Conjugacy in the discretized fold bifurcation

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Abstract

In this paper we construct a conjugacy between the time-1-map of the solution flow generated by an ordinary differential equation and its numerical approximation in a neighborhood of a fold bifurcation point. Our main result is that the conjugacy is $O(h^p)$-close to the identity on the center manifold where h is the step-size and p is the order of the numerical method.
keywords: generalized Hartman-Grobman theorem, fold bifurcation, discretization
AMS Subject Classification: 34C30, 65L05

1 Introduction

It is well known that conjugacies play a fundamental role in the qualitative theory of ordinary differential equations. Indeed, when a conjugacy exists between two dynamical systems then the dynamical systems have the same orbit structure, they are qualiatively the same.

The discretization of a dynamical system is a family of maps (depending on the step size h) which is close to the time-h-map of the dynamical system. We want to claim that under certain conditions the dynamics of the discretization considered as a discrete dynamical system and of the original

*supported by DAAD project 323–PPP, Qualitative Theory of Numerical Methods for Evolution Equations in Infinite Dimensions

†This work was done while the author was a visitor at the University of Bielefeld. The author would like to thank SFB 343 for the hospitality and Prof. W.-J. Beyn for the stimulating discussions.

system are the same. Thus it is natural to seek for conjugacies between a dynamical system and its numerical approximation.

In the vicinity of a hyperbolic equilibrium point this was done in [5] by putting the problem in the general framework of Hartman-Grobman theorem. A similar approach was carried out in [3] in the case of delay differential equations. Structural stability results were obtained in [6] (for Morse-Smale systems without periodic orbits) and in [10] (for systems satisfying Axiom A and the strong transversality condition). The construction of the conjugacies uses the various type of hyperbolicity conditions of the dynamical system.

However, hyperbolicity is usually lost in a bifurcation point. So these results cannot be applied to a bifurcation problem. We note that in general we cannot expect that a conjugacy exists in a neighborhood of a nonhyperbolic equilibrium point, as the simple example of the planar linear center and the Euler method shows. On the other hand under certain conditions the existence of a conjugacy can be saved. Namely, we show in this paper that in the neighborhood of a fold bifucation point the desired conjugacy exists. Moreover, the conjugacy is $O(h^p)$-close to the identity on the center manifold where p is the order of the method.

The proof of our main result works via the generalized Hartman-Grobman theorem (see [8], [12]), the center manifold reduction (see [9], [13]) and the method of fundamental domains. The use of fundamental domains was inspired by a lecture by Y.A. Kuznetsov where the topological normal form of the fold bifurcation was constructed in a similar way. The center manifold reduction played a fundamental role in [11] where a numerical Hopf bifurcation theorem was proved for partial differential equaitons.

The paper is organized as follows. Preliminaries are placed into Section 2. Section 3 contains our main result. We end this note with some final remarks.

2 Preliminaries

Let $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ be a globally Lipschitzian C^j function with $j \geq 4$. Consider the following ordinary differential equation depending on a single parameter α

$$\dot{z} = f(z; \alpha). \tag{1}$$

Denote the solution flow of (1) with parameter value α by $\Phi(\cdot, \cdot; \alpha) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$.

By the h-discretized equation of (1) we mean equation

$$Z = \phi(h, z; \alpha) \quad (z, Z \in \mathbf{R}^n,\ h > 0), \tag{2}$$

where ϕ is a fixed one-step method with step size h. Assume that ϕ is smooth and is of order $p \geq 1$, i.e. there exist a constant h_0 and a constant K_1 (depending only on f) such that

$$|\Phi(h, z; \alpha) - \phi(h, z; \alpha)|_j \leq K_1 h^{p+1} \quad \text{for all} \quad h \in (0, h_0],\ z \in \mathbf{R}^n, \tag{3}$$

where $\Phi(h, \cdot; \alpha) : \mathbf{R}^n \to \mathbf{R}^n$ is the time-h-map of the induced solution flow of (1) with parameter value α and $|\cdot|_j$ denotes the usual C^j-norm of the space $C^j(\mathbf{R}^n \times \mathbf{R}, \mathbf{R}^n)$.

In the usual definition of the order of the method the $|\cdot|_0$ norm is used instead of the $|\cdot|_j$ norm. Since property (3) is a consequence of the C^0-closeness for sufficiently smooth systems we use (3) as a definition of the order of the method. A more detailed treatement of this property can be found in [5].

With $[\cdot]$ denoting the integer part, for fixed $t > 0$ the approximation of the time-t-map of the induced solution flow, i.e. $\Phi(t)$, is

$$\phi^{[t/h]}(h, \cdot; \alpha),$$

and if $t/h \in \mathbf{N}$ then

$$|\Phi(t, z; \alpha) - \phi^{[t/h]}(h, z; \alpha)|_j \leq K_2 h^p \tag{4}$$

holds with some constant $K_2 > 0$ (depending only on f).

Assume that $\Phi(t, 0; 0) = 0$ and $\phi(h, 0; 0) = 0$ for all $t \in \mathbf{R}$ and all $h \in (0, h_0]$, respectively. Assume further that $\alpha = 0$ is a fold bifurcation point for both (1) and (2). To be concrete assume that there are no equilibria for $\alpha > 0$ and there are two equilibria for $\alpha < 0$. We note that a simple analysis of (4) shows that ϕ must have a nearby fold bifurcation point whenever $\alpha = 0$ is a fold bifurcation point for Φ. We only assume for simplicity that this point is shifted into 0.

By enlarging the dimension by 1, i.e. by adding $\dot{\alpha} = 0$ and $A = \alpha$ to (1) and to (2), respectively, we have local center manifolds around 0 in the enlarged phase space denoted by

$$W^C_{loc}(0) = \{(x, \xi(x, \alpha), \alpha) \ : \ x \in \mathbf{R},\ |x|, |\alpha| \text{ are sufficiently small } \}$$

and

$$W^{C_h}_{loc}(0) = \{(x, \xi_h(x, \alpha), \alpha) \ : \ x \in \mathbf{R},\ |x|, |\alpha| \text{ are sufficiently small } \}$$

where $\xi, \xi_h : \mathbf{R} \times \mathbf{R} \to \mathbf{R}^{n-1}$ are C^j functions.

Applying the result of [2] (or of [5]) we have that these manifolds are C^j-close, i.e. the functions ξ and ξ_h are C^j-close, moreover their C^j-distance is bounded by $O(h^p)$. For sake of simplicity we denote the solution flow of the enlarged system and its discretization simply by Φ and ϕ, respectively. Finally, denote the reduced maps on their center manifolds by Φ_C and ϕ_{C_h}, respectively, i.e.

$$\Phi_C(t,x;\alpha) = y, \quad \text{where } (y, \xi(y,\alpha), \alpha) = \Phi(t,(x,\xi(x,\alpha));\alpha)$$

and

$$\phi_{C_h}(h,x;\alpha) = y, \quad \text{where } (y, \xi_h(y,\alpha), \alpha) = \phi(h,(x,\xi_h(x,\alpha));\alpha).$$

From the C^j-closeness of the center manifolds and from (4) it follows that

$$|\Phi_C(t,x;\alpha) - \phi_{C_h}^{[t/h]}(h,x;\alpha)|_j = O(h^p) \tag{5}$$

where $t/h \in \mathbf{N}$. From now on we restrict ourselves to the case $1/h \in \mathbf{N}$.

Following [9] we see that the construction of the normal form of the fold bifurcation works via Taylor expansion, implicit function theorem (to eliminate the parameter dependent first order term) and inverse function theorem (to introduce a new parameter). Thus our closeness property (5) yields the following Lemma.

Lemma 1 *There are positive numbers ε, α_0 and smooth invertible coordinate transforms τ and τ_h, such that τ transforms $\Phi_C(1)$ into*

$$X = x + \alpha + ax^2 + x^3\psi(x,\alpha) =: f^1(x;\alpha) \tag{6}$$

while τ_h transforms $\phi_{C_h}^{[1/h]}(h)$ into

$$X = x + \alpha + a_h x^2 + x^3\psi_h(x,\alpha) =: f_h^2(x;\alpha) \tag{7}$$

where $a > 0$, ψ and ψ_h are smooth functions of x and α provided $|x| < \varepsilon$ and $|\alpha| < \alpha_0$ holds. Moreover, we have that

$$|a - a_h| \le K_3 h^p, \quad |\psi(x,\alpha) - \psi_h(x,\alpha)| \le K_3 h^p, \quad |\tau(x,\alpha) - \tau_h(x,\alpha)| \le K_3 h^p$$

for all $|x| < \varepsilon$, $|\alpha| < \alpha_0$.

3 Main result

Assume all the conditions listed in Section 2 hold true. We prove the following Theorem.

Theorem 1 *There are positive numbers h_1, ε_1, α_1 and a real function J defined on $(0,h_1] \times (-\varepsilon_1, \varepsilon_1) \times (-\alpha_1, \alpha_1)$ such that $J(h,\cdot,\alpha)$ is a homeomorphism,*

$$f^1(J(h,x,\alpha);\alpha) = J(h, f_h^2(x;\alpha), \alpha) \tag{8}$$

and

$$|J(h,\cdot,\alpha) - \mathrm{id}|_0 \le Kh^p \tag{9}$$

holds with some constant $K > 0$ independent of h and α.

Proof. Set

$$X = x + \alpha + ax^2 =: g(x;\alpha).$$

Our method is to construct homeomorphisms $H(\cdot,\alpha)$ and $G(h,\cdot,\alpha)$ such that

$$f^1(H(x,\alpha);\alpha) = H(g(x;\alpha),\alpha), \tag{10}$$

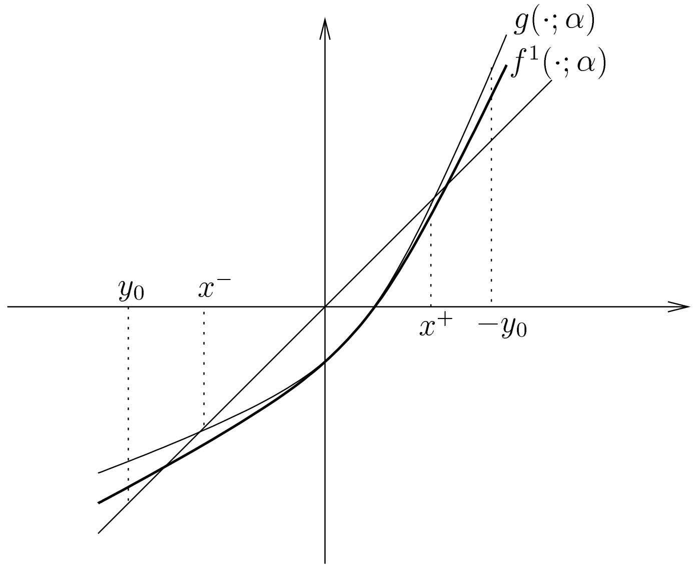
$$f_h^2(G(h,x,\alpha);\alpha) = G(h, g(x;\alpha), \alpha) \tag{11}$$

and

$$|H(\cdot,\alpha) - G(h,\cdot,\alpha)|_0 \le Kh^p \tag{12}$$

hold. Then it remains to set $J = H \circ G^{-1}$.

Let N be a neighborhood of $x = 0$ and $0 < h \le h_2$ such that f^1, f_h^2 and g have the same number of fixed points with the same stability, provided $|\alpha|$ is sufficiently small. Fix $0 > y_0 \in N$ such that $g(y_0;\alpha) < f^1(g(y_0;\alpha);\alpha)$, $g(y_0;\alpha) < f_h^2(g(y_0;\alpha);\alpha)$ and if $\alpha \le 0$ then $g(-y_0;\alpha) \in N$, $g(-y_0;\alpha) > (f^1)^{-1}(g(-y_0;\alpha);\alpha)$, $g(-y_0;\alpha) > (f_h^2)^{-1}(g(-y_0;\alpha);\alpha)$. We divide the construction of H and G into three parts according to $\alpha < 0$, $\alpha = 0$ or $\alpha > 0$.



Case $\alpha < 0$. Fix $x_0 = 0$ and set $x_k = g^k(x_0;\alpha)$, $k \in \mathbf{Z}$. Note that $x_1 = \alpha$. Set $H(x_0,\alpha) = G(h,x_0,\alpha) = g(x_0;\alpha)$ and $H(x_k,\alpha) = (f^1)^k(x_1;\alpha)$, $G(h,x_k,\alpha) = (f_h^2)^k(x_1;\alpha)$, $k \in \mathbf{Z}$. On $[x_1,x_0]$ extend both H and G linearly. For $y \in [x_2,x_1]$ set $H(y,\alpha) = f^1(H(g^{-1}(y;\alpha),\alpha);\alpha)$ and $G(h,y,\alpha) = f_h^2(G(h,g^{-1}(y;\alpha),\alpha);\alpha)$. Recursively, in both direction, we see that H and G extend continuously to the intervall (x^-,x^+), where x^-, x^+ are the negative and positive fixed points of g, respectively. See the figure above. Finally set $H(x^-,\alpha) = x_1^-$, $G(h,x^-,\alpha) = x_2^-$, $H(x^+,\alpha) = x_1^+$ and $G(h,x^+,\alpha) = x_2^+$, where x_1^-, x_1^+ are the negative and positive fixed points of f^1; x_2^-, x_2^+ are the negative and positive fixed points of f_h^2, respectively.

From initial points y_0 and $-y_0$ the same construction can be carried out (by taking the inverse when necessary). Note that here the assumptions on y_0 enter. As a result we obtain functions H and G defined on some neighborhood of $x = 0$ for all $\alpha < 0$, $|\alpha|$ sufficiently small, and all $0 < h \le h_2$.

From the construction it is easy to see that H and G are homeomorphisms (since they are continuous, strictly monotone functions) and are indeed the desired conjugacies, i.e. (10) and (11) hold.

It remains to prove the closeness of H and G, i.e. inequality (9). We restrict ourselves to estimate the distance between H and G on $[y_0,0]$, the complementary part can be treated similarily.

First we estimate $|H-G|$ on $[x^-,0]$. It is clear that $|H(x,\alpha)-G(h,x,\alpha)| \le K_4h^p$ holds for $x \in [x_1,x_0]$. Note that

$$|f^1(x;\alpha)-f_h^2(x;\alpha)| \le |a-a_h|\cdot|x|^2+|\psi(x,\alpha)-\psi_h(x,\alpha)|\cdot|x|^3 \le K_5h^p|x|^2 \quad (13)$$

provided N and α_1 are sufficiently small. Consequently,

$$|f^1(x;\alpha) - f_h^2(x;\alpha)| \le K_5h^p|x^-|^2 = K_5h^p(-\alpha/a) \quad (14)$$

for all $x \in [x^-,x_0]$. On the other hand, the derivative of f^1 (and f_h^2) is strictly monotone increasing, thus

$$|(f^1)'_x(y;\alpha)| \le |(f^1)'_x(x_1;\alpha)| \le (1+2\tilde{a}\alpha) < 1 \quad (15)$$

with some nonzero constant $\tilde{a}$, for all $y \le x_1$ (provided $|\alpha|$ small enough).

Now estimate $|H-G|$ on $[x_2,x_1]$ as

$$\begin{aligned}
&\sup_{y\in[x_2,x_1]} |H(y,\alpha) - G(h,y,\alpha)| \\
\le &\sup_{y\in[x_2,x_1]} |f^1(H(g^{-1}(y;\alpha),\alpha);\alpha) - f^1(G(h,g^{-1}(y;\alpha),\alpha);\alpha)| \\
&+ \sup_{y\in[x_1,x_0]} |f^1(y;\alpha) - f_h^2(y;\alpha)|
\end{aligned}$$

$$\leq (1+2\tilde{a}\alpha) \sup_{y\in[x_1,x_0]} |H(y,\alpha)-G(h,y,\alpha)| + K_5 h^p(-\alpha/a).$$

Repeating inductively we see that

$$|H(y,\alpha)-G(h,y,\alpha)| \leq K_4 h^p + \frac{K_5(-\alpha/a)}{-2\tilde{a}\alpha} h^p = K_6 h^p$$

for all $y \in (x^-, x_0]$. Finally, at x^- this inequality holds as well.

Finally we estimate $|H-G|$ on $[y_0, x^-]$. By setting $y_k = g^k(y_0;\alpha)$, $k \in \mathbf{N}$ we have that $\sup_{y\in[y_0,y_1]} |H(y,\alpha)-G(h,y,\alpha)| \leq K_6 h^p$, on the other hand

$$\sup_{y\in[y_1,y_2]} |H(y,\alpha)-G(h,y,\alpha)|$$

$$\leq \sup_{y\in[y_1,y_2]} |f^1(H(g^{-1}(y;\alpha),\alpha);\alpha) - f^1(G(h,g^{-1}(y;\alpha),\alpha);\alpha)|$$

$$+ \sup_{y\in[y_0,y_1]} |f^1(y;\alpha) - f_h^2(y;\alpha)|.$$

Define $a_k = |y_k|^2$. Since $|(f^1)'_x(y;\alpha)| \leq |(f^1)'_x(x^-;\alpha)| \leq q < 1$ (by (15)) and $\sup_{y\in[y_k,y_{k+1}]} |f^1(y;\alpha) - f_h^2(y;\alpha)| \leq K_5 h^p |y_k|^2 = a_k K_5 h^p$ (by (13)) and inductive application of the above estimate yields

$$\sup_{y\in[y_0,y_{k+1}]} |H(y,\alpha)-G(h,y,\alpha)| \leq q^k K_6 h^p + (q^{k-1}a_0 + q^{k-2}a_1 + \cdots + a_{k-1}) K_5 h^p.$$

Set $c_k = q^k a_0 + \cdots + a_k$ and $b_k = a_k - |x^-|^2$. Then $b_k \to 0$ as $k \to \infty$ and $c_k \leq |x^-|^2/(1-q) + \sum_{i=0}^{\infty} b_i$ (for all k). We show that $\sum_{k=0}^{\infty} b_k \leq K_7$ with some constant $K_7 > 0$ independent of α and h. This will finish the proof of case $\alpha < 0$ since $|x^-|^2/(1-q) \leq K_8$ with some constant K_8 independent of α (and h). We note here that the trivial estimate $c_k \leq a_0/(1-q)$ does not work since $1/(1-q) \to \infty$ as $\alpha \to 0$.

We construct a sequence z_k of negative numbers such that $z_0 = y_0$, $z_k > -1/(2a)$ for all $k \in \mathbf{N}$ and

$$z_{k+1} \leq z_k + \alpha + a z_k^2 \quad k = 0, 1, \ldots \tag{16}$$

hold. With such a sequence in hand (by using that $g(x;\alpha)$ is strictly monotone increasing for $x > -1/(2a)$) we get that $y_k \geq z_k$ and thus $a_k = |y_k|^2 \leq |z_k|^2$. To this end let $z_0 = y_0$, $z_k = -\sqrt{-\alpha/a} + \delta y_0/k^{1-\gamma}$, where $0 < \gamma < 1/2$ and $\delta \geq 1$ will be chosen later. It is easy to see (note that $\delta \geq 1$) that the desired inequality (16) holds for $k = 0$ provided $|\alpha|$ is sufficiently small. It remains to check that

$$-\sqrt{-\alpha/a} + \delta y_0/(k+1)^{1-\gamma} \leq -\sqrt{-\alpha/a} + \delta y_0/k^{1-\gamma} + \alpha + a(-\sqrt{-\alpha/a} + \delta y_0/k^{1-\gamma})^2$$

or equivalently

$$k^{2(1-\gamma)}/(k+1)^{1-\gamma} \geq (1 - 2a\sqrt{-\alpha/a})k^{1-\gamma} + a\delta y_0 \tag{17}$$

holds. We show a slightly stronger inequality, namely

$$k^{2(1-\gamma)}/(k+1)^{1-\gamma} \geq k^{1-\gamma} + a\delta y_0.$$

It is easy to see that (since $a > 0$)

$$d_k(\gamma) := \frac{k^{1-\gamma}(k^{1-\gamma} - (k+1)^{1-\gamma})}{a(k+1)^{1-\gamma}} \to 0 \text{ as } k \to \infty.$$

On the other hand, if γ is sufficiently close to $1/2$ then $d_k(\gamma)$ is strictly monotone increasing with respect to k ($\gamma = 0.4$ works). Note that $d_1(\gamma) > -1/(2a)$. With such a fixed γ now choose $\delta \geq 1$ such that $d_1(\gamma) \geq \delta y_0$ holds. Note that δ is independent of α. Thus

$$d_k(\gamma) \geq d_1(\gamma) \geq \delta y_0$$

and the desired inequality (16) follows. We remark that similarly the exact asymptotic behavior can be studied about nonhyperbolic equilibria, see [7].

Now we are in a position to prove the convergence of $\sum b_k$. Since $a_k \leq |z_k|^2 \leq |x^-|^2 + |\delta y_0|^2/k^{2(1-\gamma)}$ we have that $b_k \leq |\delta y_0|^2/k^{2(1-\gamma)}$ and the convergence of $\sum b_k$ follows from $2(1-\gamma) > 1$. Finally, note that δ and γ were chosen independently of α which completes the proof of case $\alpha < 0$.

Case $\alpha = 0$. The construction of H and G is the same as in the case $\alpha < 0$. The only difference is that $[x^-, x^+] = \{0\}$. Since $|(f^1)'_x(y;0)| \leq 1$ for all $y \in [y_0, 0]$ we arrive at the following estimate:

$$\sup_{y \in [y_0, y_{k+1}]} |H(y,0) - G(h,y,0)| \leq K_8 h^p + (a_{k-1} + \cdots + a_0) K_5 h^p$$

where as before $a_k = |y_k|^2 = |g^k(y_0;0)|^2$. By using the $\alpha = 0$ variant of the estimate of b_k from case $\alpha < 0$ we obtain $a_k \leq |\delta y_0|^2/k^{2(1-\gamma)}$ with suitably chosen $0 < \gamma < 1/2$ and $\delta \geq 1$, and thus

$$\sup_{y \in [y_0, 0]} |H(y,0) - G(h,y,0)| \leq K_9 h^p.$$

Case $\alpha > 0$. The construction of H and G is the same as in the case $\alpha < 0$. The only difference is that we do not make use of $-y_0$, i.e. only one initial point is necessary. Although we reach $x = 0$ in a finite number of steps for all $\alpha > 0$, the number of these steps tends to infinity as $\alpha > 0$ tends to zero.

Since $|(f^1)'_x(y;\alpha)| \le 1$ for all $y \in [y_0, 0]$ we arrive at the following estimate:

$$\sup_{y\in[y_0,y_{k+1}]} |H(y,\alpha) - G(h,y,\alpha)| \le K_8 h^p + (a_{k-1} + \cdots + a_0) K_5 h^p$$

where $a_k = |y_k|^2 = |g^k(y_0;\alpha)|^2$. For $y_k \le 0$ we show that $a_k \ge |g^k(y_0;0)|^2$. But this holds because $g^k(y_0;\alpha) > g^k(y_0;0)$. (Case $k = 0$ is clear ($\alpha > 0$). By induction, using that $g(x;\alpha)$ is monotone increasing, we have that $g^{k+1}(y_0;\alpha) = g^k(y_0;\alpha) + \alpha + a(g^k(y_0,\alpha))^2 > g^k(y_0;0) + \alpha + a(g^k(y_0,0))^2 > g^k(y_0;0) + a(g^k(y_0,0))^2 = g^{k+1}(y_0,0)$.) Thus $a_{k-1} + \cdots + a_0 \le \sum_{k=0}^{\infty} |g^k(y_0;0)|^2$ and as a result

$$\sup_{y\in[y_0,0]} |H(y,\alpha) - G(h,y,\alpha)| \le K_{10} h^p$$

which completes the proof of the Theorem. □

We end this section with a consequence of Theorem 1 claiming that $\Phi(1)$ and $\phi^{[1/h]}$ conjugate.

Corollary 1 *$\Phi(1)$ and $\phi^{[1/h]}$ conjugate in a neighborhood of the 0 equilibrium in $\mathbf{R}^n \times \mathbf{R}$.*

Proof. By using the generalized Hartman-Grobman theorem for maps, see e.g. [12], [8], we get that $\Phi(1)$ conjugates with $\Phi_C(1)$ times a standard linear saddle and $\phi^{[1/h]}$ conjugates with $\phi_{C_h}^{[1/h]}$ times a standard linear saddle. Moreover, using the C^j-closeness the linear saddles are the same. From Theorem 1 it follows that $\Phi_C(1)$ and $\phi_{C_h}^{[1/h]}$ conjugate since their normal forms conjugate. Thus we obtain the desired result. □

4 Final remarks

We conjecture that the conjugacy appearing in Corollary 1 is $O(h^p)$-close to the identity. However, we admit that we cannot prove this closeness result by using the techniques of [12] or [8]. On the other hand it is proved that partial linearization, see [1], can be carried out within the order of $O(h^p)$. Moreover, certain invariant foliations (which are the main tool in proving the generalized Hartman-Grobman theorem) are preserved by the numerical method in the C^j-norm to the order of $O(h^p)$, see [4].

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