Spectral Analysis of Localized Rotating Waves in Parabolic Systems

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Abstract. In this paper we study spectra and Fredholm properties of Ornstein-Uhlenbeck operators

$$\mathcal{L}v(x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\ x\in\mathbb{R}^d,\ d\geqslant 2,$$

where $v_\star:\mathbb{R}^d\to\mathbb{R}^m$ is the profile of a rotating wave satisfying $v_\star(x)\to v_\infty\in\mathbb{R}^m$ as $|x|\to\infty$, the map $f:\mathbb{R}^m\to\mathbb{R}^m$ is smooth, the matrix $A\in\mathbb{R}^{m,m}$ has eigenvalues with positive real parts and commutes with the limit matrix $Df(v_\infty)$. The matrix $S\in\mathbb{R}^{d,d}$ is assumed to be skew-symmetric with eigenvalues $(\lambda_1,\ldots,\lambda_d)=(\pm i\sigma_1,\ldots,\pm i\sigma_k,0,\ldots,0)$. The spectra of these linearized operators are crucial for the nonlinear stability of rotating waves in reaction diffusion systems. We prove under appropriate conditions that every $\lambda\in\mathbb{C}$ satisfying the dispersion relation

$$\det\Big(\lambda I_m + \eta^2 A - Df(v_\infty) + i\langle n,\sigma\rangle I_m\Big) = 0 \quad \text{for some } \eta\in\mathbb{R} \text{ and } n\in\mathbb{Z}^k,\ \sigma=(\sigma_1,\ldots,\sigma_k)^\top\in\mathbb{R}^k$$

belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ in L^p. For values $\operatorname{Re}\lambda$ to the right of the spectral bound for $Df(v_\infty)$ we show that the operator $\lambda I-\mathcal{L}$ is Fredholm of index 0, solve the identification problem for the adjoint operator $(\lambda I-\mathcal{L})^*$, and formulate the Fredholm alternative. Moreover, we show that the set

$$\sigma(S)\cup\{\lambda_i+\lambda_j:\ \lambda_i,\lambda_j\in\sigma(S),\ 1\leqslant i<j\leqslant d\}$$

belongs to the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ in L^p. We determine the associated eigenfunctions and show that they decay exponentially in space. As an application we analyze spinning soliton solutions which occur in the Ginzburg-Landau equation and compute their numerical spectra as well as associated eigenfunctions. Our results form the basis for investigating nonlinear stability of rotating waves in higher space dimensions and truncations to bounded domains.

Key words. Rotating wave, systems of reaction-diffusion equations, Fredholm theory, L^p-spectrum, essential spectrum, point spectrum, Ornstein-Uhlenbeck operator.
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1. Introduction

In the present paper we study operators obtained from linearizing reaction-diffusion systems

$$(1.1)\qquad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)), && t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) &= u_0(x), && t=0,\ x\in\mathbb{R}^d, \end{aligned}$$

where $A\in\mathbb{R}^{m,m}$ has eigenvalues with positive real part, $f:\mathbb{R}^m\to\mathbb{R}^m$ is sufficiently smooth, $u_0:\mathbb{R}^d\to\mathbb{R}^m$ are initial data and $u:\mathbb{R}^d\times[0,\infty)\to\mathbb{R}^m$ denotes a vector-valued solution.
Our main interest is in rotating wave solutions of (1.1) of the form

$$(1.2)\qquad u_\star(x,t) = v_\star(e^{-tS}x),\ t\geqslant 0,\ x\in\mathbb{R}^d,\ d\geqslant 2$$

with space-dependent profile $v_\star:\mathbb{R}^d\to\mathbb{R}^m$ and skew-symmetric matrix $S\in\mathbb{R}^{d,d}$. The term e^{-tS} describes rotations in $\mathbb{R}^d$, and hence $u_\star$ is a solution rotating at constant velocity while maintaining its shape determined by $v_\star$. In general, this motion will be periodic only if the eigenvalues of S are rationally dependent but quasiperiodic otherwise. The profile $v_\star$ is called (exponentially) localized, if it tends (exponentially) to some constant vector $v_\infty\in\mathbb{R}^m$ as $|x|\to\infty$.
Transforming (1.1) via $u(x,t)=v(e^{-tS}x,t)$ into a co-rotating frame yields the evolution equation

$$(1.3)\qquad \begin{aligned} v_t(x,t) =& A\triangle v(x,t) + \langle Sx,\nabla v(x,t)\rangle + f(v(x,t)), && t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) =& u_0(x), && t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

The diffusion and drift term are given by

$$(1.4)\qquad A\triangle v(x) := A\sum_{i=1}^d \frac{\partial^2 v}{\partial x_i^2}(x) \quad\text{and}\quad \langle Sx,\nabla v(x)\rangle := \sum_{i=1}^d\sum_{j=1}^d S_{ij}x_j\frac{\partial v}{\partial x_i}(x).$$

The pattern $v_\star$ itself appears as a stationary solution of (1.3), i.e. $v_\star$ solves the rotating wave equation

$$(1.5)\qquad A\triangle v_\star(x) + \langle Sx,\nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

We write (1.5) as $[\mathcal{L}_0 v_\star](x) + f(v_\star(x)) = 0$ with the Ornstein-Uhlenbeck operator defined by

$$(1.6)\qquad [\mathcal{L}_0 v](x) := A\triangle v(x) + \langle Sx,\nabla v(x)\rangle,\ x\in\mathbb{R}^d.$$

When proving nonlinear stability of the rotating wave, a crucial role is played by the linearized operator

$$(1.7)\qquad [\mathcal{L}v](x) := [\mathcal{L}_0 v](x) + Df(v_\star(x))\,v(x),\ x\in\mathbb{R}^d.$$

The aim of this paper is to analyze Fredholm properties and spectra of the L^p-eigenvalue problem associated with the linearization $\mathcal{L}$,

$$(1.8)\qquad [(\lambda I-\mathcal{L})\,v](x) = 0,\ x\in\mathbb{R}^d.$$

As usual, the L^p-spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ is decomposed into the disjoint union of point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ and essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$, cf. Definition 2.6,

$$(1.9)\qquad \sigma(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L})\,\dot\cup\,\sigma_{\mathrm{pt}}(\mathcal{L}).$$

In Section 3 we evaluate the dispersion relation associated with the limit operator,

$$(1.10)\qquad \mathcal{L}_\infty = \mathcal{L}_0 + Df(v_\infty)$$

and show that its solutions belong to $\sigma_{\mathrm{ess}}(\mathcal{L})$. For every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda$ larger than $\operatorname{Re}\sigma(Df(v_\infty))$ we prove in Section 4 that the operator $\lambda I-\mathcal{L}$ is Fredholm of index 0 in L^p-spaces. Finally, in Section 5 we compute those eigenvalues on the imaginary axis which are caused by Euclidean equivariance of the underlying equation, and we prove exponential decay in space for the corresponding eigenfunctions. The whole approach makes extensive use of our previous results on the identification problem and on exponential decay estimates for the wave itself and for solutions of the linearized equation, see [5, 30, 31]. While there is a well-developed theory of stability of closed operators and their spectra under small and relatively compact perturbations [24, Ch.4], [15, Ch. IX], the main work in this paper is to identify suitable perturbations of the Ornstein-Uhlenbeck operator (1.6) which allow to determine parts of the essential spectrum and of the point spectrum for the general linear operator (1.7). Let us finally note that the results of the paper are extensions of the PhD thesis [29], in particular concerning the Fredholm properties of the linearized operator in Section 4.

2. Assumptions and main results

2.1. Assumptions. The following conditions will be needed in this paper and relations among them will be discussed below. The conditions are essential for applying previous results from [5, 29, 30, 31].

Assumption 2.1. For $A\in\mathbb{K}^{m,m}$ with $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $1<p<\infty$ and $q=\frac{p}{p-1}$ consider the conditions

(A1) A is diagonalizable (over $\mathbb{C}$),

(A2) $\mathrm{Re}\,\sigma(A)>0$,

(A3) $\mathrm{Re}\,\langle w,Aw\rangle\geqslant\beta_A|w|^2\;\forall\,w\in\mathbb{K}^m,$ for some $\beta_A>0$,

(A4$_p$) $|z|^2\mathrm{Re}\,\langle w,Aw\rangle+(p-2)\mathrm{Re}\,\langle w,z\rangle\mathrm{Re}\,\langle z,Aw\rangle\geqslant\gamma_A|z|^2|w|^2\;\forall w,z\in\mathbb{K}^m,$ for some $\gamma_A>0$,

(A4$_q$) $|z|^2\mathrm{Re}\,\langle w,A^{\mathsf{H}}w\rangle+(q-2)\mathrm{Re}\,\langle w,z\rangle\mathrm{Re}\,\langle z,A^{\mathsf{H}}w\rangle\geqslant\delta_A|z|^2|w|^2\;\forall w,z\in\mathbb{K}^m,$ for some $\delta_A>0$,

Assumption (A1) is a *system condition* and ensures that all results for scalar equations can be extended to system cases. This condition is independent of all other conditions and is used in [29, 30] to derive an explicit formula for the heat kernel of $\mathcal{L}_0$. A typical case where (A1) holds, is a scalar complex-valued equation when transformed into a real-valued system of dimension 2. The *positivity condition* (A2) guarantees that the diffusion part $A\triangle$ is an elliptic operator. The *strict accretivity condition* (A3) is more restrictive than (A2). In (A3) we use $\langle u,v\rangle:=u^{\mathsf{H}}v$ with $u^{\mathsf{H}}:=\bar{u}^{\top}$ to denote the standard inner product on $\mathbb{K}^m$. Recall that condition (A2) is satisfied iff there exists an inner product $[\cdot,\cdot]$ and some $\beta_A>0$ such that $\mathrm{Re}\,[w,Aw]\geqslant\beta_A$ for all $w\in\mathbb{K}^m$ with $[w,w]=1$. Condition (A3) ensures that the differential operator $\mathcal{L}_0$ is closed on its (local) domain $\mathcal{D}^p(\mathcal{L}_0)$, see [29, 31]. The L^p*-dissipativity condition* (A4$_p$) is more restrictive than (A3) and imposes additional requirements on the spectrum of A. This condition originating from [11, 12], is used in [29, 31] to prove L^p-resolvent estimates for $\mathcal{L}_0$. A geometric meaning of (A4$_p$) can be given in terms of the *first antieigenvalue of* A (see [21, 22]), defined by

$$\mu_1(A):=\inf\left\{\frac{\mathrm{Re}\,\langle w,Aw\rangle}{|w||Aw|}:w\in\mathbb{K}^m,w\neq0,Aw\neq0\right\}.$$

It is shown in [29, 32] that conditions (A4$_p$), (A4$_q$) are equivalent to lower bounds for $\mu_1(A)$

(A5$_p$) A is invertible and $p\mu_1(A)>|p-2|$ (to be read as $A>0$ in case $m=1$, $\mathbb{K}=\mathbb{R}$),

(A5$_q$) A is invertible and $q\mu_1(A^{\mathsf{H}})>|q-2|$ (to be read as $A^{\mathsf{H}}>0$ in case $m=1$, $\mathbb{K}=\mathbb{R}$).

The first antieigenvalue $\mu_1(A)$ can be considered as the cosine of the maximal (real) turning angle of vectors mapped by the matrix A. Some special cases in which the first antieigenvalue can be given explicitly are treated in [29, 32]. However, we emphasize that L^p-dissipativity (A4$_p$) and L^q-dissipativity (A4$_q$) for the conjugate index q are generally unrelated except in case $p=q=2$ and $A=A^{\mathsf{H}}$.
We continue with the *rotational condition* (A6) and a *smoothness condition* (A7).

Assumption 2.2. The matrix $S\in\mathbb{R}^{d,d}$ satisfies

(A6) S is skew-symmetric, i.e. $S=-S^T$.

Assumption 2.3. The function $f:\mathbb{R}^m\to\mathbb{R}^m$ satisfies

(A7) $f\in C^2(\mathbb{R}^m,\mathbb{R}^m)$.

Later on in Section 6 we apply our results to complex-valued nonlinearities of the form

$$f:\mathbb{C}^m\to\mathbb{C}^m,\quad f(u)=g\left(|u|^2\right)u, \tag{2.1}$$

where $g:\mathbb{R}\to\mathbb{C}^{m,m}$ is a sufficiently smooth function. Such nonlinearities arise for example in Ginzburg-Landau equations, Schrödinger equations, $\lambda-\omega$ systems and many other equations from physical sciences, see [29] and references therein. Note, that the real-valued version of f in $\mathbb{R}^{2m}$ satisfies (A7) if g is in C^2. For differentiable functions $f:\mathbb{R}^m\to\mathbb{R}^m$ we denote by Df the *Jacobian matrix* in the real sense.

Assumption 2.4. For $v_\infty \in \mathbb{R}^m$ consider the following conditions:

(A8) $f(v_\infty) = 0$,

(A9) $A, Df(v_\infty) \in \mathbb{R}^{m,m}$ are simultaneously diagonalizable (over $\mathbb{C}$),

(A10) $\operatorname{Re}\sigma\left(Df(v_\infty)\right) < 0$,

(A11) $\operatorname{Re}\langle w, Df(v_\infty)w\rangle \leqslant -\beta_\infty \;\forall\, w \in \mathbb{K}^m$, $|w| = 1$ for some $\beta_\infty > 0$.

Condition (A8) requires v_∞ to be a steady state of the nonlinear equation while condition (A9) extends Assumption (A1). As above, the *coercivity condition* (A11) is more restrictive than the *spectral condition* (A10).

For matrices $C \in \mathbb{K}^{m,m}$ with spectrum $\sigma(C)$ we denote by $\rho(C) := \max_{\lambda\in\sigma(C)} |\lambda|$ its *spectral radius* and by $s(C) := \max_{\lambda\in\sigma(C)} \operatorname{Re}\lambda$ its *spectral abscissa* (or *spectral bound*). With this notation, we define the following constants which appear in the linear theory from [5, 29, 30, 31]:

$$a_{\min} = \left(\rho\left(A^{-1}\right)\right)^{-1},\; a_{\max} = \rho(A),\; a_0 = -s(-A),\; a_1 = \left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}},\; b_0 = -s(Df(v_\infty)). \tag{2.2}$$

Recall the relations $0 < a_0 \leqslant \beta_A$ and $0 < b_0 \leqslant \beta_\infty$ to the coercivity constants from (A3) and (A11).

The theory in this paper is partially developed for more general differential operators, see (2.7) below. For this purpose we transfer Assumption 2.4 to general matrices B_∞. Later on, we apply the results to $B_\infty = -Df(v_\infty)$.

Assumption 2.5. For $B_\infty \in \mathbb{K}^{m,m}$ consider the conditions

($\mathrm{A9}_{B_\infty}$) $A, B_\infty \in \mathbb{K}^{m,m}$ are simultaneously diagonalizable (over $\mathbb{C}$) with transformation $Y \in \mathbb{C}^{m,m}$,

($\mathrm{A10}_{B_\infty}$) $\operatorname{Re}\sigma(B_\infty) > 0$,

($\mathrm{A11}_{B_\infty}$) $\operatorname{Re}\langle w, B_\infty w\rangle \geqslant \beta_\infty \;\forall\, w \in \mathbb{K}^m$, $|w| = 1$ for some $\beta_\infty > 0$.

Let us recall for closed, densely defined operators $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq X \to X$ on a complex Banach space X the standard notions of resolvent set $\operatorname{res}(\mathcal{A})$ and spectrum $\sigma(\mathcal{A}) := \mathbb{C}\backslash\operatorname{res}(\mathcal{A})$, see [24, Ch.III.6]. For the essential spectrum there is quite a variety of definitions, see [24, 23, 15]. From the hierarchy of 5 versions studied in [15, Ch.I.4,IX.1], we choose $\sigma_{\mathrm{e},4}(\mathcal{A})$ which refers to the complement of the set of values where the operator is Fredholm of index 0.

Definition 2.6. The *point spectrum of* $\mathcal{A}$ is defined by

$$\sigma_{\mathrm{pt}}(\mathcal{A}) := \{\lambda \in \sigma(\mathcal{A}) \mid \lambda I - \mathcal{A} \text{ is Fredholm of index } 0\},$$

and the *essential spectrum of* $\mathcal{A}$ by

$$\sigma_{\mathrm{ess}}(\mathcal{A}) = \mathbb{C} \setminus \left(\sigma_{\mathrm{pt}}(\mathcal{A}) \cup \operatorname{res}(\mathcal{A})\right).$$

In Theorem 3.2 we will identify points $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{A})$ by constructing a singular sequence $v_n \in \mathcal{D}(\mathcal{A})$, i.e. $\|v_n\| = 1$, $(\lambda I - \mathcal{A})v_n \to 0$ and v_n has no convergent subsequence, see [15, Ch.IX, Def.1.2]. According to [15, Ch.IX, Thm.1.3] this information guarantees λ to be even in $\sigma_{\mathrm{e},2}(\mathcal{A})$ which is a subset of our $\sigma_{\mathrm{ess}}(\mathcal{A})$. However, this is not sufficient to conclude that λ belongs to $\sigma_{\mathrm{e},1}(\mathcal{A})$ in the terminology [15, Ch.IX] which agrees with the essential spectrum used in [24, Ch.IV.6].

2.2. Outline and main results. In Section 3 we investigate the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ of $\mathcal{L}$ from (1.7), which is determined by the limiting behavior of $v_\star$ at infinity. By a far-field linearization and an angular Fourier decomposition, bounded eigenfunctions of the problem (1.8) are obtained from the m-dimensional eigenvalue problem (Section 3.1)

$$\Big(\lambda I_m + \eta^2 A + i\langle n, \sigma\rangle I_m - Df(v_\infty)\Big)z = 0 \quad \text{for some } \eta \in \mathbb{R} \text{ and } n \in \mathbb{Z}^k, \sigma = (\sigma_1, \ldots, \sigma_k)^\top \tag{2.3}$$

where $\pm i\sigma_1, \ldots, \pm i\sigma_k$ are the nonzero eigenvalues of S and $1 \leqslant k \leqslant \lfloor\frac{d}{2}\rfloor$. Obviously, (2.3) has a solution $0 \neq z \in \mathbb{C}^m$ if and only if $\lambda \in \mathbb{C}$ satisfies the *dispersion relation for localized rotating waves*

$$\det\Big(\lambda I_m + \eta^2 A - Df(v_\infty) + i\langle n, \sigma\rangle I_m\Big) = 0, \quad \sigma = (\sigma_1, \ldots, \sigma_k)^\top \tag{2.4}$$

for some $\eta \in \mathbb{R}$, $n \in \mathbb{Z}^k$. Therefore, we define the *dispersion set* as follows:

$$\sigma_{\mathrm{disp}}(\mathcal{L}) := \{\lambda \in \mathbb{C} : \ \lambda \text{ satisfies (2.4) for some } \eta \in \mathbb{R} \text{ and } n \in \mathbb{Z}^k\}. \tag{2.5}$$

Theorem 2.7 (Essential spectrum at localized rotating waves)**.** *Let $f \in C^{\max\{2,r-1\}}(\mathbb{R}^m, \mathbb{R}^m)$ for some $r \in \mathbb{N}$ and let the assumptions (A4$_p$), (A6), (A8), (A9) and (A11) be satisfied for $\mathbb{K} = \mathbb{C}$ and for some $1 < p < \infty$ with $\frac{d}{p} < r$ if $r \leqslant 2$ and $\frac{d}{p} \leqslant 2$ if $r \geqslant 3$. Moreover, let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of S. Then there is a constant $K_1 = K_1(A, f, v_\infty, d, p) > 0$ with the following property: For every classical solution $v_\star \in C^{r+1}(\mathbb{R}^d, \mathbb{R}^m)$ of*

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0, \; x \in \mathbb{R}^d,$$

satisfying

$$\sup_{|x| \geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0,$$

the dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L})$ from (2.5) belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ of the linearized operator $\mathcal{L}$ from (1.7) in $L^p(\mathbb{R}^d, \mathbb{C}^m)$.

For the proof of Theorem 2.7, we consider differential operators

$$\mathcal{L}_Q : \left(\mathcal{D}^p(\mathcal{L}_0), \|\cdot\|_{\mathcal{L}_0}\right) \to \left(L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p}\right), \quad 1 < p < \infty \tag{2.6}$$

of the form

$$[\mathcal{L}_Q v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - B_\infty v(x) + Q(x)v(x), \; x \in \mathbb{R}^d, \; d \geqslant 2, \tag{2.7}$$

where $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{m,m})$ and $\mathcal{D}^p(\mathcal{L}_0)$, $1 < p < \infty$ is the domain of the Ornstein-Uhlenbeck operator $\mathcal{L}_0$ from (1.6)

$$\mathcal{D}^p(\mathcal{L}_0) = \left\{ v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^m) \cap L^p(\mathbb{R}^d, \mathbb{C}^m) : \ \mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^m) \right\}. \tag{2.8}$$

which becomes a Banach space with respect to the graph norm

$$\|v\|_{\mathcal{L}_0} = \|\mathcal{L}_0 v\|_{L^p(\mathbb{R}^d, \mathbb{C}^m)} + \|v\|_{L^p(\mathbb{R}^d, \mathbb{C}^m)}, \quad v \in \mathcal{D}^p(\mathcal{L}_0). \tag{2.9}$$

If Q vanishes at infinity, i.e.

$$\operatorname*{ess\,sup}_{|x| \geqslant R} |Q(x)| \to 0 \text{ as } R \to \infty, \tag{2.10}$$

then [31, Thm. 5.1] asserts that $\mathcal{L}_Q$ has the same maximal domain $\mathcal{D}^p(\mathcal{L}_0)$. According to [31, Thm. 6.1]) it even agrees with

$$\mathcal{D}^p_{\max}(\mathcal{L}_0) = \left\{ v \in W^{2,p}(\mathbb{R}^d, \mathbb{C}^m) : \ \langle S\,\cdot\,, \nabla v\rangle \in L^p(\mathbb{R}^d, \mathbb{C}^m) \right\},$$

and the graph norm $\|\cdot\|_{\mathcal{L}_0}$ is equivalent to $\|\cdot\|_{W^{2,p}} + \|\langle S\,\cdot\,, \nabla\rangle\|_{L^p}$, see [29, Cor. 5.26]. Such a strong characterization of the domain is rather involved to prove, but will not be needed here. The differential operator $\mathcal{L}_Q$ is a variable coefficient perturbation of the (complex-valued) Ornstein-Uhlenbeck operator $\mathcal{L}_0$, which is studied in depth in [5, Sec. 3] and [29, Sec. 7]. In Section 3.2 we continue this study and determine the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$ in L^p (see Theorem 3.2). An application of Theorem 3.2 to $-B_\infty = Df(v_\infty)$ and $Q(x) = Df(v_\star(x)) - Df(v_\infty)$ completes the proof of Theorem 2.7. For the proof of the decay (2.10) we use [5, Cor. 4.3] to deduce that $v_\star(x) \to v_\infty$ as $|x| \to \infty$.
In Section 4 we analyze Fredholm properties of the linearized operator

$$\lambda I - \mathcal{L} : (\mathcal{D}^p(\mathcal{L}_0), \|\cdot\|_{\mathcal{L}_0}) \to (L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p}) \tag{2.11}$$

with $\mathcal{L}$ given by (1.7), and of its adjoint operator

$$(\lambda I - \mathcal{L})^* : (\mathcal{D}^q(\mathcal{L}_0^*), \|\cdot\|_{\mathcal{L}_0^*}) \to (L^q(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^q}), \quad q = \frac{p}{p-1}, \tag{2.12}$$

defined by

$$[\mathcal{L}^* v](x) = A^{\mathsf{H}}\triangle v(x) - \langle Sx, \nabla v(x)\rangle + Df\left(v_\star(x)\right)^{\mathsf{H}} v(x), \; x \in \mathbb{R}^d, \; d \geqslant 2, \tag{2.13}$$

For values $\operatorname{Re}\lambda > -b_0$, with b_0 the spectral bound from (2.2), we show that the operator $\lambda I - \mathcal{L}$ is Fredholm of index 0. Moreover, we prove that its formal adjoint operator $(\lambda I - \mathcal{L})^*$ from (2.13) and its abstract adoint operator (see Definition A.1), coincide on their common domain

$$\mathcal{D}^q(\mathcal{L}_0^*) = \left\{ v \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^m) \cap L^q(\mathbb{R}^d,\mathbb{C}^m) :\ \mathcal{L}_0^* v \in L^q(\mathbb{R}^d,\mathbb{C}^m) \right\}. \tag{2.14}$$

Then the Fredholm alternative applies and leads to the following result.

Theorem 2.8 (Fredholm properties of the linearization $\mathcal{L}$)**.** *Let the assumptions* ($A4_p$) *and* (A6)*–*(A9) *be satisfied for* $\mathbb{K} = \mathbb{C}$ *and for some* $1 < p < \infty$*. Moreover, let* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geqslant -b_0 + \gamma$ *for some* $\gamma > 0$*, where* $-b_0 = s(Df(v_\infty))$ *denotes the spectral bound of* $Df(v_\infty)$*. Then there is a constant* $K_1 = K_1(A, f, v_\infty, \gamma, d, p) > 0$ *such that for every classical solution* $v_\star \in C^2(\mathbb{R}^d,\mathbb{R}^m)$ *of*

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d, \tag{2.15}$$

satisfying

$$\sup_{|x|\geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0, \tag{2.16}$$

the following statements hold:

a) *(Fredholm properties of* $\mathcal{L}$*). The linearized operator*

$$\lambda I - \mathcal{L} : (\mathcal{D}^p(\mathcal{L}_0), \|\cdot\|_{\mathcal{L}_0}) \to (L^p(\mathbb{R}^d,\mathbb{C}^m), \|\cdot\|_{L^p})$$

is Fredholm of index 0*.*

b) *(Eigenvalues of* $\mathcal{L}$*). In addition to a), let Assumption* ($A4_q$) *be satisfied for* $q = \frac{p}{p-1}$*, and let* $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L})$ *with geometric multiplicity* $1 \leqslant n := \dim \mathcal{N}(\lambda I - \mathcal{L}) < \infty$*. Then* $\mathcal{N}((\lambda I - \mathcal{L})^*) \subseteq \mathcal{D}^q(\mathcal{L}_0^*)$ *also has dimension* n *and the inhomogenous equation*

$$(\lambda I - \mathcal{L})v = g \in L^p(\mathbb{R}^d,\mathbb{C}^m) \tag{2.17}$$

has at least one (not necessarily unique) solution $v \in \mathcal{D}^p(\mathcal{L}_0)$ *iff* $g \in (\mathcal{N}((\lambda I - \mathcal{L})^*))^\perp$*, i.e.*

$$\langle \psi, g\rangle_{q,p} = 0,\ \text{for all}\ \psi \in \mathcal{N}((\lambda I - \mathcal{L})^*). \tag{2.18}$$

If the orthogonality condition (2.18) *is satisfied, then one can select a solution* v *of* (2.17) *with*

$$\|v\|_{\mathcal{L}_0} \leqslant C\,\|g\|_{L^p},\quad \|v\|_{W^{1,p}} \leqslant C\,\|g\|_{L^p}, \tag{2.19}$$

where C *denotes a generic constant which does not depend on* g*.*

An extension of Theorem 2.8 shows exponential decay of eigenfunctions and of adjoint eigenfunctions for eigenvalues $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant -b_0 + \gamma$ (Theorem 4.6), cf. [5, Thm. 3.5] for the case of eigenfunctions.
The idea of proof for Theorem 2.8 is to write $\lambda = \lambda_1 + \lambda_2$ with $\lambda_2 := -b_0 + \gamma$, $\lambda_1 := \lambda - \lambda_2$, and to decompose the variable coefficient $Q = Q_{\mathrm{s}} + Q_{\mathrm{c}}$ into the sum of a function Q_{s} which is small with respect to $\|\cdot\|_{L^\infty}$ and a function Q_{c} which is compactly supported on $\mathbb{R}^d$. This allows us to decompose the differential operator $\lambda I - \mathcal{L}_Q$ as follows

$$\lambda I - \mathcal{L}_Q = (I - Q_{\mathrm{c}}(\cdot)(\lambda_1 I - \widetilde{\mathcal{L}}_{\mathrm{s}})^{-1})(\lambda_1 I - \widetilde{\mathcal{L}}_{\mathrm{s}}), \tag{2.20}$$

where $\widetilde{\mathcal{L}}_{\mathrm{s}} := \mathcal{L}_{\mathrm{s}} - \lambda_2 I$ and $\mathcal{L}_{\mathrm{s}}$ denotes a small variable coefficient perturbation, defined by

$$[\mathcal{L}_{\mathrm{s}} v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - B_\infty v(x) + Q_{\mathrm{s}}(x)v(x),\ x \in \mathbb{R}^d. \tag{2.21}$$

For a similar decomposition under more restrictive assumptions on B_∞ see [4, 5, 29]. Then we show that $Q_{\mathrm{c}}(\cdot)(\lambda_1 I - \widetilde{\mathcal{L}}_{\mathrm{s}})^{-1}$ is compact and $\lambda_1 I - \widetilde{\mathcal{L}}_{\mathrm{s}}$ is Fredholm of index 0, which implies $\lambda I - \mathcal{L}_Q$ to be Fredholm of index 0. A crucial ingredient for the proof of these two statements is the inclusion $\mathcal{D}^p(\mathcal{L}_0) \subset W^{1,p}(\mathbb{R}^d,\mathbb{C}^m)$, proved in [29, Thm. 5.8 & 6.8], [30, Thm. 5.7]. Further, it is essential to solve the identification problem for the adjoint operator of $\mathcal{L}_Q$ in $L^q(\mathbb{R}^d,\mathbb{C}^m)$ along the lines of [31] (Lemma A.2). We show the existence and uniqueness of a solution $\tilde{v} \in \mathcal{D}^q(\mathcal{L}_0^*)$ of the resolvent equation $(\lambda I - \mathcal{L}_Q)^*\tilde{v} = g \in L^q(\mathbb{R}^d,\mathbb{C}^m)$, using the corresponding result from [5, Thm. 3.1]. For this we employ the L^q-dissipativity condition ($A4_q$) for the adjoint operator. Finally, the Fredholm alternative is applied to $\lambda I - \mathcal{L}_Q$ and $(\lambda I - \mathcal{L}_Q)^*$ (Theorem 4.3) and exponential decay of (adojoint) eigenfunctions is shown (Theorem 4.4). These results hold for

$-B_\infty = Df(v_\infty)$ and $Q(x) = Df(v_\star(x)) - Df(v_\infty)$ and thus complete the proof of Theorem 2.8 (and Theorem 4.6). Note that a similar reasoning is used in [5, 29] to prove exponential decay of the wave profile $v_\star$ itself.

In Section 5 we investigate the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ of $\mathcal{L}$, which is determined by the symmetries of the underlying SE(d)-group action of dimension $\frac{d(d+1)}{2}$. By the ansatz $v = (Dv_\star)(Ex+b)$ for $E \in \mathbb{C}^{d,d}$, $E^\top = -E$ and $b \in \mathbb{C}^d$, eigenfunctions of the problem (1.8) (in the classical sense) are obtained from the $\frac{d(d+1)}{2}$-dimensional eigenvalue problem (Section 5.1)

$$\lambda E = [E, S], \quad \lambda b = -Sb,$$

which is to be solved for (λ, E, b) with $\lambda \in \mathbb{C}$, $E \in \mathbb{C}^{d,d}$, $E^\top = -E$ and $b \in \mathbb{C}^d$. Let $\lambda_j^S, j = 1, \dots, d$ be the eigenvalues of S repeated according to their multiplicity. Then we define the *symmetry set*

$$\sigma_{\mathrm{sym}}(\mathcal{L}) := \sigma(S) \cup \{\lambda_i^S + \lambda_j^S : 1 \leqslant i < j \leqslant d\}, \tag{2.22}$$

and show that $\sigma_{\mathrm{sym}}(\mathcal{L})$ belongs to the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ of $\mathcal{L}$ in L^p. We determine their associated eigenfunctions and show that they decay exponentially in space.

Theorem 2.9 (Eigenvalues on the imaginary axis and the shape of eigenfunctions)**.** *Let $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, $S \in \mathbb{R}^{d,d}$ be skew-symmetric, and let $U \in \mathbb{C}^{d,d}$ denote the unitary matrix satisfying $\Lambda_S = U^{\mathsf{H}} S U$ with diagonal matrix $\Lambda_S = \mathrm{diag}(\lambda_1^S, \dots, \lambda_d^S)$ and eigenvalues $\lambda_1^S, \dots, \lambda_d^S \in \sigma(S)$. Moreover, let $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^m)$ be a classical solution of (1.5), then the function $v : \mathbb{R}^d \to \mathbb{C}^m$ given by*

$$v(x) = \langle Ex + b, \nabla v_\star(x)\rangle = (Dv_\star(x))(Ex + b) \tag{2.23}$$

is a classical solution of the eigenvalue problem (1.8) if $E \in \mathbb{C}^{d,d}$ and $b \in \mathbb{C}^d$ either satisfy

$$\lambda = -\lambda_l^S, \quad E = 0, \quad b = Ue_l \tag{2.24}$$

for some $l = 1, \dots, d$, or

$$\lambda = -(\lambda_i^S + \lambda_j^S), \quad E = U(I_{ij} - I_{ji})U^T, \quad b = 0 \tag{2.25}$$

for some $i = 1, \dots, d-1$ and $j = i+1, \dots, d$. Here, $I_{ij} \in \mathbb{R}^{d,d}$ denotes the matrix having the entries 1 at the i-th row and j-th column and 0 otherwise. All the eigenvalues above lie on the imaginary axis.

Theorem 2.10 (Point spectrum at localized rotating waves)**.** *Let $f \in C^{r-1}(\mathbb{R}^m, \mathbb{R}^m)$ for some $r \in \mathbb{N}$ with $r \geqslant 3$ and let the assumptions (A4$_p$), (A6), (A8), (A9) and (A11) be satisfied for $\mathbb{K} = \mathbb{C}$ and for some $1 < p < \infty$ with $\frac{d}{p} \leqslant 2$. Then, for every $0 < \varepsilon < 1$ there is a constant $K_1 = K_1(A, f, v_\infty, d, p, \varepsilon) > 0$ with the following property: For every classical solution $v_\star \in C^{r+1}(\mathbb{R}^d, \mathbb{R}^m)$ of*

$$A \triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,$$

satisfying

$$\sup_{|x| \geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0,$$

the symmetry set $\sigma_{\mathrm{sym}}(\mathcal{L})$ from (2.22) belongs to the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ of the linearized operator $\mathcal{L}$ from (1.7) in $L^p(\mathbb{R}^d, \mathbb{C}^m)$.

The eigenvalue problem for the commutator generated by a skew-symmetric matrix, is analyzed for example in [10, Lem. 4 & 5] and [40, Thm. 2]. Further, we mention that the asymptotic behavior of adjoint eigenfunctions plays a role in the study of response functions, see [9].

For the proof of Theorem 2.10, we apply Theorem 2.8a) to eigenvalues $\lambda \in \sigma_{\mathrm{sym}}(\mathcal{L})$. Here, exponential decay of the rotating wave $v_\star$, proved in [5, Cor. 4.1], implies that v from (2.23) belongs to $\mathcal{D}^p(\mathcal{L}_0)$, and hence is an eigenfunction of $\mathcal{L}$ in L^p. The Fredholm property of $\lambda I - \mathcal{L}$ follows from the spectral stability of $Df(v_\infty)$ assured by (A11) and Theorem 2.8a).

In Section 6 we apply our results to the cubic-quintic complex Ginzburg-Landau equation

$$u_t = \alpha \triangle u + u\left(\delta + \beta|u|^2 + \gamma|u|^4\right) \tag{2.26}$$

which is known to exhibit spinning soliton solutions. We rewrite (2.26) as a 2-dimensional real-valued system and formulate the eigenvalue problem for the associated linearization at the spinning soliton. We then compute numerical spectra and eigenfunctions using the freezing method from [6, 8] and the software COMSOL, [1]. This allows to compare exact and numerical spectra as well as their associated eigenfunctions. Let us finally discuss some related results from the literature. Spectra of Ornstein-Uhlenbeck operators in various function spaces are studied in [14, 25, 27, 28, 42], spectra at localized rotating waves in [4, 29], and spectra at spiral waves (nonlocalized rotating waves) in [18, 29, 36, 37, 38, 45]. For scroll waves we refer to [2, 19]. Exponential decay is proved in [5, 29] for solutions of nonlinear problems for Ornstein-Uhlenbeck operators (with unbounded coefficients of ∇u), while [20, 35] treat solutions of real-valued quasilinear second-order equations (with bounded coefficients of ∇u). We also refer to [20, 33, 44] for various results on Fredholm properties of elliptic partial differential on unbounded domains in settings different from ours. Nonlinear stability of rotating waves is investigated in [4, 39]. For numerical approximations of rotating waves (including wave profiles, velocities and spectra), based on the freezing method from [6, 8], we refer to [5, 29]. Numerical results on rotating waves are studied in [17] for scalar excitable media, and in [7] for second order evolution equations. Interactions of several rotating waves is analyzed numerically in [6, 29].

3. Essential spectrum and dispersion relation

3.1. Formal derivation of the dispersion relation. In this section we discuss the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ of the linearization $\mathcal{L}$ from (1.7). We compute eigenvalues and bounded eigenfunctions of (1.8) with $\mathcal{L}$ replaced by its far-field limit. These eigenvalues are determined by the dispersion relation (2.4). By a standard truncation procedure we then show that bounded eigenfunctions lead to singular sequences in L^p and hence belong to values in the essential spectrum. We proceed in several steps:

1. The far-field operator. Let $v_\infty \in \mathbb{R}^m$ denote the constant asymptotic state of the wave profile $v_\star$. i.e. $f(v_\infty) = 0$ and $v_\star(x) \to v_\infty \in \mathbb{R}^m$ as $|x| \to \infty$. Assuming $f \in C^1$ and introducing $Q(x) \in \mathbb{R}^{m,m}$ via

$$Q(x) := Df(v_\star(x)) - Df(v_\infty), \quad x \in \mathbb{R}^d,$$

allows us to write (1.8) as

$$(\lambda I - \mathcal{L}_Q)v = 0,\ x \in \mathbb{R}^d \tag{3.1}$$

with $\mathcal{L}_Q = \mathcal{L}_\infty + Q(x)$ and far-field operator

$$\mathcal{L}_\infty v = A\triangle v + \langle Sx, \nabla v\rangle + Df(v_\infty)v.$$

Obviously, $f \in C^1$ and $v_\star(x) \to v_\infty$ imply $Q(x) \to 0$ as $|x| \to \infty$, i.e. Q vanishes at infinity.

2. Orthogonal transformation. We next transfer the skew-symmetric matrix S into quasi-diagonal real form which allows us to separate the axes of rotations in (3.1). Let $S \in \mathbb{R}^{d,d}$ be skew-symmetric, then $\sigma(S) \subset i\mathbb{R}$ with nonzero eigenvalues $\pm i\sigma_1, \ldots, \pm i\sigma_k$ and semisimple eigenvalue 0 of multiplicity $d-2k$. Here, σ_l denotes the angular velocity in the (y_{2l-1}, y_{2l})-plane in one of the k different planes of rotation. Moreover, there is an orthogonal matrix $P \in \mathbb{R}^{d,d}$ such that

$$S = P\Lambda P^\top \quad \text{with} \quad \Lambda = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_k, \mathbf{0}), \quad \Lambda_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad \mathbf{0} \in \mathbb{R}^{d-2k,d-2k}.$$

The orthogonal transformation $\tilde{v}(y) := v(T_1(y))$ with $x = T_1(y) := Py$ for $y \in \mathbb{R}^d$ transfers (3.1) into

$$(\lambda I - \tilde{\mathcal{L}}_Q)\tilde{v} = 0,\ y \in \mathbb{R}^d \tag{3.2}$$

with $\tilde{\mathcal{L}}_Q = \tilde{\mathcal{L}}_\infty + Q(T_1(y))$ and

$$\tilde{\mathcal{L}}_\infty \tilde{v} = A\sum_{j=1}^{d} \partial_{y_j}^2 \tilde{v} + \sum_{l=1}^{k} \sigma_l \left(y_{2l}\partial_{y_{2l-1}} - y_{2l-1}\partial_{y_{2l}}\right)\tilde{v} + Df(v_\infty)\tilde{v}.$$

3. Transformation into several planar polar coordinates. Since we have k angular derivatives in k different planes it is advisable to transform each plane into planar polar coordinates via

$$\begin{pmatrix} y_{2l-1} \\ y_{2l} \end{pmatrix} = T(r_l, \phi_l) := \begin{pmatrix} r_l\cos\phi_l \\ r_l\sin\phi_l \end{pmatrix}, \quad r_l > 0,\ \phi_l \in [-\pi, \pi),\ l = 1, \ldots, k.$$

All further coordinates, $y_{2k+1},\dots,y_d$, remain fixed. The multiple planar polar coordinates transformation $\hat{v}(\psi):=\tilde{v}(T_2(\psi))$ with $T_2(\psi)=(T(r_1,\phi_1),\dots,T(r_k,\phi_k),y_{2k+1},\dots,y_d)$, $\psi=(r_1,\phi_1,\dots,r_k,\phi_k,y_{2k+1},\dots,y_d)$ in the domain $\Omega:=((0,\infty)\times[-\pi,\pi))^k\times\mathbb{R}^{d-2k}$, transfers (3.2) into

$$(\lambda I-\hat{\mathcal{L}}_Q)\hat{v}=0,\ \psi\in\Omega \tag{3.3}$$

with $\hat{\mathcal{L}}_Q=\hat{\mathcal{L}}_\infty+Q(T_1(T_2(\psi)))$ and

$$\hat{\mathcal{L}}_\infty\hat{v}=A\left[\sum_{l=1}^{k}\left(\partial_{r_l}^2+\frac{1}{r_l}\partial_{r_l}+\frac{1}{r_l^2}\partial_{\phi_l}^2\right)+\sum_{l=2k+1}^{d}\partial_{y_l}^2\right]\hat{v}-\sum_{l=1}^{k}\sigma_l\partial_{\phi_l}\hat{v}+Df(v_\infty)\hat{v}.$$

4. Simplified operator (limit operator, far-field operator). Since the essential spectrum depends on the limiting equation for $|x|\to\infty$, we formally let $r_l\to\infty$ for any $1\leqslant l\leqslant k$. This turns (3.3) into

$$(\lambda I-\mathcal{L}_\infty^{\mathrm{sim}})\hat{v}=0,\quad \psi\in\Omega \tag{3.4}$$

with the simplified far-field operator

$$\mathcal{L}_\infty^{\mathrm{sim}}\hat{v}=A\left[\sum_{l=1}^{k}\partial_{r_l}^2+\sum_{l=2k+1}^{d}\partial_{y_l}^2\right]\hat{v}-\sum_{l=1}^{k}\sigma_l\partial_{\phi_l}\hat{v}+Df(v_\infty)\hat{v}.$$

Note that we used the property $|Q(x)|\to 0$ as $|x|\to\infty$ which was established in step 1.

5. Angular Fourier transform. Finally, we solve for eigenvalues and eigenfunctions of $\mathcal{L}_\infty^{\mathrm{sim}}$ by an angular Fourier decomposition (separation of variables) with $\omega\in\mathbb{R}^k$, $\rho,y\in\mathbb{R}^{d-2k}$, $n\in\mathbb{Z}^k$, $z\in\mathbb{C}^m$, $|z|=1$, $r\in(0,\infty)^k$, $\phi\in[-\pi,\pi)^k$:

$$\hat{v}(\psi)=\exp\left(i\sum_{l=1}^{k}\omega_l r_l\right)\exp\left(i\sum_{l=1}^{k}n_l\phi_l\right)\exp\left(i\sum_{l=2k+1}^{d}\rho_l y_l\right)z=\exp\Big(i\langle\omega,r\rangle+i\langle n,\phi\rangle+i\langle\rho,y\rangle\Big)z. \tag{3.5}$$

Inserting (3.5) into (3.4) leads to the m-dimensional eigenvalue problem

$$\big(\lambda I_m+(|\omega|^2+|\rho|^2)A+i\langle n,\sigma\rangle I_m-Df(v_\infty)\big)z=0,\quad \sigma=(\sigma_1,\dots,\sigma_k)^\top. \tag{3.6}$$

6. Dispersion relation and dispersion set. The *dispersion relation* for localized rotating waves of (1.1) now states that every $\lambda\in\mathbb{C}$ satisfying

$$\det\big(\lambda I_m+(|\omega|^2+|\rho|^2)A+i\langle n,\sigma\rangle I_m-Df(v_\infty)\big)=0 \tag{3.7}$$

for some $\omega\in\mathbb{R}^k$, $\rho\in\mathbb{R}^{d-2k}$ and $n\in\mathbb{Z}^k$ belongs to the essential spectrum of $\mathcal{L}$, i.e. $\lambda\in\sigma_{\mathrm{ess}}(\mathcal{L})$. Of course, one can replace $|\omega|^2+|\rho|^2$ by any nonnegative real number, so that the set of λ-values satisfying (3.7) agrees with the dispersion set from (2.4). A rigorous proof of $\sigma_{\mathrm{disp}}(\mathcal{L})\subseteq\sigma_{\mathrm{ess}}(\mathcal{L})$ will be given in the next section (see Theorem 2.7).

Remark 3.1. a) (Several axes of rotation). While the axis of rotation is unique in dimension $d=3$, the pattern can rotate about several axes simultaneously in dimension $d\geqslant 4$. The orientation of these axes is determined by the similarity transformatiom in step 2.

b) (Dispersion relation for spiral waves). The dispersion relation for nonlocalized rotating waves, such as spiral waves and scroll waves, is harder to derive and differs from (3.7). A dispersion relation for spiral waves is developed in [18, 36, 37]. Their approach is based on a Bloch wave transformation and on an application of Floquet theory. A summary of these results, which is structured similar to the derivation above, can be found in [29, Sec. 9.5]. The angular Fourier decomposition is also used in [18] for investigating essential spectra of spiral waves. For spectra of spiral waves in the FitzHugh-Nagumo system we refer to [38, 45]. Results on essential spectra of nonlocalized rotating waves for space dimensions $d\geqslant 3$, such as scroll waves, are quite rare in the literature and we refer to [18, 19] and references therein.

3.2. Essential spectrum in L^p. Consider the general linear differential operator

$$\mathcal{L}_Q v = A\triangle v + \langle Sx, \nabla v\rangle - B_\infty v + Q(x)v,$$

satisfying $|Q(x)| \to 0$ as $|x| \to \infty$. In this case the dispersion relation reads

$$\det\left(\lambda I_m + (|\omega|^2 + |\rho|^2)A + i\langle n, \sigma\rangle I_m + B_\infty\right) = 0 \tag{3.8}$$

for some $\omega \in \mathbb{R}^k$, $\rho \in \mathbb{R}^{d-2k}$ and $n \in \mathbb{Z}^k$ and the corresponding dispersion set is given by

$$\sigma_{\mathrm{disp}}(\mathcal{L}_Q) := \left\{\lambda \in \mathbb{C} :\ \lambda \text{ satisfies (3.8) for some } \omega \in \mathbb{R}^k,\ \rho \in \mathbb{R}^{d-2k} \text{ and } n \in \mathbb{Z}^k\right\}. \tag{3.9}$$

Theorem 3.2 (Essential spectrum of $\mathcal{L}_Q$)**.** *Let the assumptions (A6), (A9$_{B_\infty}$) and*

$$Q \in L^\infty(\mathbb{R}^d, \mathbb{K}^{m,m}) \quad \text{with} \quad \eta_R := \operatorname*{ess\,sup}_{|x|\geqslant R} |Q(x)| \to 0 \text{ as } R \to \infty \tag{3.10}$$

be satisfied for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$. Moreover, let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ with $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, $1 \le k \le \lfloor \frac{d}{2} \rfloor$ denote the nonzero eigenvalues of S. Then the dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ from (3.9) belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$ of $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d, \mathbb{C}^m)$.

Proof. Let $\chi_R \in C_c^\infty([0,\infty), \mathbb{R})$ be cut-off functions with uniformly bounded derivatives for $R \geqslant 2$ and

$$\chi_R(r) = 0,\ r \in I_1 \cup I_5, \quad \chi_R(r) = 1,\ r \in I_3, \quad \chi_R(r) \in [0,1],\ r \in I_2 \cup I_4,$$

$I_1 = [0, R-1]$, $I_2 = [R-1, R]$, $I_3 = [R, 2R]$, $I_4 = [2R, 2R+1]$, $I_5 = [2R+1, \infty)$. With this define

$$w_R := \frac{v_R}{\|v_R\|_{L^p}}, \quad v_R(T_1(T_2(\psi))) := \hat{v}_R(\psi), \quad \hat{v}_R(\psi) := \left(\prod_{l=1}^{k} \chi_R(r_l)\right)\chi_R(|\tilde{y}|)\hat{v}(\psi), \quad \hat{v} \text{ from (3.5)},$$

for $\psi = (r_1, \phi_1, \ldots, r_k, \phi_k, \tilde{y})$, $\tilde{y} = (y_{2k+1}, \ldots, y_d)$, $\phi = (\phi_1, \ldots, \phi_k) \in [-\pi, \pi)^k$, $\mathbf{r} = (r_1, \ldots, r_k) \in (0,\infty)^k$, and T_1, T_2 as in Section 3.1. Obviously, we have $w_R \in \mathcal{D}^p(\mathcal{L}_0)$ and $\|w_R\|_{L^p} = 1$. We further claim that

$$\|v_R\|_{L^p}^p \geqslant CR^d, \qquad \|(\lambda I - \mathcal{L}_Q)\, v_R\|_{L^p}^p \leqslant CR^{d-1} + CR^d\eta_R. \tag{3.11}$$

Verifying these estimates is a somewhat lengthy but standard computation, which the reader may find in the supplement 2. As a consequence of (3.11) we obtain $\|(\lambda I - \mathcal{L}_Q)\, w_R\|_{L^p}^p \leqslant C(\frac{1}{R} + \eta_R) \to 0$ as $R \to \infty$. In order for w_R to be a singular sequence in the sense of [15, Ch.IX, Def.1.2] it is sufficient to show $w_R \rightharpoonup 0$ as $R \to \infty$ since L^p is reflexive ([15, Ch.IX (1.2)]. In fact, for $u \in L^q(\mathbb{R}^d, \mathbb{K}^m)$, $q = \frac{p}{p-1}$ we find from Hölder's inequality with $S_R = \{x \in \mathbb{R}^d : R-1 \leqslant |x| \leqslant 2R+1\}$

$$|\langle u, w_R\rangle| = |\int_{S_R} \langle u(x), w_R(x)\rangle dx| \leqslant \|u\|_{L^q(S_R)}\|w_R\|_{L^p(S_R)} \leqslant \|u\|_{L^q(S_R)} \to 0 \quad \text{as } R \to \infty.$$

An application of [15, Ch.IX, Thm.1.3] shows that $\lambda I - \mathcal{L}_Q$ is not semi-Fredholm with finite-dimensional kernel, in particular λ belongs to $\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$ according to Definition 2.6. $\square$

Proof (of Theorem 2.7). We apply Theorem 3.2 with the matrices $B_\infty = -Df(v_\infty)$ and $Q(x) = Df(v_\star(x)) - Df(v_\infty)$ for $x \in \mathbb{R}^d$. Since (A4$_p$), (A6) and (A9) are satisfied, it remains to check (3.10). From Taylor's theorem we obtain

$$|Q(x)| \leqslant \int_0^1 \left|D^2 f(v_\infty + s(v_\star(x) - v_\infty))\right| ds\, |v_\star(x) - v_\infty| \quad \forall\, x \in \mathbb{R}^d. \tag{3.12}$$

Since $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $v_\star \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^m)$, estimate (3.12) implies $Q \in L^\infty(\mathbb{R}^d, \mathbb{R}^{m,m})$. Then an application of [5, Cor. 4.3] for the trivial multi-index $\alpha = 0$ yields a pointwise exponential estimate

$$|v_\star(x) - v_\infty| \leqslant C_1 \exp\left(-\mu\sqrt{|x|^2+1}\right) \quad \forall\, x \in \mathbb{R}^d\ \forall\, \mu \in [0, \mu_{\max}) \quad \text{with} \quad \mu_{\max} = \frac{\sqrt{a_0 b_0}}{a_{\max} p}, \tag{3.13}$$

where $a_{\max} = \rho(A)$ denotes the spectral radius of A, $-a_0 = s(-A)$ the spectral bound of $-A$, and $-b_0 = s(Df(v_\infty))$ the spectral bound of $Df(v_\infty)$. Combining (3.12) and (3.13) yields

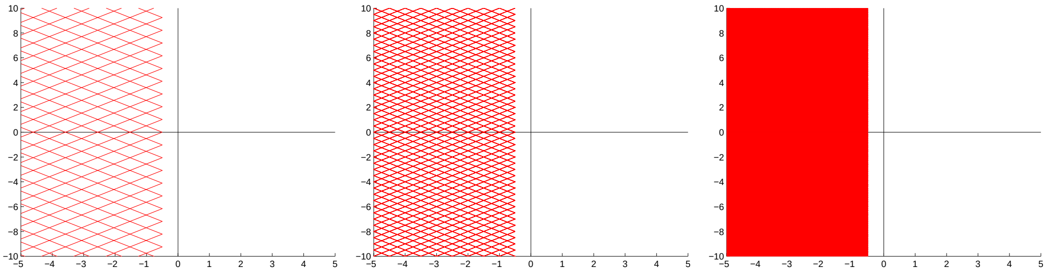
$$|Q(x)| \leqslant C \exp\left(-\mu\sqrt{|x|^2+1}\right) \quad \forall\, x \in \mathbb{R}^d\ \forall\, \mu \in [0, \mu_{\max}) \tag{3.14}$$

with $C = C_1 \sup_{|y-v_\infty|\leqslant C_1} |D^2 f(y)|$. Take a fixed $\mu \in (0, \mu_{\max})$ so that (3.14) implies

$$\eta_R := \operatorname*{ess\,sup}_{|x|\geqslant R} |Q(x)| \leqslant C \operatorname*{ess\,sup}_{|x|\geqslant R} \exp\left(-\mu\sqrt{|x|^2+1}\right) = C\exp\left(-\mu\sqrt{R^2+1}\right) \to 0 \quad \text{as} \quad R \to \infty.$$

This proves the second condition in (3.10). ☐

Let us discuss some consequences of Theorem 3.2 for the position and the structure of the essential spectrum.
From the dispersion relation (3.8) and conditions (A3), (A9$_{B_\infty}$) one infers $\sigma_{\mathrm{disp}}(\mathcal{L}_Q) \subseteq \mathbb{C}_{b_0}$, where $\mathbb{C}_{b_0} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant -b_0\}$ and $-b_0 = s(-B_\infty)$ is the spectral bound of $-B_\infty$. If in addition, the stability condition (A10$_{B_\infty}$) holds, then $-b_0 = s(-B_\infty) < 0$ and $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ is located in the left half-plane.
If there exist indices $n, j \in \{1, \ldots, k\}$ such that $\sigma_j \neq 0$ and $\sigma_n\sigma_j^{-1} \notin \mathbb{Q}$, then $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ is dense in the half-plane $\mathbb{C}_{b_0}$, which implies $\sigma_{\mathrm{ess}}(\mathcal{L}_Q) = \mathbb{C}_{b_0}$. If on the other hand, $\sigma_n\sigma_j^{-1} \in \mathbb{Q}$ for all n, j, then the dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ is a discrete subgroup of $\mathbb{C}_{b_0}$ which is independent of p. The reason for this conclusion is given by Metafune in [27, Thm. 2.6]. Therein it is proved that the essential spectrum of the drift term $v \mapsto \langle S\cdot, \nabla v\rangle)$, agrees with $i\mathbb{R}$, if and only if there exists $0 \neq \sigma_n, \sigma_j \in \mathbb{R}$ such that $\sigma_n\sigma_j^{-1} \notin \mathbb{Q}$. Otherwise, the essential spectrum is a discrete subgroup of $i\mathbb{R}$ which is independent of p.



(a) $d \in \{2, 3\}$, not dense (b) $d = 4$, not dense (c) $d = 4$, dense

FIGURE 3.1. Dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ of $\mathcal{L}_Q$ from (2.7) for parameters $A = \frac{1}{2}(1+i)$, $B_\infty = \frac{1}{2}$ and $Q = 0$.

Figure 3.1 illustrates the set $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ in the scalar complex case for $A = \frac{1}{2}(1+i)$, $B_\infty = \frac{1}{2}$ and $Q = 0$. Figure 3.1(a) shows $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ for $\sigma_1 = 1.027$ and space dimension $d = 2$ and $d = 3$ (see the examples in Section 6). In this case $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ forms a zig-zag curve, see [4] for $d = 2$, and is not dense in $\mathbb{C}_{-\frac{1}{2}}$. Note that density of $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ can only occur for space dimensions $d \geqslant 4$. Figures 3.1(b)(c) show two such cases for $d = 4$. In the first case $\sigma_1 = 1$, $\sigma_2 = 1.5$, hence $\sigma_1\sigma_2^{-1} \in \mathbb{Q}$ and $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ is not dense in $\mathbb{C}_{-\frac{1}{2}}$. The second case belongs to $\sigma_1 = 1$, $\sigma_2 = \frac{1}{2}\exp(1)$ for which density occurs. This shows that $\sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ may change dramatically with the eigenvalues of S.
For $S \neq 0$, Theorem 3.2 implies that the operator $\mathcal{L}_Q$ is not sectorial in $L^p(\mathbb{R}^d, \mathbb{C}^m)$, and the corresponding semigroup is not analytic on $L^p(\mathbb{R}^d, \mathbb{C}^m)$ for every $1 < p < \infty$, see [29, Cor. 7.10]. For the scalar real-valued case we refer to [27, 34, 43].

4. Application of Fredholm theory in $L^p(\mathbb{R}^d, \mathbb{C}^m)$

4.1. Fredholm operator of index 0. In this section we show that the differential operator

$$\lambda I - \mathcal{L}_Q : \left(\mathcal{D}^p(\mathcal{L}_0), \|\cdot\|_{\mathcal{L}_0}\right) \to \left(L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p}\right), \quad 1 < p < \infty \tag{4.1}$$

from (2.7)–(2.9) is Fredholm of index 0 provided that $\operatorname{Re}\lambda > -b_0$. The matrix-valued function $Q \in C(\mathbb{R}^d, \mathbb{C}^{m,m})$ is assumed to be *asymptotically small*, i.e. for $|x|$ large it falls below a certain computable threshold similar to (2.16). We further need the following Lemma (see [4, Lem.4.1]) which is a consequence of Sobolev imbedding and the compactness criterion in L^p-spaces.

Lemma 4.1 (Compactness of multiplication operator)**.** *Let*

$$M \in C(\mathbb{R}^d, \mathbb{C}^{m,m}) \quad \textit{with} \quad \lim_{R\to\infty} \sup_{|x|\geqslant R} |M(x)| \to 0. \tag{4.2}$$

Then the operator of multiplication

$$\widetilde{M} : (W^{1,p}(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{W^{1,p}}) \to (L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p}), \quad u(\cdot) \longmapsto \widetilde{M}u(\cdot) := M(\cdot)u(\cdot),$$

is compact for any $1 < p < \infty$.

We are now ready to prove that $\lambda I - \mathcal{L}_Q$ is Fredholm of index 0.

Theorem 4.2. *Let the assumptions* (A4$_p$), (A6) *and* (A9$_{B_\infty}$) *be satisfied for* $\mathbb{K} = \mathbb{C}$ *and for some* $1 < p < \infty$. *Moreover, let* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geqslant -b_0 + \gamma$ *for some* $\gamma > 0$, *let* $0 < \varepsilon < 1$, *and let* $Q \in C(\mathbb{R}^d, \mathbb{C}^{m,m})$ *satisfy*

$$\sup_{|x|\geqslant R_0} |Q(x)| \leqslant \frac{\varepsilon\gamma}{2} \min\left\{\frac{1}{\kappa a_1}, \frac{1}{C_{0,\varepsilon}}\right\} \quad \textit{for some } R_0 > 0, \tag{4.3}$$

where $-b_0 = s(-B_\infty)$ *denotes the spectral bound of* $-B_\infty$, $\kappa = \operatorname{cond}(Y)$ *the condition number of* Y *from* (A9$_{B_\infty}$), a_1 *the constant from* (2.2), *and* $C_{0,\varepsilon} = C_{0,\varepsilon}(d, p, \varepsilon, \kappa, a_1) > 0$ *the constant from* [5, Thm.2.10]. *Then, the operator*

$$\lambda I - \mathcal{L}_Q : (\mathcal{D}^p(\mathcal{L}_0), \|\cdot\|_{\mathcal{L}_0}) \to (L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p})$$

is Fredholm of index 0. *In particular, the operator* $\lambda I - \mathcal{L}_Q$ *has finite-dimensional kernel and cokernel.*

Proof. The proof follows our outline in formulas (2.20), (2.21).

1. Let us write $\lambda = \lambda_1 + \lambda_2$ with $\lambda_2 := -b_0 + \gamma$ and $\lambda_1 := \lambda - \lambda_2$. Further, take cut-off functions $\chi_R \in C_c^\infty(\mathbb{R}^d, [0,1]), R > 0$ satisfying

$$\chi_R(x) = \chi_1\left(\frac{x}{R}\right), \quad \chi_1(x) = 1\ (|x| \leq 1), \quad \chi_1(x) = 0\ (|x| \geqslant 2). \tag{4.4}$$

   With R_0 from (4.3) we then write

$$Q(x) = Q_s(x) + Q_c(x), \quad Q_s(x) := (1 - \chi_{R_0}(x))Q(x), \quad Q_c(x) := \chi_{R_0}(x)Q(x)$$

   and define $\mathcal{L}_s$ as in (2.21). Then Q_c has compact support in $B_{2R_0}(0)$ and Q_s satisfies due to (4.3)

$$\|Q_s\|_{L^\infty} \leqslant \|1 - \chi_{R_0}\|_\infty \sup_{|x|\geqslant R_0} |Q(x)| \leqslant \frac{\varepsilon\gamma}{2} \min\left\{\frac{1}{\kappa a_1}, \frac{1}{C_{0,\varepsilon}}\right\}. \tag{4.5}$$

   Setting $\widetilde{\mathcal{L}}_s := \mathcal{L}_s - \lambda_2 I$ we can factorize $\lambda I - \mathcal{L}_Q$ as in (2.20),

$$\lambda I - \mathcal{L}_Q = \lambda_1 I - (\mathcal{L}_s - \lambda_2 I) - Q_c(\cdot) = \left(I - Q_c(\cdot)(\lambda_1 I - \widetilde{\mathcal{L}}_s)^{-1}\right)(\lambda_1 I - \widetilde{\mathcal{L}}_s).$$

2. We verify the following two conditions
   a) $\lambda_1 I - \widetilde{\mathcal{L}}_s : (\mathcal{D}^p(\mathcal{L}_0), \|\cdot\|_{\mathcal{L}_0}) \to (L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p})$ is a linear homeomorphism,
   b) $\widetilde{Q}(\lambda_1 I - \widetilde{\mathcal{L}}_s)^{-1} : (L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p}) \to (L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p})$ is a compact operator,

   where $\widetilde{Q}_c$ denotes the operator of multiplication by Q_c which is defined by

$$\widetilde{Q}_c : (W^{1,p}(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{W^{1,p}}) \to (L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p}), \quad [\widetilde{Q}_c v](x) = Q_c(x)v(x). \tag{4.6}$$

   Then standard results from Fredholm theory [3, Satz 9.8],[41, Thm.13.1] imply that both operators

$$I - Q_c(\cdot)(\lambda_1 I - \widetilde{\mathcal{L}}_s)^{-1} : (L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p}) \to (L^p(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^p})$$

   and $\lambda I - \mathcal{L}_Q$ are Fredholm of index 0. It remains to check whether the conditions a) and b) are satisfied.

a) The boundedness of $\lambda_1 I-\widetilde{\mathcal{L}}_{\mathrm{s}}$ with respect to the graph norm $\|v\|_{\mathcal{L}_0} := \|\mathcal{L}_0 v\|_{L^p} + \|v\|_{L^p}$ follows from

$$\|(\lambda_1 I-\widetilde{\mathcal{L}}_{\mathrm{s}})v\|_{L^p} \leqslant \|\mathcal{L}_0 v\|_{L^p} + |\lambda I + B_\infty|\,\|v\|_{L^p} + \|Q_{\mathrm{s}}\|_{L^\infty}\|v\|_{L^p} \leqslant C\,\|v\|_{\mathcal{L}_0} \quad \forall\, v\in D^p(\mathcal{L}_0).$$

The unique solvability of $(\lambda_1 I-\widetilde{\mathcal{L}}_{\mathrm{s}})v=g$ and thus continuity of the inverse, is a consequece of [5, Thm.3.2]. Note that [5, Thm.3.2] is formulated for $(\lambda I-\mathcal{L}_{\mathrm{s}})v=g$ and must be applied to $(\lambda_1 I-\widetilde{\mathcal{L}}_{\mathrm{s}})v=g$, using the shifted data

$$(\widetilde{\mathcal{L}}_{\mathrm{s}}=\mathcal{L}_{\mathrm{s}}-\lambda_2 I, \lambda_1=\lambda-\lambda_2, \widetilde{B}_\infty=B_\infty+\lambda_2 I, \tilde{b}_0=b_0+\lambda_2=\gamma) \quad \text{instead of} \quad (\mathcal{L}_{\mathrm{s}},\lambda,B_\infty,b_0).$$

The assumptions of [5, Thm.3.2] are $\tilde{b}_0 := -s(-B_\infty-\lambda_2 I)>0$ and $\mathrm{Re}\,\lambda_1 \geqslant -(1-\varepsilon)\tilde{b}_0$, which in our case follow from

$$\begin{aligned}\tilde{b}_0 &:= -s(-B_\infty-\lambda_2 I) = -s(-B_\infty)-s(-\lambda_2 I) = b_0+\lambda_2=\gamma>0,\\ \mathrm{Re}\,\lambda_1 &= \mathrm{Re}\,\lambda-\lambda_2 \geqslant -b_0+\gamma-\lambda_2 = 0 > -(1-\varepsilon)\gamma = -(1-\varepsilon)\tilde{b}_0.\end{aligned}$$

b) The operator $\widetilde{Q}_c(\lambda_1 I-\widetilde{\mathcal{L}}_s)^{-1}$ is a linear bounded and compact operator. While linearity and boundedness are clear, compactness follows by an application of Lemma 4.1 with $M=Q_{\mathrm{c}}$. The condition (4.2) is obviously satisfied since Q_{c} is continuous and has compact support. As shown in a), $(\lambda_1 I-\widetilde{\mathcal{L}}_{\mathrm{s}})^{-1}: L^p(\mathbb{R}^d,\mathbb{C}^m)\to\mathcal{D}^p(\mathcal{L}_0)$ is a linear bounded operator with dense range $\mathcal{D}^p(\mathcal{L}_0)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^m)$. Moreover, a key result from [31, Sec.5] guarantees a continuous imbedding $\mathcal{D}^p(\mathcal{L}_0)\subseteq W^{1,p}(\mathbb{R}^d,\mathbb{C}^m)$. Hence Lemma 4.1 shows that $\widetilde{Q}_c(\lambda_1 I-\widetilde{\mathcal{L}}_{\mathrm{s}})^{-1}: L^p(\mathbb{R}^d,\mathbb{C}^m)\to L^p(\mathbb{R}^d,\mathbb{C}^m)$ is compact.

□

4.2. Fredholm alternative. In order to apply the abstract Fredholm alternative to $\lambda I-\mathcal{L}_Q$ from (4.1) we need to identify the abstract adjoint and its domain as a differential operator. As for the operator itself, this can be done along the lines of [29, 30, 31, 32], and we refer to Appendix A for details.
Let $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset X\to Y$ be a closed densely defined Fredholm operator of index 0 between Banach spaces and let $\mathcal{A}^*: Y^*\supset\mathcal{D}(\mathcal{A}^*)\to X^*$ be its adjoint. Then the *Fredholm alternative* states that either the homogeneous equations $\mathcal{A}x=0$ and $\mathcal{A}^*x^*=0$ have only the trivial solutions $x=0\in\mathcal{D}(\mathcal{A})$ and $x^*=0\in\mathcal{D}(\mathcal{A}^*)$, (in which case the inhomogeneous equations $\mathcal{A}x=y$ and $\mathcal{A}^*x^*=y^*$ have unique solutions $x\in\mathcal{D}(\mathcal{A})$ and $x^*\in\mathcal{D}(\mathcal{A}^*)$ for any $y\in Y$ and $y^*\in X^*$), or the homogeneous equations $\mathcal{A}x=0$ and $\mathcal{A}^*x^*=0$ have exactly $1\leqslant n:=\dim\mathcal{N}(\mathcal{A})<\infty$ linearly independent solutions $x_1,\ldots,x_n\in\mathcal{D}(\mathcal{A})$ and $x_1^*,\ldots,x_n^*\in\mathcal{D}(\mathcal{A}^*)$ (in which case the inhomogeneous equation $\mathcal{A}x=y$, $y\in Y$ admits at least one solution $x\in\mathcal{D}(\mathcal{A})$ if and only if $y\in(\mathcal{N}(\mathcal{A}^*))^\perp$) Let us formulate this alternative to the operator $\mathcal{A}=\lambda I-\mathcal{B}_p$ and its adjoint $\mathcal{A}^*=(\lambda I-\mathcal{B}_p)^*$ for $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,\lambda>-b_0$.

Lemma 4.3. *Let the assumptions* (A4$_p$), (A4$_q$), (A6), (A9$_{B_\infty}$) *be satisfied for* $\mathbb{K}=\mathbb{C}$, *for some* $1<p<\infty$ *and for* $q=\frac{p}{p-1}$. *Moreover, let* $\lambda\in\mathbb{C}$ *with* $\mathrm{Re}\,\lambda\geqslant -b_0+\gamma$ *for some* $\gamma>0$, *let* $0<\varepsilon<1$, *and let* $Q\in C(\mathbb{R}^d,\mathbb{C}^{m,m})$ *satisfy* (4.3), *where* $-b_0=s(-B_\infty)$ *denotes the spectral bound of* $-B_\infty$. *Then*

- ***either*** *the homogeneous equations*

$$(\lambda I-\mathcal{L}_Q)v=0 \quad \textit{and} \quad (\lambda I-\mathcal{L}_Q)^*\psi=0$$

have only the trivial solutions $v=0\in\mathcal{D}^p(\mathcal{L}_0)$ *and* $\psi=0\in\mathcal{D}^q(\mathcal{L}_0^*)$, *in which case the inhomogeneous equations*

$$(\lambda I-\mathcal{L}_Q)v=h \quad \textit{and} \quad (\lambda I-\mathcal{L}_Q)^*\psi=\phi$$

have unique solutions $v\in\mathcal{D}^p(\mathcal{L}_0)$ *and* $\psi\in\mathcal{D}^q(\mathcal{L}_0^*)$ *for any* $h\in L^p(\mathbb{R}^d,\mathbb{C}^m)$ *and* $\phi\in L^q(\mathbb{R}^d,\mathbb{C}^m)$.

- ***or*** *the homogeneous equations*

$$(\lambda I-\mathcal{L}_Q)v=0 \quad \textit{and} \quad (\lambda I-\mathcal{L}_Q)^*\psi=0$$

have exactly $1\leqslant n:=\dim\mathcal{N}(\lambda I-\mathcal{L}_Q)<\infty$ *(nontrivial) linearly independent solutions* $v_1,\ldots,v_n\in\mathcal{D}^p(\mathcal{L}_0)$ *and* $\psi_1,\ldots,\psi_n\in\mathcal{D}^q(\mathcal{L}_0^*)$, *in which case the inhomogeneous equation*

$$(\lambda I-\mathcal{L}_Q)v=h, \quad h\in L^p(\mathbb{R}^d,\mathbb{C}^m)$$

admits at least one (not necessarily unique) solution $v\in\mathcal{D}^p(\mathcal{L}_0)$ *if and only if* $h\in(\mathcal{N}((\lambda I-\mathcal{L}_Q)^*))^\perp$.

Proof. The assertion follows from Fredholm's alternative applied to $(\mathcal{A},\mathcal{D}(\mathcal{A}))=(\lambda I-\mathcal{B}_p,\mathcal{D}(\lambda I-\mathcal{B}_p))$ and its adjoint $(\mathcal{A}^*,\mathcal{D}(\mathcal{A}^*))=((\lambda I-\mathcal{B}_p)^*,\mathcal{D}((\lambda I-\mathcal{B}_p)^*))$. For this purpose recall $\mathcal{D}(\lambda I-\mathcal{B}_p)=\mathcal{D}^p(\mathcal{L}_0)$ and $\lambda I-\mathcal{B}_p=\lambda I-\mathcal{L}_Q$ from above as well as $\mathcal{D}((\lambda I-\mathcal{B}_p)^*)=\mathcal{D}^q(\mathcal{L}_0^*)$ and $(\lambda I-\mathcal{B}_p)^*=(\lambda I-\mathcal{L}_Q)^*$ from Lemma A.2. Finally, Theorem 4.2 shows that $\lambda I-\mathcal{B}_p=\lambda I-\mathcal{L}_Q$ is Fredholm of index 0. □

4.3. Exponential decay. Next we prove that any solution v of $(\lambda I-\mathcal{L}_Q)v=0$ decays exponentially in space. The proof is based on an application of [5, Theorem 3.5]. The result is formulated in terms of radial weight functions

$$\theta(x,\mu)=\exp\left(\mu\sqrt{|x|^2+1}\right),\ x\in\mathbb{R}^d,\ \mu\in\mathbb{R} \tag{4.7}$$

and the associated *exponentially weighted Lebesgue* and *Sobolev spaces* for $1\le p<\infty$ and $k\in\mathbb{N}_0$

$$\begin{aligned}L^p_\theta(\mathbb{R}^d,\mathbb{K}^N):=&\{u\in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{K}^N):\|u\|_{L^p_\theta}=\|\theta u\|_{L^p}<\infty\},\\ W^{k,p}_\theta(\mathbb{R}^d,\mathbb{K}^N):=&\{u\in L^p_\theta(\mathbb{R}^d,\mathbb{K}^N):\|u\|^p_{W^{k,p}_\theta}=\sum_{|\beta|\le k}\|D^\beta u\|^p_{L^p_\theta}<\infty\}.\end{aligned}$$

The following theorem also uses the constants $a_0,a_1,a_{\max}$ from (2.2), γ_A from (A4$_p$), δ_A from (A4$_q$), b_0,κ as in Theorem 4.2 and $\beta_\infty\in\mathbb{R}$ such that $\mathrm{Re}\,\langle w,B_\infty w\rangle\geqslant\beta_\infty|w|^2$ for all $w\in\mathbb{C}^m$.

Theorem 4.4 (A-priori estimates in weighted L^p-spaces)**.** *Let the assumptions* (A4$_p$), (A4$_q$), (A6), (A9$_{B_\infty}$) *be satisfied for* $\mathbb{K}=\mathbb{C}$, *for some* $1<p<\infty$ *and for* $q=\frac{p}{p-1}$. *Moreover, let* $\lambda\in\mathbb{C}$ *with* $\mathrm{Re}\,\lambda\geqslant-b_0+\gamma$ *for some* $\gamma>0$, *let* $0<\varepsilon<1$ *and let* $Q\in C(\mathbb{R}^d,\mathbb{C}^{m,m})$ *satisfy*

$$\operatorname*{ess\,sup}_{|x|\geqslant R_0}|Q(x)|\leqslant\frac{\varepsilon\gamma}{2}\min\left\{\frac{1}{\kappa a_1},\frac{1}{C_{0,\varepsilon}},\frac{\beta_\infty-b_0}{\gamma}+1\right\}\ \text{for some } R_0>0. \tag{4.8}$$

Consider weight functions $\theta_j(x)=\theta(x,\mu_j)$, $j=1,2,3,4$ *with exponents* μ_j *satisfying*

$$-\sqrt{\varepsilon\frac{\gamma_A(\beta_\infty-b_0+\gamma)}{2d|A|^2}}\leqslant\mu_1\leqslant0\leqslant\mu_2\leqslant\varepsilon\frac{\sqrt{a_0\gamma}}{a_{\max}p}, \tag{4.9}$$

and

$$-\sqrt{\varepsilon\frac{\delta_A(\beta_\infty-b_0+\gamma)}{2d|A|^2}}\leqslant\mu_3\leqslant0\leqslant\mu_4\leqslant\varepsilon\frac{\sqrt{a_0\gamma}}{a_{\max}q}. \tag{4.10}$$

Then every solution $v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^m)\cap L^p_{\theta_1}(\mathbb{R}^d,\mathbb{C}^m)$ *resp.* $\psi\in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^m)\cap L^q_{\theta_3}(\mathbb{R}^d,\mathbb{C}^m)$ *of*

$$(\lambda I-\mathcal{L}_Q)v=g\quad\text{in } L^p_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^m)\quad\text{resp.}\quad(\lambda I-\mathcal{L}_Q)^*\psi=\phi\quad\text{in } L^q_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^m)$$

with $g\in L^p_{\theta_2}(\mathbb{R}^d,\mathbb{C}^m)$ *resp.* $\phi\in L^q_{\theta_4}(\mathbb{R}^d,\mathbb{C}^m)$ *satisfies* $v\in W^{1,p}_{\theta_2}(\mathbb{R}^d,\mathbb{C}^m)$ *resp.* $\psi\in W^{1,q}_{\theta_4}(\mathbb{R}^d,\mathbb{C}^m)$. *Moreover, the following estimates hold:*

$$\|v\|_{W^{k,p}_{\theta_2}}\leqslant C_1\left(\mathrm{Re}\,\lambda+b_0\right)^{-\frac{k}{2}}\left(\|v\|_{L^p_{\theta_1}}+\|g\|_{L^p_{\theta_2}}\right),\quad k=0,1, \tag{4.11}$$

$$\|\psi\|_{W^{k,q}_{\theta_4}}\leqslant C_3\left(\mathrm{Re}\,\lambda+b_0\right)^{-\frac{k}{2}}\left(\|\psi\|_{L^q_{\theta_3}}+\|\phi\|_{L^q_{\theta_4}}\right),\quad k=0,1. \tag{4.12}$$

Remark 4.5. Due to the choice of exponents in (4.9), (4.10), the main effect is to show that solutions of inhomogeneous equations lie in a small space of exponentially decaying solution, provided they come from a large space exponentially growing solutions and provided the inhomogeneity belongs to the same small space of exponentially decreasing functions.

Proof. Decompose $\lambda\in\mathbb{C}$ into $\lambda=\lambda_1+\lambda_2$ with $\lambda_2:=-b_0+\gamma$, $\lambda_1:=\lambda-\lambda_2$, and write $\lambda I-\mathcal{L}_Q=\lambda_1 I-\widetilde{\mathcal{L}}_Q$ with $\widetilde{\mathcal{L}}_Q:=\mathcal{L}_Q-\lambda_2 I$. This implies

$$g=(\lambda I-\mathcal{L}_Q)v=(\lambda_1-\widetilde{\mathcal{L}}_Q)v. \tag{4.13}$$

Introducing the matrix $\widetilde{B}_\infty:=B_\infty+\lambda_2 I$ and the two quantities $\tilde{b}_0:=b_0+\lambda_2=\gamma$, $\tilde{\beta}_\infty:=\beta_\infty+\lambda_2$, which satisfy $0<\tilde{b}_0\leqslant\tilde{\beta}_\infty$, we now apply [5, Thm.3.5] to (4.13) with

$$(\widetilde{L}_Q,\lambda_1,\widetilde{B}_\infty,\tilde{b}_0,\tilde{\beta}_\infty)\quad\text{instead of}\quad(\mathcal{L}_Q,\lambda,B_\infty,b_0,\beta_\infty).$$

For this purpose, one must check the following three properties

$$\tilde{\beta}_\infty > 0, \quad \operatorname{Re}\lambda_1 \geqslant -(1-\varepsilon)\tilde{\beta}_\infty, \quad \text{and} \quad \operatorname*{ess\,sup}_{|x|\geqslant R_0} |Q(x)| \leqslant \frac{\varepsilon}{2}\min\left\{\frac{\tilde{b}_0}{\kappa a_1}, \frac{\tilde{b}_0}{C_{0,\varepsilon}}, \tilde{\beta}_\infty\right\}.$$

Using $0 < b_0 \leqslant \tilde{\beta}_\infty$, $0 < \varepsilon < 1$, $\operatorname{Re}\lambda \geqslant -b_0 + \gamma$, $\lambda = \lambda_1 + \lambda_2$ and $\lambda_2 = -b_0 + \gamma$, we obtain

$$\tilde{\beta}_\infty = \beta_\infty - b_0 + \gamma \geqslant \gamma < 0, \qquad \operatorname{Re}\lambda_1 = \operatorname{Re}\lambda - \lambda_2 \geqslant 0 \geqslant -(1-\varepsilon)\tilde{\beta}_\infty.$$

The Q-estimate follows from (4.8) using $\gamma = \tilde{b}_0$ and $\beta_\infty - b_0 + \gamma = \tilde{\beta}_\infty - \tilde{b}_0 + \gamma = \tilde{\beta}_\infty$. Replacing $((\text{A4}_p), p, v, g, \mu_1, \mu_2)$ by $((\text{A4}_q), q, \psi, \phi, \mu_3, \mu_4)$, the same approach yields the assertion for solutions ψ of the adjoint problem $(\lambda I - \mathcal{L}_Q^*)\psi = \phi$. □

4.4. Fredholm properties of the linearized operator and exponential decay of eigenfunctions. We now apply the previous results from Section 4 to

$$-B_\infty = Df(v_\infty) \quad \text{and} \quad Q(x) = Df(v_\star(x)) - Df(v_\infty)$$

in which case the linearization $\mathcal{L}$ from (1.7) coincides with the variable coefficient operator $\mathcal{L}_Q$ from (2.7). This allows us to transfer the Fredholm alternative (Lemma 4.3) and the exponential decay (Theorem 4.4) to the linearized operator $\mathcal{L}$ and its adjoint $\mathcal{L}^*$.
In the following, let $a_{\max} = \rho(A)$ denote the spectral radius of A, $-a_0 = s(-A)$ the spectral bound of $-A$, $-b_0 = s(Df(v_\infty))$ the spectral bound of $Df(v_\infty)$ and let β_∞ be from (A11).

Proof (of Theorem 2.8). With $-B_\infty = Df(v_\infty)$ and $Q(x) = Df(v_\star(x)) - Df(v_\infty)$ we obtain $\mathcal{L} = \mathcal{L}_Q$.

a) An application of Theorem 4.2 proves that $\lambda I - \mathcal{L}$ is Fredholm of index 0 with finite-dimensional kernel and cokernel. In order to apply Theorem 4.2, note that the assumptions (A4_p) and (A6) are directly satisfied, and (A9_{B_∞}) follows from (A9). The property $Q \in C(\mathbb{R}^d, \mathbb{C}^{m,m})$ follows from (A7) and $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$. Similarly, condition (4.3) follows for $\varepsilon = \frac{1}{2}$ from (A7) and (2.16)

$$\begin{aligned} |Q(x)| = |Df(v_\star(x)) - Df(v_\infty)| &\leqslant \int_0^1 \left|D^2 f(v_\infty + s\,(v_\star(x) - v_\infty)\right| ds\, |v_\star(x) - v_\infty| \\ &\leqslant K_1 \Big(\sup_{z \in B_{K_1}(v_\infty)} \left|D^2 f(z)\right| \Big) \leqslant \frac{\gamma}{4} \min\left\{\frac{1}{\kappa a_1}, \frac{1}{C_{0,1/2}}\right\}, \end{aligned}$$

provided we choose $K_1 = K_1(A, f, v_\infty, \gamma, d, p) > 0$ such that

$$K_1 \Big(\sup_{z \in B_{K_1}(v_\infty)} \left|D^2 f(z)\right| \Big) \leqslant \frac{\gamma}{4} \min\left\{\frac{1}{\kappa a_1}, \frac{1}{C_{0,1/2}}\right\}. \tag{4.14}$$

b) Since $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L})$ has geometric multiplicity $n = \dim \mathcal{N}(\lambda I - \mathcal{L})$ for some $n \in \mathbb{N}$, we deduce from Lemma 4.3 that the homogeneous equations $(\lambda I - \mathcal{L})v = 0$ and $(\lambda I - \mathcal{L})^*\psi = 0$ have exactly n linearly independent solutions $v_1, \ldots, v_n \in \mathcal{D}^p(\mathcal{L}_0)$ and $\psi_1, \ldots, \psi_n \in \mathcal{D}^q(\mathcal{L}_0^*)$. Further, Lemma 4.3 implies that for any $g \in L^p(\mathbb{R}^d, \mathbb{C}^m)$ the inhomogeneous equation $(\lambda I - \mathcal{L})v = g$ has at least one (not necessarily unique) solution $v \in \mathcal{D}^p(\mathcal{L}_0)$ if and only if $g \in (\mathcal{N}((\lambda I - \mathcal{L})^*))^\perp$, which corresponds (2.18). Finally, the estimates from (2.19) follow from abstract results of Fredholm theory.

□

Theorem 4.6 (Exponential decay of eigenfunctions)**.** *Let all assumptions of Theorem 2.8 a)-b) hold.*

a) *(Exponential decay of eigenfunctions in weighted L^p-spaces). Consider weight functions $\theta_j(x) = \theta(x, \mu_j)$, $j = 1, \ldots, 4$ with exponents that satisfy (4.9) and (4.10).*
Then every classical solution $v \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ of the eigenvalue problems

$$(\lambda I - \mathcal{L})v = 0 \quad \text{and} \quad (\lambda I - \mathcal{L})^*\psi = 0, \tag{4.15}$$

such that $v \in L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in L^q_{\theta_3}(\mathbb{R}^d, \mathbb{C}^m)$ satisfies $v \in W^{1,p}_{\theta_2}(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in W^{1,q}_{\theta_4}(\mathbb{R}^d, \mathbb{C}^m)$.

b) *(Pointwise exponential decay of eigenfunctions). In addition to a), let* $p \geqslant \frac{d}{2}$*,* $f \in C^k(\mathbb{R}^m,\mathbb{R}^m)$*,* $v_\star \in C^{k+1}(\mathbb{R}^d,\mathbb{R}^m)$ *and* $v \in C^{k+1}(\mathbb{R}^d,\mathbb{C}^m)$ *for some* $k\in\mathbb{N}$ *with* $k\geqslant 2$*. Then* v *belongs to* $W^{k,p}_{\theta_2}(\mathbb{R}^d,\mathbb{C}^m)$ *and satisfies the pointwise estimate*

$$|D^\alpha v(x)| \leqslant C\exp\left(-\mu_2\sqrt{|x|^2+1}\right),\quad x\in\mathbb{R}^d \tag{4.16}$$

for every exponential decay rate $0\leqslant\mu_2\leqslant\varepsilon\frac{\sqrt{a_0\gamma}}{a_{\max}p}$ *and for every multi-index* $\alpha\in\mathbb{N}_0^d$ *satisfying* $d<(k-|\alpha|)p$*.*

c) *(Pointwise exponential decay of adjoint eigenfunctions). In addition to b) let* $\min\{p,q\}\geqslant\frac{d}{2}$ *and* $\psi\in C^{k+1}(\mathbb{R}^d,\mathbb{C}^m)$*. Then* ψ *belongs to* $W^{k,q}_{\theta_4}(\mathbb{R}^d,\mathbb{C}^m)$ *and satisfies the pointwise estimate*

$$|D^\alpha \psi(x)| \leqslant C\exp\left(-\mu_4\sqrt{|x|^2+1}\right),\quad x\in\mathbb{R}^d \tag{4.17}$$

for every decay rate $0\leqslant\mu_4\leqslant\varepsilon\frac{\sqrt{a_0\gamma}}{a_{\max}q}$ *and for every multi-index* $\alpha\in\mathbb{N}_0^d$ *with* $d<(k-|\alpha|)q$*.*

Proof. As in the proof of Theorem 2.8. let $-B_\infty = Df(v_\infty)$, $Q(x)=Df(v_\star(x))-Df(v_\infty)$. Assertion a) follows directly from an application of Theorem 4.4 if K_1 from (4.14) is chosen such that

$$K_1\left(\sup_{z\in B_{K_1}(v_\infty)}\left|D^2f(z)\right|\right)\leqslant\frac{\varepsilon\gamma}{2}\min\left\{\frac{1}{\kappa a_1},\frac{1}{C_{0,\varepsilon}},\frac{\beta_\infty-b_0}{\gamma}+1\right\}.$$

The proof of b) works in quite an analogous fashion as in [5, Thm.5.1(2)] and will not be repeated here. Similarly, assertion c) follows when applying the theory from [5] to the adjoint operator. □

5. Point spectrum on the imaginary axis

5.1. Eigenvalues and eigenvectors in L^p. Let us first compute the eigenvalues λ and associated eigenfunctions v of (1.8) caused by the symmetriy w.r.t. the SE(d)-group action. These eigenvalues belong to the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ of the linearization $\mathcal{L}$ (Theorem 2.9).Consider the equation

$$0=(\lambda I-\mathcal{L})v=\lambda v-A\triangle v-(Dv)(Sx)+Df(v_\star)v,\ x\in\mathbb{R}^d. \tag{5.1}$$

Assume an eigenfunction v of the form

$$v=(Dv_\star)(Ex+b)\qquad\text{for some } E\in\mathbb{C}^{d,d},\ b\in\mathbb{C}^d,\ E^\top=-E,\ v_\star\in C^3(\mathbb{R}^d,\mathbb{R}^m). \tag{5.2}$$

Plugging (5.2) into (5.1) and using the equalities

$$\lambda v=(Dv_\star)(\lambda(Ex+b)), \tag{5.3}$$

$$A\triangle v=(D(A\triangle v_\star))(Ex+b), \tag{5.4}$$

$$(Dv)(Sx)=(D((Dv_\star)(Sx)))(Ex+b)+(Dv_\star)([E,S]x-Sb), \tag{5.5}$$

$$Df(v_\star)v=(D(f(v_\star)))(Ex+b), \tag{5.6}$$

with Lie brackets $[E,S]:=ES-SE$, we obtain

$$0=(Dv_\star)\left((\lambda E-[E,S])x+(\lambda b+Sb)\right)-D\left(A\triangle v_\star+(Dv_\star)(Sx)+f(v_\star)\right)(Ex+b). \tag{5.7}$$

Since $v_\star$ satisfies the rotating wave equation

$$0=A\triangle v_\star+(Dv_\star)(Sx)+f(v_\star),\ x\in\mathbb{R}^d, \tag{5.8}$$

the second term in (5.7) vanishes and we end up with

$$0=(Dv_\star)\left((\lambda E-[E,S])x+(\lambda b+Sb)\right),\ x\in\mathbb{R}^d. \tag{5.9}$$

Comparing coefficients in (5.9) yields the finite-dimensional eigenvalue problem

$$\lambda E=[E,S], \tag{5.10a}$$

$$\lambda b=-Sb, \tag{5.10b}$$

which is be solved for (λ,E,b). Since E is required to be skew-symmetric, we expect $\frac{d(d+1)}{2}$ nontrivial solutions. If (λ,E) is a solution of (5.10a), then $(\lambda,E,0)$ solves (5.10). Similarly, if (λ,b) is a solution of

(5.10b), then $(\lambda, 0, b)$ solves (5.10). We solve (5.10) by using that S is unitarily diagonalizable over $\mathbb{C}$, i.e. there is a unitary matrix $U \in \mathbb{C}^{d,d}$ such that $S = U\Lambda_S U^{\mathsf{H}}$, where $\Lambda_S = \operatorname{diag}(\lambda_1^S, \ldots, \lambda_d^S)$ and $\lambda_1^S, \ldots, \lambda_d^S$ denote the eigenvalues of S. In particular, this implies $S^\top = \overline{U}\Lambda_S U^\top$.

- Solving (5.10b): Multiplying (5.10b) from left by U^{H} and defining $\tilde{b} = U^{\mathsf{H}} b$ we obtain

$$\lambda\tilde{b} = \lambda U^{\mathsf{H}} b = -U^{\mathsf{H}} S b = -U^{\mathsf{H}} U \Lambda_S U^{\mathsf{H}} b = -\Lambda_S \tilde{b}. \tag{5.11}$$

  Equation (5.11) has solutions $(\lambda, \tilde{b}) = (-\lambda_l^S, e_l)$, hence (5.10b) has solutions $(\lambda, b) = (-\lambda_l^S, Ue_l)$, and (5.10) has solutions $(\lambda, E, b) = (-\lambda_l^S, 0, Ue_l)$ for $l = 1, \ldots, d$.
- Solving (5.10a): Multiplying (5.10a) from left by U^{H}, from right by $\bar{U}$, defining $\tilde{E} = U^{\mathsf{H}} E\overline{U}$, and using skew-symmetry of S and $\tilde{E}$, we obtain

$$\lambda\tilde{E} = \lambda U^{\mathsf{H}} E\overline{U} = U^{\mathsf{H}}[E,S]\overline{U} = -U^{\mathsf{H}} E\overline{U}\Lambda_S U^\top\overline{U} - U^{\mathsf{H}} U\Lambda_S U^{\mathsf{H}} E\overline{U} = \tilde{E}^\top\Lambda_S - \Lambda_S\tilde{E}. \tag{5.12}$$

  Equation (5.12) has solutions $(\lambda, \tilde{E}) = (-(\lambda_i^S + \lambda_j^S), I_{ij} - I_{ji})$, hence (5.10a) has solutions $(\lambda, E) = (-(\lambda_i^S + \lambda_j^S), U(I_{ij} - I_{ji})U^\top)$, and (5.10) has solutions $(\lambda, E, b) = (-(\lambda_i^S + \lambda_j^S), U(I_{ij} - I_{ji})U^\top, 0)$ for $i = 1, \ldots, d-1$, $j = i+1, \ldots, d$, where I_{ij} has entry 1 in the ith row and jth column and 0 otherwise.

Collecting these eigenvalues (and using skew-symmetry of S once more) we find the *symmetry set*

$$\sigma_{\text{sym}}(\mathcal{L}) = \sigma(S) \cup \{\lambda_i^S + \lambda_j^S : 1 \leqslant i < j \leqslant d\}. \tag{5.13}$$

A rigorous proof of the relation $\sigma_{\text{sym}}(\mathcal{L}) \subseteq \sigma_{\text{pt}}(\mathcal{L})$ (see Theorem 2.10) is proved next.

Proof (of Theorem 2.10). For $\lambda \in \sigma_{\text{sym}}(\mathcal{L})$ the function v from (2.23) is a classical solution of the eigenvalue problem (1.8) by Theorem 2.9. An application of [5, Cor. 4.1] implies $v_\star \in W^{3,p}_\theta(\mathbb{R}^d, \mathbb{R}^m)$. Thus we have $v \in W^{2,p}(\mathbb{R}^d, \mathbb{C}^m)$ and $\mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^m)$, and hence $v \in \mathcal{D}^p(\mathcal{L}_0)$ solves (1.8) in L^p. Therefore, v is an eigenfunction of $\mathcal{L}$ in L^p with eigenvalue $\lambda \in \sigma(\mathcal{L})$. By Theorem 2.8a) spectral stability of $Df(v_\infty)$ from (A11) implies that $\lambda I - \mathcal{L}$ is Fredholm of index 0, so that $\lambda \in \sigma_{\text{pt}}(\mathcal{L})$ holds according to Definition 2.6. ☐

5.2. Multiplicities of eigenvalues. Let us discuss some consequences of Theorem 2.10. Since $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^m)$, the function $v(x) = \langle Sx, \nabla v_\star(x)\rangle$, $x \in \mathbb{R}^d$ containing angular derivatives is in C^2 and a classical solution of $\mathcal{L}v = 0$, i.e. v is an eigenfunction of $\mathcal{L}$ with eigenvalue $\lambda = 0$. This can either be shown directly by differentiating (1.5) (cf. [4] for $d = 2$) or it can be deduced from Theorem 2.9 with $(\lambda, E, b) = (0, S, 0)$. Theorem 2.9 gives also information about the multiplicity of the isolated eigenvalues of $\mathcal{L}$. More precisely, for any fixed skew-symmetric $S \in \mathbb{R}^{d,d}$, Theorem 2.9 yields a lower bound for the geometric and hence for the algebraic multiplicities. In general, multiplicities will depend on the eigenvalues of S and higher multiplicities may arise from resonances such as $\sigma_1 + \sigma_2 = \sigma_3$ or the like.
Figure 5.1 shows the eigenvalues $\lambda \in \sigma_{\text{sym}}(\mathcal{L})$ from Theorem 2.9 and lower bounds of their multiplicities for different space dimensions $d = 2, 3, 4, 5$. In line with formula (5.13), the eigenvalues $\lambda \in \sigma(S)$ are indicated by blue circles, the eigenvalues $\lambda \in \{\lambda_i + \lambda_j \mid \lambda_i, \lambda_j \in \sigma(S), 1 \leqslant i < j \leqslant d\}$ by green crosses. The imaginary values to the right of the symbols denote the eigenvalues and the numbers to the left the lower bounds for their corresponding multiplicities. We observe that for space dimension d there are $\frac{d(d+1)}{2}$ eigenvalues on the imaginary axis that are caused by the symmetries of the SE(d)-group action.

Example 5.1 (Point spectrum of $\mathcal{L}$ for $d = 2$)**.** In case $d = 2$ the skew-symmetric matrix $S \in \mathbb{R}^{2,2}$, the diagonal matrix $\Lambda_S \in \mathbb{C}^{2,2}$ and the unitary matrix $U \in \mathbb{C}^{2,2}$, satisfying $S = U\Lambda_S U^{\mathsf{H}}$, are given by

$$S = \begin{pmatrix} 0 & S_{12} \\ -S_{12} & 0 \end{pmatrix}, \quad \Lambda_S = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

with $\sigma_1 = S_{12}$, $k = 1$, $\lambda_1^S = i\sigma_1$, $\lambda_2^S = -i\sigma_1$. Therefore, using the relation $U(I_{12} - I_{21})U^T = -i(I_{12} - I_{21})$, Theorem 2.9 implies the following eigenvalues and eigenfunctions of $\mathcal{L}$, cf. [4, Lem. 2.3],

$$\begin{aligned} \lambda_1 &= 0, & v_1 &= D^{(1,2)} v_\star, \\ \lambda_{2,3} &= \pm i\sigma_1, & v_{2,3} &= D_1 v_\star \pm i D_2 v_\star \end{aligned} \tag{5.14}$$

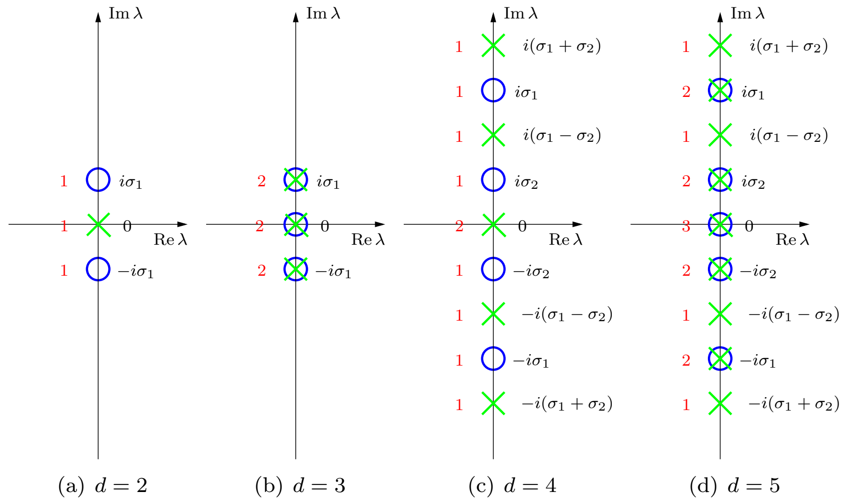


FIGURE 5.1. Point spectrum of the linearization $\mathcal{L}$ on the imaginary axis $i\mathbb{R}$ for space dimension $d = 2, 3, 4, 5$ and group dimension $\frac{1}{2}d(d+1)$ as given by Theorem 2.9.

Example 5.2 (Point spectrum of $\mathcal{L}$ for $d = 3$)**.** In case $d = 3$ the skew-symmetric matrix $S \in \mathbb{R}^{3,3}$, the diagonal matrix $\Lambda_S \in \mathbb{C}^{3,3}$ and the unitary matrix $U \in \mathbb{C}^{3,3}$, satisfying $S = U\Lambda_S U^{\mathsf{H}}$, are given by $\Lambda_S = \mathrm{diag}(i\sigma_1, -i\sigma_1, 0)$ and

$$S = \begin{pmatrix} 0 & S_{12} & S_{13} \\ -S_{12} & 0 & S_{23} \\ -S_{13} & -S_{23} & 0 \end{pmatrix}, U = \frac{1}{q\sigma_1}\begin{pmatrix} \sigma_1 S_{13} - iS_{12}S_{23} & \sigma_1 S_{13} + iS_{12}S_{23} & qS_{23} \\ \sigma_1 S_{23} + iS_{12}S_{13} & \sigma_1 S_{23} - iS_{12}S_{13} & -qS_{13} \\ i(S_{13}^2 + S_{23}^2) & -i(S_{13}^2 + S_{23}^2) & qS_{12} \end{pmatrix},$$

with $\sigma_1 = \sqrt{S_{12}^2 + S_{13}^2 + S_{23}^2}$, $q = \sqrt{2(S_{13}^2 + S_{23}^2)}$, $\lambda_1^S = i\sigma_1$, $\lambda_2^S = -i\sigma_1$ and $\lambda_3^S = 0$. From this one calculates the matrices $U(I_{12} - I_{21})U^T$, $U(I_{13} - I_{31})U^T$, $U(I_{23} - I_{32})U^T$ and finds from Theorem 2.9 the following eigenvalues and eigenfunctions of $\mathcal{L}$,

$$\begin{aligned} \lambda_1 &= 0, & v_1 &= S_{12}D^{(1,2)}v_\star + S_{13}D^{(1,3)}v_\star + S_{23}D^{(2,3)}v_\star, \\ \lambda_2 &= 0, & v_2 &= S_{23}D_1 v_\star - S_{13}D_2 v_\star + S_{12}D_3 v_\star, \\ \lambda_{3,4} &= \pm i\sigma_1, & v_{3,4} &= (\sigma_1 S_{13} \pm iS_{12}S_{23})D_1 v_\star + (\sigma_1 S_{23} \pm iS_{12}S_{13})D_2 v_\star \pm i(S_{13}^2 + S_{23}^2)D_3 v_\star, \\ \lambda_{5,6} &= \pm i\sigma_1, & v_{5,6} &= -(S_{13}^2 + S_{23}^2)D^{(1,2)}v_\star - (-S_{12}S_{13} \pm i\sigma_1 S_{23})D^{(1,3)}v_\star \\ & & &\quad + (S_{12}S_{23} \pm i\sigma_1 S_{13})D^{(2,3)}v_\star. \end{aligned} \tag{5.15}$$

6. Numerical spectra and eigenfunctions of spinning solitons

Consider the *cubic-quintic complex Ginzburg-Landau equation (QCGL)*, [26],

$$u_t = \alpha \triangle u + u\left(\delta + \beta |u|^2 + \gamma |u|^4\right) \tag{6.1}$$

where $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{C}$, $d \in \{2, 3\}$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 0$ and $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(u) := u\left(\delta + \beta |u|^2 + \gamma |u|^4\right). \tag{6.2}$$

For the parameters, see [13],

$$\alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10}i, \quad \mu = -\frac{1}{2} \tag{6.3}$$

this equation exhibits so called *spinning soliton* solutions.
Figure 6 shows the isosurfaces of $\mathrm{Re}\,v_\star(x) = \pm 0.5$ (left), $\mathrm{Im}\,v_\star(x) = \pm 0.5$ (middle), and $|v_\star(x)| = 0.5$ (right) of a spinning soliton profile $v_\star$ for $d = 3$. The rotational velocity matrix S from Example 5.2 takes the

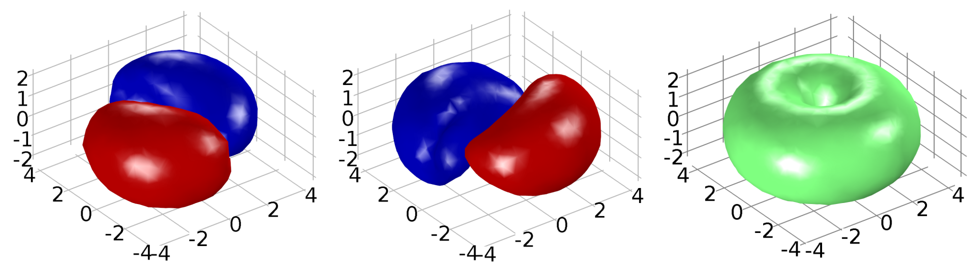


FIGURE 6.1. Isosurfaces of spinning solitons of QCGL (6.1) for parameters (6.3) and $d = 3$: $\mathrm{Re}\, v_\star(x) = \pm 0.5$ (left), $\mathrm{Im}\, v_\star(x) = \pm 0.5$ (middle), $|v_\star(x)| = 0.5$ (right).

values $(S_{12}, S_{13}, S_{23}) = (0.6888, -0.0043, -0.0043)$. Therefore, the eigenvalues of S are $\sigma(S) = \{0, \pm i\sigma_1\}$ with $\sigma_1 = \sqrt{S_{12}^2 + S_{13}^2 + S_{23}^2} = 0.6888$. Moreover, the temporal period T^{3d} that the soliton need for exact one rotation is $T^{3d} = \frac{2\pi}{|\sigma_1|} = 9.1216$. The profile $v_\star$ and the velocity matrix S of the spinning soliton are computed simultaneously by the *freezing method* from [6, 8]. For more detailed information concerning the computation of $v_\star$ and S we refer to [29]. We suggest that there is no explicit formula for spinning soliton solutions of (6.1), only implicit formulas and numerical approximations are available.
The real-valued version of (6.1) reads as follows

$$\mathbf{u}_t = \mathbf{A}\triangle\mathbf{u} + \mathbf{f}(\mathbf{u}) \quad \text{with} \quad \mathbf{A} := \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{6.4}$$

and $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{f}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} (u_1\delta_1 - u_2\delta_2) + (u_1\beta_1 - u_2\beta_2)\left(u_1^2 + u_2^2\right) + (u_1\gamma_1 - u_2\gamma_2)\left(u_1^2 + u_2^2\right)^2 \\ (u_1\delta_2 + u_2\delta_1) + (u_1\beta_2 + u_2\beta_1)\left(u_1^2 + u_2^2\right) + (u_1\gamma_2 + u_2\gamma_1)\left(u_1^2 + u_2^2\right)^2 \end{pmatrix}, \tag{6.5}$$

where $u = u_1 + iu_2$, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$, $\delta = \delta_1 + i\delta_2$.
For the real-valued formulation (6.4) the constants from (2.2) are given by

$$a_0 = \mathrm{Re}\,\alpha, \quad a_{\min} = a_{\max} = |\mathbf{A}| = |\alpha|, \quad a_1 = \left(\frac{|\alpha|}{\mathrm{Re}\,\alpha}\right)^{\frac{d}{2}}, \quad b_0 = \beta_\infty = -\mathrm{Re}\,\delta, \quad v_\infty = 0.$$

They satisfy our assumptions (A1)–(A11) provided that

$$\mathrm{Re}\,\alpha > 0, \quad \mathrm{Re}\,\delta < 0, \quad p_{\min} = \frac{2|\alpha|}{|\alpha| + \mathrm{Re}\,\alpha} < p < \frac{2|\alpha|}{|\alpha| - \mathrm{Re}\,\alpha} = p_{\max}. \tag{6.6}$$

In particular, (A5$_p$) and (A5$_q$) for $1 < p < \infty$ and $q = \frac{p}{p-1}$ lead to the same restriction on p, namely

$$\frac{\mathrm{Re}\,\alpha}{|\alpha|} = \mu_1(\alpha) = \mu_1(\bar{\alpha}) > \frac{|q-2|}{2} = \frac{|p-2|}{2},$$

which is equivalent to the latter condition in (6.6) and $q_{\min} := p_{\min} < q < p_{\max} =: q_{\max}$. In particular, if p approaches $p_{\max}$ (or $p_{\min}$) then q approaches $p_{\min}$ (or $p_{\max}$). Note that the application of Theorem 2.7 additionally requires $p \geqslant \frac{d}{2}$. For our parameters (6.3) this allows us to choose p such that

$$1.1716 \approx \frac{4}{2+\sqrt{2}} = p_{\min} < p < p_{\max} = \frac{4}{2-\sqrt{2}} \approx 6.8284 \quad \text{and} \quad p \geqslant \frac{d}{2}, \tag{6.7}$$

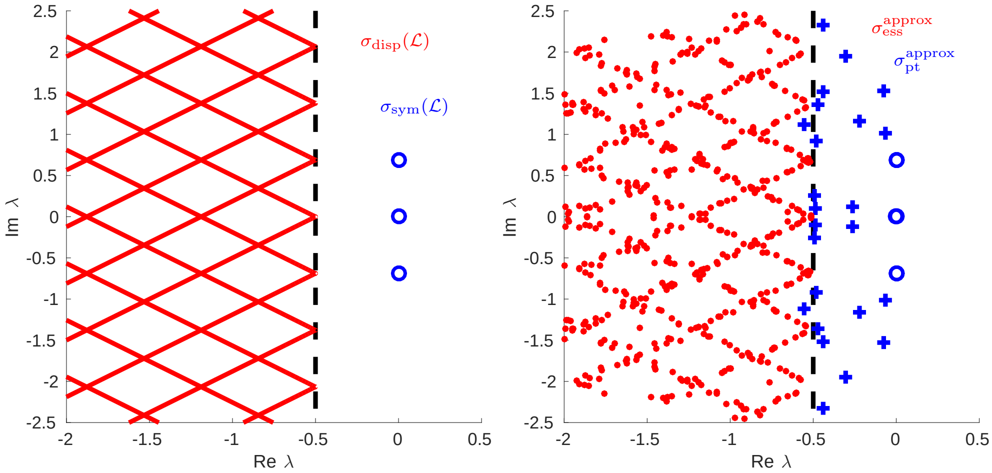
e.g. $p = 2, 3, 4, 5, 6$. A more detailed discussion of the assumptions (A1)–(A11) is worked out in [5, 29].
Linearizing at a rotating wave solution in a co-rotating frame leads to the operator

$$\mathcal{L}\mathbf{v}(x) = \mathbf{A}\triangle\mathbf{v}(x) + \langle Sx, \nabla\mathbf{v}(x)\rangle + D\mathbf{f}(\mathbf{v}_\star(x))\mathbf{v}(x)$$

and its adjoint

$$\mathcal{L}^*\psi(x) = \mathbf{A}^T\triangle\psi(x) - \langle Sx, \nabla\psi(x)\rangle + D\mathbf{f}(\mathbf{v}_\star(x))^T\psi(x).$$

We next numerically solve the eigenvalues problems $(\lambda I - \mathcal{L})\mathbf{v} = 0$ and $(\lambda I - \mathcal{L})^{*}\psi = 0$, obtained from the QCGL (6.4). We compare analytical and numerical spectrum, and compute eigenfunctions and their adjoints numerically. For the computations below, realized by the CAE software Comsol Multiphysics [1], we used continuous piecewise linear finite elements with maximal stepsize $\triangle x = 0.8$ (if $d = 3$), and homogeneous Neumann boundary conditions to compute 800 eigenvalues which are located near $\sigma = -b_0$ (measured radially) and satisfy an eigenvalue tolerance of 10^{-7}. The profile $v_\star$ and the velocity matrix S are determined from a simulation with the so-called freezing method, see [29] for a detailed discussion of how to get these quantities numerically.



(a) Analytical part of the spectrum (b) Numerical spectrum

FIGURE 6.2. Spectrum of QCGL linearized at a spinnig soliton for parameters (6.3) and $d = 3$.

Figure 6.2(a) shows the dispersion set $\sigma_{\mathrm{disp}}(\mathcal{L})$ (red lines) and the symmetry set $\sigma_{\mathrm{sym}}(\mathcal{L})$ (blue circles). Both of them belong to the analytical parts of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$. The entries and the spectrum of the velocity matrix $S \in \mathbb{R}^{3,3}$ are given by

(6.8)
$$(S_{12}, S_{13}, S_{23}) = (0.6888, -0.0043, -0.0043), \quad \sigma(S) = \{0, \pm i\sigma_1\}, \quad \sigma_1 = \sqrt{S_{12}^2 + S_{13}^2 + S_{23}^2} = 0.6888.$$

Therefore, the symmetry set (5.13) reads as follows

(6.9) $$\sigma_{\mathrm{sym}}(\mathcal{L}) = \{0, \pm i\sigma_1\},$$

and the dispersion set (2.4) is given by

(6.10) $$\sigma_{\mathrm{disp}}(\mathcal{L}) = \{\lambda = -\eta^2\alpha_1 + \delta_1 + i(\pm\eta^2\alpha_2 \mp \delta_2 - n\sigma_1) : \eta \in \mathbb{R},\, n \in \mathbb{Z}\}.$$

As shown in Theorem 2.9, Example 5.2 and Figure 5.1(b), all eigenvalues from $\sigma_{\mathrm{sym}}(\mathcal{L})$ lie on the imaginary axis have geometric multiplicity at least 2, and belong to the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ in L^p. Similarly, as shown in Theorem 2.7 and Figure 3.1(a), the eigenvalues from $\sigma_{\mathrm{disp}}(\mathcal{L})$ belong to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ in L^p and form a zig zag structure consisting of infinitely many copies of cones. The cones open to the left and their tips are located at $-b_0 + in\sigma_1$, $n \in \mathbb{Z}$, as can be seen from (6.10). The distance of two neighboring tips of the cones equals $\sigma_1 = 0.6888$. Theorem 2.8 shows that $\lambda I - \mathcal{L}$ is Fredholm of index 0 for $\mathrm{Re}\,\lambda > -b_0$, i.e. to the right of the dashed line, and hence there is no essential spectrum. We believe that the operator $\lambda I - \mathcal{L}$ is also Fredholm of index 0 for λ-values between the dashed line $\mathrm{Re}\,\lambda = -b_0$ and $\sigma_{\mathrm{disp}}(\mathcal{L})$, but we do not have a proof of this. Similarly, the Fredholm properties for λ-values inside the rhombic regions of $\sigma_{\mathrm{disp}}(\mathcal{L})$ are unknown. To conclude, we suggest that both spectral subsets $\sigma_{\mathrm{pt}}(\mathcal{L})$ and $\sigma_{\mathrm{ess}}(\mathcal{L})$ are generally strictly larger than $\sigma_{\mathrm{sym}}(\mathcal{L})$ and $\sigma_{\mathrm{disp}}(\mathcal{L})$, respectively. Finally, the spectrum of the adjoint $\mathcal{L}^*$ coincides with the spectrum of $\mathcal{L}$.

Figure 6.2(b) shows an approximation σ^{approx} of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ linearized about the spinning soliton $v_\star$ for $d = 3$. The numerical spectrum σ^{approx} of $\mathcal{L}$ is divided into the approximation of the essential spectrum $\sigma^{\mathrm{approx}}_{\mathrm{ess}}$ (red dots) and the approximation of the point spectrum $\sigma^{\mathrm{approx}}_{\mathrm{pt}}$ (blue circles and plus signs). The set $\sigma^{\mathrm{approx}}_{\mathrm{ess}}$ lies close to $\sigma_{\mathrm{disp}}(\mathcal{L})$ which suggests that numerical computations seem to capture this part of the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$. Similarly, the set $\sigma^{\mathrm{approx}}_{\mathrm{pt}}$ is an approximation of $\sigma_{\mathrm{pt}}(\mathcal{L})$, which contains approximate values from $\sigma_{\mathrm{sym}}(\mathcal{L})$ (blue circles) and 12 additional complex-conjugate pairs of isolated eigenvalues satisfying $\mathrm{Re}\,\lambda > -b_0$. In particular, one of these pairs lie between the black dashed line and the essential spectrum. Further computations show that they seem to persist under spatial mesh refinement and also when enlargeing the spatial domain. The case $d = 2$ is also treated in [4, Sec.8].
Let us briefly come back to Figure 6.2(a) to discuss analytical results on eigenfunctions and their adjoints. In Theorem 2.9 we derived explicit formulas for the eigenfunctions associated to eigenvalues from the symmetry set $\sigma_{\mathrm{sym}}(\mathcal{L})$. In case $d = 3$, the six eigenfunctions are those from Example 5.2. We recall that each λ which is located to the right of $-b_0$, either belongs to $\mathrm{res}(\mathcal{L})$ or to $\sigma_{\mathrm{pt}}(\mathcal{L})$. In case $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L})$ with $\mathrm{Re}\,\lambda > -b_0$, Theorem 4.6 shows that the associated eigenfunction and adjoint eigenfunction decay exponentially in space with exponential decay rates given by, see (4.10) with $\gamma = \mathrm{Re}\,\lambda + b_0$,

$$0 \leqslant \mu_2 \leqslant \varepsilon \frac{\sqrt{\mathrm{Re}\,\alpha(\mathrm{Re}\,\lambda - \mathrm{Re}\,\delta)}}{|\alpha|p} < \frac{\sqrt{\mathrm{Re}\,\alpha(\mathrm{Re}\,\lambda - \mathrm{Re}\,\delta)}}{|\alpha|\max\{p_{\min}, \frac{d}{2}\}}, \tag{6.11}$$

and

$$0 \leqslant \mu_4 \leqslant \varepsilon \frac{\sqrt{\mathrm{Re}\,\alpha(\mathrm{Re}\,\lambda - \mathrm{Re}\,\delta)}}{|\alpha|q} < \frac{\sqrt{\mathrm{Re}\,\alpha(\mathrm{Re}\,\lambda - \mathrm{Re}\,\delta)}}{|\alpha|p_{\min}}. \tag{6.12}$$

The upper bounds show that the decay rates are affected by the spectral gap $\mathrm{Re}\,\lambda - \mathrm{Re}\,\delta$ between the eigenvalue $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L})$ and the spectral bound $b_0 = -\mathrm{Re}\,\delta$ (black dashed line) of $D\mathbf{f}(\mathbf{v}_\infty)$. Therefore, the decay rates are large for eigenvalues far away to the right of b_0, and they become smaller the closer $\mathrm{Re}\,\lambda$ lies to the spectral bound $b_0 = -\mathrm{Re}\,\delta$. For the eigenvalues from the symmetry set $\sigma_{\mathrm{sym}}(\mathcal{L})$, parameters from (6.3), and $d = 3$, we obtain the following upper bounds for the exponential decay rates of the eigenfunctions and their adjoints

$$0 \leqslant \mu_2 < \frac{\sqrt{2}}{3} \approx 0.4714 \quad \text{and} \quad 0 \leqslant \mu_4 < \frac{4}{1+\sqrt{2}} \approx 1.6569.$$

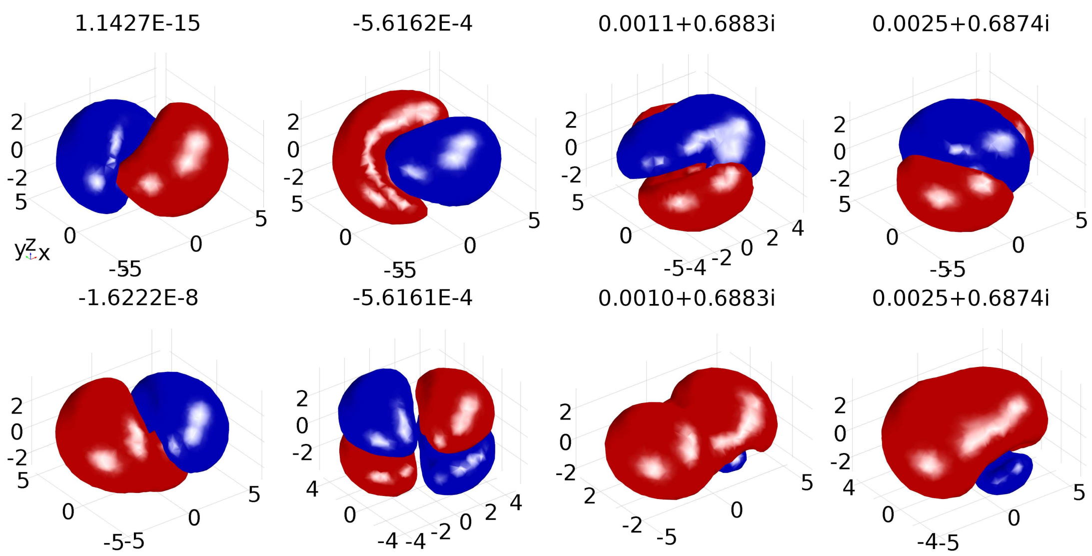


FIGURE 6.3. Isosurfaces of 4 eigenfunctions (top row) and 4 adjoint eigenfunctions (bottom row) of QCGL linearized at a spinning soliton with $d = 3$

In Figure 6.3 we visualize isosurfaces of numerical eigenfunctions $\mathbf{v}:\mathbb{R}^3\to\mathbb{C}^2$ (upper row) which belong to the eigenvalues $\lambda_1,\lambda_2,\lambda_3,\lambda_5$ from (5.15). The lower row shows the isosurfaces of adjoint eigenfunctions $\psi:\mathbb{R}^3\to\mathbb{C}^2$ for the same eigenvalues. More precisely, the red surfaces are given by $\operatorname{Re}\mathbf{v}_j(x)=-\frac{1}{2}$ and $\operatorname{Re}\psi_j(x)=-\frac{1}{2}$ while the blue surfaces are given by $\operatorname{Re}\mathbf{v}_j(x)=\frac{1}{2}$ and $\operatorname{Re}\psi_j(x)=\frac{1}{2}$. The corresponding numerical eigenvalues are provided in the title of each subfigure. Instead of double eigenvalues at 0 and $i\sigma_1$ as in the theory, one obtains two closely spaced simple eigenvalues in each case. The slight difference in the values of the top and bottom row is due to the independent runs used for the original and the adjoint eigenvalue problem. A detailed investigation of numerical decay rates of eigenfunctions and a comparison with the theory can be found in [5, Sec.6.3].

Appendix A. Identification of adjoint operator

We analyze and identify the abstract adoint operator of $\mathcal{L}_Q$ from (4.1). Let us first review some results from [29, 30, 31, 32] for the complex-valued Ornstein-Uhlenbeck operator $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^m)$ and its constant coefficient perturbation $\mathcal{L}_\infty$. Assuming (A2), (A6), (A9$_{B_\infty}$) for $\mathbb{K}=\mathbb{C}$ it is shown in [29, Thm.4.4], [30, Thm.3.1] that the function $H_\infty:\mathbb{R}^d\times\mathbb{R}^d\times(0,\infty)\to\mathbb{C}^{m,m}$ defined by

$$H_\infty(x,\xi,t)=(4\pi tA)^{-\frac{d}{2}}\exp\left(-B_\infty t-(4tA)^{-1}\left|e^{tS}x-\xi\right|^2\right), \tag{1.13}$$

is a heat kernel of the perturbed Ornstein-Uhlenbeck operator $\mathcal{L}_\infty$ from (1.10). Under the same assumptions it is proved in [30, Thm.5.3] that the family of mappings

$$[T_\infty(t)v](x):=\begin{cases}\int_{\mathbb{R}^d}H_\infty(x,\xi,t)v(\xi)d\xi & ,\ t>0\\ v(x) & ,\ t=0\end{cases}\quad ,\ x\in\mathbb{R}^d, \tag{1.14}$$

defines a strongly continuous semigroup $T_\infty(t):L^p(\mathbb{R}^d,\mathbb{C}^m)\to L^p(\mathbb{R}^d,\mathbb{C}^m)$, $t\geqslant 0$, for each $1\leqslant p<\infty$. The semigroup $(T_\infty(t))_{t\geqslant 0}$ is called an *Ornstein-Uhlenbeck semigroup*. The infinitesimal generator of this semigroup $\mathcal{A}_p:L^p(\mathbb{R}^d,\mathbb{C}^m)\supseteq\mathcal{D}(\mathcal{A}_p)\to L^p(\mathbb{R}^d,\mathbb{C}^m)$ has domain of definition

$$\mathcal{D}(\mathcal{A}_p):=\left\{v\in L^p(\mathbb{R}^d,\mathbb{C}^m)\mid \mathcal{A}_pv:=\lim_{t\downarrow 0}t^{-1}(T_\infty(t)v-v)\text{ exists in }L^p(\mathbb{R}^d,\mathbb{C}^m)\right\},$$

and satisfies resovent estimates, see [29, Cor.6.7], [30, Cor.5.5]. The identification problem requires to represent the maximal domain $\mathcal{D}(\mathcal{A}_p)$ in terms of Sobolev spaces, and to show that the generator $\mathcal{A}_p$ and the differential operator $\mathcal{L}_\infty$ conincide on this domain. This problem is solved in [31]. Assuming (A4$_p$), (A6), (A9$_{B_\infty}$) for $\mathbb{K}=\mathbb{C}$ and for some $1<p<\infty$, it is shown in [31, Thm.5.1] that

$$\mathcal{D}(\mathcal{A}_p)=\mathcal{D}^p(\mathcal{L}_0)\quad\text{and}\quad\mathcal{A}_pv=\mathcal{L}_\infty v\text{ for all }v\in\mathcal{D}(\mathcal{A}_p). \tag{1.15}$$

Moreover, in [30, Thm.5.7] a-priori estimates are used to show $\mathcal{D}^p(\mathcal{L}_0)\subseteq W^{1,p}(\mathbb{R}^d,\mathbb{C}^m)$.
Next consider the variable coefficient operator $\mathcal{L}_Q$ and assumes (A2), (A6) and (A9$_{B_\infty}$). For $Q\in L^\infty(\mathbb{R}^d,\mathbb{C}^{m,m})$ let $\widetilde{Q}$ denote the multiplication operator in $L^p(\mathbb{R}^d,\mathbb{C}^m)$ as in (4.6) and apply the bounded perturbation theorem [16, III.1.3] to conclude that $\mathcal{B}_p:=\mathcal{A}_p+\widetilde{Q}$ with $\mathcal{D}(\mathcal{B}_p):=\mathcal{D}(\mathcal{A}_p)$ generates a strongly continuous semigroup $(T_Q(t))_{t\geqslant 0}$ in $L^p(\mathbb{R}^d,\mathbb{C}^m)$. If we restrict $1<p<\infty$ and assume the stronger assumption (A4$_p$) (or equivalently (A5$_p$)) instead of (A2), an application of [31, Thm.5.1] solves the identification problem for $\mathcal{B}_p$, namely $\mathcal{D}(\mathcal{B}_p)=\mathcal{D}^p(\mathcal{L}_0)$ and $\mathcal{B}_pv=\mathcal{L}_\infty v+Qv=\mathcal{L}_Qv$ for all $v\in\mathcal{D}(\mathcal{B}_p)$. In particular, we obtain from [30, Thm.5.7] that $\mathcal{D}^p(\mathcal{L}_0)\subseteq W^{1,p}(\mathbb{R}^d,\mathbb{C}^m)$.
In the following we continue the process of identification for the adjoint differential operator and relate it to its abstract definition [24, Ch.III.3].

Definition A.1 (Adjoint operator)**.** Let X,Y be Banach spaces over $\mathbb{C}$ with dual spaces X^*,Y^* and duality pairings $\langle\cdot,\cdot\rangle_Y:Y^*\times Y\to\mathbb{C}$ and $\langle\cdot,\cdot\rangle_X:X^*\times X\to\mathbb{C}$. For a densely defined operator $\mathcal{A}:X\supseteq\mathcal{D}(\mathcal{A})\to Y$ the *abstract adjoint operator* $\mathcal{A}^*:Y^*\supseteq\mathcal{D}(\mathcal{A}^*)\to X^*$ is defined by

$$\mathcal{D}(\mathcal{A}^*)=\{y^*\in Y^*\mid\ \exists x^*\in X^*:\ \langle y^*,\mathcal{A}x\rangle_1=\langle x^*,x\rangle_2\ \forall x\in\mathcal{D}(\mathcal{A})\},\quad \mathcal{A}^*y^*:=x^*.$$

Let us assume ($A4_p$), (A6), ($A9_{B_\infty}$), $Q \in C(\mathbb{R}^d, \mathbb{C}^{m,m})$, let $1 < p < \infty$ and apply Definition A.1 to the infinitesimal generator $\mathcal{A} = \mathcal{B}_p$ using the setting

$$X = Y = L^p(\mathbb{R}^d, \mathbb{C}^m), \quad X^* = Y^* = L^q(\mathbb{R}^d, \mathbb{C}^m), \quad 1 < p, q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.16}$$
$$\langle w, v\rangle_{q,p} = \langle w, v\rangle_X = \langle w, v\rangle_Y = \int_{\mathbb{R}^d} w(x)^{\mathsf{H}} v(x)dx, \quad w \in L^q, \quad v \in L^p.$$

The abstract adjoint operator $\mathcal{A}^* = \mathcal{B}_p^*$ has maximal domain

$$\mathcal{D}(\mathcal{B}_p^*) = \left\{v \in L^q(\mathbb{R}^d, \mathbb{C}^m) \mid \exists\, w \in L^q(\mathbb{R}^d, \mathbb{C}^m) : \langle v, \mathcal{L}_Q u\rangle_{q,p} = \langle w, u\rangle_{q,p} \;\forall\, u \in \mathcal{D}^p(\mathcal{L}_0)\right\}, \tag{1.17}$$

and is defined through

$$\mathcal{B}_p^* : \left(\mathcal{D}(\mathcal{B}_p^*), \|\cdot\|_{\mathcal{B}_p^*}\right) \to \left(L^q(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^q}\right), \quad \mathcal{B}_p^* v := w, \quad w \text{ from (1.17)}. \tag{1.18}$$

Note that the element $w \in L^q(\mathbb{R}^d, \mathbb{C}^m)$ from (1.17) is uniquely determined. We compare this with the *formal adjoint (differential) operator* $\mathcal{L}_Q^* : \left(\mathcal{D}^q(\mathcal{L}_0^*), \|\cdot\|_{\mathcal{L}_0^*}\right) \to \left(L^q(\mathbb{R}^d, \mathbb{C}^m), \|\cdot\|_{L^q}\right)$, defined by

$$[\mathcal{L}_Q^* v](x) = A^{\mathsf{H}} \triangle v(x) - \langle Sx, \nabla v(x)\rangle - B_\infty^{\mathsf{H}} v(x) + Q(x)^{\mathsf{H}} v(x), \; x \in \mathbb{R}^d \tag{1.19}$$

on its domain

$$\mathcal{D}^q(\mathcal{L}_0^*) = \left\{v \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^m) \cap L^q(\mathbb{R}^d, \mathbb{C}^m) : \mathcal{L}_0^* v = A^{\mathsf{H}} \triangle v - \langle S\cdot, \nabla v\rangle \in L^q(\mathbb{R}^d, \mathbb{C}^m)\right\}, 1 < q < \infty.$$

Definition (1.19) is motivated by the following relation obtained via integration by parts

$$\langle v, \mathcal{L}_Q u\rangle_{q,p} = \langle \mathcal{L}_Q^* v, u\rangle_{q,p} \quad \forall\, u \in \mathcal{D}^p(\mathcal{L}_0)\;\forall\, v \in \mathcal{D}^q(\mathcal{L}_0^*). \tag{1.20}$$

The following result solves the identification problem for the adjoint operator. The proof is based on an application of [5, Thm.3.1] to $(A^{\mathsf{H}}, -S, B_\infty^{\mathsf{H}}, Q(x)^{\mathsf{H}}, q = \frac{p}{p-1})$ instead of $(A, S, B_\infty, Q(x), p)$. This requires the matrix A^{H} to additonally satisfy the L^q-dissipativity condition ($A4_q$) for the conjugate index $q := \frac{p}{p-1}$.

Lemma A.2 (Identification of adjoint operator)**.** *Let the assumptions* ($A4_p$), ($A4_q$), (A6), ($A9_{B_\infty}$) *and* $Q \in L^\infty(\mathbb{R}^d, \mathbb{K}^{m,m})$ *be satisfied for* $\mathbb{K} = \mathbb{C}$, *for some* $1 < p < \infty$ *and* $q = \frac{p}{p-1}$. *Then the formal adjoint operator* $\mathcal{L}_Q^*$ *and the abstract adjoint operator* $\mathcal{B}_p^*$ *coincide, i.e.*

$$\mathcal{D}(\mathcal{B}_p^*) = \mathcal{D}^q(\mathcal{L}_0^*) \quad \text{and} \quad \mathcal{B}_p^* = \mathcal{L}_Q^*.$$

In particular, the corresponding graph norms are equivalent.

Proof. For the proof we abbreviate $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{q,p}$.

- $\mathcal{D}^q(\mathcal{L}_0^*) \subseteq \mathcal{D}(\mathcal{B}_p^*)$: Let $v \in \mathcal{D}^q(\mathcal{L}_0^*)$ and choose $w = \mathcal{L}_Q^* v \in L^q(\mathbb{R}^d, \mathbb{C}^m)$, then (1.20) implies
$$\langle v, \mathcal{L}_Q u\rangle = \langle \mathcal{L}_Q^* v, u\rangle = \langle w, u\rangle \quad \forall\, u \in \mathcal{D}^p(\mathcal{L}_0),$$
which yields $v \in \mathcal{D}(\mathcal{B}_p^*)$.
- $\mathcal{D}^q(\mathcal{L}_0^*) \supseteq \mathcal{D}(\mathcal{B}_p^*)$: Let $v \in \mathcal{D}(\mathcal{B}_p^*)$ and let $w \in L^q(\mathbb{R}^d, \mathbb{C}^m)$ be defined according to (1.17). By an application of [5, Theorem 3.1] we have a unique solution $\tilde{v} \in \mathcal{D}^q(\mathcal{L}_0^*)$ of $(\bar{\lambda} I - \mathcal{L}_Q^*)\tilde{v} = \bar{\lambda} v - w$ in $L^q(\mathbb{R}^d, \mathbb{C}^m)$ for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -b_0 + \kappa a_1 \|Q\|_{L^\infty}$. Therefore, from (1.20) and (1.17) we obtain
$$\begin{aligned}\langle v, (\lambda I - \mathcal{L}_Q)u\rangle &= \langle v, \lambda u\rangle - \langle v, \mathcal{L}_Q u\rangle = \lambda\langle v, u\rangle - \langle w, u\rangle = \langle \bar{\lambda} v - w, u\rangle = \langle (\bar{\lambda} I - \mathcal{L}_Q^*)\tilde{v}, u\rangle \\ &= \lambda\langle \tilde{v}, u\rangle - \langle \mathcal{L}_Q^* \tilde{v}, u\rangle = \lambda\langle \tilde{v}, u\rangle - \langle \tilde{v}, \mathcal{L}_Q u\rangle = \langle \tilde{v}, (\lambda I - \mathcal{L}_Q)u\rangle \quad \forall\, u \in \mathcal{D}^p(\mathcal{L}_0).\end{aligned}$$
Since $\lambda I - \mathcal{L}_Q$ is onto, this implies $v = \tilde{v} \in \mathcal{D}^q(\mathcal{L}_0^*) \subset L^q(\mathbb{R}^d, \mathbb{C}^m)$.

□

References

[1] COMSOL MULTIPHYSICS 5.2, 2016. http://www.comsol.com.

[2] S. Alonso, M. Bär, and A. V. Panfilov. Negative tension of scroll wave filaments and turbulence in three-dimensional excitable media and application in cardiac dynamics. *Bull. Math. Biol.*, 75(8):1351–1376, 2013.

[3] H. W. Alt. *Lineare Funktionalanalysis: Eine anwendungsorientierte Einführung*. Springer-Verlag Berlin Heidelberg, Berlin, Heidelberg, 2006.

[4] W.-J. Beyn and J. Lorenz. Nonlinear stability of rotating patterns. *Dyn. Partial Differ. Equ.*, 5(4):349–400, 2008.

[5] W.-J. Beyn and D. Otten. Spatial Decay of Rotating Waves in Reaction Diffusion Systems. *Dyn. Partial Differ. Equ.*, 13(3):191–240, 2016.

[6] W.-J. Beyn, D. Otten, and J. Rottmann-Matthes. Stability and Computation of Dynamic Patterns in PDEs. In *Current Challenges in Stability Issues for Numerical Differential Equations*, Lecture Notes in Mathematics, pages 89–172. Springer International Publishing, 2014.

[7] W.-J. Beyn, D. Otten, and J. Rottmann-Matthes. Freezing traveling and rotating waves in second order evolution equations. pages 1–20. 2016.

[8] W.-J. Beyn and V. Thümmler. Freezing solutions of equivariant evolution equations. *SIAM J. Appl. Dyn. Syst.*, 3(2):85–116 (electronic), 2004.

[9] I. V. Biktasheva, A. V. Holden, and V. N. Biktashev. Localization of response functions of spiral waves in the FitzHugh-Nagumo system. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 16(5):1547–1555, 2006.

[10] A. M. Bloch and A. Iserles. Commutators of skew-symmetric matrices. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 15(3):793–801, 2005.

[11] A. Cialdea and V. Maz'ya. Criterion for the L^p-dissipativity of second order differential operators with complex coefficients. *J. Math. Pures Appl. (9)*, 84(8):1067–1100, 2005.

[12] A. Cialdea and V. Maz′ya. Criteria for the L^p-dissipativity of systems of second order differential equations. *Ric. Mat.*, 55(2):233–265, 2006.

[13] L.-C. Crasovan, B. A. Malomed, and D. Mihalache. Spinning solitons in cubic-quintic nonlinear media. *Pramana-journal of Physics*, 57:1041–1059, 2001.

[14] G. Da Prato and A. Lunardi. On the Ornstein-Uhlenbeck operator in spaces of continuous functions. *J. Funct. Anal.*, 131(1):94–114, 1995.

[15] D. E. Edmunds and W. D. Evans. *Spectral theory and differential operators*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1987. Oxford Science Publications.

[16] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[17] B. Ermentrout and B. I. S. van der Ventel. Rotating waves in simple scalar excitable media: approximations and numerical solutions. *J. Math. Biol.*, 73(6-7):1321–1351, 2016.

[18] B. Fiedler and A. Scheel. Spatio-temporal dynamics of reaction-diffusion patterns. In *Trends in nonlinear analysis*, pages 23–152. Springer, Berlin, 2003.

[19] A. J. Foulkes, D. Barkley, V. N. Biktashev, and I. V. Biktasheva. Alternative stable scroll waves and conversion of autowave turbulence. *Chaos*, 20(4):043136, 13, 2010.

[20] H. G. Gebran and C. A. Stuart. Exponential decay and Fredholm properties in second-order quasilinear elliptic systems. *J. Differential Equations*, 249(1):94–117, 2010.

[21] K. Gustafson. The angle of an operator and positive operator products. *Bull. Amer. Math. Soc.*, 74:488–492, 1968.

[22] K. Gustafson. *Antieigenvalue analysis : with applications to numerical analysis, wavelets, statistics, quantum mechanics, finance and optimization*. World Scientific, Hackensack, NJ [u.a.], 2012.

[23] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[24] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[25] R. V. Kozhan. L^1-spectrum of Banach space valued Ornstein-Uhlenbeck operators. *Semigroup Forum*, 78(3):547–553, 2009.

[26] L. D. Landau and V. L. Ginzburg. On the theory of superconductivity. *Journal of Experimental and Theoretical Physics (USSR)*, 20:1064, 1950.

[27] G. Metafune. L^p-spectrum of Ornstein-Uhlenbeck operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 30(1):97–124, 2001.

[28] G. Metafune, D. Pallara, and E. Priola. Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures. *J. Funct. Anal.*, 196(1):40–60, 2002.

[29] D. Otten. *Spatial decay and spectral properties of rotating waves in parabolic systems*. PhD thesis, Bielefeld University, 2014, www.math.uni-bielefeld.de/~dotten/files/diss/Diss_DennyOtten.pdf. Shaker Verlag, Aachen.

[30] D. Otten. Exponentially weighted resolvent estimates for complex Ornstein-Uhlenbeck systems. *J. Evol. Equ.*, 15(4):753–799, 2015.

[31] D. Otten. The Identification Problem for complex-valued Ornstein-Uhlenbeck Operators in $L^p(\mathbb{R}^d, \mathbb{C}^N)$. *Semigroup Forum*, pages 1–38, 2016.

[32] D. Otten. A new L^p-Antieigenvalue Condition for Ornstein-Uhlenbeck Operators. *J. Math. Anal. Appl.*, pages 1–16, 2016.

[33] A. Pogan and A. Scheel. Fredholm properties of radially symmetric, second order differential operators. *Int. J. Dyn. Syst. Differ. Equ.*, 3(3):289–327, 2011.

[34] J. Prüss, A. Rhandi, and R. Schnaubelt. The domain of elliptic operators on $L^p(\mathbb{R}^d)$ with unbounded drift coefficients. *Houston J. Math.*, 32(2):563–576 (electronic), 2006.

[35] P. J. Rabier and C. A. Stuart. Exponential decay of the solutions of quasilinear second-order equations and Pohozaev identities. *J. Differential Equations*, 165(1):199–234, 2000.

[36] B. Sandstede and A. Scheel. Absolute versus convective instability of spiral waves. *Phys. Rev. E (3)*, 62(6, part A):7708–7714, 2000.

[37] B. Sandstede and A. Scheel. Superspiral structures of meandering and drifting spiral waves. *Phys. Rev. Lett.*, 86:171–174, Jan 2001.

[38] B. Sandstede and A. Scheel. Curvature effects on spiral spectra: generation of point eigenvalues near branch points. *Phys. Rev. E (3)*, 73(1):016217, 8, 2006.

[39] B. Sandstede, A. Scheel, and C. Wulff. Dynamics of spiral waves on unbounded domains using center-manifold reductions. *J. Differential Equations*, 141(1):122–149, 1997.

[40] O. Taussky and H. Wielandt. On the matrix function $AX + X'A'$. *Arch. Rational Mech. Anal.*, 9:93–96, 1962.

[41] A. E. Taylor and D. C. Lay. *Introduction to functional analysis.* John Wiley & Sons, New York-Chichester-Brisbane, second edition, 1980.

[42] J. M. A. M. van Neerven. Second quantization and the L^p-spectrum of nonsymmetric Ornstein-Uhlenbeck operators. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 8(3):473–495, 2005.

[43] J. M. A. M. van Neerven and J. Zabczyk. Norm discontinuity of Ornstein-Uhlenbeck semigroups. *Semigroup Forum*, 59(3):389–403, 1999.

[44] V. Volpert. *Elliptic partial differential equations. Volume 1: Fredholm theory of elliptic problems in unbounded domains*, volume 101 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2011.

[45] P. Wheeler and D. Barkley. Computation of spiral spectra. *SIAM J. Appl. Dyn. Syst.*, 5(1):157–177 (electronic), 2006.

2. Supplement

In this supplement we provide a detailed proof of the estimate (3.11) needed to construct a singular sequence for points in the essential spectrum of the linearized operator in Theorem 3.2.

Proof. 1. The property $\chi_R(r) = 0$ for $r \in I_1 \cup I_5$ implies

$$(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}})\hat{v}_R(\psi) = 0, \quad \text{if } |\tilde{y}| \in I_1 \cup I_5 \text{ or } r_l \in I_1 \cup I_5 \text{ for some } 1 \leqslant l \leqslant k. \tag{2.21}$$

Similarly, $\lambda \in \sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ and $\chi_R''(r) = \chi_R'(r) = 0$ for $r \in I_3$ imply

$$(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}})\hat{v}_R(\psi) = 0, \quad \text{if } |\tilde{y}| \in I_3 \text{ and } r_l \in I_3 \text{ for every } 1 \leqslant l \leqslant k. \tag{2.22}$$

Next compute the partial derivatives

$$\partial_{r_l}^2 \hat{v}_R(\psi) = \left[\frac{\chi_R''(r_l)}{\chi_R(r_l)} + 2i\omega_l \frac{\chi_R'(r_l)}{\chi_R(r_l)} - \omega_l^2\right]\hat{v}_R(\psi),\ l = 1,\ldots,k,$$
$$\partial_{y_l}^2 \hat{v}_R(\psi) = \left[\frac{y_l^2}{|\tilde{y}|^2}\frac{\chi_R''(|\tilde{y}|)}{\chi_R(|\tilde{y}|)} + \left(\frac{|\tilde{y}|^2 - y_l^2}{|\tilde{y}|^3} + 2i\rho_l\frac{y_l}{|\tilde{y}|}\right)\frac{\chi_R'(|\tilde{y}|)}{\chi_R(|\tilde{y}|)} - \rho_l^2\right]\hat{v}_R(\psi),\ l = 2k+1,\ldots,d,$$

and consider the case $|\tilde{y}| \in I_2 \cup I_3 \cup I_4$ and $r_l \in I_2 \cup I_3 \cup I_4$ for all $1 \leqslant l \leqslant k$. Then we use $\lambda \in \sigma_{\mathrm{disp}}(\mathcal{L}_Q)$ and the estimates $|\chi_R(r)| \leqslant 1$, $\chi_R'(r) \leqslant \|\chi_R\|_{C_b^2}$, $\chi_R''(r) \leqslant \|\chi_R\|_{C_b^2}$, $|\hat{v}(\psi)| = 1$, $\left|\frac{\hat{v}_R(\psi)}{\chi_R(r_l)}\right| \leqslant 1$, $\left|\frac{\hat{v}_R(\psi)}{\chi_R(|\tilde{y}|)}\right| \leqslant 1$, $\frac{1}{|\tilde{y}|} \leqslant \frac{1}{R-1} \leqslant 1$ to obtain

$$\begin{aligned}\left|\left(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}}\right) v_R(\psi)\right| &= \left|\left(\lambda I - A\left[\sum_{l=1}^k \partial_{r_l}^2 + \sum_{l=2k+1}^d \partial_{y_l}^2\right] + \sum_{l=1}^k \sigma_l \partial_{\phi_l} + B_\infty\right)\hat{v}_R(\psi)\right| \\ &= \Bigg|\left(\lambda I_m - (|\omega|^2 + |\rho|^2)A + i\langle n,\sigma\rangle + B_\infty\right)v_R(\psi) - A\sum_{l=1}^k \left(\chi_R''(r_l) + 2i\omega_l\chi_R'(r_l)\right)\frac{\hat{v}_R(\psi)}{\chi_R(r_l)} \\ &\quad - A\sum_{l=2k+1}^d \left(\frac{y_l^2}{|\tilde{y}|^2}\chi_R''(|\tilde{y}|) + \left(\frac{|\tilde{y}|^2 - y_l^2}{|\tilde{y}|^3} + 2i\rho_l\frac{y_l}{|\tilde{y}|}\right)\chi_R'(|\tilde{y}|)\right)\frac{\hat{v}_R(\psi)}{\chi_R(|\tilde{y}|)}\Bigg|\end{aligned} \tag{2.23}$$

$$\leqslant |A| \sum_{l=1}^{k} (1+2|\omega_l|) \, \|\chi_R\|_{C_{\mathrm{b}}^2} + |A| \sum_{l=2k+1}^{d} (3+2|\rho_l|) \, \|\chi_R\|_{C_{\mathrm{b}}^2}$$
$$\leqslant |A| \left(k + 2|\omega|\sqrt{k} + 3(d-2k) + 2|\rho|\sqrt{d-2k}\right) \|\chi_R\|_{C_{\mathrm{b}}^2} =: C.$$

2. Transforming variables, setting $\langle \mathbf{r} \rangle = \prod_{l=1}^{k} r_l$ and using $|\hat{v}(\psi)| = 1$, $\chi_R(r) \geqslant 0$ ($r \in I_2 \cup I_4$), $\chi_R(r) = 1$ ($r \in I_3$) leads to

$$\|v_R\|_{L^p}^p = \int_{\mathbb{R}^d} |v_R(x)|^p \, dx = \int_0^\infty \int_{-\pi}^{\pi} \cdots \int_0^\infty \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-2k}} \langle \mathbf{r} \rangle \, |\hat{v}_R(\psi)|^p \, d\psi$$
$$= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \int_{R-1 \leqslant |\tilde{y}| \leqslant 2R+1} \langle \mathbf{r} \rangle \, |\hat{v}_R(\psi)|^p \, d\psi$$
$$= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \int_{R-1 \leqslant |\tilde{y}| \leqslant 2R+1} \langle \mathbf{r} \rangle \left(\prod_{l=1}^{k} \chi_R^p(r_l) \right) \chi_R^p(|\tilde{y}|) d\psi$$
$$= \int_{R-1 \leqslant |\tilde{y}| \leqslant 2R+1} \chi_R^p(|\tilde{y}|) d\tilde{y} \prod_{l=1}^{k} \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} r_l \chi_R^p(r_l) d\phi_l dr_l$$
$$= \left(\int_{R-1 \leqslant |\tilde{y}| \leqslant R} \chi_R^p(|\tilde{y}|) d\tilde{y} + \int_{R \leqslant |\tilde{y}| \leqslant 2R} \chi_R^p(|\tilde{y}|) d\tilde{y} + \int_{2R \leqslant |\tilde{y}| \leqslant 2R+1} \chi_R^p(|\tilde{y}|) d\tilde{y} \right)$$
$$\cdot \prod_{l=1}^{k} 2\pi \left(\int_{R-1}^{R} r_l \chi_R^p(r_l) dr_l + \int_{R}^{2R} r_l \chi_R^p(r_l) dr_l + \int_{2R}^{2R+1} r_l \chi_R^p(r_l) dr_l \right)$$
$$\geqslant \left(\int_{R \leqslant |\tilde{y}| \leqslant 2R} 1 d\tilde{y} \right) \cdot \left(\prod_{l=1}^{k} 2\pi \int_{R}^{2R} r_l dr_l \right) = C R^{\tilde{d}} \prod_{l=1}^{k} 3\pi R^2 = (3\pi)^k C R^{2k+\tilde{d}} = C R^d,$$

where C is independent of R, $d\psi := d\tilde{y} d\phi_k dr_k \cdots d\phi_1 dr_1$ and $\tilde{d} := d - 2k$. In the trivial case $\tilde{d} = 0$ the first integral is set to 1, while in case $\tilde{d} \geqslant 1$ the term $CR^{\tilde{d}}$ follows from the well-known formula

$$\tilde{d}\Gamma\left(\tfrac{\tilde{d}}{2}\right) \int_{a \leqslant |\tilde{y}| \leqslant b} 1 d\tilde{y} = 2\pi^{\frac{\tilde{d}}{2}} (b^{\tilde{d}} - a^{\tilde{d}}) \quad \text{for } 0 < a < b < \infty. \tag{2.24}$$

3. The transformation theorem and (2.21) imply

$$\left\| \left(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}}\right) v_R \right\|_{L^p}^p = \int_{\mathbb{R}^d} \left| \left(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}}\right) v_R(x) \right|^p dx$$
$$= \int_0^\infty \int_{-\pi}^{\pi} \cdots \int_0^\infty \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-2k}} \langle \mathbf{r} \rangle \left| \left(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}}\right) \hat{v}_R(\psi) \right|^p d\psi$$
$$= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \int_{R-1 \leqslant |\tilde{y}| \leqslant 2R+1} \langle \mathbf{r} \rangle \left| \left(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}}\right) \hat{v}_R(\psi) \right|^p d\psi.$$

We distinguish the following cases for $\tilde{d} = d - 2k$.
Case 1: ($\tilde{d} = 0$). From (2.23), (2.22), the multinomial theorem and

$$\int_{R-1}^{R} r_l dr_l = \frac{1}{2}(2R-1), \qquad \int_{R}^{2R} r_l dr_l = \frac{1}{2} 3R^2, \qquad \int_{2R}^{2R+1} r_l dr_l = \frac{1}{2}(4R+1), \tag{2.25}$$

we further obtain

$$= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \langle \mathbf{r} \rangle \left| \left(\lambda I - \mathcal{L}_\infty^{\mathrm{sim}}\right) \hat{v}_R(\psi) \right|^p d\phi_k dr_k \cdots d\phi_1 dr_1$$
$$\leqslant \sum_{\substack{j_1+j_2+j_3=k \\ j_2 \neq k}} \binom{k}{j_1, j_2, j_3} \left(\int_{R-1}^{R} \right)^{j_1} \left(\int_{R}^{2R} \right)^{j_2} \left(\int_{2R}^{2R+1} \right)^{j_3} C^p \langle \mathbf{r} \rangle (2\pi)^k dr_1 \cdots dr_k$$

$$= \sum_{\substack{j_1+j_2+j_3=k \\ j_2\neq k}} \binom{k}{j_1, j_2, j_3} \frac{C^p(2\pi)^k}{2^k} (2R-1)^{j_1}(3R^2)^{j_2}(4R+1)^{j_3} \leqslant CR^{d-1}.$$

For the last inequality we estimate powers of R by $j_1 + 2j_2 + j_3 = k + j_2 \leqslant 2k - 1 = d - 1$ for $j_2 \neq k$.
Case 2: ($\tilde{d} \geqslant 1$). Similarly, using (2.23) and (2.22), (2.24), the multinomial theorem gives (abbreviating $d\mathbf{r} := dr_1 \cdots dr_k$)

$$\begin{aligned}
&\leqslant \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3} \int_{R-1\leqslant|\tilde{y}|\leqslant R} C^p\langle\mathbf{r}\rangle(2\pi)^k d\tilde{y}d\mathbf{r} \\
&\quad + \sum_{\substack{j_1+j_2+j_3=k \\ j_2\neq k}} \binom{k}{j_1, j_2, j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3} \int_{R\leqslant|\tilde{y}|\leqslant 2R} C^p\langle\mathbf{r}\rangle(2\pi)^k d\tilde{y}d\mathbf{r} \\
&\quad + \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3} \int_{2R\leqslant|\tilde{y}|\leqslant 2R+1} C^p\langle\mathbf{r}\rangle(2\pi)^k d\tilde{y}d\mathbf{r} \\
&= \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} \frac{C^p(2\pi)^k}{2^k}(2R-1)^{j_1}(3R^2)^{j_2}(4R+1)^{j_3} \frac{2\pi^{\frac{\tilde{d}}{2}}}{\tilde{d}\Gamma\left(\frac{\tilde{d}}{2}\right)}(R^{\tilde{d}} - (R-1)^{\tilde{d}}) \\
&\quad + \sum_{\substack{j_1+j_2+j_3=k \\ j_2\neq k}} \binom{k}{j_1, j_2, j_3} \frac{C^p(2\pi)^k}{2^k}(2R-1)^{j_1}(3R^2)^{j_2}(4R+1)^{j_3} \frac{2\pi^{\frac{\tilde{d}}{2}}}{\tilde{d}\Gamma\left(\frac{\tilde{d}}{2}\right)}((2R)^{\tilde{d}} - R^{\tilde{d}}) \\
&\quad + \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} \frac{C^p(2\pi)^k}{2^k}(2R-1)^{j_1}(3R^2)^{j_2}(4R+1)^{j_3} \frac{2\pi^{\frac{\tilde{d}}{2}}}{\tilde{d}\Gamma\left(\frac{\tilde{d}}{2}\right)}((2R+1)^{\tilde{d}} - (2R)^{\tilde{d}}) \\
&\leqslant \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} CR^{j_1+2j_2+j_3+\tilde{d}-1} + \sum_{\substack{j_1+j_2+j_3=k \\ j_2\neq k}} \binom{k}{j_1, j_2, j_3} CR^{j_1+2j_2+j_3+\tilde{d}} \\
&\quad + \sum_{j_1+j_2+j_3=k} \binom{k}{j_1, j_2, j_3} CR^{j_1+2j_2+j_3+\tilde{d}-1} \leqslant CR^{d-1}.
\end{aligned}$$

This shows that $\|(\lambda I - \mathcal{L}_\infty^{\text{sim}})v_R\|_{L^p}^p \leqslant CR^{d-1}$.

4. For the operator $\hat{\mathcal{L}}_Q = \hat{\mathcal{L}}_\infty + Q(T_1(T_2(\psi)))$, equation (2.21) and $\chi_R(r) = 0$ for $r \in I_1 \cup I_5$ imply

$$(\lambda I - \hat{\mathcal{L}}_Q)\hat{v}_R(\psi) = 0, \quad \text{if } |\tilde{y}| \in I_1 \cup I_5 \text{ or } r_l \in I_1 \cup I_5 \text{ for some } 1 \leqslant l \leqslant k. \tag{2.26}$$

Moreover, if $|\tilde{y}| \in I_3$ and $r_l \in I_3$ for every $1 \leqslant l \leqslant k$, then we obtain from (2.22), $\chi_R'(r) \leqslant \|\chi_R\|_{C_b^2}$, $|\hat{v}_R(\psi)| \leqslant 1$, and $\frac{1}{r_l} \leqslant \frac{1}{R} \leqslant 1$,

$$\begin{aligned}
|(\lambda I - \hat{\mathcal{L}}_Q)\hat{v}_R(\psi)| &= \left| \left(\lambda I - \mathcal{L}_\infty^{\text{sim}}\right) \hat{v}_R(\psi) - A\sum_{l=1}^{k} \left(\frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}^2\right) \hat{v}_R(\psi) - Q(T_1(T_2(\psi)))\hat{v}_R(\psi) \right| \\
=& \left| A\sum_{l=1}^{k} \left(\frac{i\omega_l}{r_l} + \frac{\chi_R'(r_l)}{r_l\chi_R(r_l)} - \frac{n_l^2}{r_l^2}\right)\hat{v}_R(\psi) + Q(T_1(T_2(\psi)))\hat{v}_R(\psi) \right| \\
\leqslant& |A|\sum_{l=1}^{k} \left(\frac{|\omega_l|}{r_l} + \frac{\|\chi_R\|_{C_b^2}}{r_l} + \frac{n_l^2}{r_l^2}\right) + |Q(T_1(T_2(\psi)))| \leqslant \left(|A|\sum_{l=1}^{k} \left(|\omega_l| + \|\chi_R\|_{C_b^2} + n_l^2\right)\frac{1}{r_l} + \eta_R\right)^{\frac{1}{p}}.
\end{aligned}$$

Similarly, from (2.23), $|\chi_R(r)| \leqslant 1$, $\chi_R'(r) \leqslant \|\chi_R\|_{C_{\rm b}^2}$, $|\hat{v}_R(\psi)| \leqslant 1$, $\left|\frac{\hat{v}_R(\psi)}{\chi_R(r_l)}\right| \leqslant 1$, $\frac{1}{r_l} \leqslant \frac{1}{R-1} \leqslant 1$, $\frac{1}{r_l^2} \leqslant 1$, and $Q \in L^\infty$ we find in case $|\tilde{y}| \in I_2 \cup I_3 \cup I_4$ and $r_l \in I_2 \cup I_3 \cup I_4$ for every $1 \leqslant l \leqslant k$:

$$
\begin{aligned}
|(\lambda I - \hat{\mathcal{L}}_Q)\hat{v}_R(\psi)| &= \left| \left(\lambda I - \mathcal{L}_\infty^{\rm sim}\right)\hat{v}_R(\psi) - A\sum_{l=1}^k \left(\frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}^2\right)\hat{v}_R(\psi) - Q(T_1(T_2(\psi)))\hat{v}_R(\psi)\right| \\
=&\left| \left(\lambda I - \mathcal{L}_\infty^{\rm sim}\right)\hat{v}_R(\psi) - A\sum_{l=1}^k \left(\frac{i\omega_l}{r_l} + \frac{\chi_R'(r_l)}{r_l\chi_R(r_l)} - \frac{n_l^2}{r_l^2}\right)\hat{v}_R(\psi) - Q(T_1(T_2(\psi)))\hat{v}_R(\psi)\right| \\
\leqslant&\left| \left(\lambda I - \mathcal{L}_\infty^{\rm sim}\right)\hat{v}_R(\psi)\right| + |A|\sum_{l=1}^k \left(\frac{|\omega_l|}{r_l} + \frac{\|\chi_R\|_{C_{\rm b}^2}}{r_l} + \frac{n_l^2}{r_l^2}\right) + \left|Q(T_1(T_2(\psi)))\right| \\
\leqslant& C + |A|\Big(|\omega|\sqrt{k} + k\,\|\chi_R\|_{C_{\rm b}^2} + |n|^2\Big) + \|Q\|_{L^\infty} = C.
\end{aligned}
$$

5. Finally, let us consider $(\lambda I - \mathcal{L}_Q)v_R$ in L^p. From the transformation theorem and (2.26) we obtain

$$
\begin{aligned}
\|(\lambda I - \mathcal{L}_Q)v_R\|_{L^p}^p &= \int_{\mathbb{R}^d} |(\lambda I - \mathcal{L}_Q)v_R(x)|^p dx \\
&= \int_0^\infty \int_{-\pi}^{\pi} \cdots \int_0^\infty \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-2k}} \langle \mathbf{r}\rangle |(\lambda I - \hat{\mathcal{L}}_Q)\hat{v}_R(\psi)|^p d\psi \\
&= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \int_{R-1\leqslant|\tilde{y}|\leqslant 2R+1} \langle \mathbf{r}\rangle |(\lambda I - \hat{\mathcal{L}}_Q)\hat{v}_R(\psi)|^p d\psi.
\end{aligned}
$$

Again we distinguish two cases for $\tilde{d} := d - 2k$:
Case 1: ($\tilde{d} = 0$). From step 4, equation (2.25), and $d = 2k$ we deduce

$$
\begin{aligned}
\leqslant& \int_R^{2R} \int_{-\pi}^{\pi} \cdots \int_R^{2R} \int_{-\pi}^{\pi} \langle \mathbf{r}\rangle \left[|A| \sum_{l=1}^k (|\omega| + \|\chi_R\|_{C_{\rm b}^2} + n_l^2)\frac{1}{r_l} + \eta_R\right] d\phi_k dr_k \cdots d\phi_1 dr_1 \\
&+ \sum_{\substack{j_1+j_2+j_3\\ j_2\neq k}} \binom{k}{j_1, j_2, j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3} C^p \langle \mathbf{r}\rangle (2\pi)^k dr_1 \cdots dr_k \\
\leqslant& (2\pi)^k \int_R^{2R} \cdots \int_R^{2R} \left[|A| \left(\sum_{l=1}^k \left(\prod_{\substack{j=1\\ j\neq l}}^k r_j\right) \left(|\omega_l| + \|\chi_R\|_{C_{\rm b}^2} + n_l^2\right)\right) + \langle \mathbf{r}\rangle \eta_R\right] d\mathbf{r} + CR^{d-1} \\
=& (2\pi)^k \left[|A| \sum_{l=1}^k \left(|\omega_l| + \|\chi_R\|_{C_{\rm b}^2} + n_l^2\right) \int_R^{2R} \cdots \int_R^{2R} \left(\prod_{\substack{j=1\\ j\neq l}}^k r_j\right) d\mathbf{r} + \eta_R \int_R^{2R} \cdots \int_R^{2R} \langle \mathbf{r}\rangle d\mathbf{r}\right] + CR^{d-1} \\
=& (2\pi)^k |A| \sum_{l=1}^k \left(|\omega_l| + \|\chi_R\|_{C_{\rm b}^2} + n_l^2\right) \left(\prod_{\substack{j=1\\ j\neq l}}^k \int_R^{2R} r_j dr_j\right) \int_R^{2R} dr_l + (2\pi)^k \eta_R \prod_{j=1}^k \int_R^{2R} r_j dr_j + CR^{d-1} \\
=& (2\pi)^k |A| \left(\sum_{l=1}^k \left(|\omega_l| + \|\chi_R\|_{C_{\rm b}^2} + n_l^2\right) \left(\frac{3}{2}\right)^{k-1} R^{2k-1}\right) + (2\pi)^k \eta_R \left(\frac{3}{2}\right)^k R^{2k} + CR^{d-1} \\
\leqslant& CR^{d-1} + CR^d \eta_R.
\end{aligned}
$$

For the first inequality we refer to case 1 of step 3.
Case 2: ($\tilde{d} \geqslant 1$). From the procedure used in case 2 of step 5 and in case 1 and (2.24) we obtain

$$
\begin{aligned}
&\leqslant \int_R^{2R} \int_{-\pi}^{\pi} \cdots \int_R^{2R} \int_{-\pi}^{\pi} \int_{R\leqslant|\tilde{y}|\leqslant 2R} \langle \mathbf{r}\rangle \left[|A| \sum_{l=1}^k (|\omega_l| + \|\chi_R\|_{C_{\rm b}^2} + n_l^2)\frac{1}{r_l} + \eta_R\right] d\psi + CR^{d-1} \\
&\leqslant \left(CR^{2k-1} + CR^{2k}\eta_R\right) \int_{R\leqslant|\tilde{y}|\leqslant 2R} d\tilde{y} + CR^{d-1} \leqslant CR^{2k-1+\tilde{d}} + CR^{d-1} + CR^{2k+\tilde{d}}\eta_R
\end{aligned}
$$

$$=CR^{d-1} + CR^d\eta_R.$$

The constant CR^{d-1} in the first inequality comes from an estimate of three sums, compare case 2 from step 3. For the second inequality compare case 1. This shows that $\|(\lambda I - \mathcal{L}_Q)v_R\|_{L^p}^p \leqslant CR^{d-1} + CR^d\eta_R$. $\square$