The numerical computation of homoclinic orbits for maps

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Abstract. Transversal homoclinic orbits of maps are known to generate shift dynamics on a set with Cantor like structure. In this paper a numerical method is developed for computation of the corresponding homoclinic orbits. They are approximated by finite orbit segments subject to asymptotic boundary conditions. We provide a detailed error analysis including a shadowing type result by which one can infer the existence of a transversal homoclinic orbit from a finite segment. This approach is applied to several examples. In some of them parameters appear and closed loops of homoclinic orbits are found by a path-following algorithm.

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1 Introduction

One of the fundamental results on chaotic behavior in discrete dynamical systems is Smale's Homoclinic Theorem, see [23], [21], [22]. For a more recent overview of homoclinic orbits, their bifurcations and the history of their discovery we refer to [19].

Consider a (time-)discrete system

$$x_{n+1} = f(x_n), \quad n \in \mathbb{Z} \tag{1.1}$$

with a C^l-diffeomorphism $f : \mathbb{R}^k \to \mathbb{R}^k$ and assume that $\xi \in \mathbb{R}^k$ is a hyperbolic fixed point of f, i.e. the Jacobian $f'(\xi)$ has no eigenvalues on the unit circle. Further assume that x_0 is a transversal homoclinic point, which means that the orbit generated by (1.1) satisfies

$$\lim_{n\to+\infty} x_n = \lim_{n\to-\infty} x_n = \xi \tag{1.2}$$

and the stable and unstable manifolds of ξ intersect transversally at x_0. Then the theorem states that there exists a compact set M and an integer p such that the p-th iterate of f, denoted by f^p, leaves M invariant and is topologically conjugate on M to the Bernoulli shift on two symbols. It is remarkable that this chaotic behavior on a certain subset is created by a homoclinic point that is transversal, a property that persists under the perturbation of system (1.1).

The perturbation stability of transversal homoclinic points suggests that we should be able to compute these points numerically in a robust and stable way. This is the topic of the

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present paper. Instead of homoclinic points our approach aims at computing the complete homoclinic orbit $x_{\mathbb{Z}} = (x_n)_{n\in\mathbb{Z}}$ by solving the 'boundary value problem' (1.1), (1.2). Any direct method for the single homoclinic point x_0 implicitly tries to solve this boundary value problem and hence is prone to the usual difficulties of shooting-type methods which are caused by exponential divergence of trajectories.

Therefore, we propose to approximate the infinite orbit by a finite orbit segment $x_J = (x_{n_-}, \ldots, x_{n_+})$, $J = \{n_-, \ldots, n_+\}$ which satisfies a 'finite boundary value problem'

$$x_{n+1} = f(x_n), \quad n = n_-, \ldots, n_+ - 1, \tag{1.3}$$

$$b(x_{n_-}, x_{n_+}) = 0. \tag{1.4}$$

Here (1.4) is a general set of boundary conditions defined by a smooth mapping $b : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$. Together, (1.3) and (1.4) comprise a set of $(n_+ - n_- + 1)k$ nonlinear equations for the same number of unknowns. We solve this system by Newton's method and take advantage of the sparsity pattern of the corresponding Jacobian.

The most important examples for (1.4) are periodic boundary conditions

$$b(x_{n_-}, x_{n_+}) = x_{n_+} - x_{n_-} = 0 \tag{1.5}$$

and projection boundary conditions

$$b(x_{n_-}, x_{n_+}) = (b_-(x_{n_-}), b_+(x_{n_-})) = 0 \tag{1.6}$$

where the zero sets of b_- and b_+ are linear approximations to the local unstable and stable manifold of the fixed point.

The whole approach closely mimics the well known methods for approximating homoclinic orbits of parametrized continuous systems $\dot{x} = f(x, \lambda)$ on finite intervals [9]. [4], [2], [11]. But there is one important difference to keep in mind. In the discrete case, stable and unstable manifolds can, and in the generic sense do, intersect transversally at a homoclinic point while in the continuous case both tangent spaces always contain the flow direction. If the manifolds intersect at all in a continuous system they do so in a whole curve obtained by shifting the phase of the orbit. Therefore, one parameter is needed for homoclinic orbits to occur generically in a continuous system and an additional constraint is needed to fix the arbitrary phase.

The paper is organized as follows. In section 2 we collect the necessary prerequisites from the theory of exponential dichotomies for difference equations. This section is largely based on the paper of Palmer [21] where exponential dichotomies are used to prove the Shadowing Lemma and Smale's Homoclinic Theorem. Earlier references on dichotomies for difference equations include [5] and [15].

Section 3 contains the basic results of this paper. In a neighborhood of a transversal homoclinic orbit we find a unique solution of the equations (1.3), (1.4) provided b satisfies a nondegeneracy condition and n_-, n_+ are taken sufficiently large. We also show error estimates which are analogous to the continuous case.

In section 4 we treat several examples including the Poincaré map for the Duffing system. We also compute branches of discrete homoclinic orbits in cases where the system (1.1) contains a parameter. Using a standard continuation procedure (see [1]) we can easily pass through turning points of these branches. At such turning points the homoclinic orbits actually become nontransversal.

In section 5 we take up the question of converse theorems which, given a solution of the finite boundary value problem (1.3), (1.4), guarantee the existence of a transversal homoclinic orbit. This is in the spirit of the recent shadowing type results as in [6], [7] except that here we conclude the existence of an infinite sequence, rather than a continuous orbit, from the knowledge of a finite orbit.

We emphasize that we do not intend a computational verification of the existence of a homoclinic orbit as in [18] where interval analysis is used to verify an exact homoclinic orbit by a pseudo orbit.

2 Exponential dichotomies for difference equations

In this section we review some basic tools from the theory of exponential dichotomies [21] and we add a few results which are useful later on. Consider a homogeneous difference equation in $\mathbb{R}^k$

$$u_{n+1} = A_n u_n, \quad n \in \mathbb{Z}, \quad A_n \in \mathbb{R}^{k,k} \text{ nonsingular} \tag{2.1}$$

with solution operator

$$\Phi(n,m) = \begin{cases} A_{n-1} \cdot \ldots \cdot A_m & \text{if } n > m \\ I & \text{if } n = m \\ A_n^{-1} \cdot \ldots \cdot A_{m-1}^{-1} & \text{if } n < m \end{cases}.$$

In the following let

$$J = \{n \in \mathbb{Z} : n_- \le n \le n_+\}, \quad n_\pm \in \mathbb{Z} \cup \{\pm\infty\}, \quad n_- \le n_+$$

be any interval in $\mathbb{Z}$. If no confusion with real intervals arises we simply write $J = [n_-, n_+]$. We also make frequent use of the Banach space of bounded sequences on J given by

$$S_J = \{u_J = (u_n)_{n\in J} \subset (\mathbb{R}^k)^J : \|u_J\|_\infty := \sup_{n\in J} \|u_n\| < \infty\}.$$

Definition 2.1 The difference equation (2.1) has an **exponential dichotomy on** J if there exist projectors $P_n, (n \in J)$ in $\mathbb{R}^k$ and constants $K, \alpha > 0$ such that

$$P_n \Phi(n,m) = \Phi(n,m) P_m \quad \text{for all } n, m \in J$$

and

$$\begin{aligned} \|\Phi(n,m)P_m\| &\le Ke^{-\alpha(n-m)} \\ \|\Phi(m,n)(I-P_n)\| &\le Ke^{-\alpha(n-m)} \end{aligned} \qquad \text{for all } n \ge m \text{ in } J.$$

For brevity we will say that (2.1) has an exponential dichotomy on J with data (K, α, P_n). It is easy to see that this implies that the adjoint equation

$$\varphi_{n+1} = (A_{n+1}^{-1})^T \varphi_n, \quad n \in \mathbb{Z} \tag{2.2}$$

has an exponential dichotomy on $J-1=\{n_- -1,\dots,n_+ -1\}$ with data $(CK,\alpha,I-P^T_{n+1})$, where C is a constant such that

$$\|A^T\| \le C\|A\| \quad \text{for all } A\in\mathbb{R}^{k,k}.$$

Notice that the solution operator of (2.2) is given by

$$\Phi^*(n,m) := \Phi(m+1,n+1)^T, \quad n,m\in\mathbb{Z}.$$

On semifinite intervals $J=[n_-,\infty)$ the ranges of the projectors P_n are uniquely determined and we can solve inhomogeneous equations of the type

$$u_{n+1}=A_n u_n + r_n, \quad n\ge n_-, \qquad P_{n_-}u_{n_-}=\mu\in\mathcal{R}(P_{n_-}) \tag{2.3}$$

according to the following lemma (cf. [21, Proposition 2.3, Lemma 2.7]).

Lemma 2.2 *Let (2.1) have an exponential dichotomy on $J=[n_-,\infty)$ with data (K,α,P_n). Then for any $n\ge n_-$ we have*

$$\mathcal{R}(P_n)=\{u\in\mathbb{R}^k : \sup_{j\ge n}\|\Phi(j,n)u\|<\infty\}.$$

Suppose that (L,β,Q_n) is another set of dichotomy data on J. Then $\mathcal{R}(Q_n)=\mathcal{R}(P_n)$ holds and we have the estimate

$$\|Q_n-P_n\|\le KLe^{-(\alpha+\beta)(n-n_-)}\|Q_{n_-}-P_{n_-}\|, \quad n\in J.$$

For any $\mu\in\mathcal{R}(P_{n_-})$ and any bounded sequence $r_J\in S_J$ the system (2.3) has a unique bounded solution $u_J\in S_J$ which satisfies

$$\|u_J\|_\infty \le K\left(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\|r_J\|_\infty+\|\mu\|\right). \tag{2.4}$$

Similarly, in the case $J=(-\infty,n_+]$ we have

$$\mathcal{N}(P_n)=\{u\in\mathbb{R}^k : \sup_{j\le n}\|\Phi(j,n)u\|<\infty\}.$$

and the inhomogeneous equation

$$u_{n+1}=A_n u_n + r_n, \quad n\le n_+-1, \qquad (I-P_{n_+})u_{n_+}=\mu\in\mathcal{N}(P_{n_+}) \tag{2.5}$$

has a unique bounded solution which again satisfies (2.4).

In the next step we consider the case $J=\mathbb{Z}$, but merely assume that (2.1) has dichotomies on $\mathbb{Z}_-$ and $\mathbb{Z}_+$ separately. The following lemma is the discrete analogue of [20, Lemma 4.2].

Lemma 2.3 *Let $u_{n+1}=A_n u_n$ have an exponential dichotomy on $\mathbb{Z}_-$ with data (K^-,α^-,P^-_n) and on $\mathbb{Z}_+$ with data (K^+,α^+,P^+_n) and assume that A_n and A_n^{-1} are uniformly bounded. Then the operator*

$$L: \begin{array}{ccc} S_\mathbb{Z} & \to & S_\mathbb{Z} \\ u_\mathbb{Z} & \mapsto & (u_{n+1}-A_n u_n)_{n\in\mathbb{Z}} \end{array}$$

is Fredholm of index $\operatorname{rank}(P_0^+) - \operatorname{rank}(P_0^-)$ *and*

$$\begin{aligned} \dim \mathcal{N}(L) &= \dim(\mathcal{R}(P_0^+) \cap \mathcal{N}(P_0^-)), \\ \operatorname{codim}\mathcal{R}(L) &= \operatorname{codim}(\mathcal{R}(P_0^+) + \mathcal{N}(P_0^-)). \end{aligned} \tag{2.6}$$

Let $L^* : S_{\mathbb{Z}} \to S_{\mathbb{Z}}$ *be defined by*

$$L^* \varphi_{\mathbb{Z}} = (\varphi_{n+1} - (A_{n+1}^{-1})^T \varphi_n)_{n \in \mathbb{Z}},$$

then $\dim \mathcal{N}(L^*) = \operatorname{codim}\mathcal{R}(L)$ *and*

$$r_{\mathbb{Z}} \in \mathcal{R}(L) \iff \sum_{n \in \mathbb{Z}} \varphi_n^T r_n = 0 \quad \textit{for all } \varphi_{\mathbb{Z}} \in \mathcal{N}(L^*). \tag{2.7}$$

Proof. From Lemma 2.2 it follows that any $\mu \in \mathcal{R}(P_0^+) \cap \mathcal{N}(P_0^-)$ defines an element

$$(\Phi(n,0)\mu)_{n \in \mathbb{Z}} \in \mathcal{N}(L)$$

and that this map is an isomorphism, hence

$$\dim \mathcal{N}(L) = \dim(\mathcal{R}(P_0^+) \cap \mathcal{N}(P_0^-)).$$

In a similar way we use the exponential dichotomy of L^* on $(-\infty, -1]$ and on $[-1, \infty)$ with the projectors $I - (P_{n+1}^-)^T$ and $I - (P_{n+1}^+)^T$ respectively. An application of Lemma 2.2 with $n_- = -1$ yields that

$$\eta \mapsto (\Phi^*(n, -1)\eta)_{n \in \mathbb{Z}}$$

is an isomorphism from

$$\begin{aligned} \mathcal{R}(I - (P_0^+)^T) \cap \mathcal{N}(I - (P_0^-)^T) &= \mathcal{R}(P_0^+)^\perp \cap \mathcal{N}(P_0^-)^\perp \\ &= (\mathcal{R}(P_0^+) + \mathcal{N}(P_0^-))^\perp \end{aligned}$$

onto $\mathcal{N}(L^*)$. Therefore

$$\dim \mathcal{N}(L^*) = \operatorname{codim}(\mathcal{R}(P_0^+) + \mathcal{N}(P_0^-)). \tag{2.8}$$

For the proof of (2.7) we first notice that $\sum_{n \in \mathbb{Z}} \varphi_n^T r_n$ exists for all $\varphi_{\mathbb{Z}} \in \mathcal{N}(L^*)$, $r_{\mathbb{Z}} \in S_{\mathbb{Z}}$ since φ_n decays exponentially as $n \to \pm\infty$. Assuming $Lu_{\mathbb{Z}} = r_{\mathbb{Z}}$ for some $u_{\mathbb{Z}} \in S_{\mathbb{Z}}$ we find

$$\sum_{n \in \mathbb{Z}} \varphi_n^T r_n = \sum_{n \in \mathbb{Z}} \varphi_n^T (u_{n+1} - A_n u_n) = \sum_{n \in \mathbb{Z}} (\varphi_{n-1}^T - \varphi_n^T A_n) u_n = 0.$$

For a proof of the converse in (2.7) we let u_n^+, $n \geq 0$ and u_n^-, $n \leq 0$ be the solutions of the initial value problem (2.3) and(2.5) with $n_- = 0$, $P_0^+ u_0^+ = 0$ and $n_+ = 0$, $(I - P_0^-)u_0^- = 0$ respectively.

Next we show

$$u_0^- - u_0^+ \in \mathcal{R}(P_0^+ - (I - P_0^-)). \tag{2.9}$$

Any $\eta \in \mathcal{R}(P_0^+ - (I - P_0^-))^\perp$ satisfies $\eta^T (P_0^+ - (I - P_0^-)) = 0$ and

$$(I - (P_0^+)^T)\eta = (P_0^-)^T \eta \in \mathcal{R}(I - (P_0^+)^T) \cap \mathcal{R}((P_0^-)^T).$$

We conclude that $\varphi_{\mathbb{Z}} = (\Phi^*(n,-1)(P_0^-)^T\eta)_{n\in\mathbb{Z}}$ is an element of $\mathcal{N}(L^*)$ and hence by our assumptions

$$\begin{aligned} 0 &= \sum_{n\in\mathbb{Z}} \varphi_n^T r_n \\ &= \sum_{n\geq 0} \varphi_n^T(u_{n+1}^+ - A_n u_n^+) + \sum_{n\leq -1} \varphi_n^T(u_{n+1}^- - A_n u_n^-) \\ &= \sum_{n\geq 0} (\varphi_n^T - \varphi_{n+1}^T A_{n+1}) u_{n+1}^+ - \varphi_0^T A_0 u_0^+ \\ &\quad + \sum_{n\leq -2} (\varphi_n^T - \varphi_{n+1}^T A_{n+1}) u_{n+1}^- + \varphi_{-1}^T u_0^- \\ &= \varphi_{-1}^T u_0^- - \varphi_{-1}^T u_0^+ \\ &= \eta^T(P_0^- u_0^- - (I - P_0^+) u_0^+) \\ &= \eta^T(u_0^- - u_0^+). \end{aligned}$$

Thus (2.9) holds and there exists some $\mu \in \mathbb{R}^k$ such that

$$(P_0^+ - (I - P_0^-))\mu = u_0^- - u_0^+.$$

From this relation we find that the sequence

$$u_n = \begin{cases} \Phi(n,0)(I - P_0^-)\mu + u_n^-, & n \leq 0 \\ \Phi(n,0)P_0^+\mu + u_n^+, & n \geq 0 \end{cases}$$

is consistently defined at $n = 0$ and satisfies $Lu_{\mathbb{Z}} = r_{\mathbb{Z}}$. Therefore $r_{\mathbb{Z}} \in \mathcal{R}(L)$ holds.

From (2.7) we obtain $\dim\mathcal{N}(L^*) = \operatorname{codim}\mathcal{R}(L)$ as follows. Let $\{\eta^i\}$ be an orthogonal basis of $\mathcal{R}(P_0^+)^\perp \cap \mathcal{N}(P_0^-)^\perp$. Then we know that $\varphi_{\mathbb{Z}}^i = (\Phi^*(n,-1)\eta^i)_{n\in\mathbb{Z}}$ form a basis of $\mathcal{N}(L^*)$ and it is easy to verify with the help of (2.7) that the sequences $u_{\mathbb{Z}}^i = (\delta_{-1,n}\eta^i)_{n\in\mathbb{Z}}$ form a basis in $S_{\mathbb{Z}}/\mathcal{R}(L)$.

Finally, we can compute the Fredholm index of L from (2.6) and (2.8),

$$\begin{aligned} \operatorname{ind}(L) &= \dim\mathcal{N}(L) - \operatorname{codim}\mathcal{R}(L) \\ &= \dim(\mathcal{R}(P_0^+) \cap \mathcal{N}(P_0^-)) - (k - \dim(\mathcal{R}(P_0^+) + \mathcal{N}(P_0^-))) \\ &= \dim\mathcal{R}(P_0^+) + \dim\mathcal{N}(P_0^-) - k \\ &= \operatorname{rank}(P_0^+) - \operatorname{rank}(P_0^-). \end{aligned}$$

■

In the special case $\operatorname{rank}(P_0^+) = \operatorname{rank}(P_0^-)$ the operator L is Fredholm of index 0. If in addition, $\mathcal{N}(L) = \{0\}$ holds then for any $r_{\mathbb{Z}} \in S_{\mathbb{Z}}$ the inhomogeneous system

$$u_{n+1} = A_n u_n + r_n, \quad n \in \mathbb{Z} \tag{2.10}$$

has a unique solution $u_{\mathbb{Z}} \in S_{\mathbb{Z}}$. More precisely by [21, Proposition 2.6, Lemma 2.7] we have

Lemma 2.4 *The difference equation (2.1) has an exponential dichotomy on $\mathbb{Z}$ if and only if it has one on $\mathbb{Z}_-$ and $\mathbb{Z}_+$ separately with projectors of equal rank and if (2.1) has no bounded* *nontrivial solution. In case one of these equivalent assertions holds then the system (2.10) has a unique solution $u_{\mathbb{Z}} \in S_{\mathbb{Z}}$ for each $r_{\mathbb{Z}} \in S_{\mathbb{Z}}$ which satisfies*

$$\|u_{\mathbb{Z}}\|_\infty \le K \frac{1+e^{-\alpha}}{1-e^{-\alpha}} \|r_{\mathbb{Z}}\|_\infty.$$

Here (K, α, P_n) are the dichotomy data on $\mathbb{Z}$.

Finally, we need the following consequence of the roughness theorem, see [21, Proposition 2.10, Remark 2.11].

Proposition 2.5 *Consider two difference equations*

$$u_{n+1} = A_n u_n, \quad u_{n+1} = B_n u_n, \quad n \ge 0$$

with nonsingular matrices $A_n, B_n \in \mathbb{R}^{k,k}$ such that

$$A_n - B_n \to 0 \quad \text{as } n \to \infty. \tag{2.11}$$

Assume that $u_{n+1} = A_n u_n$ has an exponential dichotomy on $\mathbb{Z}_+$ with data (K, α, P_n) and let $0 < \beta < \alpha$. Then the system $u_{n+1} = B_n u_n$ has an exponential dichotomy on $\mathbb{Z}_+$ with suitable data (L, β, Q_n) where Q_n has the same rank as P_n and

$$P_n - Q_n \to 0 \quad \text{as } n \to \infty. \tag{2.12}$$

Proof. Because of (2.11) we can choose N_0 such that the roughness theorem [21, Proposition 2.10, Remark 2.11] yields an exponential dichotomy for $u_{n+1} = B_n u_n$ on any interval $[N, \infty)$, $N \ge N_0$ with data $(K_1, \beta, Q_n^{(N)})$ as well as an estimate

$$\|P_n - Q_n^{(N)}\| \le C_1 \sup_{\nu \ge N} \|A_\nu - B_\nu\| \quad \text{for } n \ge N.$$

We notice that the constants C_1, K_1 and β are independent of N, and hence the projectors P_n and $Q_n^{(N)}$ are of equal rank if N_0 is taken sufficiently large. For a given $\varepsilon > 0$ we may choose $N_1 \ge N_0$ such that

$$C_1 \sup_{\nu \ge N_1} \|A_\nu - B_\nu\| \le \frac{\varepsilon}{2}.$$

An application of Lemma 2.2 with $J = [N_1, \infty)$ yields

$$\|Q_n^{(N_1)} - Q_n^{(N_0)}\| \le K_1^2 e^{-2\beta(n-N_1)} \|Q_{N_1}^{(N_1)} - Q_{N_1}^{(N_0)}\| \quad \text{for } n \ge N_1.$$

Therefore, we can select $N_2 \ge N_1$ such that

$$\|Q_n^{(N_1)} - Q_n^{(N_0)}\| \le \frac{\varepsilon}{2} \quad \text{for } n \ge N_2.$$

By the triangle inequality we obtain

$$\|P_n - Q_n^{(N_0)}\| \le \varepsilon \quad \text{for all } n \ge N_2$$

which proves (2.12). Finally, the exponential dichotomy on $[N_0, \infty)$ easily carries over to $[0, \infty)$ after adjusting the constant K_1 (cf. [21, Remark 2.2]). ■

3 Approximation of transversal homoclinic orbits

Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a C^1-diffeomorphism with a fixed point $\xi \in \mathbb{R}^k$. We call $x_{\mathbb{Z}} = (x_n)_{n\in\mathbb{Z}}$ a **homoclinic orbit with respect to** ξ if the following holds

$$\begin{aligned} x_{n+1} &= f(x_n) \quad \text{for } n \in \mathbb{Z}, \\ \lim_{n\to\pm\infty} x_n &= \xi. \end{aligned} \tag{3.1}$$

We may write (3.1) as an operator equation

$$\Gamma(x_{\mathbb{Z}}) = 0$$

where the C^1-operator $\Gamma : S_{\mathbb{Z}} \to S_{\mathbb{Z}}$ is defined by

$$\Gamma(x_{\mathbb{Z}}) = (x_{n+1} - f(x_n))_{n\in\mathbb{Z}}.$$

In our definition we have included the trivial case $x_n = \xi$ for all $n \in \mathbb{N}$. All the results of the following sections also hold for this trivial case. Of course, in the applications we are only interested in nontrivial orbits.

Assume that ξ is a hyperbolic fixed point and consider $\lambda_s < 1 < \lambda_u$ such that each stable eigenvalue λ of $f'(\xi)$ satisfies $|\lambda| < \lambda_s$ and each unstable one satisfies $|\lambda_u| < |\lambda|$. Let $\mathbb{R}^k = X_s \oplus X_u$ be the corresponding decomposition into the stable and unstable subspaces of $f'(\xi)$ and let us choose the norm in $\mathbb{R}^k$ such that the following holds (see [16])

$$\|x\| = \max(\|x_s\|, \|x_u\|), \quad x = x_s + x_u, x_s \in X_s, x_u \in X_u, \tag{3.2}$$

$$\|f'(\xi)^n x_s\| \le \lambda_s^n \|x_s\|, \quad x_s \in X_s, n \ge 0, \tag{3.3}$$

$$\|f'(\xi)^n x_u\| \le \lambda_u^n \|x_u\|, \quad x_u \in X_u, n \le 0. \tag{3.4}$$

This implies that $u_{n+1} = f'(\xi)u_n$ has an exponential dichotomy on any interval $J \subset \mathbb{Z}$ with solution operator

$$\Phi(n,m) = f'(\xi)^{n-m}$$

and data $(1, \alpha, P_s)$ where

$$\alpha := \min(-\ln\lambda_s, \ln\lambda_u) \tag{3.5}$$

and P_s is the projector onto X_s along X_u.

The local stable and unstable manifolds of ξ are defined as

$$\begin{aligned} W^s_{loc} &:= \{x \in \xi + V : f^n(x) \in \xi + V \text{ for all } n \ge 0 \text{ and } f^n(x) \to \xi \text{ as } n \to \infty\}, \\ W^u_{loc} &:= \{x \in \xi + V : f^{-n}(x) \in \xi + V \text{ for all } n \ge 0 \text{ and } f^{-n}(x) \to \xi \text{ as } n \to \infty\} \end{aligned}$$

where $V \subset \mathbb{R}^k$ is a sufficiently small neighborhood of 0. By definition these sets are positive and negative invariant respectively. It is well known (see [16, Chapter 6]) that we may take $V = V_s \times V_u$ for suitable neighborhoods $V_s \subset X_s$, $V_u \subset X_u$ of the origin such that the local stable and unstable manifold of ξ may be represented as graphs. We get

$$W^s_{loc} = \{\xi + x_s + q_s(x_s) : x_s \in V_s\}, \quad W^u_{loc} = \{\xi + x_u + q_u(x_u) : x_u \in V_u\} \tag{3.6}$$

where the functions $q_s \in C^1(V_s, X_u)$, $q_u \in C^1(V_u, X_s)$ satisfy

$$q_\kappa(0) = 0, \quad q'_\kappa(0) = 0, \quad \kappa = s, u. \tag{3.7}$$

The global stable and unstable manifolds are then given by

$$W^s = \bigcup_{n \leq 0} f^n(W^s_{loc}), \quad W^u = \bigcup_{n \geq 0} f^n(W^u_{loc})$$

with the differential structure induced by $W^{s,u}_{loc}$ via the mapping f.

Theorem 3.1 *Let $x_\mathbb{Z}$ be a homoclinic orbit of (3.1) with respect to a hyperbolic fixed point ξ. Then the following assertions are equivalent.*

(i) x_0 is a transversal homoclinic point, i.e. the tangent spaces to the stable and unstable manifolds at x_0 satisfy

$$T_{x_0} W^s \cap T_{x_0} W^u = \{0\}. \tag{3.8}$$

(ii) The linear system

$$u_{n+1} = f'(x_n) u_n \tag{3.9}$$

has an exponential dichotomy on $\mathbb{Z}$.

(iii) $x_\mathbb{Z}$ is a regular zero of the operator $\Gamma : S_\mathbb{Z} \to S_\mathbb{Z}$, i.e. $\Gamma(x_\mathbb{Z}) = 0$ and $\Gamma'(x_\mathbb{Z}) : S_\mathbb{Z} \to S_\mathbb{Z}$ is a homeomorphism.

Proof. In [21, Definition 4.2] condition (ii) is used to define a transversal homoclinic point and the equivalence with (3.8) is proved later on [21, Proposition 5.6]. The implication (ii)⇒(iii) is immediate from Lemma 2.4 since

$$\Gamma'(x_\mathbb{Z}) u_\mathbb{Z} = (u_{n+1} - f'(x_n) u_n)_{n \in \mathbb{Z}}.$$

Let us assume (iii). Then we have

$$f'(x_n) - f'(\xi) \to 0 \quad \text{as } n \to \pm\infty$$

and Proposition 2.5 shows that (3.9) has exponential dichotomies on both $\mathbb{Z}_-$ and $\mathbb{Z}_+$ with projectors of the same rank. By our assumption equation (3.9) has no nontrivial bounded solution on $\mathbb{Z}$, so Lemma 2.4 proves assertion (ii). ■

Definition 3.2 A homoclinic orbit $(x_n)_{n \in \mathbb{Z}}$ with respect to a hyperbolic fixed point ξ is called **transversal** iff it satisfies one of the equivalent condition of Theorem 3.1.

Next we consider the approximate equations (1.3), (1.4) on a finite interval $J = [n_-, n_+]$ and write them as

$$\Gamma_J(x_J) = 0, \quad x_J \in S_J \tag{3.10}$$

where $\Gamma_J : S_J \to S_J$ is defined by

$$\Gamma_J(x_J) = (x_{n+1} - f(x_n)(n = n_-, \ldots, n_+ - 1), b(x_{n_-}, x_{n_+})).$$

We treat equation (3.10) with the help of the following approximate inverse function theorem (cf. [24, §3], [4, Lemma 3.1], [17, Lemma 4.7])

Proposition 3.3 *Let $F \in C^1(Y, Z)$ with Banach spaces Y, Z and assume that $F'(y_0)$ is a homeomorphism for some $y_0 \in Y$. Further, let constants $\rho, \kappa, \sigma > 0$ be given with the following properties*

$$\begin{aligned} \|F'(y) - F'(y_0)\| \le \kappa < \sigma \le \tfrac{1}{\|F'(y_0)^{-1}\|} \qquad & \text{for all } y \in B_\rho(y_0), \\ \|F(y_0)\| \le (\sigma - \kappa)\rho. \end{aligned} \tag{3.11}$$

Then $F(y) = 0$ has a unique solution in $B_\rho(y_0)$ and the following estimates hold

$$\|y_1 - y_2\| \le \frac{1}{\sigma - \kappa}\|F(y_1) - F(y_2)\| \quad \text{for all } y_1, y_2 \in B_\rho(y_0), \tag{3.12}$$

$$\|F'(y)^{-1}\| \le \frac{1}{\sigma - \kappa} \quad \text{for all } y \in B_\rho(y_0). \tag{3.13}$$

We can now state our main approximation theorem

Theorem 3.4 *Let $\bar{x}_{\mathbb{Z}}$ be a transversal homoclinic orbit with respect to a hyperbolic fixed point ξ of a diffeomorphism $f \in C^1(\mathbb{R}^k, \mathbb{R}^k)$. Assume $b \in C^1(\mathbb{R}^{2k}, \mathbb{R}^k)$ satisfies*

$$b(\xi, \xi) = 0 \tag{3.14}$$

and the map $B \in L(X_s \oplus X_u, \mathbb{R}^k)$ defined by

$$B(x_s + x_u) = D_1 b(\xi, \xi)x_s + D_2 b(\xi, \xi)x_u \tag{3.15}$$

is nonsingular.

Then there exist constants $\delta, C_1, C_2 > 0$ and $N \in \mathbb{N}$ such that (3.10) has a unique solution x_J in

$$B_\delta(\bar{x}_{|J}) = \{x \in S_J : \|\bar{x}_{|J} - x\|_\infty \le \delta\}$$

for all $J = [n_-, n_+]$, $-n_-, n_+ \ge N$ and the following estimates hold

$$\|\Gamma_J'(y_J)^{-1}\|_\infty \le C_1 \quad \text{for all } y_J \in B_\delta(\bar{x}_{|J}), \tag{3.16}$$

$$\|\bar{x}_{|J} - x_J\|_\infty \le C_2\|b(\bar{x}_{n_-}, \bar{x}_{n_+})\|. \tag{3.17}$$

Proof. We apply Proposition 3.3 with

$$Y = Z = S_J, \quad y_0 = \bar{x}_{|J}, \quad F = \Gamma_J.$$

We consider first

$$\Gamma_J'(\bar{x}_{|J})u_J = (u_{n+1} - f'(\bar{x}_n)u_n (n \in J), B_J u_J)$$

where $B_J u_J = D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})u_{n_-} + D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})u_{n_+}$ and show

$$\|\Gamma_J'(\bar{x}_{|J})^{-1}\|_\infty \le C \tag{3.18}$$

for $n_\pm$ sufficiently large. This will be done in two steps.

Step 1: For any sequence z_n, $n \in \tilde{J} = [n_-, n_+ - 1]$ there exist u_n, $n \in J$ such that

$$u_{n+1} - f'(\bar{x}_n)u_n = z_n, \quad n \in \tilde{J} \tag{3.19}$$

and $\|u_J\|_\infty \le C\|z_{\tilde{J}}\|_\infty$.

Step 2: $\Gamma'_J(\bar{x}_{|J})v_J = (0, r)$ has a unique solution $v_J \in S_J$ for each $r \in \mathbb{R}^k$ and $\|v_J\|_\infty \leq C\|r\|$.

Here $C > 0$ denotes some generic constant independent of $n_\pm$ and the right hand sides. The constants are chosen within the proof.

Suppose we have accomplished these two steps. Then for given $z_{\tilde{J}}$ and $r \in \mathbb{R}^k$ we choose u_J as in step 1 and let v_J solve

$$\Gamma'_J(\bar{x}_{|J})v_J = (0, r - B_J u_J)$$

as in step 2. This implies

$$\Gamma'_J(\bar{x}_{|J})(u_J + v_J) = (z_{\tilde{J}}, r)$$

as well as

$$\begin{aligned}
\|u_J + v_J\|_\infty &\leq \|u_J\|_\infty + \|v_J\|_\infty \\
&\leq C(\|z_{\tilde{J}}\|_\infty + \|r - B_J u_J\|_\infty) \\
&\leq C(\|z_{\tilde{J}}\|_\infty + \|r\| + \|B_J\|\,\|u_J\|_\infty) \\
&\leq C(\|z_{\tilde{J}}\|_\infty + \|r\|).
\end{aligned}$$

Thus we have shown the estimate (3.18).

For $y_J \in B_\delta(\bar{x}_{|J})$ and δ sufficiently small we find

$$\begin{aligned}
\|\Gamma'_J(y) - \Gamma'_J(\bar{x}_{|J})\|_\infty &\leq \sup_{n \in J} \|f'(y_n) - f'(\bar{x}_n)\| \\
&\quad + \|D_1 b(y_{n_-}, y_{n_+}) - D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})\| \\
&\quad + \|D_2 b(y_{n_-}, y_{n_+}) - D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})\| \\
&\leq \frac{1}{2C}.
\end{aligned}$$

Taking $\sigma = \frac{1}{C}$, $\kappa = \frac{1}{2C}$ and $\rho = \delta$ in (3.11) we finally obtain from assumption (3.14)

$$\|\Gamma_J(\bar{x}_{|J})\|_\infty = \|b(\bar{x}_{n_-}, \bar{x}_{n_+})\| \leq \frac{\delta}{2C}$$

for $n_\pm$ sufficiently large. Proposition 3.3 then yields the existence of the solution x_J in $B_\delta(\bar{x}_{|J})$ and the estimate (3.17) follows from (3.12) by setting $y_1 = \bar{x}_{|J}$, $y_2 = x_J$.

Proof of step 1. Let (K, α, P_n) be the dichotomy data associated by Theorem 3.1 with the linear system

$$u_{n+1} = f'(\bar{x}_n)u_n, \quad n \in \mathbb{Z}$$

We extend $z_{\tilde{J}}$ by setting $z_n = 0$ for $n \in \mathbb{Z} \setminus \tilde{J}$. Then we use Theorem 3.1(iii) and solve

$$\Gamma'(\bar{x})u_{\mathbb{Z}} = z_{\mathbb{Z}}.$$

Obviously, $u_n (n \in J)$ solves (3.19) and satisfies

$$\|u_{|J}\|_\infty \leq \|u_{\mathbb{Z}}\|_\infty \leq C\|z_{\mathbb{Z}}\|_\infty = C \sup_{n \in \tilde{J}} \|z_n\|.$$

Proof of step 2. We notice that Proposition 2.5 implies

$$P_n \to P_s \quad \text{as } n \to \pm\infty.$$

For $|n|$ large we have $||P_n - P_s|| < 1$, hence the matrices $E_n = I + P_s - P_n$ and $D_n = I - P_s + P_n$ are nonsingular with

$$||E_n^{-1}||, ||D_n^{-1}|| \le \frac{1}{1 - ||P_n - P_s||} \tag{3.20}$$

and we find $\mathbb{R}^k = \mathcal{R}(P_s) \oplus \mathcal{N}(P_n) = \mathcal{N}(P_s) \oplus \mathcal{R}(P_n)$. Therefore, $E_n : \mathcal{R}(P_n) \to \mathcal{R}(P_s) = X_s$ is bijective and satisfies

$$||(I - P_s)E_n^{-1}x_s|| = ||(P_n - P_s)E_n^{-1}x_s|| \le \frac{||P_n - P_s||}{1 - ||P_n - P_s||}||x_s|| \quad \text{for } x_s \in X_s. \tag{3.21}$$

Similarly, $D_n : \mathcal{N}(P_n) \to \mathcal{N}(P_s) = X_u$ is bijective and

$$||P_s D_n^{-1} x_u|| \le \frac{||P_n - P_s||}{1 - ||P_n - P_s||}||x_u|| \quad \text{for } x_u \in X_u. \tag{3.22}$$

Using the solution operator $\Phi(n, m)$ for equation (3.9) we may write v_J as

$$v_n = \Phi(n, n_-)\eta_- + \Phi(n, n_+)\eta_+, \quad n \in J \tag{3.23}$$

where $\eta_- \in \mathcal{R}(P_{n_-})$, $\eta_+ \in \mathcal{N}(P_{n_+})$ are to be determined from the boundary conditions

$$\begin{aligned} r &= B_J v_J \\ &= D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})(\eta_- + \Phi(n_-, n_+)\eta_+) \\ &\quad + D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})(\eta_+ + \Phi(n_+, n_-)\eta_-). \end{aligned} \tag{3.24}$$

By the exponential dichotomy we have

$$||\Phi(n, n_\pm)\eta_\pm|| \le K e^{\pm\alpha(n - n_\pm)}||\eta_\pm||. \tag{3.25}$$

It is convenient to change coordinates in (3.24) via

$$E_{n_-}\eta_- = x_s \in X_s, \quad D_{n_+}\eta_+ = x_u \in X_u. \tag{3.26}$$

We employ the linear map B from (3.15) and rewrite (3.24) as

$$r = B(x_s + x_u) + \rho_+ + \rho_- \tag{3.27}$$

where

$$\begin{aligned} \rho_+ &= (D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+}) - D_1 b(\xi, \xi))x_s \\ &\quad + D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})((I - P_s)E_{n_-}^{-1}x_s + \Phi(n_-, n_+)\eta_+) \\ \rho_- &= (D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+}) - D_2 b(\xi, \xi))x_u \\ &\quad + D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})(P_s D_{n_+}^{-1}x_u + \Phi(n_+, n_-)\eta_-). \end{aligned}$$

Using (3.20), (3.21), (3.22) and (3.25), (3.26) we obtain

$$||\rho_+|| + ||\rho_-|| \le \varepsilon_J(||x_s|| + ||x_u||) \quad \text{with } \varepsilon_J \to 0 \text{ as } n_\pm \to \pm\infty.$$

Therefore, assumption (3.15) implies that the linear equation (3.27) has a unique solution (x_s, x_u) which satisfies

$$||x_s|| + ||x_u|| \le C||r||$$

with a constant C independent of $n_\pm$. We then define η_-, η_+ by (3.26) and v_n by (3.23). From (3.20) and the exponential dichotomies we obtain the estimate of step 2. ■

Remark 3.5 *Assuming that* $||\Gamma_J'(x_J)||_\infty$ *is moderate, the constant* C_1 *in (3.16) is the crucial quantity which measures the condition of the nonlinear system (3.10). A more quantitative analysis (see [17]) shows that* C_1 *depends linearly on* $\frac{1}{\gamma}$*, where* γ *is the angle between the stable and unstable manifold at* $\bar{x}_0$*. Therefore, this constant may also be taken as a measure of the transversality of the orbit.*

Inequality (3.17) shows that the approximation error $||\bar{x}_{|J} - x_J||_\infty$ is determined by the boundary error

$$||b(\bar{x}_{n_-}, \bar{x}_{n_+})|| = ||b(\bar{x}_{n_-}, \bar{x}_{n_+}) - b(\xi, \xi)||.$$

We say that the boundary conditions (1.4) are **of order** (p_-, p_+) if there exists a constant C such that for all $-n_-$, n_+ sufficiently large

$$||b(\bar{x}_{n_-}, \bar{x}_{n_+}) - b(\xi, \xi)|| \le C(||\bar{x}_{n_-} - \xi||^{p_-} + ||\bar{x}_{n_+} - \xi||^{p_+}). \tag{3.28}$$

In the hyperbolic case we have

$$||\bar{x}_n - \xi|| \le C\lambda_s^n,\ n \ge 0\ ||\bar{x}_n - \xi|| \le C\lambda_u^n,\ n \le 0 \tag{3.29}$$

where $\lambda_s < 1 < \lambda_u$ are the spectral bounds for $f'(\xi)$ as in the beginning of this section.

From (3.17), (3.28) and (3.29) we then obtain the estimate

$$||\bar{x}_{|J} - x_J||_\infty \le C(\lambda_s^{p_+ n_+} + \lambda_u^{p_- n_-}).$$

Clearly, the periodic boundary conditions (1.5) are of order $(1, 1)$ since

$$||b(\bar{x}_{n_-}, \bar{x}_{n_+})|| = ||\bar{x}_{n_+} - \bar{x}_{n_-}|| \le ||\bar{x}_{n_+} - \xi|| + ||\bar{x}_{n_-} - \xi||.$$

Moreover, they always satisfy conditions (3.14) and (3.15).

Approximations of second order are obtained from **projection boundary conditions** given by

$$b(x_{n_-}, x_{n_+}) = (b_-(x_{n_-}), b_+(x_{n_+})) = (Q_s^T(x_{n_-} - \xi), Q_u^T(x_{n_+} - \xi)) \tag{3.30}$$

where the columns of $Q_s \in \mathbb{R}^{k,k_s}$ and $Q_u \in \mathbb{R}^{k,k_u}$, $k_s + k_u = k$ provide a basis of the stable and unstable subspace of $f'(\xi)^T$ respectively. More formally, we have a block diagonalization

$$\begin{pmatrix} Q_s^T \\ Q_u^T \end{pmatrix} f'(\xi) = \begin{pmatrix} L_s & 0 \\ 0 & L_u \end{pmatrix} \begin{pmatrix} Q_s^T \\ Q_u^T \end{pmatrix} \tag{3.31}$$

where $|\lambda| < \lambda_s < 1$ for all eigenvalues λ of $L_s \in \mathbb{R}^{k_s,k_s}$ and $1 < \lambda_u < |\lambda|$ for all eigenvalues λ of $L_u \in \mathbb{R}^{k_u,k_u}$. Transforming (3.31) into

$$f'(\xi)(E_s\, E_u) = (E_s\, E_u) \begin{pmatrix} L_s & 0 \\ 0 & L_u \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} Q_s^T \\ Q_u^T \end{pmatrix} (E_s\, E_u) = I$$

we find for the projectors

$$P_s = E_s Q_s^T, \quad P_u = E_u Q_u^T.$$

Therefore, the projection boundary conditions require

$$x_{n_-} \in \xi + X_u, \quad x_{n_+} \in \xi + X_s$$

which means that x_{n_-}, x_{n_+} lie in the linear approximation of the stable and unstable manifold at ξ.

The second order easily follows from (3.7). For $f \in C^l(\mathbb{R}^k, \mathbb{R}^k)$ the functions q_s, q_u in (3.6) are C^l-smooth. Hence for $l \geq 2$ we find for $x \in W^u_{lox}$ by a Taylor expansion

$$\begin{aligned} \|Q_s^T(x-\xi)\| &= \|Q_s^T(x_u + q_u(x_u))\| = \|Q_s^T q_u(x_u)\| \\ &\leq C\|x_u\|^2 \leq C(\|x-\xi\| + \|q_u(x_u)\|)^2 \leq C\|x-\xi\|^2 \end{aligned}$$

and a similar estimate holds for $x \in W^s_{loc}$.

We notice that the projection boundary conditions also satisfy (3.14), (3.15) since

$$B(x_s + x_u) = \begin{pmatrix} Q_s^T x_s \\ Q_u^T x_u \end{pmatrix}, \quad x_s \in X_s = \mathcal{R}(E_s),\ x_u \in X_u = \mathcal{R}(E_u)$$

is nonsingular.

Corollary 3.6 *Let $\bar{x}_{\mathbb{Z}}$ be a transversal homoclinic orbit as in Theorem 3.4. Then the conclusions of Theorem 3.4 hold for periodic as well as for projection boundary conditions. The corresponding finite orbits $x^\pi_{[n_-,n_+]}$ (periodic) and $x^p_{[n_-,n_+]}$ (projection) satisfy for all $n \in \{n_-, \ldots, n_+\}$*

$$\|\bar{x}_n - x^\pi_n\| \leq C(\lambda_s^{n_+} + \lambda_u^{n_-}), \quad \|\bar{x}_n - x^p_n\| \leq C(\lambda_s^{2n_+} + \lambda_u^{2n_-}).$$

For the second estimate we assume $f \in C^2$.

If, in the periodic case , we take $n_\pm = \pm N$ then (x_0^π, x_N^π) is a period-2 orbit of f^N. It is contained in the invariant set M on which f^N exhibits shift dynamics according to Smale's Theorem.

In conclusion of this section we outline the extension of our results to orbits that connect two fixed points of a parametrized system

$$x_{n+1} = f(x_n, \mu), \quad n \in \mathbb{Z}, f \in C^\infty(\mathbb{R}^k \times \mathbb{R}^p, \mathbb{R}^k) \tag{3.32}$$

where $f(\cdot, \mu)$ is a C^1-diffeomorphism for all $\mu \in \mathbb{R}^p$.

Assume that $(\bar{x}_{\mathbb{Z}}, \bar{\mu}) \in S_{\mathbb{Z}} \times \mathbb{R}^p$ is a solution of (3.32) such that

$$\bar{x}_n \to \xi_\pm \quad \text{as } n \to \pm\infty \tag{3.33}$$

where $\xi_\pm$ are hyperbolic fixed points of $f(\cdot, \bar{\mu})$ with stable and unstable dimensions $k_{\pm s}$, $k_{\pm u} = k - k_{\pm s}$. Introducing the operator

$$\Gamma: \begin{array}{ccc} S_{\mathbb{Z}} \times \mathbb{R}^p & \to & S_{\mathbb{Z}} \\ (x_{\mathbb{Z}}, \mu) & \mapsto & (x_{n+1} - f(x_n, \mu))_{n\in\mathbb{Z}} \end{array}$$

we find

$$\Gamma'(\bar{x}_{\mathbb{Z}}, \bar{\mu})(x_{\mathbb{Z}}, \mu) = Lx_{\mathbb{Z}} - R\mu$$

where

$$Lx_{\mathbb{Z}} = (x_{n+1} - f_x(\bar{x}_n, \bar{\mu})x_n)_{n\in\mathbb{Z}}, \quad R\mu = (f_\mu(\bar{x}_n, \bar{\mu})\mu)_{n\in\mathbb{Z}}.$$

By Proposition 2.5 we have exponential dichotomies for L on $\mathbb{Z}_\pm$ with data $(K^\pm, \alpha^\pm, P_n^\pm)$ and $\operatorname{rank} P_n^\pm = k_{\pm s}$. By Lemma 2.3 the operator is Fredholm of index $k_{+s} - k_{-s}$ in the space $S_\mathbb{Z}$ and hence by the bordering lemma ([4]) $\Gamma'(\bar{x}_\mathbb{Z}, \bar{\mu})$ is Fredholm of index $k_{+s} - k_{-s} + p$. We want $(\bar{x}_\mathbb{Z}, \bar{\mu})$ to be a regular zero of Γ which leads us to the following definition.

Definition 3.7 The connecting orbit $(\bar{x}_\mathbb{Z}, \bar{\mu})$ is called **transversal**, if the following conditions hold

(i) $p = k_{-s} - k_{+s} \geq 0$,

(ii) $Lx_\mathbb{Z} + R\mu = 0, x_\mathbb{Z} \in S_\mathbb{Z}, \mu \in \mathbb{R}^p \Rightarrow x_\mathbb{Z} = 0, \mu = 0$.

Condition (i) yields that $\Gamma'(\bar{x}_\mathbb{Z}, \bar{\mu})$ is Fredholm of index 0 and (ii) implies that its null space is trivial. We notice that this definition is in accordance with the homoclinic case where $k_{-s} = k_{+s}$, $p = 0$, see Definition 3.2.

Assuming (i) we may express (ii) in Definition 3.7 equivalently by the following two conditions

$$\mathcal{R}(P_0^+) \cap \mathcal{N}(P_0^-) = \{0\}, \tag{3.34}$$

the matrix

$$E = \sum_{n \in \mathbb{Z}} \Phi_n^T f_\mu(\bar{x}_n, \bar{\mu}) \in \mathbb{R}^{p,p} \tag{3.35}$$

is nonsingular where the columns of $\Phi_n \in \mathbb{R}^{k,p}$, $(n \in \mathbb{Z})$ form a basis of $\mathcal{N}(L^*)$.

It is clear that (ii) implies $\mathcal{N}(L) = \{0\}$ and hence (3.34) by (2.6). From (i) and Lemma 2.3 we then find

$$\dim \mathcal{N}(L^*) = \operatorname{codim} \mathcal{R}(L) = k - (k_{+s} + (k - k_{-s})) = p$$

and (3.35) is a consequence of (ii) and (2.7). Conversely, assume (3.34), (3.35) and let $Lx_\mathbb{Z} + R\mu = 0$ hold for some $x_\mathbb{Z} \in S_\mathbb{Z}$, $\mu \in \mathbb{R}^p$. Then (2.7) implies $E\mu = 0$ and hence $\mu = 0$ by (3.35). Therefore, $x_\mathbb{Z} \in \mathcal{N}(L)$ holds and (3.34), (2.6) yield $x_\mathbb{Z} = 0$.

For the extension of Theorem 3.4 to the general case (3.32), (3.33) we consider the approximate system

$$\Gamma_J(x_J, \mu) := (x_{n+1} - f(x_n, \mu)(n = n_-, \dots, n_+), b(x_{n_-}, x_{n_+}, \mu)) = 0 \tag{3.36}$$

where $b \in C^1(\mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^p, \mathbb{R}^{k+p})$.

Theorem 3.8 *Let $(\bar{x}_\mathbb{Z}, \bar{\mu})$ be a transversal connecting orbit of the parametrized system (3.32) with hyperbolic endpoints $\xi_\pm$. Assume $b \in C^1(\mathbb{R}^{2k+p}, \mathbb{R}^{k+p})$ satisfies*

$$b(\xi_-, \xi_+, \bar{\mu}) = 0$$

and the map $B \in L(X_s^- \times X_u^+, \mathbb{R}^{k+p})$ defined by

$$B(x_s, x_u) = D_1 b(\xi_-, \xi_+, \bar{\mu}) x_s + D_2 b(\xi_-, \xi_+, \bar{\mu}) x_u, \quad x_s \in X_s^-, x_u \in X_u^+$$

is nonsingular. Here X_s^- is the stable subspace of $f_x(\xi_-, \bar\mu)$ and X_u^+ is the unstable subspace of $f_x(\xi_+, \bar\mu)$.

Then there exist constants δ, $C > 0$ and $N \in \mathbb{N}$ such that (3.36) has a unique solution (x_J, μ_J) in $B_\delta((\bar x_{|J}, \bar\mu))$ for all $J = [n_-, n_+]$, $-n_-, n_+ \geq N$ and the following estimate holds

$$||\bar x_{|J} - x_J||_\infty + ||\bar\mu - \mu_J|| \leq C||b(\bar x_{n_-}, \bar x_{n_+}, \bar\mu)||.$$

The proof is quite similar to that of Theorem 3.4 and will be omitted. We notice that B can be represented by a quadratic matrix, since by the transversality condition (i)

$$\dim(X_s^- \times X_u^+) = k_{-s} + k_{+u} = k_{-s} - k_{+s} + k = p + k.$$

The projection boundary condition (3.30) now depends on the parameter μ

$$b(x_{n_-}, x_{n_+}, \mu) = (Q_s(\mu)^T(x_{n_-} - \xi_-(\mu)), Q_u(\mu)^T(x_{n_+} - \xi_+(\mu))).$$

Here $\xi_\pm(\mu)$ are hyperbolic fixed points of $f(\cdot, \mu)$ and the columns of $Q_s(\mu) \in \mathbb{R}^{k,k_{-s}}$, $Q_u(\mu) \in \mathbb{R}^{k,k_{+u}}$ form bases of the stable subspace of $f_x(\xi_-(\mu), \mu)^T$ and the unstable subspace of $f_x(\xi_+(\mu), \mu)^T$ and all the quantities depend smoothly on μ.

So far we considered the parameters μ in (3.32), (3.33) to be part of the unknowns which have to be determined by the numerical calculation, that is by Newton's method applied to (3.36). If there are more than $k_{-s} - k_{+s}$ parameters in the system then we can use these for the continuation of connecting orbits of type (k_{-s}, k_{+s}). This will be done in the next section for the continuation of homoclinic orbits.

4 Numerical implementation and applications

In the following examples we compute approximate homoclinic orbits by applying Newton's method to the nonlinear system

$$\Gamma_J(y_J) := (b(y_{n_-}, y_{n_+}), y_{n+1} - f(y_n)(n = n_-, \ldots, n_+ - 1)) = 0. \tag{4.1}$$

This is justified by Theorem 3.4.

An iteration vector $y_J \in S_J$ for (4.1) is accepted if for a given tolerance $\varepsilon > 0$ the inequality

$$||\Gamma_J'(y_J)^{-1}\Gamma_J'(y_J)||_\infty \leq \varepsilon(1 + ||y_J||_\infty)$$

holds. In each Newton step we use LU decomposition with absolute column pivoting and take advantage of the sparsity pattern of $\Gamma_J'(y_J)$. With exception of Example 2 (Duffing map) we use $\varepsilon = 10^{-15}$. The machine precision used in all computations is about 10^{-16}.

We mainly use projection boundary conditions as in (3.30) except for the first example where we also test the periodic boundary conditions (1.5).

In all examples to follow we have no explicitly given transversal homoclinic orbit. Actually, we do not know of any suitable example with explicit homoclinic orbits and an analytic map f. Therefore, we take a zero of $\Gamma_{[\bar n_-, \bar n_+]}$ with very large values of $-\bar n_-$, $\bar n_+$ as 'exact orbit'. For the error analysis we then compare this orbit to various shorter orbits obtained for moderate values of n_-, n_+. We always use the Euclidean norm in $\mathbb{R}^k$.

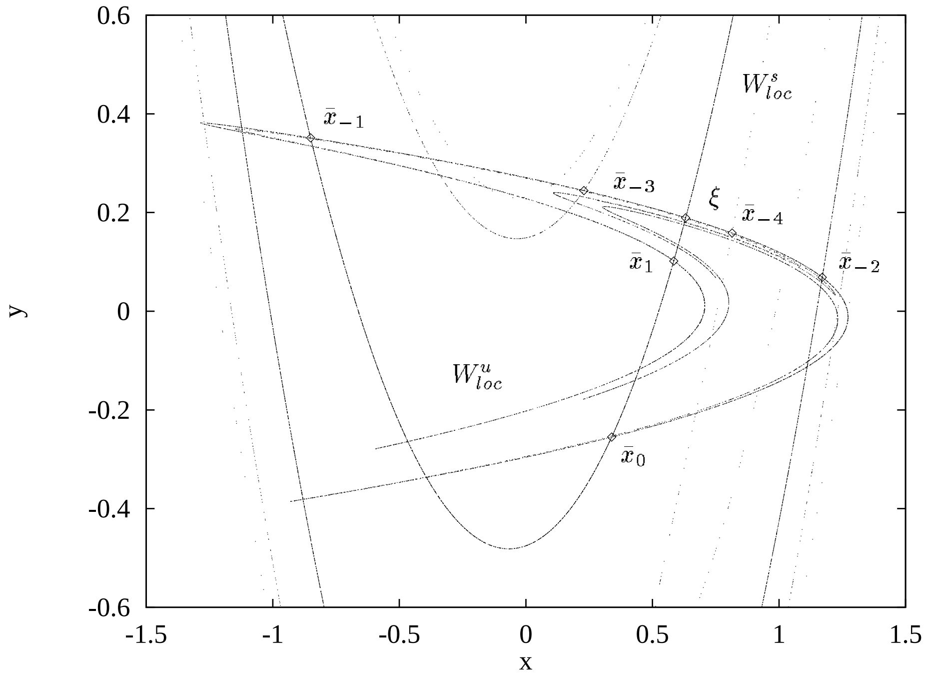

Figure 1: Parts of the stable and unstable manifold for Hénon's map together with points of the 'exact orbit'.

Example 1 (Hénon map)

For $b \neq 0$ the map

$$f(x, y) = (1 + y - ax^2, bx)$$

is a C^∞ diffeomorphism and it has two fixed points for $(b+1)^2 > 4a \neq 0$ given by

$$x_\pm = \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2a}, \quad y_\pm = bx_\pm.$$

In the following we take $\xi = (x_+, y_+)$ and choose the parameters

$$a = 1.4, \quad b = 0.3. \tag{4.2}$$

We get $\xi = (0.631354, 0.189406)$ as hyperbolic fixed point and

$$\sigma(f'(\xi)) = \{0.1559463, -1.923739\}$$

as eigenvalues of $f'(\xi)$. We set $\lambda_s = 0.155947$ and $\lambda_u = 1.92373$. For more details about the Hénon map see [14, §15.2], [8, §2.9].

Figure 1 shows parts of the stable and unstable manifolds of ξ. They are obtained by starting forward and backward iterations on the linear approximation of the unstable and stable manifolds at ξ in some neighborhood of ξ. We show the orbit segment $\bar{x}_{-100,100}$ that was taken as 'exact orbit' for comparison. It is calculated using the starting vector $v_{[-100,100]}$ with

$$v_i = \begin{cases} 0 & \text{if } i = 0 \\ \xi & \text{otherwise} \end{cases}. \tag{4.3}$$

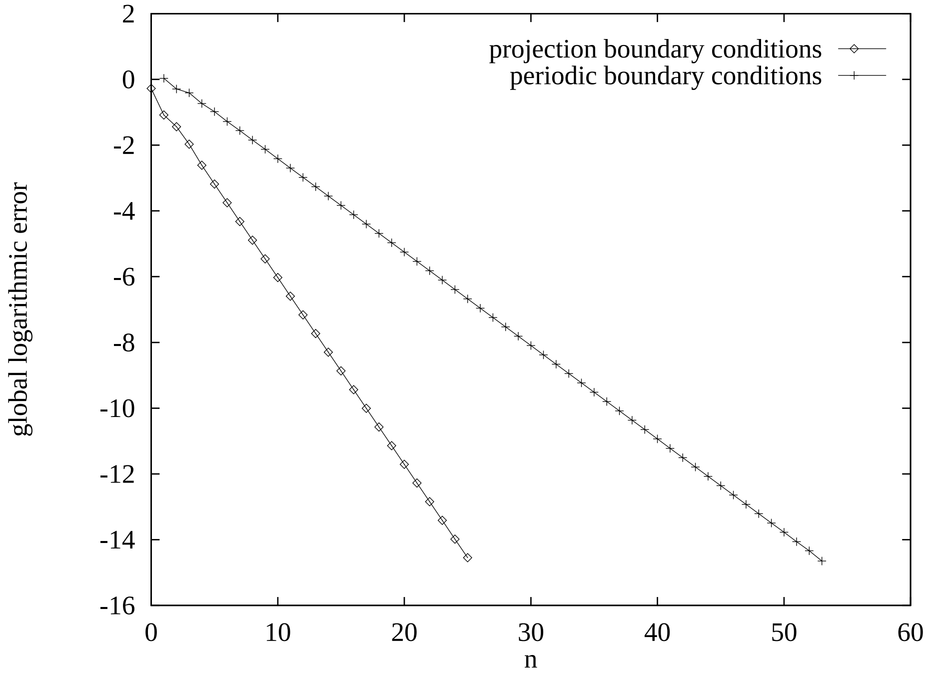

Figure 2: Global error $e_\infty(-n, n)$ with n varied using periodic and projection boundary conditions for Hénon's map.

To analyze the approximation error we compute zeros $y_{[n_-,n_+]}$ of $\Gamma_{[n_-,n_+]}$ starting Newton's methods with the truncated 'exact orbit' $\bar{x}_{[n_-,n_+]}$. For these solutions let

$$\mathrm{e}(i, n_-, n_+) := \log_{10} ||y_i - \bar{x}_i||_\infty$$

be the local logarithmic error at $n_- \leq i \leq n_+$ and let

$$\mathrm{e}_\infty(n_-, n_+) := \max_{n_- \leq i \leq n_+} \mathrm{e}(i, n_-, n_+)$$

be the global logarithmic error.

In Figure 2 we compare the global error $\mathrm{e}_\infty(-n, n)$ obtained for the projection and the periodic boundary conditions. For the projection boundary conditions we get a slope of nearly $2\log_{10}\lambda_u$, while the slope for periodic boundary conditions is nearly $\log_{10}\lambda_u$ as it is predicted by Corollary 3.6. For the remaining tests we always use projection boundary conditions.

Figure 3 shows the behavior of the local error along the approximate homoclinic orbits. The right boundary is fixed and the left boundary is varied. We notice that the maximal error occurs at the boundaries. A more detailed study of the local error at the boundaries is given in Figure 4. The local error $\mathrm{e}(n, n, 4)$ at the left boundary has nearly the expected slope of $2\log_{10}\lambda_u$ while the local error $\mathrm{e}(4, n, 4)$ at the right boundary is nearly constant. So the global error cannot pass below this value $\mathrm{e}(4, n, 4)$. The dependence of the global error on both variables n_- and n_+ is illustrated in Figure 5.

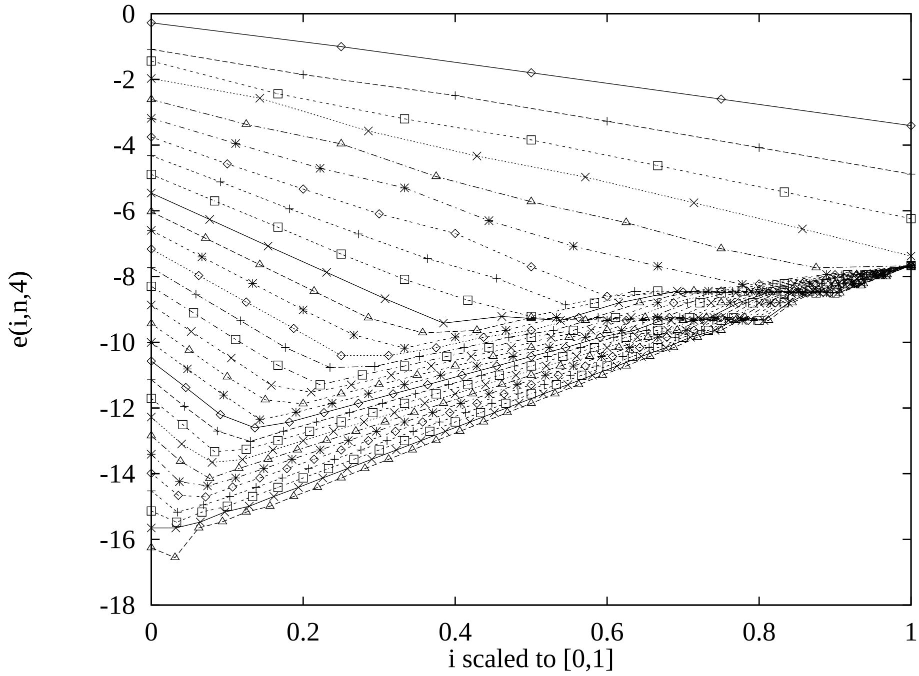

Figure 3: Local error $e(i,n,4)$ with $n = 0,\ldots,29$ varied and i scaled to $[0,1]$ for Hénon's map.

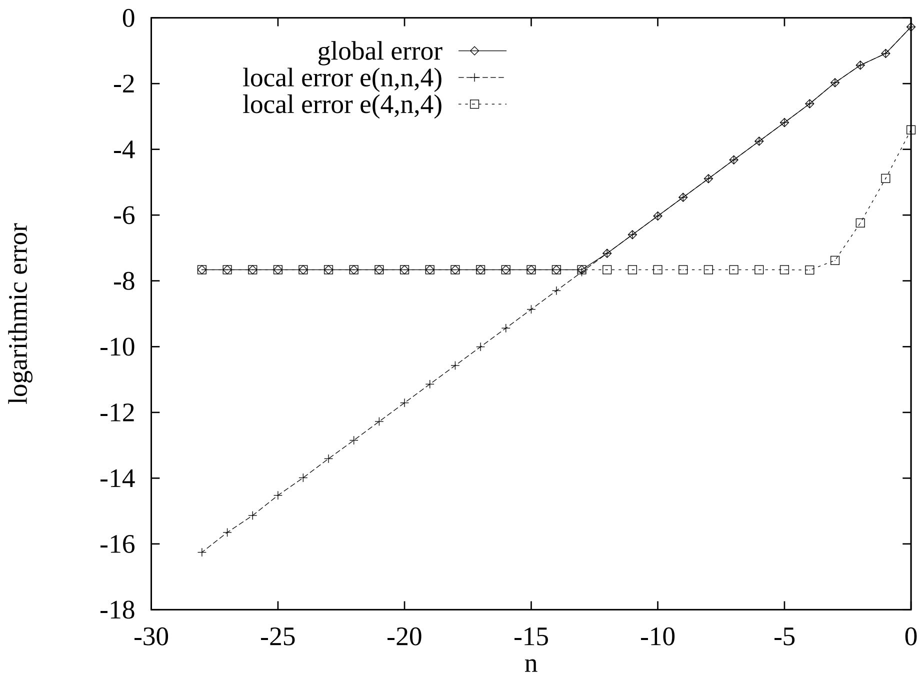

Figure 4: Global error and local errors at the boundaries varying the left boundary for Hénon's map.

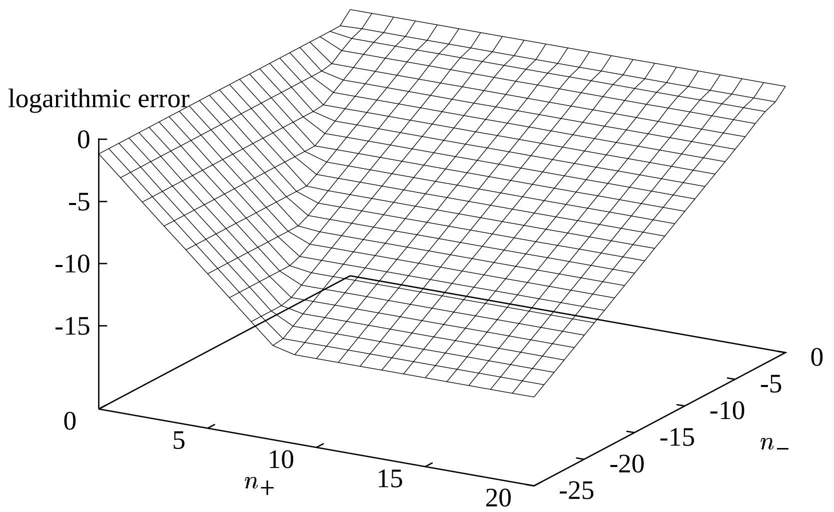

Figure 5: Global error with varied left and right boundary for Hénon's map.

Example 2 (Duffing map)

We consider the forced Duffing equation

$$\dot{u} = v, \quad \dot{v} = u - u^3 - \beta v + \gamma \cos(\omega t) \tag{4.4}$$

with $\omega \neq 0$ and define f as the period map, that is, $f(x, y)$ is the value of the solution of (4.4) at $t = \frac{2\pi}{\omega}$ with initial values $(u(0), v(0)) = (x, y)$. Because of the smoothness of (4.4) the map f is a C^∞-diffeomorphism. We choose the parameters as

$$\beta = 0.25, \quad \gamma = 0.3, \quad \omega = 1.0.$$

The initial value problems are solved with a code called 'ode' from NETLIB. This code is an implementation of an Adams' method by Gordon and Shampine and its tolerance is set to 10^{-15}. The Jacobian of f is calculated by solving the variational equation of (4.4) and the fixed point

$$\xi = (-0.14898822176, 0.018883591437)$$

of f is approximated using Newton's method with tolerance 10^{-14}. For the eigenvalues of $f'(\xi)$ we get the bounds

$$\lambda_s = 0.000901886, \quad \lambda_u = 230.494.$$

For more details about the Duffing map see [13, §2.2].

Due to the extreme values of the Floquet multipliers λ_s λ_u it is sufficient to take $\bar{x}_{[-15,15]}$

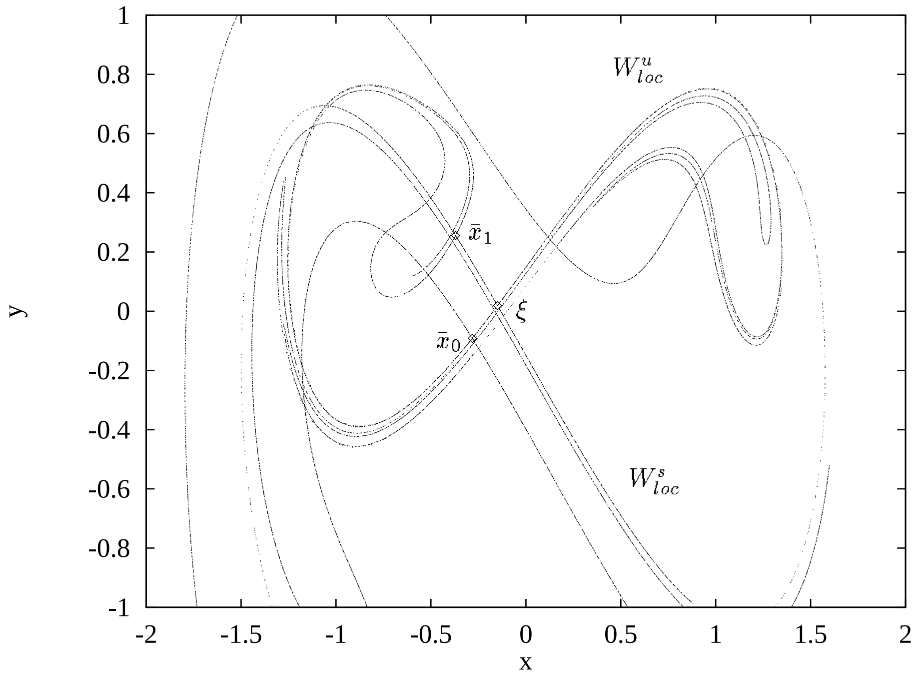

Figure 6: Parts of the stable and unstable manifold together with points of the 'exact orbit' for the Duffing map.

as 'exact orbit'. To compute it, Newton's method is started at $v_{[-15,15]}$ with

$$v_i = \begin{cases} (-0.284, -0.0961) & \text{if } i = 0 \\ (-0.3725, 0.254) & \text{if } i = 1 \\ \xi & \text{otherwise} \end{cases}.$$

We set the tolerance to 10^{-13} and use projection boundary conditions.

In Figure 6 we show parts of the stable and unstable manifolds of ξ together with the 'exact orbit' $\bar{x}_{[-15,15]}$. The behavior of the approximation errors was quite similar to Hénon's map (Figures 2 - 5), so we do not display them here.

We investigate for this example how the error depends on the perturbation of the boundary condition. Let Q_s, Q_u and b_+ be as in (3.30). We set

$$b_-(y_{n_-}, \varphi) = Q_s^T \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} (y_{n_-} - \xi)$$

and use the 'rotated' boundary condition

$$b_\varphi(y_{n_-}, y_{n_+}) = (b_-(y_{n_-}, \varphi), b_+(y_{n_+})).$$

Calculating zeros of $\Gamma_{[-2,5]}$ for various values of φ gives the global errors shown in Figure 7. We observe a sharp peak at $\varphi = 0$ which clearly shows that the higher order convergence from Section 3 only occurs at the accurate projection angle.

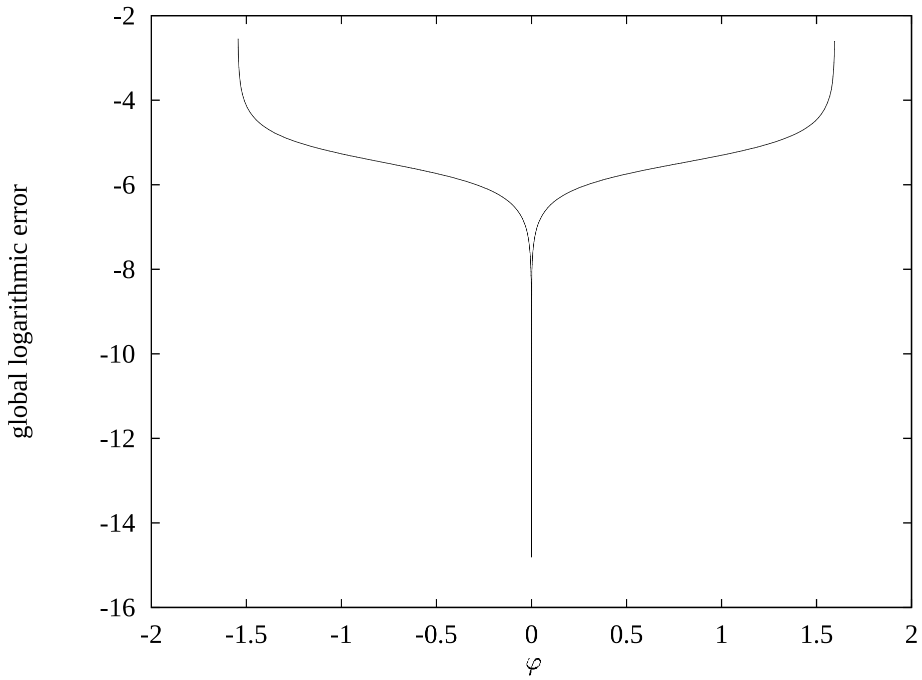

Figure 7: Global error $e_\infty(-2,5)$ using a perturbed left projection boundary condition for the Duffing map.

Example 3 (3D-Hénon map)

Changing variables $x \mapsto \frac{\tilde{x}}{a}$, $y \mapsto \frac{b\tilde{y}}{a}$ for $a \neq 0$ Hénon's map is transformed to $\tilde{f}(\tilde{x},\tilde{y}) = (a + b\tilde{y} - \tilde{x}^2, \tilde{x})$, see [14, Examples 15.11]. We define a **3D-Hénon map** as

$$f(x,y,z) = (a + bz - x^2, x, y).$$

As the original Hénon map it is a C^∞-diffeomorphism for $b \neq 0$ and has two fixed points for $(b+1)^2 > 4a \neq 0$ given by

$$x_\pm = \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2}, \quad z_\pm = y_\pm = x_\pm.$$

We use the same parameter values as above, see (4.2), and consider the fixed point

$$\xi = (0.883896, 0.883896, 0.883896).$$

This fixed point is hyperbolic with eigenvalues

$$\sigma(f'(\xi)) = \{0.374238, -0.483270, -1.65876\},$$

so we use

$$\lambda_s = 0.48328, \quad \lambda_u = 1.6587.$$

For these calculations we use projection boundary conditions and the 'exact orbit' is computed by starting at $v_{[-200,200]}$ defined as in (4.3). Some numbers are given in Table 1.

In Figure 8 we show the global error and the local errors at the boundaries when both boundaries are varied, like Figure 4 for the original Hénon. We notice that the slope of $e(-n,-n,n)$ is $2\log_{10}\lambda_u$ while the slope of $e(n,-n,n)$ is oscillating around $2\log_{10}\lambda_s$.

n	$\bar{x}_n$	$\bar{y}_n$	$\bar{z}_n$
-6	0.807797	0.928589	0.856507
-5	1.004416	0.807797	0.928589
-4	0.669725	1.004416	0.807797
-3	1.193808	0.669725	1.004416
-2	0.276148	1.193808	0.669725
-1	1.524660	0.276148	1.193808
0	-0.566445	1.524660	0.276148
1	1.161985	-0.566445	1.524660
2	0.507189	1.161985	-0.566445
3	0.972826	0.507189	1.161985
4	0.802205	0.972826	0.507189
5	0.908624	0.802205	0.972826
6	0.866250	0.908624	0.802205

Table 1: Some numbers of the 'exact orbit' $(\bar{x}_n, \bar{y}_n, \bar{z}_n)_{n\in\{-200,\ldots,200\}}$ for a 3D-Hénon map.

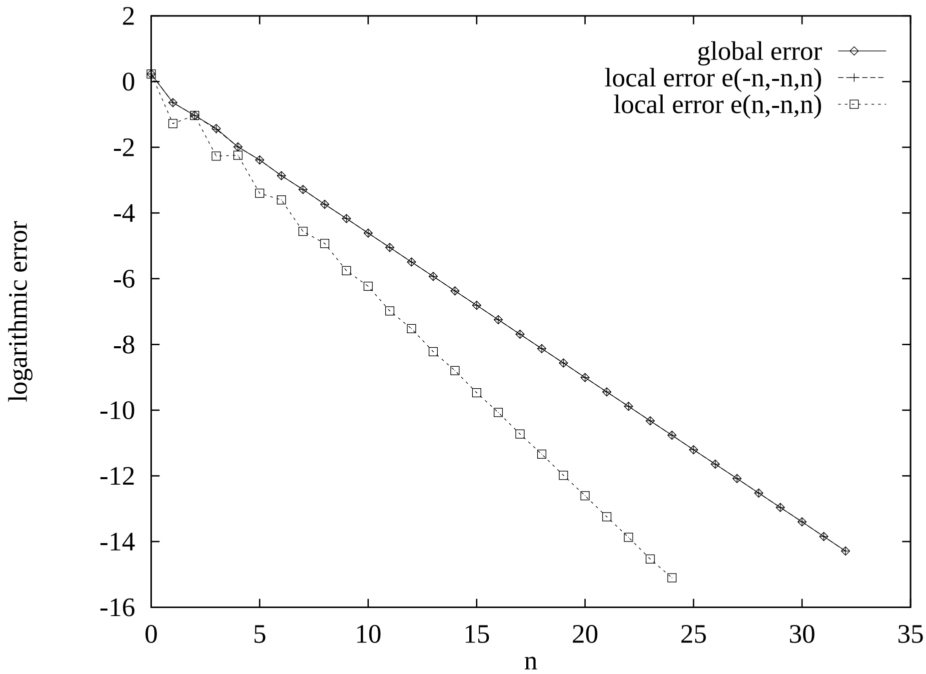

Figure 8: Local errors for the approximate homoclinic orbits, obtained with projection boundary conditions for a 3D-Hénon map. The error behavior for the 3D-Hénon map varying both boundaries.

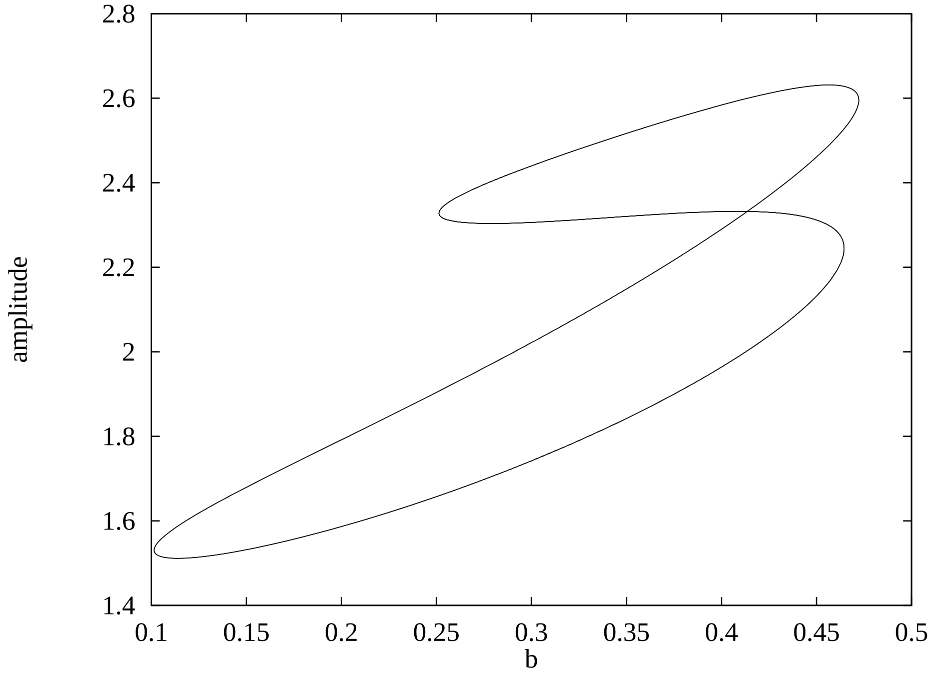

Figure 9: Following a branch for the Hénon map.

Continuation of homoclinic orbits

Let us assume that the map f in Theorem 3.1 depends smoothly on a parameter $\lambda \in \mathbb{R}$ (see also (3.32))

$$x_{n+1} = f(x_n, \lambda), \quad n \in \mathbb{Z}. \tag{4.5}$$

If this system has a transversal homoclinic orbit at some value of λ then it is easy to show by Theorem 3.1 and the implicit function theorem that a local branch of transversal homoclinic orbits passes through the given one. Moreover, by an extension of Theorem 3.4 one can show that there is a corresponding branch of approximating finite orbit segments.

To follow such branches numerically we employ a predictor-corrector method from [1, Algorithm 3.3.7] and take care of the sparse matrix structure, see [1, Chapter 10]. The corrector is of Gauss-Newton type using Moore-Penrose inverses. Its tolerance is set to 10^{-15}.

In the following let

$$\mathrm{Ampl}(y_{[n_-,n_+]}) = \left(\sum_{i=n_-}^{n_+} ||y_i - \xi||^2 \right)^{\frac{1}{2}}$$

be the **amplitude** of an orbit segment $y_{[n_-,n_+]}$.

As a first example we revisit Hénon's map. We take the same parameters and starting vector for the first corrector step as in Example 1. The parameter b is varied and projection boundary conditions are used. Figure 9 shows the result of the numerical computation. In an amplitude versus parameter diagram we get a closed loop of homoclinic orbits. A closer look at the numerical values shows that each point of the homoclinic orbit describes a closed loop in phase space upon variation of b.

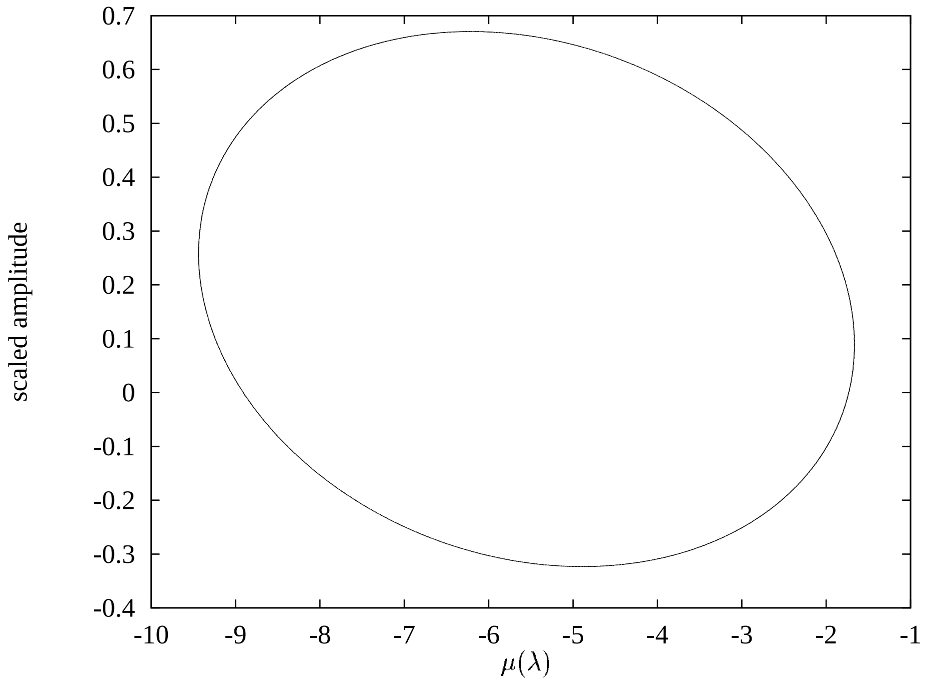

Figure 10: Closed loop of homoclinic orbits for Euler's map. The amplitude is scaled to $(\mathrm{Ampl}(y_{[-300,300]}) - 4.3065375071) \cdot 10^{10}$.

A remarkable feature is that nontransversal homoclinic orbits occur at the turning points of the homoclinic loop. Due to the standard reparametrization during the continuation procedure there are no problems to pass through these points. We notice however, that the approximation properties of the finite orbits close to these turning points are not covered by Theorem 3.4.

Example 4 (Euler map)

Consider the continuous system

$$\dot{u} = v, \quad \dot{v} = u - u^2 + \lambda v + \frac{1}{2}uv. \tag{4.6}$$

Let $f(x, y, \lambda, h)$ be the result of one Euler step for (4.6) with steplength h starting at (x, y), that is

$$f(x, y, \lambda, h) = (x + hy, y + h(x - x^2 + \lambda y + \frac{1}{2}xy)). \tag{4.7}$$

In [3]the question was raised, how such a map behaves when the continuous system passes through a homoclinic bifurcation. For the continuous system (4.6) there exists a homoclinic orbit based at $\xi = (0, 0)$ for $\lambda \approx -0.429505849$, see [4, Example 1].

The above question was largely solved in [10]. There it is shown that transversal homoclinic orbits for the one-step map can occur for all sufficiently small step sizes h and they do so at most within a wedge in (h, λ)-plane that is exponentially small. We will explore this wedge to some extent.

In the following numerical experiment we fix $h = 0.4$ and we start the branch of homoclinic orbits for the Euler map (4.7) at the value $\lambda = -0.7089493$ suggested by [3, §2]. For

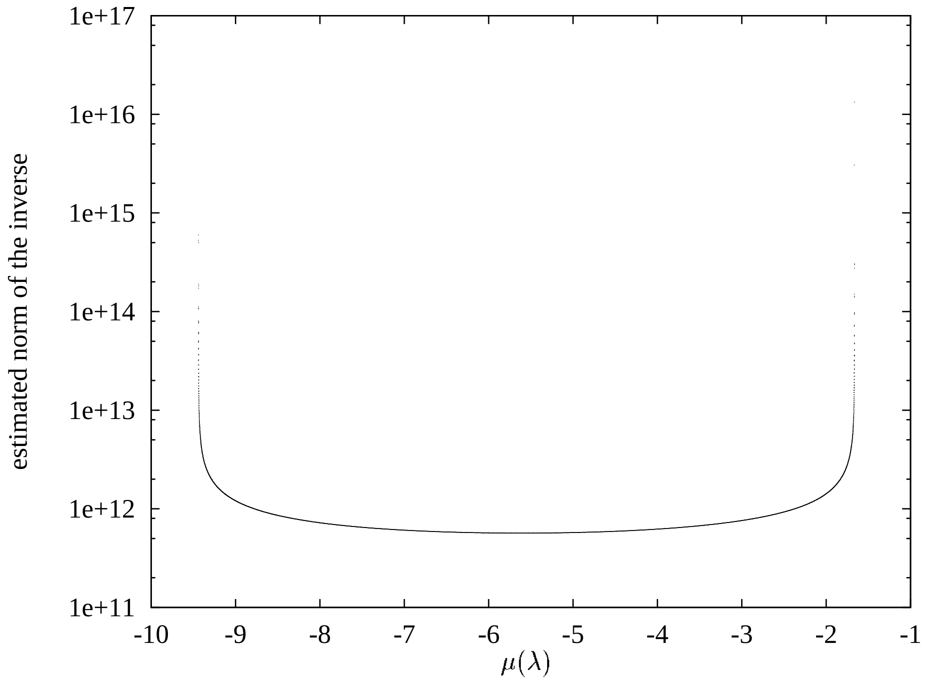

Figure 11: The estimated norms of the inverses of the linearized operator obtained by fixing the parameter λ for the Euler map.

the numerical results shown in Figure 10 we used projection boundary conditions and orbit segments with $-n_- = n_+ = 300$. Since the homoclinic orbits occur only in a very small λ-interval we used

$$\mu(\lambda) = (\lambda + 0.708949271858) \cdot 10^{13}.$$

for the horizontal axis.

Here, as with the Hénon map we find a closed loop of homoclinic orbits. The two turning points mark the boundary of the above mentioned wedge. Although the homoclinic orbits have extremely weak transversality the continuation method has no problems since λ is never taken for parametrization.

In order to demonstrate the weak transversality we estimate (following [12, §3.5.4]) at each point of the branch the norms of the inverses of the linearized operator obtained by fixing the parameter λ, that is

$$\Gamma'_{[n_-,n_+]}(y_{[n_-,n_+]}(\lambda))^{-1}.$$

These estimates are shown in Figure 11. The norms are quite large and, as we expect, they grow towards the turning points. This indicates the small angle between stable and unstable manifolds, see Remark 3.5. In contrast to this, the norms of the inverses of the matrices used to calculate the Moore-Penrose inverses which occur during the continuation are quite moderate (≤ 20).

A remarkable feature of the current loop is that after one turning round with the continuation procedure we get the same orbit in phase space but with the points y_n being shifted to y_{n+1}. In fact this is only true in a strict sense for the infinite orbit $x_{\mathbb{Z}}$ of (4.5) while there

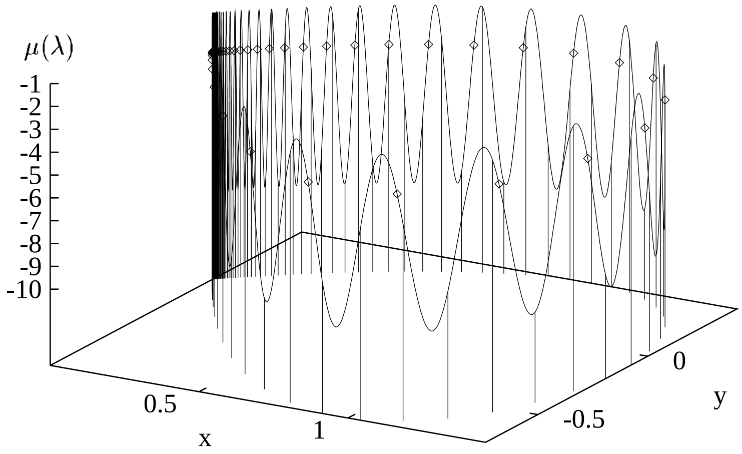

Figure 12: The homoclinic crown: A curve followed by one point of a homoclinic orbit for the Euler map after several turns of the closed loop. The marks indicate the initial homoclinic orbit.

must be a deviation for the finite orbits obtained numerically. However, this deviation is so small that it does not show up in the graphics in Figure 10.

A consequence of the shift after one turn is that all points of the homoclinic orbits lie on a common curve in (x, y, λ)-space. This curve is shown in Figure 12. It was obtained by plotting the points y_{-40} during several turns of the continuation method. If λ is between the two turning points any horizontal cut through this curve gives two orbits which exist for the particular parameter value. For the sake of visualization in Figure 12 we have deactivated the stepsize adaption during the continuation and we have drawn vertical lines at every 40th point.

We finally remark that the same phenomenon of a closed loop with shifted orbits occurs for Hénon's map when we choose parameter values different from those in (4.2), for example $a = 1.4$, $b = -0.3$.

5 A shadowing result for discrete homoclinic orbits

In this section we reverse the question considered in the theorems of section 3. We assume that finite orbit segments which satisfy (3.10) are given and we want to conclude the existence of a transversal homoclinic orbit.

One such result has been derived in [17, §4.4]. It assumes a sequence of solutions x_J which converges uniformly as J grows such that the linearizations about x_J have uniformly bounded inverses, see (3.16). In this way Theorem 3.4 can be turned into an equivalence result.

Our aim here is to avoid these somewhat restrictive assumptions on an infinite sequence of orbits and try to work with a single finite orbit instead. For that purpose we specialize (3.10) to have separated boundary conditions as in (1.6)

$$\Gamma_J(x_J) = (x_{n+1} - f(x_n)(n = n_-, \dots, n_+), b_-(x_{n_-}), b_+(x_{n_+})) = 0. \tag{5.1}$$

We assume that $b_- \in C^1(\mathbb{R}^k, \mathbb{R}^{k_s})$, $b_+ \in C^1(\mathbb{R}^k, \mathbb{R}^{k_u})$ where k_s and k_u are the dimensions of the stable and unstable subspace X_s and X_u of $f'(\xi)$ and ξ is a hyperbolic fixed point of the C^1-diffeomorphism f. Moreover we assume

$$b_-(\xi) = 0, \quad b_+(\xi) = 0, \tag{5.2}$$

$$\mathcal{N}(b'_-(\xi)) = X_u, \quad \mathcal{N}(b'_+(\xi)) = X_s, \tag{5.3}$$

$$b'_-(\xi)_{|X_s}, \quad b'_+(\xi)_{|X_u} \text{ are nonsingular.} \tag{5.4}$$

Assumptions (5.3) and (5.4) imply that the boundary conditions are at least of order $(2, 2)$. In particular, (5.2) - (5.4) are satisfied for the projection boundary conditions (3.30).

In order to avoid interrupting the argument we begin with a perturbation lemma for the linear case which will be an important tool in the proof of our main result.

Lemma 5.1 *Let $A \in \mathbb{R}^{k,k}$ be hyperbolic with stable and unstable projectors P_s and $P_u = I - P_s$ and let $\alpha > 0$ and the norm $\|\,\|$ be chosen so that (3.2), (3.3), (3.4) and (3.5) hold with A instead of $f'(\xi)$. Furthermore, let $\rho > 0$ and matrices A_n, $n \le -1$ be given with*

$$\|A_n - A\| \le \rho \quad \textit{for all } n \le -1.$$

Then any bounded solution $x_{\mathbb{Z}_-}$ of

$$x_{n+1} - A_n x_n = 0, \quad n \le -1$$

satisfies

$$\|x_n\| \le \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \rho \|x_{\mathbb{Z}_-}\|_\infty + e^{\alpha n} \|P_u x_0\| \quad \textit{for all } n \le 0 \tag{5.5}$$

and

$$\|P_s x_0\| \le \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \rho \|x_{\mathbb{Z}_-}\|_\infty.$$

Proof. By Lemma 2.2 with $J = \mathbb{Z}_-$ the equation

$$y_{n+1} - A y_n = (A_n - A) x_n, \; n \le -1, \quad P_u y_0 = 0.$$

has a unique bounded solution $y_{\mathbb{Z}_-}$ and for $n \le 0$ we have the estimate

$$\|y_n\| \le \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \sup_{n \le 0} \|(A_n - A) x_n\| \le \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \rho \|x_{\mathbb{Z}_-}\|_\infty. \tag{5.6}$$

The sequence $z_n = y_n - x_n$, $n \le 0$ is bounded and satisfies $z_{n+1} - A z_n = 0$, $n \le -1$ hence $P_s z_0 = 0$ and

$$\|z_n\| \le e^{\alpha n} \|z_0\| = e^{\alpha n} \|P_u x_0\|, \quad n \le 0.$$

Combining this with (5.6) gives the desired estimates. ∎

Theorem 5.2 *Let the above assumptions hold. Then for any constant $C > 0$ there exists an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$ we find a $\delta = \delta(\varepsilon, C) > 0$ with the following conclusions. If $\bar{x}_J$, $J = [n_-, n_+]$ is a solution of (5.1) which satisfies*

$$\begin{aligned} \|\bar{x}_J\|_\infty &\leq C, & (5.7)\\ \|\Gamma'_J(\bar{x}_J)^{-1}\|_\infty &\leq C, & (5.8)\\ \|\bar{x}_{n_\pm} - \xi\| &\leq \delta & (5.9) \end{aligned}$$

then there exists a unique transversal homoclinic orbit $x_{\mathbb{Z}}$ of f with respect to ξ satisfying

$$\|\bar{x}_J - x_{|J}\| \leq \varepsilon. \tag{5.10}$$

Remark 5.3 1. *There are no explicit assumptions made on $-n_-$, n_+ being large but, of course, the inequality (5.9) requires this in practise. Moreover the homoclinic orbit $x_{\mathbb{Z}}$ may be trivial unless we know that $\|\bar{x}_n - \xi\| > \varepsilon$ holds for at least one $n \in J$.*

2. *As the proof will show, we can weaken the assumption $\Gamma_J(\bar{x}_J) = 0$ by $\|\Gamma_J(\bar{x}_J)\|_\infty \leq \delta$. In this sense the theorem gives a shadowing result (see [21, Theorem 3.5]). It is of a special type, however, since we shadow a finite orbit by an infinite one.*

3. *The assumptions (5.3), (5.4) will be crucial in our proof. It seems likely that condition (5.3) can be dropped and more general boundary conditions can be allowed as in Theorem 3.4 and its converse in [17]. But we don't know of any proof or counterexample.*

Proof. We assume $C > 0$ to be given as in the theorem and we will collect the various conditions on $\varepsilon_0 = \varepsilon_0(C)$ and $\delta = \delta(\varepsilon, C)$, $0 < \varepsilon \leq \varepsilon_0$ during the proof. Let $\delta_1 > 0$ be such that the local stable and unstable manifolds of ξ in the ball

$$B_{\delta_1}(\xi) = \{x \in \mathbb{R}^k : \|x - \xi\| \leq \delta_1\}$$

have a representation as in (3.6) with $V_s = B_{\delta_1}(0) \cap X_s$ and $V_u = B_{\delta_1}(0) \cap X_u$. Let P_s be the projector onto X_s along X_u and $P_u = I - P_s$. By the choice of norms we have $\|P_s\|$, $\|P_u\| \leq 1$.

Then an element $x \in B_{\delta_1}(\xi)$ belongs to W^u_{loc} iff it satisfies the boundary condition

$$P_s(x - \xi) - q_u(P_u(x - \xi)) = 0.$$

Because of (5.4) this is equivalent to

$$B_-(x) := b'_-(\xi)(P_s(x - \xi) - q_u(P_u(x - \xi))) = 0. \tag{5.11}$$

With (5.3) we have

$$B'_-(x) = b'_-(\xi)(P_s - q'_u(P_u(x - \xi))P_u) = b'_-(\xi) - b'_-(\xi)q'_u(P_u(x - \xi))P_u \tag{5.12}$$

and (3.7) implies

$$B'_-(\xi) = b'_-(\xi).$$

Similarly, $x \in W^s_{loc}$ is characterized by

$$B_+(x) := b'_+(\xi)(P_u(x - \xi) - q_s(P_s(x - \xi))) = 0 \tag{5.13}$$

and

$$B'_+(\xi) = b'_+(\xi)$$

holds.

We construct the homoclinic orbit by solving the 'boundary value problem'

$$\tilde{\Gamma}_J(x_J) = 0 \tag{5.14}$$

where $\tilde{\Gamma} : S_J \to S_J$ is defined by

$$\tilde{\Gamma}_J(x_J) = (x_{n+1} - f(x_n)(n = n_-, \ldots, n_+ - 1), B(x_{n_-}, x_{n_+}))$$

with $B(x_{n_-}, x_{n_+}) = (B_-(x_{n_-}), B_+(x_{n_+}))$. We apply Proposition 3.3 with $Y = Z = S_J$ with norm $|| \, ||_\infty$, $F = \tilde{\Gamma}_J$, $y_0 = \bar{x}_J$, $\sigma = \frac{5}{8C}$, $\kappa = \frac{1}{4C}$ and $\rho = \varepsilon$. We have

$$\begin{aligned}
||\tilde{\Gamma}'_J(\bar{x}_J) - \Gamma'_J(\bar{x}_J)||_\infty &= \max(||B'_-(\bar{x}_{n_-}) - b'_-(\bar{x}_{n_-})||, ||B'_+(\bar{x}_{n_+}) - b'_+(\bar{x}_{n_+})||) \\
&\leq \max\big(||b'_-(\xi) - b'_-(\bar{x}_{n_-})|| + ||b'_-(\xi)|| \, ||q'_u(P_u(\bar{x}_{n_-} - \xi))||, \\
&\qquad ||b'_+(\xi) - b'_+(\bar{x}_{n_+})|| + ||b'_+(\xi)|| \, ||q'_s(P_s(\bar{x}_{n_+} - \xi))||\big).
\end{aligned}$$

We choose ε_0 such that $\varepsilon_0 \leq \frac{1}{2}\delta_1$ and such that the following estimates hold for all $x \in B_{2\varepsilon_0}(0)$

$$\begin{aligned}
||b'_-(\xi)|| \, ||q'_u(x)|| \leq \frac{1}{8C}, \qquad & ||b'_+(\xi)|| \, ||q'_s(x)|| \leq \frac{1}{8C}, \\
||b'_-(\xi) - b'_-(\xi + x)|| \leq \frac{1}{4C}, \qquad & ||b'_+(\xi) - b'_+(\xi + x)|| \leq \frac{1}{4C}
\end{aligned} \tag{5.15}$$

where C is the constant in (5.8). For $0 < \varepsilon \leq \varepsilon_0$ and $\delta \leq \varepsilon$ we obtain by using (5.9)

$$||\tilde{\Gamma}'_J(\bar{x}_J) - \Gamma'_J(\bar{x}_J)||_\infty \leq \frac{3}{8C}$$

and thus by (5.8) and the Banach Lemma

$$||\tilde{\Gamma}'_J(\bar{x}_J)^{-1}||_\infty \leq \frac{8}{5}C = \frac{1}{\sigma}.$$

For $||y - \bar{x}_J||_\infty \leq \varepsilon$ we use (5.9), (5.15) and find the estimate

$$\begin{aligned}
||\tilde{\Gamma}'_J(y) - \tilde{\Gamma}'_J(\bar{x}_J)||_\infty &\leq \max\big(||b'_-(\xi)|| \, ||q'_u(P_u(y_{n_-} - \xi)) - q'_u(P_u(\bar{x}_{n_-} - \xi))||, \\
&\qquad ||f'(y_n) - f'(\bar{x}_n)||(n = n_-, \ldots, n_+ - 1), \\
&\qquad ||b'_+(\xi)|| \, ||q'_s(P_s(y_{n_+} - \xi)) - q'_s(P_s(\bar{x}_{n_+} - \xi))||\big) \\
&\leq \frac{1}{4C} = \kappa
\end{aligned}$$

provided ε_0 satisfies

$$\max(||f'(x) - f'(z)|| : ||x|| \leq C, ||x - z|| \leq \varepsilon_0) \leq \frac{1}{4C}.$$

Finally, we obtain

$$\begin{aligned}
||\tilde{\Gamma}_J(\bar{x}_J)||_\infty &= \max(||B_-(\bar{x}_{n_-})||, ||B_+(\bar{x}_{n_+})||) \\
&\leq \max\big(||b'_-(\xi)||(||\bar{x}_{n_-} - \xi|| + ||q_u(P_u(\bar{x}_{n_-} - \xi))||), \\
&\qquad ||b'_+(\xi)||(||\bar{x}_{n_+} - \xi|| + ||q_s(P_s(\bar{x}_{n_+} - \xi))||)\big) \\
&\leq \frac{3\varepsilon}{8C} = (\sigma - \kappa)\rho
\end{aligned}$$

if δ satisfies

$$||b'_-(\xi)||\delta \leq \frac{\varepsilon}{4C}, \quad ||b'_+(\xi)||\delta \leq \frac{\varepsilon}{4C}.$$

Notice that by (5.15) and (5.9)

$$\begin{aligned} ||b'_-(\xi)||\,||q_u(P_u(\bar{x}_{n_-} - \xi))|| &\leq ||b'_-(\xi))||\delta \max(||q'_u(x)|| : ||x|| \leq \delta) \\ &\leq \frac{\delta}{8C} \leq \frac{\varepsilon}{8C}. \end{aligned}$$

From Proposition 3.3 we conclude the existence of a unique solution y_J of (5.14) in $B_\varepsilon(\bar{x}_J)$. Since

$$||y_{n_\pm} - \xi|| \leq ||y_{n_\pm} - \bar{x}_{n_\pm}|| - ||\bar{x}_{n_\pm} - \xi|| \leq \varepsilon + \delta \leq 2\varepsilon_0 \leq \delta_1$$

the extended orbit $y_\mathbb{Z}$ obtained by iterating $y_{n+1} = f(y_n)$, $n < n_-$ or $n \geq n_+$ is homoclinic to ξ. Moreover, any homoclinic orbit $x_\mathbb{Z}$ with the property (5.10) satisfies $||x_{n_\pm} - \xi|| \leq \delta_1$ and hence satisfies the boundary conditions (5.11), (5.13). This proves the uniqueness.

To show that $y_\mathbb{Z}$ is transversal let us assume that it is nontransversal. By Definition 3.2, Theorem 3.1 and Lemma 2.4 there is a sequence $x_\mathbb{Z} \in S_\mathbb{Z}$ such that

$$||x_\mathbb{Z}||_\infty = 1, \quad x_{n+1} - f'(y_n)x_n = 0, \quad n \in \mathbb{Z}. \tag{5.16}$$

Applying Lemma 5.1 to $A = f'(\xi)$, $A_n = f'(y_{n+n_-})$ and with x_{n+n_-} instead of x_n we get

$$||P_s x_{n_-}|| \leq \frac{1+e^{-\alpha}}{1-e^{-\alpha}} \sup_{n \leq n_-} ||f'(y_n) - f'(\xi)|| =: \beta_-. \tag{5.17}$$

In a similar way

$$||P_u x_{n_+}|| \leq \frac{1+e^{-\alpha}}{1-e^{-\alpha}} \sup_{n \geq n_+} ||f'(y_n) - f'(\xi)|| =: \beta_+.$$

By the positive invariance of W^s_{loc} and the negative invariance of W^u_{loc} we have

$$||y_n - \xi|| \leq 2\varepsilon_0 \quad \text{for all } n \leq n_- \text{ or } n \geq n_+.$$

Furthermore

$$\tilde{\Gamma}'_J(y_J)x_{|J} = (0, \ldots, 0, B'_-(y_{n_-})x_{n_-}, B'_+(y_{n_+})x_{n_+}) \tag{5.18}$$

and as a byproduct of Proposition 3.3 (see (3.13))

$$||\tilde{\Gamma}'_J(y_J)^{-1}||_\infty \leq \frac{1}{\rho - \kappa} = \frac{8}{3}C. \tag{5.19}$$

From (5.12), (5.15), (5.16) and (5.17) we obtain

$$||B'_-(y_{n_-})x_{n_-}|| = ||b'_-(\xi)(P_s - q'_u(P_u(y_{n_-} - \xi))P_u)x_{n_-}|| \leq ||b'_-(\xi)||\beta_- + \frac{1}{8C}.$$

Finally, we impose on ε_0 the restriction

$$\frac{1+e^{-\alpha}}{1-e^{-\alpha}} \max_{||x|| \leq 2\varepsilon_0} (||f'(\xi) - f'(\xi + x)||) < \min(\frac{1}{16C||b'_-(\xi)||}, \frac{1}{16C||b'_+(\xi)||}, \frac{1}{2}) \tag{5.20}$$

which yields $||B'_-(y_{n_-})x_{n_-}|| < \frac{3}{16C}$ and similarly $||B'_+(y_{n_+})x_{n_+}|| < \frac{3}{16C}$. Using these estimates and (5.19) in (5.18) we obtain

$$||x_{|J}||_\infty < \frac{1}{2}.$$

In particular, $||P_u x_{n_-}|| < \frac{1}{2}$ and the inequality (5.5) from Lemma 5.1 and (5.20) yield

$$||x_n|| \le \beta_- + e^{\alpha(n-n_-)}||P_u x_{n_-}|| \le \beta_- + \frac{1}{2} < 1 \quad \text{for } n \le n_- .$$

In an analogous way we find $||x_n|| < 1$ for $n \ge n_+$ and thus arrive at a contradiction to $||x_{\mathbb{Z}}||_\infty = 1$. ■

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