Polynomial Estimates and Discrete Saddle-node Homoclinic Orbits

Thorsten Hüls* and Yongkui Zou*†

January 7, 2000

Abstract

We derive polynomial rates of convergence for orbits of maps that converge to an equilibrium via the center manifold. Similar estimates are obtained for the variational equation along these orbits. We show how these results apply to the analysis of discrete saddle-node homoclinics.

Keywords: Polynomial rate of convergence, saddle-node homoclinic orbit, center manifold.

AMS Classification: Primary 58F08. Secondary 58F14, 58F36.

1 Introduction

Consider a discrete dynamical system depending on a parameter

$$x_{n+1} = f(x_n, \lambda), \quad x \in \mathbb{R}^m, \quad \lambda \in \mathbb{R},$$

where $m \geq 1$ and the map f is smooth with respect to both x and λ. The behaviour of orbits near a hyperbolic fixed point, and the convergent or divergent properties along the stable or unstable manifold are known to be exponential. At a critical fixed point $(\bar{\xi}, \bar{\lambda})$, where the Jacobian matrix $D_x f(\bar{\xi}, \bar{\lambda})$ has an eigenvalue of the central type, i.e. its norm is 1, bifurcations of fixed points may occur under small perturbation. On the other hand, there are also many bifurcation phenomena of connecting orbits related to a critical fixed point, e.g. saddle-node homoclinic or heteroclinic orbit, which plays an important role in global bifurcation analysis, especially for the analysis of

*Supported by SFB 343 "Diskrete Strukturen in der Mathematik", Univ. of Bielefeld.
†Financial supported by Volkswagen foundation of Germany in 1999.

appearance of chaos. All of these motivate us to study the exact behaviour of convergence of the orbit along the center manifold of the critical point, which is the main objective of this work. Instead of the exponential behaviour near a hyperbolic fixed point, we will prove the polynomial rate of decay for orbits on the center manifold.

The solutions of the associated variational equation are the basic tools for detailed analysis near the orbit. They show exponential dichotomies on a hyperbolic set, see [12]. In the case of the saddle-node homoclinic orbit, there is a nontrivial solution which has the rate of $1/n^{\sigma}$ as $n \to -\infty$ or $+\infty$ for some $\sigma > 1$. For flows this has been used by Schecter in [14] with $\sigma = 2$ to analyze numerical methods for a saddle-node homoclinic bifurcation point. In this work, we will derive a general analysis for the discrete variational equation along an orbit which converges to a critical point.

In section 2, we study the rate of convergence for a general map with a simple eigenvalue 1. This case occurs for simple bifurcations, such as fold bifurcation, pitchfork bifurcation and cusp bifurcation, see [11]. In applications, many phenomena, e.g. the flip bifurcation point (with a simple eigenvalue -1) and Neimark-Sacker bifurcation point (with a pair of purely imaginary eigenvalues) can be reduced to this case, by looking at the map f^2 and using polar coordinates, respectively. The solutions of the variational equations are also of importance for further global analysis near the orbit. The polynomial rate is derived for both the orbit and the solutions of the variational equation.

Many bifurcation properties on the saddle-node homoclinic orbits have been summarized in [1, 9]. In section 3, we study the exact behaviour of the discrete transversal saddle-node homoclinic orbit. As an application of the results in section 2, the rate of convergence of the homoclinic orbit along the center manifold and of the solutions of the associated variational equation is shown to be of polynomial type. Together with the exponential estimates along stable and unstable directions, this provides us with appropriate techniques for analyzing numerical approximations of saddle-node homoclinic orbits.

2 Basic estimates and a model function

In this section, we will introduce the basic lemma which is used to estimate the rate of convergence of a map along the one dimensional center manifold in section 3.

Lemma 2.1 *For any $\alpha > 0$ and integer $k \geq 1$, define*

$$g(x, \alpha) = \frac{x}{(1 + \alpha x^k)^{1/k}}.$$

Then

$$\begin{aligned} &g(0, \alpha) = 0, \quad g'_x(0, \alpha) = 1, \quad g_x^{(i)}(0, \alpha) = 0, \quad i = 2, \cdots, k, \\ &g_x^{(k+1)}(0, \alpha) = -(k+1)(k-1)!\alpha, \end{aligned}$$

and for any $0 < x_1 \leq \frac{1}{\alpha^{1/k}}$, the sequence generated by

$$x_{n+1} = g(x_n, \alpha), \quad n \in \mathbb{N}, \tag{2.1}$$

converges to 0 as $n \to +\infty$. Furthermore,

$$\lim_{n \to +\infty} \alpha^{1/k} n^{1/k} x_n = 1.$$

Proof. The values and derivatives of the function g at $x = 0$ with respect to x follow from direct computation. Let $x_1 = \frac{1}{\gamma \alpha^{1/k}}$ with $\gamma \geq 1$, by induction we can prove that the sequence with initial value x_1 created by (2.1) exactly is $x_n = \frac{1}{\alpha^{1/k}(n - 1 + \gamma^k)^{1/k}}$, which gives us the the desired rate of convergence. ∎

Let $f(x)$ be a C^{k+2} smooth function and satisfy

$$f(0) = 0, \; f'(0) = 1, \; f^{(i)}(0) = 0, \; i = 2, \cdots, k, \; f^{(k+1)}(0) < 0, \tag{2.2}$$

which indicates that the function f has the form

$$f(x) = x - \beta x^{k+1} + O(x^{k+2}), \tag{2.3}$$

where $\beta > 0$.

Mostly, in bifurcation analysis, the map restricted to its one-dimensional center manifold usually has the form of (2.3). It is of interest for us to understand the behaviour of the f-orbit and solutions of the associated variational equation near a critical fixed point. We start to show the convergent rate of orbits generated by f.

Lemma 2.2 *There exists an $\bar{x}_0 > 0$ small, such that for any $0 < x_1 < \bar{x}$, the orbit defined by*

$$x_{n+1} = f(x_n), \quad n \in \mathbb{N} \tag{2.4}$$

satisfies

$$\lim_{n \to +\infty} (\beta k)^{1/k} n^{1/k} x_n = 1. \tag{2.5}$$

Proof. Define $h(x,\alpha) := f(x) - g(x,\alpha)$, we get

$$\begin{aligned} h_x^{(i)}(0,\alpha) &= 0,\ i = 0,\dots,k, \\ h_x^{(k+1)}(0,\alpha) &= -(k+1)!\beta + (k+1)(k-1)!\alpha. \end{aligned}$$

For $\bar\alpha = k\beta$ it follows $h_x^{(k+1)}(0,\bar\alpha) = 0$ and for any $\varepsilon > 0$ we have

$$h_x^{(k+1)}(0,\bar\alpha+\varepsilon) > 0, \quad h_x^{(k+1)}(0,\bar\alpha-\varepsilon) < 0.$$

Let $\varepsilon_0 > 0$ fixed. Thus there exists an $\bar x > 0$, such that for $0 < x < \bar x$

$$h_x^{(k+1)}(x,\bar\alpha+\varepsilon_0) > 0, \quad h_x^{(k+1)}(x,\bar\alpha-\varepsilon_0) < 0.$$

Noticing $h_x^{(i)}(0,\alpha) = 0$ for $i = 0,1,\cdots,k$, by induction from k to 0, we find for $0 < x < \bar x$, $\bar x$ can be reduced if necessary,

$$h(x,\bar\alpha+\varepsilon_0) > 0, \quad h(x,\bar\alpha-\varepsilon_0) < 0,$$

which implies

$$g(x,\bar\alpha+\varepsilon_0) < f(x) < g(x,\bar\alpha-\varepsilon_0). \tag{2.6}$$

For any fixed $0 < \tilde x_1 < \bar x$, let $x_1^- = x_1^+ = \tilde x_1$, define for $n \in \mathbb{N}$ the sequences

$$x_{n+1}^- = g(x_n^-,\bar\alpha+\varepsilon_0), \quad \tilde x_{n+1} = f(\tilde x_n), \quad x_{n+1}^+ = g(x_n^+,\bar\alpha-\varepsilon_0).$$

(2.6) implies $x_2^- < \tilde x_2 < x_2^+$. Assume $x_n^- < \tilde x_n < x_n^+$ for $n \geq 2$. Due to the strictly monotone increasing properties of f near $x = 0$, we get

$$x_{n+1}^- = g(x_n^-,\bar\alpha+\varepsilon_0) < f(x_n^-) < f(\tilde x_n) = \tilde x_{n+1} < f(x_n^+) < g(x_n^+,\bar\alpha-\varepsilon_0) = x_{n+1}^+.$$

By induction, we obtain $x_n^- < \tilde x_n < x_n^+$ for all $n \in \mathbb{N}$, and from Lemma 2.1 we know

$$\lim_{n\to+\infty} n^{1/k} x_n^\pm = \frac{1}{(\bar\alpha \mp \varepsilon_0)^{1/k}}.$$

Thus there exists an $N(\varepsilon_0)$ such that for all $n \geq N(\varepsilon_0)$

$$a_n := n^{1/k}\tilde x_n \in \left[\frac{1}{(\bar\alpha+\varepsilon_0)^{1/k}} - \varepsilon_0, \frac{1}{(\bar\alpha-\varepsilon_0)^{1/k}} + \varepsilon_0\right],$$

which means that the sequence a_n is bounded, hence $\limsup\limits_{n\to+\infty} a_n$ and $\liminf\limits_{n\to+\infty} a_n$ exist. Next, we want to prove that $\lim\limits_{n\to+\infty} a_n = \dfrac{1}{\bar\alpha^{1/k}}$. For contradiction we make the assumption $\limsup_{n\to+\infty} a_n = \dfrac{1}{\bar{\alpha}^{1/k}} + c$ with $c > 0$. Then we choose $\varepsilon_1 < \varepsilon_0$ such that

$$\frac{1}{(\bar{\alpha}-\varepsilon_1)^{1/k}} + \varepsilon_1 < \frac{1}{\bar{\alpha}^{1/k}} + \frac{1}{2}c$$

holds. Similarly, we find an $\bar{x}^*$ such that

$$g^n(x, \bar{\alpha}+\varepsilon_1) < f^n(x) < g^n(x, \bar{\alpha}-\varepsilon_1)$$

for all $0 < x < \bar{x}^*$ and all $n \in \mathbb{N}$. Because $\tilde{x}_n$ converges to 0 there exists an $l > 0$ such that $\tilde{x}_n < \bar{x}^*$ for all $n \geq l$. Defining $\tilde{y}_n := \tilde{x}_{n+l}$ for $n \in \mathbb{N}$, we get

$$\limsup_{n\to+\infty} n^{1/k}\tilde{x}_n = \limsup_{n\to+\infty} n^{1/k}\tilde{y}_n.$$

Set $y_1^- = \tilde{y}_1 = y_1^+ \in (0, \bar{x}^*)$. Similarly $y_n^- \leq \tilde{y}_n \leq y_n^+$ holds for all $n \in \mathbb{N}$ and there exists an $N(\varepsilon_1)$ such that the following holds:

$$n^{1/k}\tilde{y}_n \in \left[\frac{1}{(\bar{\alpha}+\varepsilon_1)^{1/k}} - \varepsilon_1, \frac{1}{(\bar{\alpha}-\varepsilon_1)^{1/k}} + \varepsilon_1\right], \ \forall n \geq N(\varepsilon_1).$$

But $\frac{1}{\bar{\alpha}^{1/k}} + c \notin \left[\frac{1}{(\bar{\alpha}+\varepsilon_1)^{1/k}} - \varepsilon_1, \frac{1}{(\bar{\alpha}-\varepsilon_1)^{1/k}} + \varepsilon_1\right]$, which is a contradiction, hence $\limsup_{n\to+\infty} a_n \leq \dfrac{1}{\bar{\alpha}^{1/k}}$. The same argumentation shows $\liminf_{n\to+\infty} a_n \geq \dfrac{1}{\bar{\alpha}^{1/k}}$.

Thus we proved $\lim_{n\to+\infty} \bar{\alpha}^{1/k} n^{1/k} x_n = 1$. ■

Remark 2.3 *In the general case of $f(x) = x - \beta x^{k+1} + O(x^{k+2})$ with $\beta \neq 0$ and for any initial value x_0 close to 0, whether the sequence $x_{n+1} = f(x_n)$ converges to 0 or not, depends on the value of k, the sign of β and x_0, and the limit process being either $n \to -\infty$ or $n \to +\infty$. For example, if $k = 2$, $\beta < 0$ and $x_0 > 0$, x_n converges to 0 only if $n \to -\infty$. In any case, if an orbit x_n converges to 0, the corresponding rate of convergence must be* (2.5) *with β and x_n replaced by their absolute values, respectively.*

The former lemma interprets the rate of convergence of the orbits created by the map (2.4). Next, we study the rate of the solution of the associated variational equation along these orbits.

Lemma 2.4 *For any $0 < x_1 < \bar{x}$, $u_1 \in \mathbb{R}$ and $0 < \delta < \frac{1}{k}$ the solutions $(u_n)_{n\in\mathbb{N}}$ of the variational equation*

$$u_{n+1} = f'(x_n)u_n$$

satisfy

$$\lim_{n\to+\infty} n^{1+\frac{1}{k}-\delta} u_n = 0,$$

where $x_{n+1} = f(x_n)$.

Proof. Without losing generality and to simplify the notations of the proof, we assume $\beta = 1$ in (2.3) and $u_1 > 0$, then $u_n > 0$ for all $n \in \mathbb{N}$. We also assume that the function f has the form $f(x) = x - x^{k+1}$. We get

$$u_{n+1} = f'(x_n)u_n = \left(1-(k+1)x_n^k\right)u_n = \prod_{i=1}^{n}\left(1-(k+1)x_i^k\right)u_1.$$

For any given $0 < \delta < \frac{1}{k}$, let $\delta = \delta_1 + \delta_2$, $\delta_1, \delta_2 > 0$. Take $\varepsilon_0 > 0$ such that $\frac{1+\frac{1}{k}}{1+\frac{1}{k}-\delta_1}\frac{k}{k+\varepsilon_0} \geq 1$.

By showing $b_n := \log\left(n^{1+\frac{1}{k}-\delta}u_{n+1}\right)$ goes to $-\infty$ as $n \to \infty$, we can finish the proof of this lemma.

$$\begin{aligned} b_n &= (1+\frac{1}{k}-\delta)\log(n) + \log u_{n+1} \\ &= (1+\frac{1}{k}-\delta)\log(n) + \sum_{i=1}^{n}\log(1-(k+1)x_i^k) + \log(u_1) \\ &= -\delta_2\log(n) + M_n + \Pi_n + \log(u_1), \end{aligned}$$

where

$$\begin{aligned} \Pi_n &= \sum_{i=1}^{n}(k+1)x_i^k + \sum_{i=1}^{n}\log(1-(k+1)x_i^k), \\ M_n &= (1+\frac{1}{k}-\delta_1)\log(n) - (1+\frac{1}{k})\sum_{i=1}^{n}kx_i^k. \end{aligned}$$

We will show that M_n is bounded from above and Π_n converges as $n \to \infty$. For $0 < x_1 \leq \bar{x}$ let $x_i^- = g^i(x_1, \bar{\alpha}+\varepsilon_0)$. Then we get $x_i^- < x_i$ and

$$\begin{aligned} M_n &\leq (1+\frac{1}{k}-\delta_1)\left[\log(n) - \frac{1+\frac{1}{k}}{1+\frac{1}{k}-\delta_1}\sum_{i=1}^{n}k(x_i^-)^k\right] \\ &= (1+\frac{1}{k}-\delta_1)\left[\log(n) - \frac{1+\frac{1}{k}}{1+\frac{1}{k}-\delta_1}\cdot\frac{k}{k+\varepsilon_0}\sum_{i=1}^{n}\frac{1}{i-1+\gamma^k}\right] \\ . &\leq (1+\frac{1}{k}-\delta_1)\left[\log(n) - \sum_{i=1}^{n}\frac{1}{i-1+\gamma^k}\right] \\ &\leq C^*, \end{aligned}$$

since $\log(n) - \sum_{i=1}^{n} \frac{1}{i}$ converges, see [13].

Next we show the convergence of Π_n:

$$\Pi_n = \sum_{i=1}^{n} \left[\log(1-(k+1)x_i^k) + (k+1)x_i^k\right].$$

We see that $x_i^k \to 0$ as $i \to \infty$. Using L'Hospitals principle twice we get

$$\lim_{x\to 0, x>0} \frac{\log(1-(k+1)x) + (k+1)x}{x^2} = -\frac{1}{2}(k+1)^2.$$

Therefore

$$\lim_{i\to\infty} \frac{\log(1-(k+1)x_i^k) + (k+1)x_i^k}{(x_i^k)^2} = -\frac{1}{2}(k+1)^2.$$

From Lemma 2.2 we know $\lim_{i\to\infty} (x_i^k)^2 i^2 = \frac{1}{k^2}$, hence $\sum_{i=1}^{\infty} (x_i^k)^2$ absolutely converges. Thus $\sum_{i=1}^{n} \left[\log(1-(k+1)x_i^k) + (k+1)x_i^k\right]$ converges. ■

Remark 2.5 *1) A similar result for ODEs is given by Schecter in* [14]. *He considered an equation of the form* $\dot{x} = f(x) = x^2 + O(x^3)$*, and he proved if* $y(t)$ *is a solution of its variational equation* $\dot{y}(t) = f'(x(t))y(t)$*, then* $\lim_{t\to\infty} t^2 y(t) = \text{constant}$.

2) For further analysis of the discrete saddle-node homoclinic orbit, we need that the series $\sum_{n=1}^{\infty} u_n$ *converges absolutely. This of course is true due to* Lemma 2.4.

3 Asymptotic estimates for saddle-node homoclinic orbits

In this section, we apply the results prepared in the previous section to the discrete dynamical systems

$$x_{n+1} = f(x_n, \lambda), \tag{3.1}$$

and study the rate of convergence of its saddle-node homoclinic orbit along the one-dimensional center manifold. Furthermore, we investigate the behaviour of solutions of the associated variational equation along this orbit. First, we start to introduce our basic assumptions.

H1 $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is smooth enough.

H2 $(\bar{\xi}, \bar{\lambda})$ is a saddle-node fixed point of the system (3.1).

Let $A = D_x f(\bar{\xi}, \bar{\lambda})$ be the Jacobian Matrix which has m_s stable and m_u unstable eigenvalues besides the simple eigenvalue 1. Denote by X^i $(i = s, u, c)$ the corresponding eigenspaces of dimension m_s, m_u and 1, respectively. The spaces X^i $(i = s, u, c)$ are equivalent to the spaces $\mathbb{R}^{m_s}$, $\mathbb{R}^{m_u}$ and $\mathbb{R}$, respectively, and without loss of generality, we identify them. Let $A^{\pm}$ be the restrictions of A to X^s and X^u. Obviously, the restriction of A to X^c is 1. Then the spectra of $A^{\pm}$ lay inside and outside the unit circle on complex plane, respectively. Assume the matrix A has left and right unit eigenvectors corresponding to eigenvalue 1, which are denoted by e_l and e_r, respectively.

H3 $e_l^T e_r = 1, \quad e_l^T D_{xx} f(\bar{\xi}, \bar{\lambda})(e_r, e_r) > 0.$

Due to H1-H3, two families of hyperbolic fixed points bifurcate from the saddle-node point $(\bar{\xi}, \bar{\lambda})$. Without loss of generality, we assume they exist for $\lambda < \bar{\lambda}$ and there is no fixed point for $\lambda > \bar{\lambda}$. For simplicity, let $(\bar{\xi}, \bar{\lambda}) = (0, 0)$.

Let η_j $(j = 1, \cdots, m-1)$ be the eigenvalues of A besides the simple eigenvalue 1. We require the following nonresonance condition.

H4 $|\eta_j| \neq |\eta_1|^{\kappa_1} \cdots |\eta_{m-1}|^{\kappa_{m-1}}$, for all j and any $\kappa \in \mathbb{Z}_+^{m-1}$, $|\kappa| \geq 2$.

H5 At $\lambda = 0$, the mapping $f(\cdot, 0)$ permits a transversal saddle-node homoclinic orbit $\bar{x}_n$, which is the intersection of the stable manifold and the unique part of the center manifold, (see Figure 3.1).

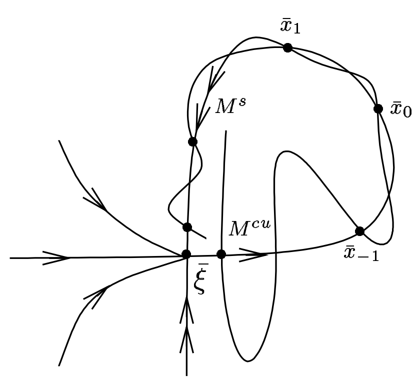

Figure 3.1 The transversal saddle-node homoclinic orbit.

For further analysis, we will rectify the local stable, unstable and center manifolds to the corresponding axes simultaneously. It is here that the non-resonance condition H4 is used. The corresponding results can be found in [8, 9] for maps, and in [15, 6, 3, 14, 9] for flows. Here we show the results that are presented in [9].

Lemma 3.1 [9, Theorem 4.1] *Assume* H1-H4. *Then there exists a neighborhood of the origin and a C^3 change of coordinates $\Phi : \mathbb{R}^m \to \mathbb{R}^{m_s} \oplus \mathbb{R}^{m_u} \oplus \mathbb{R}$ such that in the new coordinates and in that neighborhood the system has the following form:*

$$s_{n+1} = A^+(w_n, \lambda)s_n, \;\; u_{n+1} = A^-(w_n, \lambda)u_n, \;\; w_{n+1} = \tilde{f}^c(w_n, \lambda) \tag{3.2}$$

where $s_n \in \mathbb{R}^{m_s}$, $u_u \in \mathbb{R}^{m_u}$, $w_n \in \mathbb{R}$ $(n \in \mathbb{Z})$, $A^\pm(0,0) = A^\pm$, $\tilde{f}^c(w, \lambda) = \lambda + w + p(w, \lambda)w^2$ and $p(w, \lambda)$ is C^3 smooth.

Assumption H3 implies $p(0,0) > 0$. Applying Lemma 2.2 and Remark 2.3 in the case of $k = 1$ to the iteration $w_{n+1} = \tilde{f}^c(w_n, 0)$ with $w_1 > 0$, we obtain $nw_n \to 1/p(0,0)$ as $n \to -\infty$.

As $n \to -\infty$, the tail of the homoclinic orbit $\bar{x}_n$ has the form $(0, 0, \bar{w}_n)$ and the associated variational equation reads

$$\begin{pmatrix} S_{n+1} \\ U_{n+1} \\ W_{n+1} \end{pmatrix} = \begin{pmatrix} A^+(\bar{w}_n, 0) & & \\ & A^-(\bar{w}_n, 0) & \\ & & D_w\tilde{f}^c(\bar{w}_n, 0) \end{pmatrix} \begin{pmatrix} S_n \\ U_n \\ W_n \end{pmatrix}.$$

Due to the roughness lemma in [5, 12], we know that the solutions of the form $(S_n, 0, 0)$ leave from 0 and of the form $(0, U_n, 0)$ tend to 0 exponentially as $n \to -\infty$, respectively. More precisely, there exist two constants $\gamma, K > 0$ such that for $-\infty < n \leq l \leq 0$

$$\|S_l\| \leq Ke^{-\gamma(l-n)}\|S_n\|, \quad \|U_n\| \leq Ke^{-\gamma(l-n)}\|U_l\|.$$

Due to Lemma 2.3, any solution of the form $(0, 0, W_n)$ satisfies for any given $0 < \delta < 1$

$$\lim_{n \to -\infty} n^{2-\delta} W_n = 0.$$

Transforming the variables (s, u, w) back to the original variable x, we get the following results for the map $f(\cdot, 0)$.

Theorem 3.2 *Assume* H1-H5. *Then the homoclinic orbit $\bar{x}_n$ of the map $f(\cdot, 0)$ satisfies*

$$\lim_{n \to -\infty} n\bar{x}_n = constant.$$

Furthermore, for any given $0 < \delta < 1$, the variational equation along this homoclinic orbit

$$X_{n+1} = D_x f(\bar{x}_n, 0) X_n, \quad X_n \in \mathbb{R}^m, \quad n \leq 0$$

possesses m linearly independent solutions $X_J^{s,j}$ ($j = 1, \cdots, m_s$), $X_J^{u,j}$ ($j = 1, \cdots, m_u$) and X_J^c with $J = \mathbb{Z}^-$, which satisfy

$$\begin{aligned} \|X_l^{s,j}\| &\leq K e^{-\gamma(l-n)} \|X_n^{s,j}\|, \quad j = 1, \cdots, m_s, \\ \|X_n^{u,j}\| &\leq K e^{-\gamma(l-n)} \|X_l^{u,j}\|, \quad j = 1, \cdots, m_u \end{aligned}$$

for some $\gamma, K > 0$ and $-\infty < n \leq l \leq 0$, and

$$\lim_{n \to -\infty} n^{2-\delta} \|X_n^c\| = 0.$$

Next, we consider the behaviour of solutions of the variational equation as $n \to +\infty$. Meanwhile, the tail of the homoclinic orbit has the form $(\bar{s}_n, 0, 0)$.

Theorem 3.3 *Assume* H1-H5. *As $n \to +\infty$, the saddle-node homoclinic orbit $\bar{x}_n$ converges to the fixed point 0 exponentially fast. Furthermore, the variational equation*

$$Z_{n+1} = D_x f(\bar{x}_n, 0) Z_n, \quad Z_n \in \mathbb{R}^m, \quad n \geq 0$$

possesses m linearly independent solutions $Z_J^{s,j}$ ($j = 1, \cdots, m_s$), $Z_J^{u,j}$ ($j = 1, \cdots, m_u$) and Z_J^c with $J = \mathbb{Z}^+$, which satisfy

$$\begin{aligned} \|Z_n^{s,j}\| &\leq K e^{-\gamma(n-l)} \|Z_l^{s,j}\|, \quad j = 1, \cdots, m_s, \\ \|Z_l^{u,j}\| &\leq K e^{-\gamma(n-l)} \|Z_n^{u,j}\|, \quad j = 1, \cdots, m_u \end{aligned}$$

for some $\gamma, K > 0$ and $0 < l \leq n \leq +\infty$, and there exists a nonzero constant c^* such that

$$\lim_{n \to \infty} Z_n^c = c^* e_r \text{ with exponential rate.}$$

Proof. As $n \to +\infty$, the homoclinic orbit $\bar{x}_n$ is on the stable manifold, then the exponential rate follows. From Lemma 3.1 we know that the corresponding variational equation along the saddle-node homoclinic orbit $(\bar{s}_n, 0, 0)$ is of the form

$$\begin{pmatrix} S_{n+1} \\ U_{n+1} \\ W_{n+1} \end{pmatrix} = \begin{pmatrix} A^+(0,0) & 0 & D_w A^+(0,0)\bar{s}_n \\ 0 & A^-(0,0) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_n \\ U_n \\ W_n \end{pmatrix}, \quad (3.3)$$

as $n \to +\infty$. Rewrite (3.3) as

$$(S_{n+1}, U_{n+1}, W_{n+1})^T = (C_n + B_n)(S_n, U_n, W_n)^T, \tag{3.4}$$

where

$$C_n = \mathrm{Diag}(A^+, A^-, 1), \quad B_n = \begin{pmatrix} 0 & 0 & D_w A^+(0,0)\bar{s}_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider the follow equation

$$(S'_{n+1}, U'_{n+1}, W'_{n+1})^T = C_n(S'_n, U'_n, W'_n)^T. \tag{3.5}$$

It follows immediately that the W'-component of its solution satisfies

$$\lim_{n\to+\infty} W'_n = c, \tag{3.6}$$

where c is a constant and the (S', U')-components have the properties of exponential dichotomies.

Obviously, the matrix B_n satisfies $\lim\limits_{n\to+\infty} |B_n| = 0$ with exponential rate.

Similar to the roughness lemma for continuous time systems, (see [4, p. 104, 106], [7, p. 305], [5], [14] and [2]), we can prove that by adding the exponentially small perturbation of B_n to the system (3.5), the similar asymptotic behaviour, i.e. the property of (3.6) and the exponential dichotomies, will be preserved by solutions of the perturbed system (3.4) as $n \to +\infty$. Transforming the variable (s, u, w) back to x leads to the estimates in the theorem. ■

Acknowledgment The authors would like to thank Prof. W.-J. Beyn and Prof. M. Huang for their stimulating discussions and suggestions. The second author also thanks Prof. W.-J. Beyn for his hospitality while his visiting University of Bielefeld.

References

[1] V. Arnol'd, V. Afraimovich, Y. Il'yashenko & L. Shil'nikov, *Bifurcation theory*, in V. Arnol'd, ed., 'Dynamical Systems V. Encyclopaedia of Mathematical Sciences', Springer-Verlag, New York, 1994.

[2] W.-J. Beyn, *On well-posed problems for connecting orbits in dynamical systems.* Contemporary Mathematics, Vol. **172**, 131-168, 1994.

[3] S.-N. Chow & X.-B. Lin, *Bifurcation of a homoclinic orbit with a saddle-node equilibrium.* Diff. and Int. Eqns., **3**. 435-466. 1990.

[4] E. A. Coddington & N. Levinson, Theory of Ordinary Differential Equations. McGraw-Hill, New York, 1955.

[5] W. A. Coppel, Dichotomies in Stability Theory. Lecture Notes in Mathematics, **629**, Springer-Verlag, New York, 1978.

[6] N. Fenichel, *Asymptotic stability with rate conditions.* Indiana Univ. Math. J., **23**, 1109-1137, 1974.

[7] P. Hartman, Ordinary Differential Equations. John Wiley & Sons, New York, 1964.

[8] M. W. Hirsch, C. C. Pugh & M. Shub, Invariant Manifolds. Lecture Notes in Mathematics, **583**, Springer-Verlag, New York, 1977.

[9] Y. Ilyashenko & W. Li, Nonlocal Bifurcations. Mathematical Surveys and Monographs, **66**, American Mathematical Society, 1999.

[10] G. Iooss, Bifurcation of Maps and Applications. North-Holland, Amsterdam, 1979.

[11] Yuri A. Kuznetsov, Elements of Applied Bifurcation Theory (Second edition). Applied Mathematical Sciences, **112**, Springer-Verlag, New York, 1998.

[12] K. J. Palmer, *Exponential dichotomies, the shadowing lemma and transversal homoclinic points*, in U. Kirchgraber & H. O. Walther, ed., 'Dynamics Reported', Vol. **1**, 265-306, Teubner, Stuttgart, 1988.

[13] E. D. Rainville, Special Functions. Chelsea Publishing Company, Bronx, New York, 1971.

[14] S. Schecter, *Numerical computation of saddle-node homoclinic bifurcation points.* Siam J. Numer. Anal., **30**, 1155-1178, 1993.

[15] F. Takens, *Partially hyperbolic fixed points.* Topology, **10**, 133-147, 1971.

T. Hüls: SFB 343, Universität Bielefeld, Postfach 100131, D–33501, Bielefeld, Germany. huels@mathematik.uni-bielefeld.de.

Y. Zou: Department of Mathematics, Jilin University, Changchun 130023, P. R. of China. yzou@mathematik.uni-bielefeld.de.