p}$ becomes uniformly bounded even for $\omega = 3$.

We have indicated (see Remark 5.1 in Section 5) that it would be enough to consider suitable modifications of the value of $J^E(0)$, for every $h > 0$ and $\alpha > 0$. Nevertheless, $J^E(0)$ and the extension on $[0, q_1]$ should be not only $\mathcal{O}(h^p)$, but $\mathcal{O}(h^p\alpha)$, since if $S_1 \not\equiv 0$ in (43), then the term $\left(\prod_{i=1}^{N^*} D_i\right) S_1 \approx \frac{\text{const}}{\alpha} S_1$ generally will not be annihilated.

We now briefly mention an attempt in this direction.

Attempts to transform the normal forms further. It can be asked whether it is possible to eliminate the cubic term in (37) (or in (38)). For simplicity, set $\widehat{\eta}_3 \equiv a \in \mathbb{R}$, $h = 1$ and $\alpha = 0$. Then we aim to find a near-identity transform *trans* : $x \mapsto x + bx^\nu$ with suitable b and ν such that it brings our mapping *map* : $x \mapsto x + x^2 + ax^3$ into a mapping with the cubic term eliminated. In other words, we would like to find b and ν such that $\mathit{elimmap} := \mathit{trans}^{[-1]} \circ \mathit{map} \circ \mathit{trans}$ contains no cubic terms.

The actual computations were performed again in *Mathematica*. If the value of ν is set, then the following command computes the series expansion of *elimmap* about the origin up to order 10:

```
ComposeSeries[InverseSeries[x+bx^ν+O[x]^10],x+x^2+ax^3+O[x]^10,
                x+bx^ν+O[x]^10]//Simplify
```

Substituting different values of ν into the above expression, the following pattern emerges. With $2 \leq \nu \in \mathbb{N}$ set, it is possible to choose b (also depending on a) such that *elimmap* contains no terms of order 2ν. This suggests trying *Puiseaux-series* instead of *Taylor-series*, however, $\nu = \frac{3}{2}$ leads to $\mathit{elimmap} \equiv x + x^2 + \frac{1}{2}bx^{5/2} + \left(a + \frac{b^2}{4}\right)x^3 + \ldots$, so an unwanted term of order $\frac{5}{2}$ enters. It was also in vain to try $\mathit{trans} \equiv x + bx^\nu + cx^\mu$ with various choices of ν and μ.

Therefore, we conclude that—at least with these type of transformations—it does not seem to be possible to convert the general $\omega = 3$ case into the $\omega = 4$ case.

2. The other question is the continuity of the conjugacy mapping. In our construction, we have assured that $x \mapsto J(h, x, \alpha)$ is a homeomorphism, for every fixed h and α. Continuity of $h \mapsto J(h, x, \alpha)$ $(0 < h \leq h_0)$ is also seen to hold. However, it would be a reasonable aim to decide whether—by possibly modifying the construction—the mapping $\alpha \mapsto J(h, x, \alpha)$ can be shown to be continuous at the critical bifurcation value $\alpha = 0$ and $x \geq 0$ as well. The reason for this discrepancy at $\alpha = 0$, $x \geq 0$ is that while in the fixed point-free $\alpha > 0$ case the conjugacy equation (40) extends $J(h, \cdot, \alpha)$ to the whole $[-\varepsilon_0, \varepsilon_0]$ interval if it is defined on *one* fundamental domain, in the case of $\alpha = 0$—due to the presence of the fixed point at $x = 0$—two fundamental domains are needed on each half-line.

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