Conjugacy in the discretized transcritical bifurcation

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Abstract

The present work can be considered as another case study—analogous to our earlier preprint [1]—in the direction of discretizing one-dimensional ordinary differential equations near non-hyperbolic equilibria. This time the hyperbolicity condition is violated due to the presence of a *transcritical bifurcation point*. The main aim is to show that the dynamics induced by the time-h-map of the original continuous system and that of the discretized one are still locally topologically equivalent, meaning that there exists a conjugacy between the corresponding phase portraits in the vicinity of the equilibrium. Besides the construction of a conjugacy map $J(h,\cdot,\alpha)$, the important point is that we also estimate the distance between $J(h,\cdot,\alpha)$ and the one-dimensional identity map.

In the first part of the paper, we derive normal forms for the time-h-map of the ordinary differential equation and its discretization near a transcritical bifurcation point at bifurcation parameter $\alpha = 0$ in one dimension and with discretization stepsize $h > 0$. We assume that the discretization method preserves equilibria. We will see that it is sufficient to construct a conjugacy between these normal forms.

In the second part, $J(h,\cdot,\alpha)$ is constructed for $0 < h \le h_0$ and $-\alpha_0 \le \alpha \le \alpha_0$ with h_0 and α_0 sufficiently small. Then the quantity $|x - J(h,x,\alpha)|$ is proved to be $\mathcal{O}(h^p)$ small, uniformly in x and α, in a small $x \in [-\varepsilon_0, \varepsilon_0]$ neighbourhood of the origin, where p denotes the order of the one-step discretization method.

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1 Introduction and notation

Suppose we have a one-dimensional ordinary differential equation

$$\dot{x} = f(x, \alpha) \tag{1}$$

and its one-step discretization

$$x_{n+1} := \varphi(h, x_n, \alpha), \qquad n = 0, 1, 2, \ldots, \tag{2}$$

where $\alpha \in \mathbb{R}$ is a scalar bifurcation parameter, $h > 0$ is the step-size of the sufficiently smooth one-step method $\varphi : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of order $p \geq 1$, and the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class C^{p+k+1} with $k \geq 5$ and uniformly bounded derivatives.

Since the numerical method is of order p, we have that

$$|\Phi(h, x, \alpha) - \varphi(h, x, \alpha)| \leq const \cdot h^{p+1}, \quad \forall h \in [0, h_0], \forall |x| \leq \varepsilon_0, \forall |\alpha| \leq \alpha_0, \tag{3}$$

where $\Phi(h, \cdot, \alpha) : \mathbb{R} \to \mathbb{R}$ is the time-h-map of the solution flow induced by (1) at parameter value α, further h_0, ε_0 and α_0 are some small positive constants. Throughout the paper, the symbols *const* will denote generic positive constants in the estimates, with dependence only on f. (These can have possibly different values at different occurrences.)

Suppose that the origin $x = 0$, $\alpha = 0$ is an equilibrium as well as a *transcritical bifurcation point* for (1), that is the following conditions hold

$$\begin{gathered} f(0, \alpha) = 0, \quad \forall |\alpha| \leq \alpha_0, \\ f_x^B = 0, \quad f_{xx}^B \neq 0, \quad f_{x\alpha}^B \neq 0, \end{gathered} \tag{4}$$

where subscripts x and α denote partial differentiation with respect to their corresponding variables, while superscript ^{B} abbreviates *evaluation at the bifurcation point*, that is, evaluation at $x = 0$ and $\alpha = 0$. (The evaluation is performed *after* taking all partial derivatives.)

The evaluation operator ^{B} will also be used for functions of three variables—h, x and α—when we evaluate a function at $h = 0$, $x = 0$ and $\alpha = 0$, as in $\Phi_{hx\alpha}^B$ abbreviating $\Phi_{hx\alpha}(0, 0, 0)$. (Here subscript h, of course, again stands for partial differentiation.)

For functions of three variables h, x and α, the evaluation operator ^{E} denotes *evaluation at general parameter values* h and α, where the dependence of ^{E} on h and α is suppressed. (Values of the parameters $h \in [0, h_0]$ and $\alpha \in [-\alpha_0, \alpha_0]$ can be arbitrary but fixed.) Thus, for example, the function $J(h, \cdot, \alpha)$ is abbreviated to J^E, if $J : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Some more notation is introduced. The symbol $g^{[-1]}$ means the *inverse* of a real function g. Similarly, $g^{[k]}$ is the k^{th} *iterate* ($k \in \mathbb{Z}$) of $f : \mathbb{R} \to \mathbb{R}$. The symbol *id* denotes the identity function of $\mathbb{R}$. Symbols $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$, as usual, denote the *floor* and the *ceiling* functions, respectively. The set of nonnegative integers is denoted by $\mathbb{N}$. Finally, for any $a, b \in \mathbb{R}$, the symbol $[\{a, b\}]$ represents the *closed interval between* the elements of the set $\{a, b\}$, that is $[\{a, b\}] := [\min(a, b), \max(a, b)]$.

Remark 1.1 Notice that instead of assumption $f(0,\alpha)=0,\ \forall|\alpha|\le\alpha_0$ in (4), [2] simply assumes $f(0,0)=0$ when it determines conditions for transcritical bifurcation of fixed points of maps. However, this is insufficient as illustrated by the map $x_{n+1}:=f(x_n,\alpha)$ with

$$f(x,\alpha):=\alpha^2+(1+\alpha)x+x^2.$$

Since $(x,\alpha)=(0,0)$ is the only fixed point of the map, clearly no bifurcation of fixed points can occur here. (The same discrepancy is present in [2] in the case of the *pitchfork bifurcation.*)

We add that [3], for example, correctly uses $f(0,0)$ *and* a kind of discriminant condition to define transcritical bifurcation of fixed points of maps. Condition $f(0,\alpha)=0$ we have adopted is more "direct" and a bit simpler to work with.

2 Construction of the normal forms

In this section, we compute normal forms for the maps

$$x\mapsto\Phi(h,x,\alpha) \tag{5}$$

and

$$x\mapsto\varphi(h,x,\alpha) \tag{6}$$

near the equilibrium being also a transcritical bifurcation point.

The properties of the solution flow together with (3)–(4) imply for $h\ge0$, $|x|\le\varepsilon_0$ and $|\alpha|\le\alpha_0$ that

$$\Phi(h,0,\alpha)=0,\quad \forall|\alpha|\le\alpha_0, \tag{7}$$
$$\varphi(0,x,\alpha)=\Phi(0,x,\alpha)=x, \tag{8}$$
$$\Phi_h(h,x,\alpha)=f(\Phi(h,x,\alpha),\alpha), \tag{9}$$
$$\varphi_h(0,x,\alpha)=\Phi_h(0,x,\alpha). \tag{10}$$

Instead of (9), the shorter form $\Phi_h=f\circ\Phi$ will be used.

To ensure that *the origin $x=0$ is a fixed point also for the discretization map* (6), we assume that

$$\varphi(h,0,\alpha)=0 \tag{11}$$

holds for sufficiently small $h\ge0$ and $|\alpha|$, which is the case, for example, for all Runge-Kutta discretizations.

Lemma 2.1 *Under the assumptions above and for $h\in[0,h_0]$, $|x|\le\varepsilon_0$, $|\alpha|\le\alpha_0$, we have that*

$$\Phi(h,x,\alpha)=f_0(h,\alpha)+f_1(h,\alpha)x+f_2(h,\alpha)x^2+\psi_3(h,x,\alpha)x^3,$$

where

$$\begin{aligned} f_0(h,\alpha) &\equiv 0,\\ f_1(h,\alpha) &\equiv 1+h\alpha\cdot f^B_{x\alpha}+h\alpha^2\cdot\psi_1(h,\alpha), \qquad f^B_{x\alpha}\ne0,\\ f_2(h,\alpha) &= \frac{1}{2}h\cdot f^B_{xx}+h\alpha\cdot\psi_2(h,\alpha), \qquad f^B_{xx}\ne0,\\ \psi_3(h,x,\alpha) &= h\cdot\widehat{\psi}_3(h,x,\alpha)\end{aligned}$$

hold with some smooth functions ψ_1,ψ_2 and $\widehat{\psi}_3$.

Proof. We expand Φ in a multivariate Taylor series about the equilibrium with the remainders in integral form.

Since $f(0,\alpha)=0$ for all $|\alpha|$ sufficiently small, we have (7), hence $f_0(h,\alpha)$ should vanish.

As for f_1, we get that

$$f_1(h,\alpha)=\Phi_x^B+\alpha\cdot \mathrm{I}_{011}(\alpha)+h\cdot \mathrm{I}_{110}(h)+h\alpha\cdot\Phi_{hx\alpha}^B+$$

$$h\alpha^2\cdot \mathrm{I}_{112}(\alpha)+h^2\alpha\cdot \mathrm{I}_{211}(h)+h^2\alpha^2\cdot \mathrm{I}_{212}(h,\alpha),$$

where $\Phi_x^B=1$,

$$\mathrm{I}_{011}(\alpha)=\int_0^1 \Phi_{x\alpha}(0,0,\tau\alpha)\mathrm{d}\tau\equiv 0,$$

$$\mathrm{I}_{110}(h)=\int_0^1 \Phi_{hx}(\tau h,0,0)\mathrm{d}\tau\equiv 0,$$

because $\Phi_{hx}=(f\circ\Phi)_x=(f_x\circ\Phi)\cdot\Phi_x$.

It is easy to verify that $\Phi_{hx\alpha}^B=f_{x\alpha}^B$. Indeed, we have that

$$\Phi_{hx\alpha}^B=(f\circ\Phi)_{x\alpha}^B=((f_x\circ\Phi)_\alpha\cdot\Phi_x+(f_x\circ\Phi)\cdot\Phi_{x\alpha})^B=(f_x\circ\Phi)_\alpha^B,$$

because $\Phi_{x\alpha}^B=0$ and $\Phi_x^B=1$. But

$$(f_x\circ\Phi)_\alpha^B=f_{xx}(\Phi^B,0)\cdot\Phi_\alpha^B+f_{x\alpha}(\Phi^B,0)=f_{x\alpha}^B,$$

since $\Phi_\alpha(0,x,\alpha)\equiv 0$.

The last three integrals read

$$\mathrm{I}_{112}(\alpha)=\int_0^1(1-\tau)\Phi_{hx\alpha\alpha}(0,0,\tau\alpha)\mathrm{d}\tau,$$

$$\mathrm{I}_{211}(h)=\int_0^1(1-\tau)\Phi_{hhx\alpha}(\tau h,0,0)\mathrm{d}\tau$$

and

$$\mathrm{I}_{212}(h,\alpha)=\int_0^1\int_0^1(1-\tau)(1-\sigma)\Phi_{hhx\alpha\alpha}(\tau h,0,\sigma\alpha)\mathrm{d}\tau\mathrm{d}\sigma.$$

We now show that $\mathrm{I}_{211}(h)$ vanishes, or, more precisely, that $\Phi_{hhx\alpha}(h,0,0)\equiv 0$ for every small $h\geq 0$. By direct differentiation we obtain that

$$\Phi_{hhx\alpha}=(f_{xx}\circ\Phi)_\alpha\cdot\Phi_x\cdot\Phi_h+(f_{xx}\circ\Phi)\cdot\Phi_{x\alpha}\cdot\Phi_h+$$

$$(f_{xx}\circ\Phi)\cdot\Phi_x\cdot\Phi_{h\alpha}+(f_x\circ\Phi)_\alpha\cdot\Phi_{hx}+(f_x\circ\Phi)\cdot\Phi_{hx\alpha}.$$

Here $\Phi_h(h,0,0)=f(\Phi(h,0,0),0)=f(0,0)=0$, so the first two terms above vanish. The third term is also zero, since

$$\Phi_{h\alpha}(h,0,0)=f_x(\Phi(h,0,0),0)\cdot\Phi_\alpha(h,0,0)+f_\alpha(\Phi(h,0,0),0)$$

but $\Phi(h,0,0)=0$ and $f_x(0,0)=0=f_\alpha(0,0)$. The fourth term is zero, because

$$\Phi_{hx}(h,0,0)=f_x(\Phi(h,0,0),0)\cdot\Phi_x(h,0,0)=0\cdot\Phi_x(h,0,0).$$

Finally, the fifth term vanishes due to the factor $f_x(\Phi(h,0,0),0)=0$.

By defining the smooth function $\psi_1(h,\alpha) := \mathrm{I}_{112}(\alpha) + h\cdot \mathrm{I}_{212}(h,\alpha)$, f_1 has the form stated above.

In the case of f_2, we have that

$$f_2(h,\alpha) = \frac{1}{2}\left(\Phi^B_{xx} + \alpha\cdot \mathrm{I}_{021}(\alpha) + h\cdot \Phi^B_{hxx} + h^2\cdot \mathrm{I}_{220}(h) + h\alpha\cdot \mathrm{I}_{121}(h,\alpha)\right),$$

where $\Phi^B_{xx} = 0$ and

$$\mathrm{I}_{021}(\alpha) = \int_0^1 \Phi_{xx\alpha}(0,0,\tau\alpha)\mathrm{d}\tau \equiv 0.$$

However,

$$\Phi^B_{hxx} = (f\circ\Phi)^B_{xx} = (f_{xx}\circ\Phi)^B\cdot\left((\Phi_x)^2\right)^B + (f_x\circ\Phi)^B\cdot\Phi^B_{xx} = f^B_{xx}\cdot 1 + 0 \neq 0.$$

Further,

$$\Phi_{hhxx} = (f_x\circ\Phi)_{xx}\cdot\Phi_h + 2(f_x\circ\Phi)_x\cdot\Phi_{hx} + (f_x\circ\Phi)\cdot\Phi_{hxx},$$

thus

$$\mathrm{I}_{220}(h) = \int_0^1 (1-\tau)\Phi_{hhxx}(\tau h,0,0)\mathrm{d}\tau \equiv 0.$$

Finally,

$$\mathrm{I}_{121}(h,\alpha) = \int_0^1\int_0^1 \Phi_{hxx\alpha}(\tau h,0,\sigma\alpha)\mathrm{d}\sigma\mathrm{d}\tau.$$

Thus, $\psi_2(h,\alpha) := \frac{1}{2}\mathrm{I}_{121}(h,\alpha)$ defines the desired smooth function.

For the remainder ψ_3, the integral formula gives

$$\psi_3(h,x,\alpha) = \frac{1}{2}\int_0^1 (1-\tau)^2\Phi_{xxx}(h,\tau x,\alpha)\mathrm{d}\tau. \tag{12}$$

But

$$\Phi_{xxx}(h,\tau x,\alpha) = \Phi_{xxx}(0,\tau x,\alpha) + h\cdot\int_0^1 \Phi_{hxxx}(\sigma h,\tau x,\alpha)\mathrm{d}\sigma$$

and $\Phi_{xxx}(0,\tau x,\alpha)\equiv 0$, so the lemma is proved. ■

Now we introduce a new parameter $\beta\equiv\beta(h,\alpha)$ by

$$\beta(h,\alpha) := \alpha\cdot f^B_{x\alpha} + \alpha^2\cdot \mathrm{I}_{112}(\alpha) + h\alpha^2\cdot \mathrm{I}_{212}(h,\alpha),$$

i.e., $\beta(h,\alpha) = \frac{f_1(h,\alpha)-1}{h}$.

We notice that $\beta(h,0)=0$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\beta(h,0) = f^B_{x\alpha}\neq 0$ independently of $h\in[0,h_0]$, thus the inverse function theorem guarantees the local existence and uniqueness of a smooth inverse function $\overline{\alpha}_0\equiv\overline{\alpha}_0(h,\beta)$ of $\alpha\mapsto\beta(h,\alpha)$. Moreover, it is easy to see that the domain of definition of this inverse function contains a neighbourhood of the origin independent of $h\in[0,h_0]$. Further, $\overline{\alpha}_0(h,0)=0$, hence

$$\overline{\alpha}_0(h,\beta) = \beta\cdot\psi_a(h,\beta) \tag{13}$$

holds for $h\in[0,h_0]$ and $|\beta|$ small with some smooth function ψ_a.

Therefore (5) is transformed into the map

$$x\mapsto(1+h\beta)x + h\cdot q(h,\beta)x^2 + h\cdot\widehat{\psi}_3(h,x,\overline{\alpha}_0(h,\beta))x^3$$

with $q(h,\beta)\equiv\frac{1}{2}f^B_{xx} + \frac{1}{2}\overline{\alpha}_0(h,\beta)\cdot \mathrm{I}_{121}(h,\overline{\alpha}_0(h,\beta))$.

A final scaling $\eta := |q(h,\beta)|x$ with $s := \operatorname{sign}(q(h,0)) = \pm1$ (being also independent of $h\in[0,h_0]$) yields the following normal form.

Lemma 2.2 *There are smooth invertible coordinate and parameter changes transforming the system*

$$x \mapsto \Phi(h, x, \alpha)$$

into

$$\eta \mapsto (1 + h\beta)\eta + s \cdot h\eta^2 + h\eta^3 \cdot \widehat{\eta}_3(h, \eta, \beta)$$

where $\widehat{\eta}_3(h, \eta, \beta) = \widehat{\psi}_3(h, x, \overline{\alpha}_0(h, \beta)) \cdot |q(h, \beta)|^{-2}$ *is a smooth function.* ■

Now let us consider the discretization map φ. We prove an analogous result to that of Lemma 2.1 first.

Lemma 2.3 *Under the assumptions of Lemma 2.1 together with (11) and for* $h \in [0, h_0]$, $|x| \leq \varepsilon_0$, $|\alpha| \leq \alpha_0$, *we have that*

$$\varphi(h, x, \alpha) = \widetilde{f}_0(h, \alpha) + \widetilde{f}_1(h, \alpha)x + \widetilde{f}_2(h, \alpha)x^2 + \chi_3(h, x, \alpha)x^3,$$

where

$$\begin{aligned}
\widetilde{f}_0(h, \alpha) &= 0, \\
\widetilde{f}_1(h, \alpha) &= 1 + h\alpha \cdot f^B_{x\alpha} + h^{p+1} \cdot \chi_{10}(h) + h\alpha \cdot \chi_{11}(h, \alpha), \\
\widetilde{f}_2(h, \alpha) &= \frac{1}{2}h \cdot f^B_{xx} + h^{p+1} \cdot \chi_{20}(h) + h\alpha \cdot \chi_{21}(h, \alpha), \\
\chi_3(h, x, \alpha) &= h \cdot \widetilde{\chi}_3(h, x, \alpha)
\end{aligned}$$

hold with some smooth functions χ_{10}, χ_{11}, χ_{20}, χ_{21} *and* $\widetilde{\chi}_3$. *Moreover, for* $h \in [0, h_0]$, $|x| \leq \varepsilon_0$ *and for* $|\alpha| \leq \alpha_0$,

$$|\psi_3(h, x, \alpha) - \chi_3(h, x, \alpha)| \leq const \cdot h^{p+1}. \tag{14}$$

Proof. By (11), we have that $\widetilde{f}_0(h, \alpha) \equiv 0$.

The remainders of the Taylor series are also represented by integrals and denoted—analogously to the proof of Lemma 2.1—by $\widetilde{\mathrm{I}}$'s. These integrals, of course, now always contain φ instead of Φ.

As for $\widetilde{f}_1$, by (8) one has that $\varphi^B_x = 1$ and $\widetilde{\mathrm{I}}_{011}(\alpha) \equiv 0$, further, we get that $\varphi^B_{hx\alpha} = \Phi^B_{hx\alpha} = f^B_{x\alpha} \neq 0$, hence

$$\widetilde{f}_1(h, \alpha) = 1 + h \cdot \widetilde{\mathrm{I}}_{110}(h) + h\alpha \cdot f^B_{x\alpha} +$$
$$h\alpha^2 \cdot \widetilde{\mathrm{I}}_{112}(\alpha) + h^2\alpha \cdot \widetilde{\mathrm{I}}_{211}(h) + h^2\alpha^2 \cdot \widetilde{\mathrm{I}}_{212}(h, \alpha).$$

Since f is at least C^{p+4}, from [4] we obtain that

$$\left| f_1(h, \alpha) - \widetilde{f}_1(h, \alpha) \right| \leq const \cdot h^{p+1}. \tag{15}$$

Evaluating this at $\alpha = 0$ yields $|h \cdot \widetilde{\mathrm{I}}_{110}(h)| \leq const \cdot h^{p+1}$. The smooth functions χ_{10} and χ_{11} are defined as

$$\chi_{10}(h) := \frac{h \cdot \widetilde{\mathrm{I}}_{110}(h)}{h^{p+1}}$$

and

$$\chi_{11}(h, \alpha) := \alpha \cdot \widetilde{\mathrm{I}}_{112}(\alpha) + h \cdot \widetilde{\mathrm{I}}_{211}(h) + h\alpha \cdot \widetilde{\mathrm{I}}_{212}(h, \alpha).$$

(It can be easily proved that $\widetilde{\mathrm{I}}_{112}(\alpha) \equiv \mathrm{I}_{112}(\alpha)$, but this property will not be needed later.)

Considering $\widetilde{f}_2$, we obtain that $\varphi^B_{xx} = 0$ and $\widetilde{\mathrm{I}}_{021}(\alpha) \equiv 0$. By differentiating (10) we see that $\varphi^B_{hxx} = \Phi^B_{hxx} = f^B_{xx} \neq 0$, thus

$$\widetilde{f}_2(h,\alpha) = \frac{1}{2}\left(h \cdot f^B_{xx} + h^2 \cdot \widetilde{\mathrm{I}}_{220}(h) + h\alpha \cdot \widetilde{\mathrm{I}}_{121}(h,\alpha)\right),$$

and again, using $f \in C^{p+5}$ and [4]

$$\left| f_2(h,\alpha) - \widetilde{f}_2(h,\alpha) \right| \leq const \cdot h^{p+1}. \tag{16}$$

Evaluating this at $\alpha = 0$, we see that $|h^2 \cdot \widetilde{\mathrm{I}}_{220}(h)| \leq const \cdot h^{p+1}$, so we can set

$$\chi_{20}(h) := \frac{1}{2} \cdot \frac{h^2 \cdot \widetilde{\mathrm{I}}_{220}(h)}{h^{p+1}}$$

and

$$\chi_{21}(h,\alpha) := \frac{1}{2} \cdot \widetilde{\mathrm{I}}_{121}(h,\alpha)$$

to obtain two smooth functions.

To prove the product form of the remainder χ_3, we use the same argument as in (12). Finally, for (14) we take into account $f \in C^{p+6}$ and [4] again to get

$$|\psi_3(h,x,\alpha) - \chi_3(h,x,\alpha)| = \left| \frac{1}{2} \int_0^1 (1-\tau)^2 \left(\Phi_{xxx}(h,\tau x,\alpha) - \varphi_{xxx}(h,\tau x,\alpha)\right) \mathrm{d}\tau \right| \leq$$

$$\leq const \cdot h^{p+1} \cdot \frac{1}{2} \int_0^1 (1-\tau)^2 \mathrm{d}\tau,$$

completing the proof of the lemma. ■

Now we the introduce the analogue of parameter β. Set

$$\widetilde{\beta} \equiv \widetilde{\beta}(h,\alpha) := \widetilde{\mathrm{I}}_{110}(h) + \alpha \cdot f^B_{x\alpha} + \alpha^2 \cdot \widetilde{\mathrm{I}}_{112}(\alpha) + h\alpha \cdot \widetilde{\mathrm{I}}_{211}(h) + h\alpha^2 \cdot \widetilde{\mathrm{I}}_{212}(h,\alpha).$$

We will show that the function $\widetilde{\beta}(h,\cdot)$ is locally invertible at the origin for every $h \geq 0$ small enough, and its inverse function, $\widetilde{\alpha}(h,\cdot)$ is $\mathcal{O}(h^p)$-close to $\overline{\alpha}_0(h,\cdot)$, *i.e.* to the inverse of $\beta(h,\cdot)$. As in [5], we will use the same quantitative inverse function theorem, see Lemma 2.4 in [5]. (Now a letter G will play the role of $\widetilde{F}$ in that lemma.) We set

$$G(h,\beta,\alpha) := \beta - \widetilde{\beta}(h,\alpha).$$

In order to check the conditions of the lemma, define $\kappa_1 := \frac{1}{2}|f^B_{x\alpha}| > 0$ and $\kappa_2 := \frac{1}{2}\kappa_1$. We have that

$$\frac{\partial G}{\partial \alpha}(h,\beta,\alpha) = f^B_{x\alpha} + 2\alpha \cdot \widetilde{\mathrm{I}}_{112}(\alpha) + \alpha^2 \frac{\mathrm{d}}{\mathrm{d}\alpha}\widetilde{\mathrm{I}}_{112}(\alpha) +$$

$$h \cdot \widetilde{\mathrm{I}}_{211}(h) + 2h\alpha \cdot \widetilde{\mathrm{I}}_{212}(h,\alpha) + h\alpha^2 \frac{\mathrm{d}}{\mathrm{d}\alpha}\widetilde{\mathrm{I}}_{212}(h,\alpha).$$

Thus

$$\left| \frac{\partial G}{\partial \alpha}(h,\beta,\alpha) - \frac{\partial G}{\partial \alpha}(h,\beta,\overline{\alpha}_0(h,\beta)) \right| \leq \kappa_2$$

holds by smoothness of the functions $\widetilde{\mathrm{I}}$'s provided that $|\alpha - \overline{\alpha}_0(h,\beta)| \leq r_1$ and $h < r_2$ are small enough. It is also seen that

$$\left| \frac{\partial G}{\partial \alpha}(h,\beta,\overline{\alpha}_0(h,\beta)) \right| \geq \kappa_1,$$

if $h, |\beta| < r_2$ are small enough, taking also into account (13). Finally, using that $\overline{\alpha}_0(h,\cdot)$ is the inverse function of $\beta(h,\cdot)$, we get that

$$|G(h,\beta,\overline{\alpha}_0(h,\beta))| = \left|\beta - \widetilde{\beta}(h,\overline{\alpha}_0(h,\beta))\right| = \left|\beta(h,\overline{\alpha}_0(h,\beta)) - \widetilde{\beta}(h,\overline{\alpha}_0(h,\beta))\right|.$$

But (15) implies that

$$|\beta(h,\alpha) - \widetilde{\beta}(h,\alpha)| \le const \cdot h^p, \tag{17}$$

hence $|G(h,\beta,\overline{\alpha}_0(h,\beta))| \le const \cdot h^p$ and also $|G(h,\beta,\overline{\alpha}_0(h,\beta))| \le (\kappa_1 - \kappa_2) \cdot r_1$ if $h < r_2$ is small enough.

Therefore, Lemma 2.4 in [5] is applicable in our situation and we get a unique zero $\widetilde{\alpha}(h,\beta)$ of $G(h,\beta,\cdot)$, which—by the construction of G—is the inverse function of $\alpha \mapsto \widetilde{\beta}(h,\alpha)$. Furthermore,

$$|\widetilde{\alpha}(h,\beta) - \overline{\alpha}_0(h,\beta)| \le const \cdot h^p \tag{18}$$

holds for $h \in [0,h_0]$ and $|\beta|$ sufficiently small.

As a conclusion, (6) becomes

$$x \mapsto (1 + h\widetilde{\beta})x + h \cdot \widetilde{q}(h,\widetilde{\beta})x^2 + h \cdot \widetilde{\chi}_3(h,x,\widetilde{\alpha}(h,\widetilde{\beta}))x^3$$

with $\widetilde{q}(h,\widetilde{\beta}) \equiv \frac{1}{2}\left(f^B_{xx} + h \cdot \widetilde{\mathrm{I}}_{220}(h) + \widetilde{\alpha}(h,\widetilde{\beta}) \cdot \widetilde{\mathrm{I}}_{121}(h,\widetilde{\alpha}(h,\widetilde{\beta}))\right)$.

We claim that

$$\left|\widetilde{q}(h,\widetilde{\beta}) - q(h,\beta)\right| \le const \cdot h^p \tag{19}$$

also holds. But this is a consequence of inequalities (18), (16) and the smoothness (and boundedness) of the functions I_{121} and $\widetilde{\mathrm{I}}_{121}$ when combined with standard triangle inequalities and the mean value theorem.

By applying a final scaling

$$\widetilde{\eta} := |\widetilde{q}(h,\widetilde{\beta})|x$$

with $s := \mathrm{sign}(\widetilde{q}(h,0)) = \pm 1$ (being independent of $h \in [0,h_0]$ for h_0 small enough, due to (18) evaluated at $\beta = 0$, (13) and the boundedness of the function $\widetilde{\mathrm{I}}_{121}$) and defining

$$\widetilde{\eta}_3(h,\widetilde{\eta},\widetilde{\beta}) := \widetilde{\chi}_3(h,x,\widetilde{\alpha}(h,\widetilde{\beta})) \cdot |\widetilde{q}(h,\widetilde{\beta})|^{-2},$$

we have derived a normal form for (6) in the theorem below.

For the closeness estimates in the theorem, we should only verify that

$$\left|\widehat{\eta}_3(h,\eta,\beta) - \widetilde{\eta}_3(h,\widetilde{\eta},\widetilde{\beta})\right| \le const \cdot h^p.$$

This estimate, however, is a simple consequence of (19) and the fact that

$$\left|\widehat{\psi}_3(h,x,\overline{\alpha}_0(h,\beta)) - \widetilde{\chi}_3(h,x,\widetilde{\alpha}(h,\widetilde{\beta}))\right| \le const \cdot h^p.$$

(For this last inequality, (14), the smoothness of $\widehat{\psi}_3$, a standard triangle inequality and the mean value theorem suffice.)

Theorem 2.4 *There are smooth invertible coordinate and parameter changes transforming the system*

$$x \mapsto \varphi(h, x, \alpha)$$

into

$$\widetilde{\eta} \mapsto (1 + h\widetilde{\beta})\widetilde{\eta} + s \cdot h\widetilde{\eta}^2 + h\widetilde{\eta}^3 \cdot \widetilde{\eta}_3(h, \widetilde{\eta}, \widetilde{\beta})$$

where $\widetilde{\eta}_3$ *is a smooth function.*

Moreover, the smooth invertible coordinate and parameter changes above and those in Lemma 2.2 are $\mathcal{O}(h^p)$*-close to each other, further*

$$|\widehat{\eta}_3 - \widetilde{\eta}_3| \le const \cdot h^p \qquad \blacksquare$$

Finally, we apply a parameter shift $\widetilde{\beta} \mapsto \beta$ to the normal form in the theorem above, being $\mathcal{O}(h^p)$-close to the identity due to (17). So from now on we will use the bifurcation parameter α again instead of β and $\widetilde{\beta}$. To simplify our notations further, instead of η and $\widetilde{\eta}$ the letter x will be used.

3 Construction of the conjugacy

We have thus the following normal forms

$$\mathcal{N}_\Phi(h, x, \alpha) = (1 + h\alpha)x + s \cdot hx^2 + hx^3\,\widehat{\eta}_3(h, x, \alpha) \tag{20}$$

$$\mathcal{N}_\varphi(h, x, \alpha) = (1 + h\alpha)x + s \cdot hx^2 + hx^3\,\widetilde{\eta}_3(h, x, \alpha) \tag{21}$$

with $s = 1$ or $s = -1$, where $\widehat{\eta}_3$ and $\widetilde{\eta}_3$ are smooth functions. Let $K > 0$ denote a uniform bound on $\left|\frac{\mathrm{d}^i}{\mathrm{d}x^i}\eta(h, \cdot, \alpha)\right|$ $(i \in \{0, 1, 2\}, \eta \in \{\widehat{\eta}_3, \widetilde{\eta}_3\})$ in a neighbourhood of the origin for any small $h > 0$ and $|\alpha|$, as well as a uniform bound on $\left|\frac{\mathrm{d}}{\mathrm{d}\alpha}\eta(h, x, \cdot)\right|$ $(\eta \in \{\widehat{\eta}_3, \widetilde{\eta}_3\})$ in a neighbourhood of the origin for any small $h > 0$ and $|x|$. We also have that there exists a constant $c > 0$ such that

$$|\mathcal{N}_\Phi(h, x, \alpha) - \mathcal{N}_\varphi(h, x, \alpha)| \le c \cdot h^{p+1}|x|^3 \tag{22}$$

holds for all sufficiently small $h > 0$, $|x| \ge 0$ and $|\alpha| \ge 0$. Throughout the section, c will denote this particular positive constant. (Other generic constants, if needed, are denoted by *const*.)

We will consider the case $s = 1$, the other one is similar. Then it is easy to see that $\omega_{\Phi,0}(h, \alpha) \equiv 0$ is an attracting fixed point of the map $\mathcal{N}_\Phi(h, \cdot, \alpha)$ for $\alpha < 0$, and repelling for $\alpha > 0$. For any fixed $h > 0$ and $\alpha \in [-\alpha_0, \alpha_0] \setminus \{0\}$, this map possesses another fixed point, denoted by $\omega_{\Phi,+} \equiv \omega_{\Phi,+}(h, \alpha) > 0$ (if $\alpha < 0$) and $\omega_{\Phi,-} \equiv \omega_{\Phi,-}(h, \alpha) < 0$ (if $\alpha > 0$). It is seen that $\omega_{\Phi,+}$ is repelling and $\omega_{\Phi,-}$ is attracting. The two branches of fixed points, $\omega_{\Phi,0}(h, \alpha)$ and $\omega_{\Phi,\pm}(h, \alpha)$ merge at $\alpha = 0$.

Analogous results hold, of course, for the map $\mathcal{N}_\varphi(h, \cdot, \alpha)$. Its fixed points are denoted by $\omega_{\varphi,0}$ and $\omega_{\varphi,-}$ (or $\omega_{\varphi,+}$).

We will construct a conjugacy in a natural way and prove optimal closeness estimates in the $x \le 0$ region—the $x > 0$ case is similar due to symmetry.

In what follows, we suppose that

$$0 < h \le h_0 := \frac{1}{5},$$

$$|x| \le \varepsilon_0 := \min\left(\frac{1}{25}, \frac{1}{25K}\right) \text{ and} \tag{23}$$

$$|\alpha| \le \alpha_0 := \min\left(\frac{1}{51}, \frac{1}{51K}\right).$$

With these values of h_0, ε_0 and α_0, all constructions and proofs below can be carried out. (There is only one constraint which has not been taken into account explicitly: if the domain of definition of the functions $\widehat{\eta}_3$ and $\widetilde{\eta}_3$ is smaller than $(0, h_0] \times [-\varepsilon_0, \varepsilon_0] \times [-\alpha_0, \alpha_0]$ given above, then h_0, ε_0 or α_0 should be decreased further suitably.)

Lemma 3.1 *For every* $0 < h \le h_0$ *and* $0 < \alpha \le \alpha_0$ *we have that*

$$\{\omega_{\varphi,-}, \omega_{\Phi,-}\} \subset \left(-\frac{3}{2}\alpha, -\frac{6}{7}\alpha\right).$$

Proof. By definition, $\omega_{\varphi,-}$ solves $\alpha + x + x^2 \cdot \widetilde{\eta}_3(h, x, \alpha) = 0$. But $|x| \le \frac{1}{6K}$ implies $\frac{2}{3} \le 1 + x\,\widetilde{\eta}_3 \le \frac{7}{6}$, so

$$-\frac{3\alpha}{2} \le \omega_{\varphi,-} = \frac{-\alpha}{1 + \omega_{\varphi,-} \cdot \widetilde{\eta}_3(h, \omega_{\varphi,-}, \alpha)} \le -\frac{6\alpha}{7}.$$

The proof for $\omega_{\Phi,-}$ is similar. ■

By iterating one of the normal forms, say $\mathcal{N}_\varphi(h, \cdot, \alpha)$, let us define three sequences x_n, y_n and z_n. For $\alpha > 0$, let $x_n \equiv x_n(h, \alpha)$ be defined as

$$x_{n+1} := \mathcal{N}_\varphi(h, x_n, \alpha), \quad n = 0, 1, 2, \ldots$$

with $x_0 := -\frac{\alpha}{3}$, further, let $y_n \equiv y_n(h, \alpha)$ be defined as

$$y_n := \left(\mathcal{N}_\varphi^E\right)^{[-n]}(x_0), \quad n = 0, 1, 2, \ldots,$$

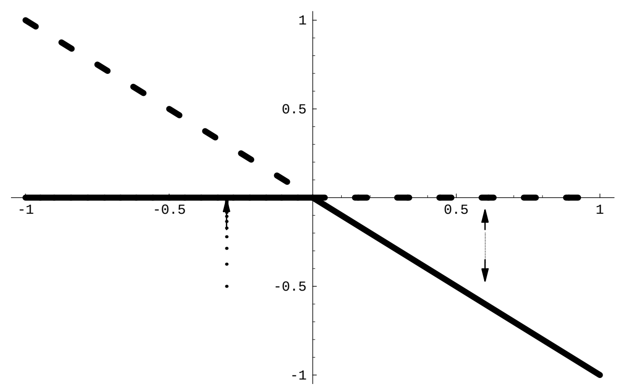
so $y_0 := x_0$, and set $y_{-1} := x_1$. Finally, for all $\alpha \in [-\alpha_0, \alpha_0]$ define $z_n \equiv z_n(h, \alpha)$ as

$$z_n := \left(\mathcal{N}_\varphi^E\right)^{[n]}(z_0), \quad n = 0, 1, 2, \ldots,$$

with $z_0 < 0$ being independent of h and α such that $2\alpha_0 < |z_0| < \frac{1}{2K}$ holds. An appropriate choice for z_0 is, *e.g.*, $z_0 := -\varepsilon_0$.

Simple calculations show that, for example, under conditions (23), both $\mathcal{N}_\varphi^E$ and $\mathcal{N}_\Phi^E$ (together with their inverses) are monotone increasing, further $|\alpha| < \frac{6}{K}$ implies $x_0(\alpha) > x_1(h, \alpha)$ and $2\alpha_0 < |z_0| < \frac{1}{2K}$ implies $z_0 < z_1(h, \alpha)$. This means that x_n is monotone decreasing, y_n is monotone increasing (if $\alpha > 0$ and $n \ge 0$), and $\lim_{n\to\infty} x_n(h, \alpha) = \omega_{\varphi,-}$, while $\lim_{n\to\infty} y_n(h, \alpha) = \omega_{\varphi,0}$. Moreover, z_n is monotone increasing, further, for $\alpha > 0$, $\lim_{n\to\infty} z_n(h, \alpha) = \omega_{\varphi,-}$ and for $\alpha \le 0$, $\lim_{n\to\infty} z_n(h, \alpha) = \omega_{\varphi,0}$.

The following figure shows the branch of stable and unstable fixed points of $\mathcal{N}_\varphi^E$ in the (α, x)-plane together with the first few terms of the inner sequences ($x_n(h,\alpha)$ and $y_n(h,\alpha)$), and the outer sequence $z_n(h,\alpha)$ with some $h > 0$ and α fixed. The arrows indicate the direction of the sequences.



A homeomorphism J^E satisfying the conjugacy equation

$$J^E \circ \mathcal{N}_\varphi^E = \mathcal{N}_\Phi^E \circ J^E \tag{24}$$

is now piecewise defined on the fundamental domains, *i.e.* on $[x_{n+1}, x_n]$, $[y_n, y_{n+1}]$ and $[z_n, z_{n+1}]$ ($n \in \mathbb{N}$), for any fixed $0 < h \le h_0$ and $-\alpha_0 \le \alpha \le \alpha_0$.

We first consider the region between the fixed points for $0 < \alpha \le \alpha_0$.

Let $J^E(x_0) := x_0$ and $J^E(x_1) := \mathcal{N}_\Phi^E(x_0)$. For $x \in [x_1, x_0]$ extend J^E linearly. For $n \ge 1$ and $x \in [x_{n+1}, x_n]$, we recursively set

$$J^E(x) := \left(\mathcal{N}_\Phi^E \circ J^E \circ \left(\mathcal{N}_\varphi^E\right)^{[-1]}\right)(x),$$

while for $n \ge 0$ and $x \in [y_n, y_{n+1}]$, we let

$$J^E(x) := \left(\left(\mathcal{N}_\Phi^E\right)^{[-1]} \circ J^E \circ \mathcal{N}_\varphi^E\right)(x).$$

(Since $[y_{-1}, y_0] \equiv [x_1, x_0]$, these two definitions are compatible.) Finally, set

$$J^E(\omega_{\varphi,-}) := \omega_{\Phi,-}$$

and

$$J^E(\omega_{\varphi,0}) := \omega_{\Phi,0}.$$

Then J^E is continuous, strictly monotone increasing on $[\omega_{\varphi,-}, 0]$, since it is a composition of three such functions, and satisfies (24).

In the outer region, *i.e.* below the fixed points, fix $z_0 < 0$ ($2\alpha_0 < |z_0| < \frac{1}{2K}$), then for $\alpha \in [-\alpha_0, \alpha_0]$ the construction of J^E is analogous to the construction above with the sequence x_n: this time z_n plays the role of x_n. (Of course, now the counterpart of the sequence y_n is not needed.) Then the function J^E becomes continuous, strictly monotone increasing on $[z_0, \omega_{\varphi,-}]$ ($0 < \alpha \le \alpha_0$) and $[z_0, \omega_{\varphi,0}]$ (for $-\alpha_0 \le \alpha \le 0$), and satisfies (24).

The construction of J^E—with the appropriate and natural modifications—in the upper half-plane $x > 0$ is analogous to the one presented above.

4 The closeness estimate for the conjugacy

4.1 Optimality at the fixed points

We first prove that the constructed conjugacy J^E is $\mathcal{O}(h^p\,\alpha^2)$-close to the identity at the fixed points $\omega_{\varphi,-}(h,\alpha)$, further, an explicit example will show that this estimate is optimal in h and α.

Since fixed points must be mapped into nearby fixed points by the conjugacy and we are going to prove $\mathcal{O}(h^p)$-closeness in the whole domain, the result above means that our estimates of $|\,id - J^E|$ near a transcritical bifurcation point are optimal in h.

The following auxiliary estimate will frequently be used.

Lemma 4.1 *For any $0 < h \le h_0$, $-\varepsilon_0 \le x < 0$ and $-\alpha_0 \le \alpha \le \alpha_0$, we have that*

$$(\mathcal{N}_\Phi^E)'(x) \le 1 + h\alpha + \frac{7}{4}hx.$$

Proof. The conditions in (23) have been set up to imply this inequality, too. ∎

Lemma 4.2 *For any $0 < h \le h_0$ and $0 < \alpha \le \alpha_0$ (satisfying (23)), we have that*

$$|\,\omega_{\varphi,-} - \omega_{\Phi,-}| \le \frac{27}{4}c \cdot h^p\,\alpha^2.$$

Proof.

$$|\,id - J^E|(\omega_{\varphi,-}(h,\alpha)) \le |\mathcal{N}_\varphi^E(\omega_{\varphi,-}) - \mathcal{N}_\Phi^E(\omega_{\varphi,-})| + |\mathcal{N}_\Phi^E(\omega_{\varphi,-}) - \mathcal{N}_\Phi^E(\omega_{\Phi,-})| \le$$

$$c \cdot h^{p+1}|\omega_{\varphi,-}|^3 + \left(\sup_{[\{\,\omega_{\varphi,-},\omega_{\Phi,-}\}]}(\mathcal{N}_\Phi^E)'\right)|\,\omega_{\varphi,-} - \omega_{\Phi,-}| \le$$

$$\frac{27}{8}c \cdot h^{p+1}\alpha^3 + \left(1 - \frac{h\alpha}{2}\right)|\,\omega_{\varphi,-} - \omega_{\Phi,-}|,$$

by Lemma 3.1, (22) and Lemma 4.1. Solving the above inequality for $|\,\omega_{\varphi,-} - \omega_{\Phi,-}| \equiv |\,id - J^E|(\omega_{\varphi,-})$ yields the desired result. ∎

Remark 4.1 on optimality. The next example shows that the distance of fixed points of normal forms satisfying (22) can be bounded from *below* by $\mathcal{O}(h^p)$ $(h \to 0)$.

Indeed, set $\mathcal{N}_\Phi(h,x,\alpha) := (1+h\alpha)x + hx^2$ and $\mathcal{N}_\varphi(h,x,\alpha) := (1+h\alpha)x + hx^2 + h^{p+1}x^3$. Then these maps satisfy (22) in a neighbourhood of the origin, further, $\omega_{\Phi,-} = -\alpha$ and $\omega_{\varphi,-} = \frac{-1+\sqrt{1-4h^p\,\alpha}}{2h^p}$. Using inequality $1+\frac{t}{2}-\frac{t^2}{4} \le \sqrt{1+t} \le 1+\frac{t}{2}-\frac{t^2}{8}$ for $-\frac{1}{2} \le t \le 0$, one sees that

$$|\,\omega_{\varphi,-} - \omega_{\Phi,-}| \ge h^p\,\alpha^2,$$

if, for example, $h \le 1$ and $\alpha \le \frac{1}{8}$.

4.2 The inner region

Now the closeness estimate in $(\omega_{\varphi,-},x_0]$ is proved for any fixed $0 < h \le h_0$ and $0 < \alpha \le \alpha_0$. It is clear that $\sup_{(\omega_{\varphi,-},x_0]} |\, id - J^E| = \sup_{n\in\mathbb{N}} \sup_{[x_{n+1},x_n]} |\, id - J^E|$.

Since $x_0 = J^E(x_0)$, we have that

$$\sup_{[x_1,x_0]} |\, id - J^E| = |x_1 - J^E(x_1)| = |\mathcal{N}_\varphi^E(x_0) - \mathcal{N}_\Phi^E(x_0)|$$

$$\le c\cdot h^{p+1}|x_0|^3 = \frac{c}{27}h^{p+1}\alpha^3,$$

while for $n \ge 1$

$$\sup_{[x_{n+1},x_n]} |\, id - J^E| \le \sup_{[x_{n+1},x_n]} \left|\mathcal{N}_\varphi^E \circ (\mathcal{N}_\varphi^E)^{[-1]} - \mathcal{N}_\Phi^E \circ (\mathcal{N}_\varphi^E)^{[-1]}\right| +$$

$$+ \sup_{[x_{n+1},x_n]} \left|\mathcal{N}_\Phi^E \circ (\mathcal{N}_\varphi^E)^{[-1]} - \mathcal{N}_\Phi^E \circ J^E \circ (\mathcal{N}_\varphi^E)^{[-1]}\right| =$$

$$= \sup_{[x_n,x_{n-1}]} \left|\mathcal{N}_\varphi^E - \mathcal{N}_\Phi^E\right| + \sup_{[x_n,x_{n-1}]} \left|\mathcal{N}_\Phi^E - \mathcal{N}_\Phi^E \circ J^E\right| \le$$

$$\le \sup_{[x_n,x_{n-1}]} \left|\mathcal{N}_\varphi^E - \mathcal{N}_\Phi^E\right| + \sup_{x\in[x_n,x_{n-1}]} \left(\left(\sup_{[\{x,J^E(x)\}]} (\mathcal{N}_\Phi^E)'\right) |\, x - J^E(x)|\right) \le$$

$$\le c\cdot h^{p+1}|x_n|^3 + \left(1 + h\alpha + \frac{7}{4}h \max\left(x_{n-1}, J^E(x_{n-1})\right)\right) \sup_{[x_n,x_{n-1}]} |\, id - J^E|,$$

the last inequality being true due to

$$\sup_{[\{x,J^E(x)\}]} (\mathcal{N}_\Phi^E)' \le \sup_{[\{x,J^E(x)\}]} (1 + h\alpha + \frac{7}{4}h\cdot id) \le 1 + h\alpha + \frac{7}{4}h \max\left(x, J^E(x)\right)$$

taking into account Lemma 4.1, then using the fact that the functions id and J^E are increasing.

From these we have for $n \ge 1$ that

$$\sup_{[x_{n+1},x_n]} |\, id - J^E| \le c\cdot h^{p+1} \sum_{i=0}^{n} |x_i|^3 \prod_{j=i}^{n-1} \left(1 + h\alpha + \frac{7}{4}h \max\left(x_j, J^E(x_j)\right)\right),$$

where $\prod_{j=n}^{n-1}$ is understood to be 1.

So in order to prove that the conjugacy J^E is $\mathcal{O}(h^p)$-close to the identity on the interval $(\omega_{\varphi,-},x_0]$ for any $h \in (0,h_0]$ and $\alpha \in (0,\alpha_0]$, it is enough to show that

$$\sup_{h\in(0,h_0]} \sup_{\alpha\in(0,\alpha_0]} \sup_{n\in\mathbb{N}} h \sum_{i=0}^{n} |x_i|^3 \prod_{j=i}^{n-1} \left(1 + h\alpha + \frac{7}{4}h \max\left(x_j, J^E(x_j)\right)\right) \le const \qquad (25)$$

holds with a suitable $const \ge 0$.

First an explicit estimate of the sequence $\max\left(x_n, J^E(x_n)\right)$ is given.

Lemma 4.3 *For $n \geq 0$, set*

$$a_n(h,\alpha) := -\frac{3}{4}\alpha \cdot \frac{(1+h\alpha)^{n+1}}{2+(1+h\alpha)^n},$$

then we have that $x_n \in (\omega_{\varphi,-}, a_n)$ and $J^E(x_n) \in (\omega_{\Phi,-}, a_n)$.

Proof. It is easily checked that, due to assumptions (23),

$$\max\left(\omega_{\varphi,-}, \omega_{\Phi,-}\right) < a_n$$

for $n \geq 0$, so the intervals in the lemma are non-degenerate. We proceed by induction.

$a_0 = -\frac{\alpha}{4}(1+h\alpha) > x_0 \equiv J^E(x_0) \equiv -\frac{\alpha}{3}$ is equivalent to $h\alpha < \frac{1}{3}$, being true by assumptions (23) on h_0 and α_0.

So suppose that the statement is true for some $n \geq 0$. Since $\mathcal{N}_\varphi^E(x) < (1+h\alpha)x + \frac{6}{5}h\,x^2$ is implied by $|x| \leq \varepsilon_0 < \frac{1}{5K}$, and $\mathcal{N}_\varphi^E$ is monotone increasing, we get that

$$x_{n+1} = \mathcal{N}_\varphi^E(x_n) < \mathcal{N}_\varphi^E(a_n) < (1+h\alpha)a_n + \frac{6}{5}h\,a_n^2,$$

thus it is enough to prove that the right-hand side above is smaller than a_{n+1}. But

$$a_{n+1} - \left((1+h\alpha)a_n + \frac{6}{5}h\,a_n^2\right) =$$

$$-\frac{3h\alpha^2(1+h\alpha)^{2+2n}\left(-2+(1+h\alpha)^n(-1+9h\alpha)\right)}{40\left(2+(1+h\alpha)^n\right)^2\left(2+(1+h\alpha)^{n+1}\right)} > 0$$

is equivalent to $-2+(1+h\alpha)^n(-1+9h\alpha) < 0$, which is implied by $h\alpha < \frac{1}{9}$.

Of course, the above inequalities remain true, if $\mathcal{N}_\varphi$ is replaced by $\mathcal{N}_\Phi$, also noticing that, by construction, $J^E(x_{n+1}) = \mathcal{N}_\Phi^E(J^E(x_n))$, so the induction is complete. ∎

Remark 4.2.1 The induction would fail, if, in estimate $\mathcal{N}_\varphi^E(x) < (1+h\alpha)x + \frac{6}{5}h\,x^2$, the constant $\frac{6}{5}$ was replaced by, say, $\frac{7}{5}$. (The explanation resides in the particular choice of the constant $\frac{3}{4}$ in the definition of a_n, since $\frac{3}{4}\cdot\frac{6}{5} < 1 < \frac{3}{4}\cdot\frac{7}{5}$.)

Remark 4.2.2 The upper estimate a_n in our first main lemma has been found by computer experiments with *Mathematica* based on the parametrized model function in [6].

In order to prove the boundedness of (25), the sum $\sum_{i=0}^{n}$ will be split into two. An appropriate index to split at is $\lceil \frac{const}{h\alpha} \rceil$, as established by the following lemma.

Lemma 4.4 *Suppose that $n > \lceil \frac{6}{h\alpha} \rceil$. Then*

$$\max\left(x_n, J^E(x_n)\right) < -\frac{2}{3}\alpha,$$

hence

$$1 + h\alpha + \frac{7}{4}h\max\left(x_n, J^E(x_n)\right) < 1 - \frac{h\alpha}{6}$$

holds for $n > \lceil \frac{6}{h\alpha} \rceil$.

Proof. By Lemma 4.3 it is sufficient to show that $n > \lceil \frac{6}{h\alpha} \rceil$ implies $a_n < -\frac{2}{3}\alpha$. This latter inequality is equivalent to $(1+h\alpha)^n(1+9h\alpha) > 16$. But if $n > \lceil \frac{6}{h\alpha} \rceil$, then

$$(1+h\alpha)^n > (1+h\alpha)^{\lceil \frac{6}{h\alpha} \rceil} = \left(1+\frac{1}{\frac{1}{h\alpha}}\right)^{(1+\frac{1}{h\alpha})\cdot\frac{h\alpha}{1+h\alpha}\cdot\lceil \frac{6}{h\alpha} \rceil}.$$

However, it is known that $\left(1+\frac{1}{A}\right)^{A+1} > e$, if $A \ge 1$, and it is easy to see that $\frac{B}{1+B}\cdot\lceil \frac{6}{B} \rceil > 3$, if $0 < B < 1$. Since $e^3 > 16$, the proof is complete. ■

Now we can turn to (25). Fix $h \in (0, h_0]$, $\alpha \in (0, \alpha_0]$ and $n \in \mathbb{N}^+$. (If $n \le \lceil \frac{6}{h\alpha} \rceil$, then the sums $\sum_{i=\lceil \frac{6}{h\alpha} \rceil+1}^{n}$ below are, of course, not present, making the proof even simpler.) Since now $\omega_{\varphi,-} < x_i < 0$, by Lemma 3.1 $|x_i| \le \frac{3}{2}\alpha$, and by monotonicity $\max\left(x_j, J^E(x_j)\right) \le x_0 \equiv J^E(x_0) \equiv -\frac{\alpha}{3}$, further, by using Lemma 4.4, assumption $h\alpha < 1$ from (23) and inequality $(1+\frac{1}{A})^A \le e$ (if $A \ge 1$), we get that

$$h\sum_{i=0}^{n} |x_i|^3 \prod_{j=i}^{n-1}\left(1+h\alpha+\frac{7}{4}h\max\left(x_j, J^E(x_j)\right)\right) \le$$

$$\frac{27h\alpha^3}{8}\sum_{i=0}^{\lceil \frac{6}{h\alpha} \rceil}\prod_{j=1}^{\lceil \frac{6}{h\alpha} \rceil-1}\left(1+h\alpha-\frac{7}{4}\cdot\frac{h\alpha}{3}\right) + \frac{27h\alpha^3}{8}\sum_{i=\lceil \frac{6}{h\alpha} \rceil+1}^{n}\prod_{j=i}^{n-1}\left(1-\frac{h\alpha}{6}\right) \le$$

$$\frac{27h\alpha^3}{8}\left(1+\frac{5}{12}h\alpha\right)^{\frac{6}{h\alpha}}\left(\left\lceil \frac{6}{h\alpha} \right\rceil + 1\right) + \frac{27h\alpha^3}{8}\sum_{i=\lceil \frac{6}{h\alpha} \rceil+1}^{n}\left(1-\frac{h\alpha}{6}\right)^{n-i} \le$$

$$\frac{27h\alpha^3}{8}\left(1+\frac{5}{12}h\alpha\right)^{\frac{12}{5h\alpha}\cdot\frac{5h\alpha}{12}\cdot\frac{6}{h\alpha}}\left(\frac{6+2h\alpha}{h\alpha}\right) + \frac{27h\alpha^3}{8}\sum_{i=0}^{\infty}\left(1-\frac{h\alpha}{6}\right)^{i} \le$$

$$\frac{27h\alpha^3}{8}\cdot e^{\frac{30}{12}}\cdot\frac{8}{h\alpha} + \frac{27h\alpha^3}{8}\cdot\frac{6}{h\alpha} \le 350\,\alpha^2.$$

Therefore, $\sup_{[x_{n+1},x_n]} |\, id - J^E| \le 350c \cdot h^p\alpha^2$ for any $h \in (0, h_0]$, $\alpha \in (0, \alpha_0]$ and $n \ge 1$, further, as we have seen, $\sup_{[x_1,x_0]} |\, id - J^E| \le \frac{c}{27}h^{p+1}\alpha^3$, which yield the following lemma.

Lemma 4.5 *Under assumption (23)*

$$\sup_{(\omega_{\varphi,-},x_0]} |\, id - J^E| \le 350c \cdot h^p\alpha^2.$$

Now the closeness estimate is proved in the interval $(y_0, \omega_{\varphi,0})$. Recall that $y_0 = x_0 = J^E(x_0) \equiv -\frac{\alpha}{3}$ and $\omega_{\varphi,0} = \omega_{\Phi,0} \equiv 0$.

Suppose that $n \ge 1$. (The case $n = 0$ will be examined later.) Then

$$\sup_{[y_n,y_{n+1}]} |\, id - J^E| = \sup_{[y_n,y_{n+1}]} \left|\left(\mathcal{N}_\Phi^E\right)^{[-1]} \circ \mathcal{N}_\Phi^E - \left(\mathcal{N}_\Phi^E\right)^{[-1]} \circ J^E \circ \mathcal{N}_\varphi^E\right| \le$$

$$\sup_{x\in[y_n,y_{n+1}]}\left[\left(\sup_{[\{\mathcal{N}_\Phi^E(x), J^E\circ\mathcal{N}_\varphi^E(x)\}]}\left(\left(\mathcal{N}_\Phi^E\right)^{[-1]}\right)'\right)\left(\left|\mathcal{N}_\Phi^E - \mathcal{N}_\varphi^E\right|(x) + \left|\mathcal{N}_\varphi^E - J^E\circ\mathcal{N}_\varphi^E\right|(x)\right)\right]$$

$$\leq \left[\sup_{x\in[y_n,y_{n+1}]} \sup_{[\{\mathcal{N}_\Phi^E(x),J^E\circ\mathcal{N}_\varphi^E(x)\}]} \left((\mathcal{N}_\Phi^E)^{[-1]} \right)' \right] \left[c\cdot h^{p+1}|y_n|^3 + \sup_{[y_{n-1},y_n]} |\, id - J^E| \right],$$

provided that $\sup_{[\{\mathcal{N}_\Phi^E(x),J^E\circ\mathcal{N}_\varphi^E(x)\}]} \left((\mathcal{N}_\Phi^E)^{[-1]} \right)'$ is nonnegative.

Lemma 4.6 *Suppose that $n \geq 1$, then under assumption (23) we have that*

$$\sup_{x\in[y_n,y_{n+1}]} \sup_{[\{\mathcal{N}_\Phi^E(x),J^E\circ\mathcal{N}_\varphi^E(x)\}]} \left((\mathcal{N}_\Phi^E)^{[-1]} \right)' \leq 1 - \frac{h\alpha}{8}.$$

Proof.

$$\sup_{x\in[y_n,y_{n+1}]} \sup_{[\{\mathcal{N}_\Phi^E(x),J^E\circ\mathcal{N}_\varphi^E(x)\}]} \left((\mathcal{N}_\Phi^E)^{[-1]} \right)' = \sup_{x\in[y_n,y_{n+1}]} \sup_{[\{\mathcal{N}_\Phi^E(x),J^E\circ\mathcal{N}_\varphi^E(x)\}]} \frac{1}{(\mathcal{N}_\Phi^E)'\circ(\mathcal{N}_\Phi^E)^{[-1]}}$$

$$= \sup_{x\in[y_n,y_{n+1}]} \sup_{[\{x,(\mathcal{N}_\Phi^E)^{[-1]}\circ J^E\circ\mathcal{N}_\varphi^E(x)\}]} \frac{1}{(\mathcal{N}_\Phi^E)'} = \dots$$

But, by definition, $(\mathcal{N}_\Phi^E)^{[-1]}\circ J^E\circ\mathcal{N}_\varphi^E(x) = J^E(x)$, if $x\in[y_n,y_{n+1}]$, and $[\{x, J^E(x)\}] = [\min(x, J^E(x)), \max(x, J^E(x))]$, further, by the monotonicity of id and J^E we obtain that

$$\dots = \sup_{[\min(y_n,J^E(y_n)),\max(y_{n+1},J^E(y_{n+1}))]} \frac{1}{(\mathcal{N}_\Phi^E)'} \leq \dots$$

By construction, however, $[\min(y_n, J^E(y_n)), \max(y_{n+1}, J^E(y_{n+1}))] \subset (y_0, 0) = (-\frac{\alpha}{3}, 0)$ and $(\mathcal{N}_\Phi^E)'$ is nonnegative here by assumption (23), justifying the computations just above the lemma. We now continue the proof of the lemma.

$$\dots \leq \sup_{(-\frac{\alpha}{3},0)} \frac{1}{(\mathcal{N}_\Phi^E)'} \leq \dots$$

It is easy to see that assumption (23) together with $x<0$ imply that $(\mathcal{N}_\Phi^E)'(x) \geq 1 + h\alpha + \frac{9}{4}hx \geq 0$. So

$$\dots \leq \sup_{x\in(-\frac{\alpha}{3},0)} \frac{1}{1+h\alpha+\frac{9}{4}hx} \leq \frac{1}{1+h\alpha+\frac{9}{4}h\left(-\frac{\alpha}{3}\right)} = \frac{1}{1+\frac{1}{4}h\alpha} \leq 1 - \frac{h\alpha}{8},$$

since $\frac{1}{1+A} \leq 1 - \frac{A}{2}$, if $A\in[0,1]$. ∎

We have thus proved (also using $|y_n| \leq \frac{\alpha}{3}$) that for $n \geq 1$

$$\sup_{[y_n,y_{n+1}]} |\, id - J^E| \leq \left(1 - \frac{h\alpha}{8}\right) \left[\frac{c}{27}\cdot h^{p+1}\alpha^3 + \sup_{[y_{n-1},y_n]} |\, id - J^E| \right] \qquad (26)$$

For $n=0$, similarly as before, we get that

$$\sup_{[y_0,y_1]} |\, id - J^E| \leq \left[\sup_{x\in[y_0,y_1]} \sup_{[\{\mathcal{N}_\Phi^E(x),J^E\circ\mathcal{N}_\varphi^E(x)\}]} \left((\mathcal{N}_\Phi^E)^{[-1]} \right)' \right] \left[c\cdot h^{p+1}|y_0|^3 + \sup_{[y_{-1},y_0]} |\, id - J^E| \right].$$

But $[y_{-1}, y_0] \equiv [x_1, x_0]$, so the second factor $[\dots]$ is bounded by $2\cdot\frac{c}{27}h^{p+1}\alpha^3$. As for the first factor $[\dots]$, we notice that $y_0 < (\mathcal{N}_\Phi^E)^{[-1]}(y_0)$ (since this is equivalent to $x_1 < x_0$), which implies that

$$\sup_{x\in[y_0,y_1]} \sup_{[\{\mathcal{N}_\Phi^E(x),J^E\circ\mathcal{N}_\varphi^E(x)\}]} \left((\mathcal{N}_\Phi^E)^{[-1]} \right)' = \sup_{x\in[y_0,y_1]} \sup_{[\{x,(\mathcal{N}_\Phi^E)^{[-1]}\circ J^E\circ\mathcal{N}_\varphi^E(x)\}]} \frac{1}{(\mathcal{N}_\Phi^E)'} =$$

$$\sup_{[y_0,y_1]\cup[y_0,(\mathcal{N}_\Phi^E)^{[-1]}(y_0)]} \frac{1}{(\mathcal{N}_\Phi^E)'} \le \sup_{[y_0,0)} \frac{1}{(\mathcal{N}_\Phi^E)'} \le 1,$$

therefore

$$\sup_{[y_0,y_1]} |\,id - J^E| \le 2\cdot \frac{c}{27} h^{p+1}\alpha^3. \tag{27}$$

Repeated application of (26), further (27) yield for $n \ge 1$ that

$$\sup_{[y_n,y_{n+1}]} |\,id - J^E| \le \left(1-\frac{h\alpha}{8}\right)^n \sup_{[y_0,y_1]} |\,id - J^E| + \frac{c}{27}h^{p+1}\alpha^3 \sum_{i=1}^{n}\left(1-\frac{h\alpha}{8}\right)^i \le$$

$$1\cdot 2\cdot \frac{c}{27}h^{p+1}\alpha^3 + \frac{c}{27}h^{p+1}\alpha^3\cdot\frac{8}{h\alpha} \le \frac{c}{3}h^p\alpha^2,$$

due to $h\alpha \le \frac{1}{2}$ by (23). The same upper estimate is valid for $n = 0$, so we have proved the following result.

Lemma 4.7 *Under assumption (23)*

$$\sup_{(x_0,0)} |\,id - J^E| \le \frac{c}{3}h^p\alpha^2.$$

5 The outer region

In this section, we first prove an $\mathcal{O}(h^p)$ closeness-estimate in the interval $[z_0, \omega_{\varphi,-})$ for $\alpha > 0$. Then, in the second part, the closeness is proved on $[z_0, \omega_{\Phi,0}) \equiv [z_0, 0)$ for $\alpha \le 0$.

The derivation of the following formulae is similar to their counterparts in the inner region, with the difference that—since this time the sequence z_n is increasing—an extra term and an index-shift occur.

For $n \ge 1$ (also using (23)) we have that

$$\sup_{[z_n,z_{n+1}]} |\,id - J^E| \le c\cdot h^{p+1}|z_0|^3 \prod_{j=1}^{n}\left(1+h\alpha+\frac{7}{4}h\max\left(z_j, J^E(z_j)\right)\right) +$$

$$c\cdot h^{p+1}\sum_{i=0}^{n-1}|z_i|^3 \prod_{j=i+2}^{n}\left(1+h\alpha+\frac{7}{4}h\max\left(z_j, J^E(z_j)\right)\right), \tag{28}$$

where, again $\prod_{j=n+1}^{n}$ above is 1, and

$$\sup_{[z_0,z_1]} |\,id - J^E| \le c\cdot h^{p+1}|z_0|^3.$$

The following main lemma, as a counterpart of Lemma 4.3, gives a lower estimate of the sequence z_n, if $\alpha > 0$.

Lemma 5.1 *For $n \ge 0$, set*

$$b_n(h,\alpha) := -2\alpha\cdot\frac{(1+h\alpha)^{n+1}}{-1+\alpha+(1+h\alpha)^n},$$

then $b_n \le \min\left(z_n, J^E(z_n)\right)$.

Proof. $b_0 = 2 - 2h\alpha < -2 \le -1 \le -\varepsilon_0 \le z_0 = J^E(z_0)$ holds due to assumption (23). Suppose that the statement is true for some $n \ge 0$. Since $\mathcal{N}_\varphi^E(x) \ge (1 + h\alpha)x + \frac{3}{5}h\,x^2$ follows from $|x| \le \varepsilon_0 < \frac{2}{5K}$, further $(1+h\alpha)id + \frac{3}{5}h\,id^2$ is monotone increasing (which is implied by, *e.g.*, $|x| \le \frac{5}{6h}$, but it is easy to see that $h \le \frac{5}{18}$ and $-3 < b_n < 0$ follows from (23), hence $|b_n| \le \frac{5}{6h}$), so we obtain that

$$z_{n+1} = \mathcal{N}_\varphi^E(z_n) \ge (1+h\alpha)z_n + \frac{3}{5}h\,z_n^2 \ge (1+h\alpha)b_n + \frac{3}{5}h\,b_n^2,$$

thus it is sufficient to show that

$$(1+h\alpha)b_n + \frac{3}{5}h\,b_n^2 \ge b_{n+1}.$$

However, this is equivalent to

$$0 \le \frac{2h\alpha^2(1+h\alpha)^{2+2n}}{5\,(-1+\alpha+(1+h\alpha)^n)^2} \cdot \frac{-1+\alpha+(1+h\alpha)^n(1+6h\alpha)}{-1+\alpha+(1+h\alpha)^{n+1}},$$

which is true since $\alpha > 0$ and $h > 0$.

The proof remains valid if $\mathcal{N}_\varphi$ is replaced by $\mathcal{N}_\Phi$ (and $J^E(z_n)$ is written instead of z_n), hence $b_n \le J^E(z_n)$ also holds. ■

Now, since $z_j < \omega_{\varphi,-}$ and $J^E(z_j) < \omega_{\Phi,-}$, by Lemma 3.1 we get that the right-hand side of (28) is at most

$$c \cdot h^{p+1}|z_0|^3 \prod_{j=1}^{n}\left(1 - \frac{h\alpha}{2}\right) + c \cdot h^{p+1}\sum_{i=0}^{n-1}|z_i|^3 \prod_{j=i+2}^{n}\left(1 - \frac{h\alpha}{2}\right) \le$$

$$c \cdot h^{p+1}|z_0|^3 + c \cdot h^{p+1}\sum_{i=0}^{n-1}|z_i|^3\left(1 - \frac{h\alpha}{2}\right)^{n-1-i}.$$

We will verify that $h\sum_{i=0}^{n}|z_i|^3\left(1 - \frac{h\alpha}{2}\right)^{n-i}$ is uniformly bounded for any $n \ge 0$, $0 < h \le h_0$ and $0 < \alpha \le \alpha_0$.

If $n \ge \lceil \frac{1}{h\alpha} \rceil$, then by Lemma 5.1 (also using that $h\alpha \le \frac{1}{9}$ and $z_j < 0$)

$$h\sum_{i=\lceil \frac{1}{h\alpha}\rceil}^{n}|z_i|^3\left(1 - \frac{h\alpha}{2}\right)^{n-i} \le h\sum_{i=\lceil \frac{1}{h\alpha}\rceil}^{n}|b_i|^3\left(1 - \frac{h\alpha}{2}\right)^{n-i} \le$$

$$11h\alpha^3\sum_{i=\lceil \frac{1}{h\alpha}\rceil}^{n}\left(\frac{(1+h\alpha)^i}{-1+\alpha+(1+h\alpha)^i}\right)^3\left(1 - \frac{h\alpha}{2}\right)^{n-i} \le \ldots$$

for these i indices however $\frac{(1+h\alpha)^i}{-1+\alpha+(1+h\alpha)^i} \le 3$ holds (since this is implied by $\frac{3}{2} \le (1+h\alpha)^i$, being true by $(1+h\alpha)^i \ge (1+h\alpha)^{\frac{1}{h\alpha}} \ge 1 + \frac{1}{h\alpha}\cdot h\alpha > \frac{3}{2}$), thus

$$\ldots \le 27 \cdot 11\alpha^2 h\alpha \sum_{i=0}^{\infty}\left(1 - \frac{h\alpha}{2}\right)^i = 594\alpha^2.$$

On the other hand, if $n < \lceil \frac{1}{h\alpha} \rceil$, then (using that $|z_i| \le 1$ and $h\alpha \le \frac{1}{9}$ again)

$$h\sum_{i=0}^{n}|z_i|^3\left(1 - \frac{h\alpha}{2}\right)^{n-i} \le h\sum_{i=0}^{n}|z_i|^2\left(1 - \frac{h\alpha}{2}\right)^{n-i} \le \tag{29}$$

$$5h\sum_{i=0}^{n}\left(\frac{\alpha(1+h\alpha)^i}{-1+\alpha+(1+h\alpha)^i}\right)^2\left(1-\frac{h\alpha}{2}\right)^{n-i}\leq\ldots$$

now using inequalities $e^{\frac{x}{2}}\leq 1+x$ ($x\in[0,1]$) and $1+x\leq e^x$ ($x\in\mathbb{R}$) we get that $(1+h\alpha)^{2i}\leq e^{h\alpha 2i}\leq e^{h\alpha 2n}\leq e^2<8$, further, $\left(1-\frac{h\alpha}{2}\right)^{n-i}\leq e^{-\frac{h\alpha}{2}(n-i)}$ and $e^{\frac{h\alpha}{2}i}\leq(1+h\alpha)^i$, therefore

$$\ldots\leq 40h\sum_{i=0}^{n}\left(\frac{\alpha e^{-\frac{h\alpha}{4}(n-i)}}{-1+\alpha+e^{\frac{h\alpha}{2}i}}\right)^2.$$

Set $g_{h,\alpha,n}(x)\equiv g(x):=\left(\frac{\alpha\exp\left(-\frac{1}{4}h\alpha(n-x)\right)}{-1+\alpha+\exp\left(\frac{1}{2}h\alpha x\right)}\right)^2$, if $x\in[0,\infty)$. Notice that g is bounded at $x=0$. For this function we have that

$$g'(x)=-\frac{1}{2}h\alpha^3\,e^{-\frac{1}{2}h\alpha(n-x)}\cdot\frac{1-\alpha+e^{\frac{1}{2}hx\alpha}}{\left(-1+\alpha+e^{\frac{1}{2}hx\alpha}\right)^3},$$

meaning that g is strictly monotone decreasing, if $\alpha<1$. Hence

$$40h\sum_{i=0}^{n}\left(\frac{\alpha e^{-\frac{h\alpha}{4}(n-i)}}{-1+\alpha+e^{\frac{h\alpha}{2}i}}\right)^2=40h+40h\sum_{i=1}^{n}g_{h,\alpha,n}(i)\leq$$

$$40h+40h\int_0^n g_{h,\alpha,n}(x)\mathrm{d}x=40h+40h\left[-2\alpha\frac{\exp\left(-\frac{1}{2}h\alpha n\right)}{h\left(-1+\alpha+\exp\left(\frac{1}{2}h\alpha x\right)\right)}\right]_{x=0}^{n}=$$

$$40h+40h\left(\frac{2\left(1-\exp\left(-\frac{1}{2}h\alpha n\right)\right)}{h\left(\exp\left(\frac{1}{2}h\alpha n\right)-1+\alpha\right)}\right)\leq 40h+80\left(\frac{1-\exp\left(-\frac{1}{2}h\alpha n\right)}{\exp\left(\frac{1}{2}h\alpha n\right)-1}\right)=$$

$$40h+80e^{-\frac{1}{2}h\alpha n}\leq 120,$$

since $h\leq 1$.

Now combining all the estimates so far in the section, under assumption (23) we get that if $\alpha>0$, then

$$\sup_{[z_0,\omega_{\varphi,-})}|id-J^E|=\sup_{n\in\mathbb{N}}\sup_{[z_n,z_{n+1}]}|id-J^E|\leq$$

$$\sup_{n\in\mathbb{N}}\max\left(c\cdot h^{p+1}|z_0|^3,\ c\cdot h^{p+1}|z_0|^3+c\cdot h^{p+1}\sum_{i=0}^{n}|z_i|^3\left(1-\frac{h\alpha}{2}\right)^{n-i}\right)\leq$$

$$c\cdot h^{p+1}|z_0|^3+c\cdot h^p\cdot(120+594\alpha^2)\leq 130c\cdot h^p.$$

Remark 5.1 If, in (29), the exponent of $|z_i|$ had not been changed to 2, then the integral of g would have been significantly more complicated. (Interestingly, similar complication occurs, if one considers simply $|z_i|$ instead of $|z_i|^2$.) The rational pair $\frac{1}{4}$ and $\frac{1}{2}$ in the definition of g has also been a fortunate choice: when working with the numbers $\frac{1}{5}$ and $\frac{1}{2}$ instead, for example, *Mathematica* produced so complicated integrals that were practically useless from the viewpoint of further analysis.

Finally, we prove a closeness estimate on $[z_0, 0)$ for $\alpha \le 0$. We begin with a simple observation on monotonicity of the sequence $z_n \equiv z_n(\alpha)$. (As before, for brevity, the dependence on h is still suppressed.)

Lemma 5.2 *Suppose that $\alpha \le 0$ and assumption (23) hold. Then for any $0 < h \le h_0$, $-\alpha_0 \le \alpha \le \beta \le 0$ and $n \in \mathbb{N}$ we have that*

$$0 > z_n(\alpha) \ge z_n(\beta).$$

Proof. By definition, we have that $z_0(\alpha) = z_0(\beta) = z_0$, so suppose that for some n we already know that $z_n(\alpha) \ge z_n(\beta)$. Then, by the definition of the sequence z_n, further by the facts that the function $z \mapsto \mathcal{N}_\varphi(h, z, \alpha)$ is monotone *increasing* and the function $\alpha \mapsto \mathcal{N}_\varphi(h, z, \alpha)$ is monotone *decreasing*, we get that

$$z_{n+1}(\alpha) = \mathcal{N}_\varphi(h, z_n(\alpha), \alpha) \ge \mathcal{N}_\varphi(h, z_n(\beta), \alpha) \ge \mathcal{N}_\varphi(h, z_n(\beta), \beta) = z_{n+1}(\beta),$$

which completes the induction. ■

This means that $0 > z_n(\alpha) \ge z_n(0)$ holds for $\alpha \le 0$, hence it is enough to give a lower estimate for $z_n(0)$. But such an estimate has been constructed in Lemma 3.3 [1], namely we recall the following.

Lemma 5.3 *Under assumption (23), we have for $n \in \mathbb{N}$ that*

$$z_n(0) \ge z_0$$

and for $n \ge \lfloor \frac{1}{h} \rfloor + 1$

$$z_n(0) \ge -\frac{2}{nh}.$$

Then we can simply estimate (28) for $\alpha \le 0$ as follows. Supposing that $n \ge 1$ we get that

$$\sup_{[z_n, z_{n+1}]} |\, id - J^E| \le c \cdot h^{p+1} |z_0|^3 \prod_{j=1}^{n} \left(1 + h\alpha + \frac{7}{4} h \max\left(z_j, J^E(z_j)\right)\right) +$$

$$c \cdot h^{p+1} \sum_{i=0}^{n-1} |z_i|^3 \prod_{j=i+2}^{n} \left(1 + h\alpha + \frac{7}{4} h \max\left(z_j, J^E(z_j)\right)\right) \le$$

$$c \cdot h^{p+1} |z_0|^3 \cdot 1^{n-1} + c \cdot h^p \cdot h \sum_{i=0}^{n} |z_i(0)|^3 \cdot 1^{n-i-1} \le$$

$$c \cdot h^p \left(h|z_0|^3 + h \sum_{i=0}^{\lfloor \frac{1}{h} \rfloor} |z_i(0)|^2 + h \sum_{i=\lfloor \frac{1}{h} \rfloor + 1}^{n} |z_i(0)|^2 \right),$$

where, of course, for $n \le \lfloor \frac{1}{h} \rfloor$, the sum above $\sum_{i=\lfloor \frac{1}{h} \rfloor + 1}^{n}$ should be omitted. But

$$h \sum_{i=0}^{\lfloor \frac{1}{h} \rfloor} |z_i(0)|^2 \le h \cdot \frac{1}{h} \cdot z_0^2 = z_0^2,$$

and

$$h \sum_{i=\lfloor \frac{1}{h} \rfloor +1}^{n} |z_i(0)|^2 \leq h \sum_{i=\lfloor \frac{1}{h} \rfloor +1}^{n} \frac{4}{i^2 h^2} \leq \frac{4}{h} \int_{\frac{1}{h}-1}^{\infty} \frac{1}{i^2} = \frac{4}{1-h} \leq 8.$$

We have thus proved that

$$\sup_{[z_0,0)} |\, id - J^E| \leq 10c \cdot h^p.$$

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