Invariant Manifolds for Non-autonomous Systems with Application to One-step Methods

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Abstract

In this paper we study the existence of invariant manifolds for a special type of nonautonomous systems which arise in the study of discretization methods. According to [10], a one-step scheme of step-size ε for an autonomous system can be interpreted as the ε-flow of a perturbed nonautonomous system. The perturbation is 'rapidly forced' in the sense that it is periodic with respect to time with period ε. Assuming a saddle node for the autonomous system, we prove that these rapidly forced perturbations have center manifolds which exist in a uniform neighborhood and which converge to a center manifold of the autonomous system as ε tends to zero. Our results are applied to obtain a smooth continuation as well as estimates of the well known center manifolds for one-step schemes. They also form the basis for studying saddle-node homoclinic orbits under discretization.

Keywords: Invariant manifolds, center manifolds, nonautonomous systems, one-step methods, centered Euler scheme.
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1 Introduction

In recent years many efforts have been directed towards a deeper understanding of the effects caused by discretizing a continuous dynamical system, see [13], [20] for recent reviews.

Consider a smooth dynamical system

$$\dot{x} = f(x),\ x \in \mathbb{R}^m, \tag{1.1}$$

and let $F(t,x)$ be the induced t-flow. We are interested in the longtime behavior of one-step methods

$$x_{n+1} = \Phi(\varepsilon, x_n),\ n = 0, 1, 2, \cdots \tag{1.2}$$

Here ε denotes a constant but small step size and $\Phi(\varepsilon, \cdot)$ is a smooth mapping in $\mathbb{R}^n$ that approximates the ε-flow $F(\varepsilon, \cdot)$ to a certain order

$$|F(\varepsilon, x) - \Phi(\varepsilon, x)| \leq C\varepsilon^{p+1}, \quad \forall x \in \mathbb{R}^m. \tag{1.3}$$

For estimates of this type we refer to [5], [11], [20].

For certain structurally stable systems (1.1), it is possible to set up an ε-dependent topological conjugacy between the maps $F(\varepsilon, \cdot)$ and $\Phi(\varepsilon, \cdot)$ and to estimate its distance to the identity in powers of ε (see [12], [13], [14]). In general such an embedding is impossible and some of the obstructions when periodic orbits are present are discussed in [12], [13], [14].

An important structurally unstable situation occurs if the system (1.1) has a saddle with a homoclinic orbit. This situation was analyzed in detail in [10] for a parametrized family. The basis assumption in [10] is that the homoclinic orbit, as a function of time, has an analytic extension to a strip in the complex plane. Then it is shown that generically the map $\Phi(\varepsilon, \cdot)$ has transversal homoclinic points and that the angle of intersection of the stable and unstable manifolds is exponentially small, that is of the order $O(e^{-c/\varepsilon})$.

The basic tool in [10] is to view the one step map $\Phi(\varepsilon, \cdot)$ of order p as the 0 to ε flow of a perturbed nonautonomous system

$$\dot{x} = f(x) + \varepsilon^p g(\varepsilon, t/\varepsilon, x) \tag{1.4}$$

where g is smooth and 1-periodic in its second variable. The perturbation term in (1.4) then has period ε in t and such perturbations are called **rapidly forced** in [10].

In this paper we analyze the situation near a nonhyperbolic equilibrium of the systems (1.1), (1.2) via the interpolated system (1.4). For simplicity we consider only a saddle node of (1.1) and a special one-step method, the centered Euler scheme or implicit mid-point rule

$$\frac{x_{n+1} - x_n}{\varepsilon} = f(\frac{x_{n+1} + x_n}{2}). \tag{1.5}$$

We show that the resulting rapidly forced system (1.4) has locally invariant (or integral) manifolds which are ε-periodic in t and which connect up to the well known center manifold at $\varepsilon = 0$.

More precisely, the invariant manifold is of the form

$$\tilde{M} = \{(\varepsilon, t, \theta, u); t \in \mathbb{R}, 0 < \varepsilon < \varepsilon_0, u = h(\varepsilon, t/\varepsilon, \theta)\} \tag{1.6}$$

where $x = (\theta, u)$ is the decomposition into the center variable θ (here $\theta \in \mathbb{R}$) and the hyperbolic variable u (here $u \in \mathbb{R}^{m-1}$) determined by the spectrum of the Jacobian of f at the equilibrium. We will show that $h(\varepsilon, \cdot, \theta)$ is 1-periodic in its second variable and converges uniformly to a function $\tilde{h}(\theta)$ as $\varepsilon \to 0$ which defines a center manifold for the unperturbed system (1.1).

In this way we obtain an invariant manifold which reflects the special rapidly forced nature of the perturbation. It seems that such a result does not follow directly from the well known general theory of center manifolds for nonautonomous systems, cf. [2], [21].

Our motivation for deriving this result is twofold. First, if we restrict the invariant manifolds for the interpolated system (1.4) to the time slice $t = 0$ or $t = \varepsilon$ we obtain invariant manifolds of the one-step mapping which approximate the center manifold of the original system (1.1). In this way we recover some well known results on the persistence of center manifolds under discretization, see [5], [18], [22] and [14] for a generalization.

As a second application we use the 'rapidly oscillating' manifold in (1.6) to study the effect of discretization on a saddle node homoclinic orbit, see [8], [19] for the unfolding theory of this case. The details of this application will be contained in a forthcoming paper.

We give a brief outline of the present paper. In section 2 we repeat the construction of the interpolated system from [10] in order to obtain the special properties of the perturbation term g when the original system (1.1) has a saddle node. As an aside we discuss in section 3 the relation of the interpolation (1.4) to the backward error analysis of numerical schemes [7], [9]. Section 4 then contains the main existence proof as well as some estimates as $\varepsilon \to 0$. Section 5 illustrates the main result by an example and, finally, some extensions to parametrized systems and more general one-step schemes are discussed in section 6.

2 Construction and properties of the interpolation

We consider the system (1.1) under the following assumptions.

(H1) f and its derivatives up to order $k \geq 4$ are continuous and uniformly bounded in $\mathbb{R}^m$.

(H2) f has an equilibrium x_0, i.e. $f(x_0) = 0$, without loss of generality we may assume $x_0 = 0$.

(H3) $D_x f(0)$ has the simple eigenvalue 0 with right eigenvector e_r and all other eigenvalues have nonzero real parts.

As a first step we follow the approach of [10] and show that the centered Euler method

$$\frac{x_{n+1} - x_n}{\varepsilon} = f(\frac{x_{n+1} + x_n}{2}) \tag{2.1}$$

may be viewed as the time-ε map of a suitable nonautonomous system

$$\dot{x}(t) = f(x(t)) + \varepsilon^2 g(\varepsilon, t/\varepsilon, x(t)). \tag{2.2}$$

Our aim here is to analyze the properties of the perturbation g under the assumptions (H1) - (H3).

Using (H1) we may solve (2.1) for x_{n+1} uniformly in $x_n \in \mathbb{R}^m$ and $|\varepsilon| < \varepsilon_0$ and write it as

$$x_{n+1} = \Phi(\varepsilon, x_n) \tag{2.3}$$

where $\Phi \in C^k((-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^m, \mathbb{R}^m)$ and $\Phi(\varepsilon, \cdot)$ is a global diffeomorphism. This can be seen from the Hadamard-Levy global inverse function theorem [1] (2.5.17) which applies to

$$T(\varepsilon; x, y) := (x, y - x - \varepsilon f(\frac{x+y}{2})) \text{ if } |\varepsilon f'(x)| \leq \frac{1}{2}, \quad \forall x \in \mathbb{R}^m.$$

Another possibility is to employ Theorem 2.2 below.

The basic approximation properties of the centered Euler map $\Phi(\varepsilon,\cdot)$ and the ε-flow $F(\varepsilon,\cdot)$ will be summarized in the following lemma. More general statements concerning higher derivatives and finite time intervals can be found in [11].

Lemma 2.1 *Assume* (H1), *then there exist constants* $\varepsilon_0, C > 0$ *such that the following estimates hold for all* $|\varepsilon| < \varepsilon_0$, $y \in \mathbb{R}^m$

$$\begin{aligned} |D_yF(\varepsilon,y) - I| + |D_{yy}F(\varepsilon,y)| &\leq C|\varepsilon|, && (2.4)\\ |F(\varepsilon,y) - \Phi(\varepsilon,y)| + |D_yF(\varepsilon,y) - D_y\Phi(\varepsilon,y)| &\leq C|\varepsilon|^3. && (2.5)\end{aligned}$$

For later use in the construction of (2.2) we quote here a global and quantitative version of the implicit function theorem.

Theorem 2.2 *Let* X, Y *be* Banach *spaces,* $\Omega \subset X$ *be open and let* $T \in C^k(Y \times \mathbb{R} \times \Omega, Y)$, $g_0 \in C^k(\Omega, Y)$, $k \geq 1$ *be given such that the following estimates hold for all* $\xi \in \Omega$, $|\varepsilon| < \varepsilon_0$ *with suitable constants* $\sigma, \kappa, \delta > 0$

$$\begin{aligned} &D_yT(g_0(\xi),0,\xi) \text{ is a homeomorphism and}\\ &\qquad |D_yT(g_0(\xi),0,\xi)^{-1}| \leq \sigma, && (2.6)\\ &|D_yT(y,\varepsilon,\xi) - D_yT(g_0(\xi),0,\xi)| \leq \kappa < \frac{1}{\sigma}, \text{ for } |y - g_0(\xi)| \leq \delta, && (2.7)\\ &|T(g_0(\xi),\varepsilon,\xi)| \leq (\frac{1}{\sigma} - \kappa)\delta. && (2.8)\end{aligned}$$

Then there is a unique function $g \in C^k((-\varepsilon_0,\varepsilon_0) \times \Omega, Y)$ *such that*

$$T(g(\varepsilon,\xi),\varepsilon,\xi) = 0 \text{ and } |g(\varepsilon,\xi) - g_0(\xi)| \leq \delta$$

and for $|\varepsilon| < \varepsilon_0$, $\xi \in \Omega$ *we have the estimate*

$$|D_\xi g(\varepsilon,\xi)| \leq \frac{\sigma}{1-\sigma\kappa}|D_\xi T(g(\varepsilon,\xi),\varepsilon,\xi)|. \tag{2.9}$$

The important point here is that Ω needs not be bounded nor a small ball due to the uniform estimates in (2.6)-(2.8). A proof of this theorem may be obtained from the parametrized contraction mapping theorem [17], Appendix (C.7) applied to the following fixed point formulation.

$$y = D_yT(g_0(\xi),0,\xi)^{-1}[D_yT(g_0(\xi),0,\xi)y - T(y + g_0(\xi),\varepsilon,\xi)].$$

The basic interpolation result is contained in the following theorem.

Theorem 2.3 *Let the conditions* (H1)-(H3) *hold. Then there exist* $\varepsilon_0 > 0$ *and a vector field* $g \in C^{k-3}((-\varepsilon_0, \varepsilon_0) \times \mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$ *such that the following properties hold for all* $|\varepsilon| < \varepsilon_0, \tau \in \mathbb{R}, x \in \mathbb{R}^m$

(i) *the map* $(\varepsilon, \tau, x) \to \varepsilon^2 g(\varepsilon, \tau, x)$ *is of class* C^{k-1}, *and* 1*-periodic in* τ,

(ii) $|g(\varepsilon, \tau, x)| + |D_x g(\varepsilon, \tau, x)| \le C$ *for some* $C > 0$,

(iii) $g(\varepsilon, \tau, 0) = 0$ *and* $D_x g(\varepsilon, \tau, 0)$ *has the simple eigenvalue* 0 *with right eigenvector* e_r *and no other eigenvalues on the imaginary axis,*

(iv) $G(\varepsilon, \varepsilon, x) = \Phi(\varepsilon, x)$ *where* $G(t, \varepsilon, x)$ *is the* t*-flow of the system* (2.2) *with* $G(0, \varepsilon, x) = x$ *and* $\varepsilon \neq 0$.

Proof. We essentially follow the construction of g in [10]. We make the implicit function argument somewhat more transparent by using Theorem 2.2 and we add some simplifications due to the global bound in (H1).

Let $\chi_0 : \mathbb{R} \to [0, 1]$ be a C^∞ cut-off function such that

$$\chi_0(\tau) = 1 \ for \ \tau \le 0, \quad \chi_0(\tau) = 0 \ for \ \tau \ge 1$$

and denote $\chi_1(\tau) = 1 - \chi_0(\tau)$.

For $|\varepsilon| < \varepsilon_0$, $\varepsilon \neq 0$, $y \in \mathbb{R}^m$ we consider the C^k-curve

$$G(t, \varepsilon, y) = \chi_0(t/\varepsilon) F(t, y) + \chi_1(t/\varepsilon) F(t - \varepsilon, \Phi(\varepsilon, y)), \quad 0 \le \varepsilon t \le \varepsilon^2 \qquad (2.10)$$

which satisfies

$$G(0, \varepsilon, y) = F(0, y) = y, \quad G(\varepsilon, \varepsilon, y) = F(0, \Phi(\varepsilon, y)) = \Phi(\varepsilon, y). \qquad (2.11)$$

g will be defined so that $G(\cdot, \varepsilon, y)$ solves (2.2). For this purpose it is convenient to introduce the scaled variable $\tau = t/\varepsilon$ and set for $0 \le \tau \le 1$

$$\tilde{G}(\tau, \varepsilon, y) := G(\varepsilon\tau, \varepsilon, y) = \chi_0(\tau) F(\varepsilon\tau, y) + \chi_1(\tau) F(\varepsilon\tau - \varepsilon, \Phi(\varepsilon, y)). \qquad (2.12)$$

By this formula we see that $\tilde{G}$ and $D_\tau \tilde{G}$ have a C^k-smooth extension to $[0, 1] \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^m$ satisfying

$$\tilde{G}(\tau, 0, y) = y, \quad D_\tau \tilde{G}(\tau, 0, y) = 0, \qquad (2.13)$$

$$\tilde{G}(0, \varepsilon, y) = y, \quad \tilde{G}(1, \varepsilon, y) = \Phi(\varepsilon, y). \qquad (2.14)$$

Next we extend $\tilde{G}$ to all $\tau \in \mathbb{R}$ by setting

$$\tilde{G}(\tau, \varepsilon, y) = \tilde{G}(\tau - [\tau], \varepsilon, \Phi^{[\tau]}(\varepsilon, y)) \qquad (2.15)$$

where $n = [\tau]$ is the largest integer not exceeding τ and $\Phi^n(\varepsilon, \cdot)$ denotes the n-th iterate of $\Phi(\varepsilon, \cdot)$.

In virtue of (2.14), (2.15) $\tilde{G}$ is then continuous in $\mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^m$ and satisfies for all $n \in \mathbb{Z}$, $\tau \in \mathbb{R}$, $|\varepsilon| < \varepsilon_0$

$$\tilde{G}(n, \varepsilon, y) = \Phi^n(\varepsilon, y), \quad \tilde{G}(\tau, \varepsilon, \tilde{G}(n, \varepsilon, y)) = \tilde{G}(\tau + n, \varepsilon, y). \tag{2.16}$$

We claim that $\tilde{G}$ and $D_\tau \tilde{G}$ are C^k-smooth. By the last relation in (2.16) it is sufficient to consider the τ-derivatives at $\tau = 0$.

Let D_+^j, $j \leq k+1$ denote the j-th derivative at $\tau = 0$ from above and D_-^j from below. Then we find from (2.12), (2.15)

$$\begin{aligned} D_+^j \tilde{G}(\tau, \varepsilon, y) &= D_+^j[\chi_0(\tau) F(\varepsilon\tau, y) + \chi_1(\tau) F(\varepsilon\tau - \varepsilon, \Phi(\varepsilon, y))] \\ &= \varepsilon^j (D_t^j F)(0, y) \\ &= D_-^j[\chi_0(\tau+1) F(\varepsilon(\tau+1), \Phi^{-1}(\varepsilon, y)) + \chi_1(\tau+1) F(\varepsilon\tau, y)] \\ &= D_-^j \tilde{G}(\tau, \varepsilon, y). \end{aligned}$$

Transforming backwards in (2.12) we may then define G for $t \in \mathbb{R}$, $\varepsilon \neq 0$ and find that (2.2) requires

$$D_t G(\varepsilon, t, y) = f(G(t, \varepsilon, y)) + \varepsilon^2 g(\varepsilon, t/\varepsilon, G(t, \varepsilon, y)).$$

In the scaled time variable $\tau = t/\varepsilon$ this leads to the setting

$$g(\varepsilon, \tau, y) := \varepsilon^{-2} \tilde{g}(\varepsilon, \tau, x), \quad \tilde{g}(\varepsilon, \tau, x) := \varepsilon^{-1} D_\tau \tilde{G}(\tau, \varepsilon, y) - f(x) \tag{2.17}$$

where $y = y(\tau, \varepsilon, x)$ is defined implicitly by

$$x = \tilde{G}(\tau, \varepsilon, y). \tag{2.18}$$

First we show that (2.18) can be solved for y uniformly in $(\tau, \varepsilon, x) \in V := [0, 2] \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^m$. We use Theorem 2.2 with $\xi = (\tau, x)$, $\Omega = (-1, 3) \times \mathbb{R}^m$ and

$$T(y, \varepsilon, \xi) = \tilde{G}(\tau, \varepsilon, y) - x, \quad g_0(\xi) = x.$$

Obviously, from (2.13) $y = x$ solves (2.18) at $\varepsilon = 0$ and $D_y T(g_0(\xi), 0, \xi) = I$, so we set $\sigma = 1$ in (2.6). Furthermore

$$\begin{aligned} &D_y T(y, \varepsilon, \xi) - D_y T(g_0(\xi), 0, \xi) = D_y \tilde{G}(\tau, \varepsilon, y) - I \\ = \; &\chi_0(\tau)(D_y F(\varepsilon\tau, y) - I) + \chi_1(\tau)(D_y F(\varepsilon\tau - \varepsilon, \Phi(\varepsilon, y)) D_y \Phi(\varepsilon, y) - I). \end{aligned}$$

By Lemma 2.1 we can estimate the first term by $C|\varepsilon\tau|$ and the second by

$$\begin{aligned}
&|(D_yF(\varepsilon\tau-\varepsilon,\Phi(\varepsilon,y))-D_yF(\varepsilon\tau-\varepsilon,F(\varepsilon,y)))D_y\Phi(\varepsilon,y)| \\
+\ &|D_yF(\varepsilon\tau-\varepsilon,F(\varepsilon,y))(D_y\Phi(\varepsilon,y)-D_yF(\varepsilon,y))|+|D_yF(\varepsilon\tau,y)-I| \\
\le\ &C|\varepsilon|^4(1+C|\varepsilon|)+(1+C|\varepsilon|)\cdot C|\varepsilon|^3+C|\varepsilon|\le C|\varepsilon|\le\frac{1}{2}=\kappa
\end{aligned}$$

if $|\varepsilon|$ is small. Finally, with $\delta=1$, we obtain (2.8) for small ε since

$$\begin{aligned}
&|T(g_0(\xi),\varepsilon,\xi)|=|\tilde{G}(\tau,\varepsilon,x)-x| \\
\le\ &|F(\varepsilon\tau,x)-x|+|\chi_1(\tau)(F(\varepsilon\tau-\varepsilon,\Phi(\varepsilon,x))-F(\varepsilon\tau-\varepsilon,F(\varepsilon,x)))| \\
\le\ &C|\varepsilon\tau|+C|\varepsilon(\tau-1)|\cdot|\varepsilon|^3\le\frac{1}{2}=(\frac{1}{\sigma}-\kappa)\delta.
\end{aligned}$$

We notice that (2.9) implies an estimate of the derivative

$$|D_xy(\tau,\varepsilon,x)|\le C,\quad(\tau,\varepsilon,x)\in V. \tag{2.19}$$

Using $D_\tau\tilde{G}(\tau,0,y)=0$, $y(\tau,0,x)=x$ and the C^k-smoothness of $D_\tau\tilde{G}$ we have thus shown that $\tilde{g}$, as defined by (2.17) is of class C^{k-1} in V. Moreover, from Taylor's theorem g is then of class C^{k-3} if we show

$$|\tilde{g}(\varepsilon,\tau,x)|\le C\varepsilon^2,\quad(\varepsilon,\tau,x)\in V. \tag{2.20}$$

In fact, from $\chi_0'+\chi_1'=0$ we find for $\varepsilon\neq0$

$$\begin{aligned}
&\tilde{g}(\varepsilon,\tau,y) \\
=\ &\frac{1}{\varepsilon}\chi_0'(\tau)F(\varepsilon\tau,y)+\frac{1}{\varepsilon}\chi_1'(\tau)F(\varepsilon\tau-\varepsilon,\Phi(\varepsilon,y)) \\
&+\chi_0(\tau)D_tF(\varepsilon\tau,y)+\chi_1(\tau)D_tF(\varepsilon\tau-\varepsilon,\Phi(\varepsilon,y))-f(x) \\
=\ &\frac{1}{\varepsilon}\chi_1'(\tau)[F(\varepsilon\tau-\varepsilon,\Phi(\varepsilon,y))-F(\varepsilon\tau-\varepsilon,F(\varepsilon,y))] \\
&+\chi_1(\tau)[f(F(\varepsilon\tau-\varepsilon,\Phi(\varepsilon,y)))-f(F(\varepsilon\tau-\varepsilon,F(\varepsilon,y)))] \\
&+f(F(\varepsilon\tau,y))-f(x).
\end{aligned} \tag{2.21}$$

Using Lemma 2.1 and (H1) we can estimate the first term by $C\varepsilon^2$, the second by $C|\varepsilon|^3$ and for the last one we obtain from (2.18)

$$\begin{aligned}
&|f(F(\varepsilon\tau,y))-f(x)|\le C|F(\varepsilon\tau,y)-x| \\
=\ &C|F(\varepsilon\tau,y)-\chi_0(\tau)F(\varepsilon\tau,y)-\chi_1(\tau)F(\varepsilon\tau-\varepsilon,\Phi(\varepsilon,y))| \\
=\ &C|\chi_1(\tau)(F(\varepsilon\tau-\varepsilon,F(\varepsilon,y))-F(\varepsilon\tau-\varepsilon,\Phi(\varepsilon,y)))| \\
\le\ &C|\varepsilon|^3.
\end{aligned}$$

In a similar way we can derive a bound

$$|D_x \tilde{g}(\varepsilon, \tau, x)| \le C\varepsilon^2, \quad (\varepsilon, \tau, x) \in V \tag{2.22}$$

by taking derivatives of the terms above and using (2.19) as well as Lemma 2.1. Property (ii) then follows from (2.20), (2.22) for $(\varepsilon, \tau, x) \in V$.

So far we have worked with $\tau \in [0, 2]$. It is now shown as in [10] that $\tilde{g}$ is 1-periodic in τ so that the properties (i), (ii) and (iv) follow by periodic continuation. From (2.15) we have for $0 \le \tau \le 1$

$$x = \tilde{G}(\tau + 1, \varepsilon, y(\tau + 1, \varepsilon, x)) = \tilde{G}(\tau, \varepsilon, \Phi(\varepsilon, y(\tau + 1, \varepsilon, x)))$$

hence $y(\tau, \varepsilon, x) = \Phi(\varepsilon, y(\tau + 1, \varepsilon, x))$ and by (2.15), (2.17)

$$\begin{aligned} & \tilde{g}(\varepsilon, \tau + 1, x) - \tilde{g}(\varepsilon, \tau, x) \\ = \;& \frac{1}{\varepsilon}(D_\tau \tilde{G}(\tau + 1, \varepsilon, y(\tau + 1, \varepsilon, x)) - D_\tau \tilde{G}(\tau, \varepsilon, y(\tau, \varepsilon, x))) \\ = \;& \frac{1}{\varepsilon}(D_\tau \tilde{G}(\tau, \varepsilon, \Phi(\varepsilon, y(\tau + 1, \varepsilon, x))) - D_\tau \tilde{G}(\tau, \varepsilon, y(\tau, \varepsilon, x))) = 0. \end{aligned}$$

Finally we prove (iii). It is easy to see that (H2) and (2.21) imply

$$\tilde{G}(\tau, \varepsilon, 0) = 0, \quad y(\tau, \varepsilon, 0) = 0, \quad \Phi(\varepsilon, 0) = 0, \quad \tilde{g}(\varepsilon, \tau, 0) = 0.$$

Without loss of generality we can choose coordinates in $\mathbb{R}^m$ so that

$$A = D_x f(0) = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \quad B \in \mathbb{R}^{m-1, m-1}$$

where the eigenvalues of B have nonzero real parts. Then

$$D_x F(t, 0) = e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & e^{Bt} \end{pmatrix}.$$

Taking x-derivatives in

$$\Phi(\varepsilon, x) - x = \varepsilon f(\frac{x + \Phi(\varepsilon, x)}{2})$$

shows that

$$D_x \Phi(\varepsilon, 0) = Q_\varepsilon(A) = \begin{pmatrix} 1 & 0 \\ 0 & Q_\varepsilon(B) \end{pmatrix} \tag{2.23}$$

where $Q_\varepsilon(M) = (I - \frac{\varepsilon}{2}M)^{-1}(I + \frac{\varepsilon}{2}M)$ for a matrix M.

By definition of $g(\varepsilon,\tau,x)$

$$D_x g(\varepsilon,\tau,0) = \varepsilon^{-2}(-A + \frac{1}{\varepsilon} D_y D_\tau \tilde{G}(\tau,\varepsilon,0) D_x y(\tau,\varepsilon,0)). \qquad (2.24)$$

With $P_\varepsilon(M) := e^{-\varepsilon\tau} Q_\varepsilon(M) - I$ we find upon differentiation of (2.18)

$$\begin{aligned} I &= e^{\varepsilon\tau A}[I + \chi_1(\tau)P_\varepsilon(A)]D_x y(\tau,\varepsilon,0), \\ D_x y(\tau,\varepsilon,0) &= \begin{pmatrix} 1 & 0 \\ 0 & (I+\chi_1(\tau)P_\varepsilon(B))^{-1}e^{-\varepsilon\tau B} \end{pmatrix}. \end{aligned}$$

Furthermore, using $\chi_0' + \chi_1' = 0$, $\chi_0 = 1 - \chi_1$ and $F_{ty}(t,0) = Ae^{At}$ we find

$$\begin{aligned} D_y D_\tau \tilde{G}(\tau,\varepsilon,0) &= \chi_0'(\tau)F_y(\varepsilon\tau,0) + \chi_1'(\tau)F_y(\varepsilon\tau-\varepsilon,0)\Phi_y(\varepsilon,0) \\ &\quad +\varepsilon[\chi_0(\tau)F_{yt}(\varepsilon\tau,0) + \chi_1(\tau)F_{yt}(\varepsilon\tau-\varepsilon,0)\Phi_y(\varepsilon,0)] \\ &= \begin{pmatrix} 0 & 0 \\ 0 & B_1(\varepsilon,\tau) \end{pmatrix} \end{aligned}$$

where $B_1(\varepsilon,\tau) = e^{\varepsilon\tau B}[\varepsilon B + (\chi_1' + \varepsilon\chi_1 B)P_\varepsilon(B)]$. Collecting terms we end up with the desired block structure

$$D_x g(\varepsilon,\tau,0) = \begin{pmatrix} 0 & 0 \\ 0 & B_2(\varepsilon,\tau) \end{pmatrix}$$

where $B_2(\varepsilon,\tau) = \varepsilon^{-2}(-B + \frac{1}{\varepsilon}(\varepsilon B + (\chi_1' + \varepsilon\chi_1 B)P_\varepsilon(B))(I + \chi_1 P_\varepsilon(B))^{-1})$. After some calculations one finds $B_2(\varepsilon,\tau) = \frac{7}{12}\chi_1'(\tau)B^3 + O(\varepsilon)$. ■

Remark 2.4 As the proof of Theorem 2.3 shows one can generalize the results to the case of m_c eigenvalues of $D_x f(0)$ with zero real parts which generate m_c eigenvalues of $D_x g(\varepsilon,\tau,0)$ on the unit circle with the same eigenvectors. Moreover this can be extended to one-step methods whose growth function Q_ε (compare (2.23)) satisfies

$$\mathrm{Re}(z) = 0 \quad \Rightarrow \quad |Q_\varepsilon(z)| = 1.$$

3 Backward error analysis

In a backward error analysis of a numerical method the approximate solution is interpreted as an exact solution of a perturbed problem. For a one-step method with step-size ε this means that we look for a perturbed dynamical system, depending on ε, such that its ε-flow is precisely the one-step mapping. Unfortunately, such a strong embedding is impossible in general (see [4], [12], [20] for the obstructions). But it is possible to set up a perturbed system the ε-flow of which approximates the one-step mapping to arbitrary order in ε. These systems are called **modified equations**. A clear exposition of their construction for Runge-Kutta methods is given in [7] (see also [9], [15]).

Now Theorem 2.3 states that we can perform an exact backward error analysis if we leave the category of autonomous systems and work with nonautonomous, in fact rapidly forced systems. This perturbed system is certainly not unique since there are many ways to choose a proper interpolation, e.g. by varying the cut-off function χ_0. It is then natural to ask how these rapidly forced perturbations relate to the autonomous modified equations.

In this section we will show that the limit of equation (2.2) as $\varepsilon \to 0$, but with t/ε fixed in $g(\varepsilon, t/\varepsilon, x)$, is a suitable nonautonomous modified equation. Indeed, from (2.21) and the fact that only the first term is of order ε^2 we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} g(\varepsilon,\tau,x) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}\chi_1'(\tau)[F(\varepsilon\tau-\varepsilon,\Phi(\varepsilon,y)) - F(\varepsilon\tau-\varepsilon,F(\varepsilon,y))]\\ &= \chi_1'(\tau)\lim_{\varepsilon\to 0}\Big\{\int_0^1 D_xF(\varepsilon\tau-\varepsilon, s\Phi(\varepsilon,y)+(1-s)F(\varepsilon,y))ds\,\cdot\\ &\qquad \cdot\frac{1}{\varepsilon^3}[\Phi(\varepsilon,y)-F(\varepsilon,y)]\Big\}.\end{aligned}$$

Now the integral converges to I since $D_xF(0,x) = I$ and $y(\tau,0,x) = x$. Moreover, from Lemma 2.1

$$g_0(x) := \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}(\Phi(\varepsilon,x) - F(\varepsilon,x)) \tag{3.1}$$

exists and together with $y(\tau,0,x) = x$ we find

$$\lim_{\varepsilon\to 0} g(\varepsilon,\tau,x) = \chi_1'(\tau)g_0(x).$$

By a standard Taylor expansion we have

$$\begin{aligned}\Phi(\varepsilon,x) &= x + \varepsilon f(x) + \frac{1}{2}\varepsilon^2 D_xf(x)f(x)\\ &\quad +\frac{1}{6}\varepsilon^3[\frac{3}{4}D_x^2f(x)[f(x),f(x)] + \frac{1}{2}(D_xf(x))^2f(x)] + O(\varepsilon^4)\\ &= F(\varepsilon,x) + \varepsilon^3 g_0(x) + O(\varepsilon^4)\end{aligned}\tag{3.2}$$

where

$$g_0(x) = -\frac{1}{12}[\frac{1}{2}D_x^2f(x)[f(x),f(x)] + (D_xf(x))^2f(x)]. \tag{3.3}$$

Our nonautonomous modified equation then takes the form

$$\dot{x} = f(x) + \varepsilon^2 \chi_1'(t/\varepsilon) g_0(x). \tag{3.4}$$

We notice that χ_1' is a bump function on $[0,1]$ with integral 1. It has an obvious 1-periodic and smooth extension to $\mathbb{R}$, so that (3.4) is defined for all $t \in \mathbb{R}$.

We will show that the ε-evolution of (3.4) is an $O(\varepsilon^4)$-approximation to the one-step map $\Phi(\varepsilon, \cdot)$.

Proposition 3.1 *Assume* (H1) *and denote by* $\Psi(t, \varepsilon, x)$ *the solution of* (3.4) *with* $\Psi(0, \varepsilon, x) = x$. *Then uniformly for* $x \in \mathbb{R}^m$

$$|\Psi(\varepsilon, \varepsilon, x) - \Phi(\varepsilon, x)| = O(\varepsilon^4).$$

Proof. With $\tau = t/\varepsilon$ and $x' = \dfrac{dx}{d\tau}$ equation (3.4) becomes

$$x' = \varepsilon f(x) + \varepsilon^3 \chi_1'(\tau) g_0(x) \tag{3.5}$$

with solution $\tilde{\Psi}(\tau, \varepsilon, x) = \Psi(\varepsilon\tau, \varepsilon, x)$. Now we expand $\tilde{\Psi}(\tau, \varepsilon, x)$ in ε at $\varepsilon = 0$ for all $\tau \in [0,1]$. Clearly, $\tilde{\Psi}(\tau, 0, x) = x$ from (3.5) and by taking ε-derivatives we find for $x_i(\tau) := D_\varepsilon^i \tilde{\Psi}(\tau, \varepsilon, x) (i \geq 0)$ the variational equations

$$\begin{aligned} x_1' &= f(x_0) + \varepsilon D_x f(x_0) x_1 + 3\varepsilon^2 \chi_1'(\tau) g_0(x_0) + O(\varepsilon^3) \\ x_2' &= 2 D_x f(x_0) x_1 + \varepsilon D_x^2 f(x_0)[x_1, x_1] \\ &\quad + \varepsilon D_x f(x_0) x_2 + 6\varepsilon \chi_1'(\tau) g_0(x_0) + O(\varepsilon^2). \end{aligned} \tag{3.6}$$

At $\varepsilon = 0$ we have $x_1(0) = x_2(0) = 0$ and hence by integration

$$D_\varepsilon \tilde{\Psi}(\tau, 0, x) = \tau f(x), \qquad D_\varepsilon^2 \Psi(\tau, 0, x) = \tau^2 D_x f(x) f(x).$$

Finally, computing the third derivative of (3.5) at $\varepsilon = 0$ yields

$$x_3' = 3\tau^2 [D_x^2 f(x)[f(x), f(x)] + (D_x f(x))^2 f(x)] + 6\chi_1' g_0(x) + O(\varepsilon),$$

and by integration

$$D_\varepsilon^3 \tilde{\Psi}(\tau, 0, x) = \tau^3 [D_x^2 f(x)[f(x), f(x)] + (D_x f(x))^2 f(x)] + 6\chi_1(\tau) g_0(x).$$

Comparing with (3.2), (3.3) we end up with

$$|\Psi(\varepsilon, \varepsilon, x) - \Phi(\varepsilon, x)| = |\tilde{\Psi}(1, \varepsilon, x) - \Phi(\varepsilon, x)| = O(\varepsilon^4)$$

which completes the proof. ■

Let us finally discuss the relation of the nonautonomous equation (3.4) to an autonomous modified equation

$$\dot{x} = f(x) + \varepsilon^2 g_1(x). \tag{3.7}$$

Suppose that the t-flow $G(t, \varepsilon, x)$ of this system satisfies

$$\Phi(\varepsilon, x) - G(\varepsilon, \varepsilon, x) = O(\varepsilon^4). \tag{3.8}$$

Expanding G with respect to the second variable one finds

$$G(\varepsilon, \varepsilon, x) = F(\varepsilon, x) + \varepsilon^3 g_1(x) + O(\varepsilon^4)$$

and by comparison with (3.1), (3.8)

$$\begin{aligned} \varepsilon^3 g_1(x) &= \Phi(\varepsilon, x) - F(\varepsilon, x) + O(\varepsilon^4) \\ g_1(x) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3}(\Phi(\varepsilon, x) - F(\varepsilon, x)) = g_0(x). \end{aligned}$$

Therefore, the nonautonomous system (3.4) is just a smeared version of the autonomous modified equation (3.7), or to put it the other way around, (3.7) is an averaged version of (3.4) over the time interval $[0, \varepsilon]$.

Remark 3.2 Several generalizations are rather obvious. We can use any p-th order one step method for (1.1) and get in the limit $\varepsilon \to 0$ the nonautonomous term $\chi_1'(t/\varepsilon) g_0(x)$ where now

$$g_0(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{p+1}}(\Phi(\varepsilon, x) - F(\varepsilon, x))$$

and the evolution of (3.4) is $O(\varepsilon^{p+2})$ close to the discrete solution. Moreover, by further expansion of $g(\varepsilon, \tau, x)$ with respect to ε we can set up nonautonomous modified equations which approximate the one-step mapping to higher order.

4 Existence and estimates of the invariant manifold

In this section we study center manifolds for the rapidly forced system (2.2) under the assumptions of Theorem 2.3. As in the proof of Theorem 2.3 it is convenient to work with the scaled time variable $\tau = t/\varepsilon$, so that (2.2) becomes

$$x' = \varepsilon f(x) + \varepsilon^3 g(\varepsilon, \tau, x). \tag{4.1}$$

Notice that the standard trick of adding the equation $\varepsilon' = 0$ to this system does not work here. The system (4.1) becomes trivial at $\varepsilon = 0$ and hence the center manifold will be the whole space.

Therefore, we are forced to repeat the center manifold proofs (for example as in [6]) and take care of the balance between the f and g-terms in (4.1) as $\varepsilon \to 0$.

As in the proof of Theorem 2.3 we use the eigenvectors of $D_x f(0)$ to define proper coordinates $\theta \in \mathbb{R}$, $u \in \mathbb{R}^{m-1}$ such that the system (4.1) has the following form

$$\begin{aligned} \theta' &= \varepsilon f_1(\theta, u) + \varepsilon^3 g_1(\varepsilon, \tau, \theta, u) \\ u' &= \varepsilon B u + \varepsilon f_2(\theta, u) + \varepsilon^3 g_2(\varepsilon, \tau, \theta, u). \end{aligned} \tag{4.2}$$

The functions f_1, f_2 and g_1, g_2 are determined by f and g respectively and satisfy the following properties (with $l = k-1 \geq 3$)

(C1) $f_i \in C^l(\mathbb{R}^m, \mathbb{R}^m)$, $\varepsilon^2 g_i \in C^l((-\varepsilon_0, \varepsilon_0) \times \mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$ for $i = 1, 2$.

(C2) $f_i(0,0) = 0$, $g_i(\varepsilon, \tau, 0, 0) = 0$, $D_\theta f_i(0,0) = 0$, $D_u f_i(0,0) = 0$,
$D_\theta g_i(\varepsilon, \tau, 0, 0) = 0$, $D_u g_1(\varepsilon, \tau, 0, 0){=}0$, for $i = 1, 2$, $|\varepsilon| < \varepsilon_0$, $\tau \in \mathbb{R}$.

For the matrix $B \in \mathbb{R}^{m-1,m-1}$ we make the assumption

(C3) $\mathrm{Re}(\lambda) < 0$ for all eigenvalues λ of B.

As usual this will simplify the construction of a center manifold compared to the general case when B has eigenvalues on both sides of the imaginary axis.

Our first aim is to construct for any $0 < \varepsilon < \varepsilon_0$ a locally invariant manifold for the system (4.2) of the form

$$M_\varepsilon = \{(\tau, \theta, u) \in \mathbb{R}^{m+1} : u = h(\varepsilon, \tau, \theta), \quad |\theta| < \delta\} \tag{4.3}$$

where $h(\varepsilon, \tau, \cdot)$ is of class C^l, 1-periodic in τ and satisfies

$$h(\varepsilon, \tau, 0) = 0, \quad D_\theta h(\varepsilon, \tau, 0) = 0. \tag{4.4}$$

Introducing $t = \tau\varepsilon$ again we then obtain a locally invariant manifold

$$\tilde{M}_\varepsilon = \{(t, \theta, u) \in \mathbb{R}^{m+1} : u = h(\varepsilon, t/\varepsilon, \theta), |\theta| < \delta\} \tag{4.5}$$

for the original system (2.2).

In a second step we investigate the behavior of M_ε as $\varepsilon \to 0$. We will show that the original system (1.1) has a center manifold of class C^l

$$M_0 = \{(\theta, u) \in \mathbb{R}^m : u = \tilde{h}(\theta), |\theta| < \delta\} \tag{4.6}$$

such that

$$|h(\varepsilon, \tau, \theta) - \tilde{h}(\theta)| \leq C\varepsilon^2 \quad \text{for all } \tau \in \mathbb{R},\ 0 < \varepsilon < \varepsilon_0,\ |\theta| < \delta. \tag{4.7}$$

Defining $h(0, \tau, \theta) = \tilde{h}(\theta)$ we will then have an invariant manifold

$$\hat{M} = \{(\varepsilon, \tau, \theta, u) : \tau \in \mathbb{R}, 0 \leq \varepsilon < \varepsilon_0, |\theta| < \delta, u = h(\varepsilon, \tau, \theta)\}.$$

We notice however that we prove only smoothness with respect to τ and θ. With respect to ε there is only continuity at $\varepsilon = 0$. Smoothness with respect to ε seems to be a delicate matter and some comments and partial results will be given in Section 6.

In the following theorem we consider a general system of the form (4.2) under the assumptions (C1) – (C3). Of course our application later on is to the interpolated system (2.2).

Theorem 4.1 *Let* (C1), (C2) *and* (C3) *hold with* $l \geq 1$. *Then there exist constants* $\varepsilon_0, \delta, c > 0$ *and functions* $h \in C((0, \varepsilon_0) \times \mathbb{R} \times (-\delta, \delta), \mathbb{R}^{m-1})$, $\tilde{h} \in C^l((-\delta, \delta), \mathbb{R}^{m-1})$ *with the following properties*

(i) *for* $\varepsilon \in (0, \varepsilon_0)$ *fixed,* $h(\varepsilon, \cdot, \cdot) \in C^1(\mathbb{R} \times (-\delta, \delta), \mathbb{R}^{m-1})$ *and also for* $\tau \in \mathbb{R}$ *fixed,* $h(\varepsilon, \tau, \cdot) \in C^l((-\delta, \delta), \mathbb{R}^{m-1})$

$$h(\varepsilon, \tau, 0) = 0, \quad D_\theta h(\varepsilon, \tau, 0) = 0, \quad \tau \in \mathbb{R}, \tag{4.8}$$

$$h(\varepsilon, \tau + 1, \theta) = h(\varepsilon, \tau, \theta), \tag{4.9}$$

(ii) $M = \{(\varepsilon, \tau, \theta, u) : u = h(\varepsilon, \tau, \theta), 0 < \varepsilon < \varepsilon_0, \tau \in \mathbb{R}, |\theta| < \delta\}$ *is a locally invariant manifold of the system* (4.2),

(iii) $\tilde{h}(0) = 0$, $\tilde{h}'(0) = 0$ *and* $M_0 = \{(\theta, u) : u = \tilde{h}(\theta), |\theta| < \delta\}$ *is a locally invariant manifold of the system* (1.1),

(iv)

$$|h(\varepsilon, \tau, \theta) - \tilde{h}(\theta)| \leq c\varepsilon^2,\ \forall\ 0 < \varepsilon < \varepsilon_0\ , \tau \in \mathbb{R},\ |\theta| < \delta. \tag{4.10}$$

Proof. Let $\psi \in C^\infty(\mathbb{R}, [0,1])$ be a cut-off function with $\psi(\theta) = 1$ for $|\theta| \leq 1$ and $\psi(\theta) = 0$ for $|\theta| \geq 2$. For $\delta > 0$, $i = 1, 2$ define

$$H_i(\varepsilon, \tau, \theta, u) = \varepsilon f_i(\theta\psi(\theta/\delta), u) + \varepsilon^3 g_i(\varepsilon, \tau, \theta\psi(\theta/\delta), u) \tag{4.11}$$

and consider the cut-off system

$$\theta' = H_1(\varepsilon, \tau, \theta, u), \tag{4.12}$$

$$u' = \varepsilon B u + H_2(\varepsilon, \tau, \theta, u). \tag{4.13}$$

Global invariant manifolds of this system will then be locally invariant for (4.2). For $p, p_1 > 0$ consider the function space

$$X = \Big\{ h \in C((0,\varepsilon_0) \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^{m-1}) : h(\varepsilon, \tau, 0) = 0,\ |h(\varepsilon, \tau, \theta)| \leq p,$$
$$|h(\varepsilon, \tau, \theta_1) - h(\varepsilon, \tau, \theta_2)| \leq p_1 |\theta_1 - \theta_2|, \text{ for } 0 < \varepsilon < \varepsilon_0, \theta, \theta_1, \theta_2 \in \mathbb{R} \Big\}.$$

It is a complete metric space under the sup-norm $\|h\| = \sup\limits_{\varepsilon,\tau,\theta} |h(\varepsilon, \tau, \theta)|$.

For given $h \in X$, $0 < \varepsilon < \varepsilon_0$, $\theta_0 \in \mathbb{R}$ and $\tau \in \mathbb{R}$ let $\theta(s) = \theta(s, \tau, \varepsilon, \theta_0, h)$ be the solution of

$$\theta' = H_1(\varepsilon, s, \theta, h(\varepsilon, s, \theta)), \quad s \in \mathbb{R},\ \theta(\tau) = \theta_0. \tag{4.14}$$

The global bounds on H_1 and h ensure existence of the solution for all $s \in \mathbb{R}$. The operator T on X is then defined by

$$(Th)(\varepsilon, \tau, \theta_0) = \int_{-\infty}^{\tau} e^{-\varepsilon B(s-\tau)} H_2(\varepsilon, s, \theta(s, \tau, \varepsilon, \theta_0, h), h(\varepsilon, s, \theta(s, \tau, \varepsilon, \theta_0, h)))ds. \tag{4.15}$$

We show that T is a contraction on X for suitably chosen p, p_1, δ and ε_0.

If $h \in X$ is the fixed point of T then the invariance of the graph of h is obtained in the following way.

For given $\theta_0, \tau \in \mathbb{R}$ we claim that

$$\tilde{\theta}(t) := \theta(t, \tau, \varepsilon, \theta_0, h), \quad \tilde{u}(t) := h(\varepsilon, t, \tilde{\theta}(t))$$

solves (4.12) and (4.13) and has initial values

$$\tilde{\theta}(\tau) = \theta_0, \quad \tilde{u}(\tau) = h(\varepsilon, \tau, \theta_0) \tag{4.16}$$

The invariance then follows by the uniqueness of the solution to the initial value problem. Since (4.16) and (4.12) hold by construction it remains to prove (4.13). By the cocycle property

$$\theta(s, t, \varepsilon, \tilde{\theta}(t), h) = \tilde{\theta}(s)$$

and hence by (4.15)

$$\begin{aligned}\tilde{u}(t) &= h(\varepsilon,t,\tilde{\theta}(t)) = (Th)(\varepsilon,t,\tilde{\theta}(t))\\ &= \int_{-\infty}^{t} e^{-\varepsilon B(s-t)} H_2(\varepsilon,s,\tilde{\theta}(s),h(\varepsilon,s,\tilde{\theta}(s)))ds\\ &= \int_{-\infty}^{t} e^{-\varepsilon B(s-t)} H_2(\varepsilon,s,\tilde{\theta}(s),\tilde{u}(s))ds.\end{aligned}$$

From this (4.13) follows by differentiation.

By the definition of H_1, H_2 and properties (C1), (C2) there is a continuous function $\omega : [0,\infty) \to [0,\infty)$, $\omega(0) = 0$ such that the following estimates hold for all $0<\varepsilon<\varepsilon_0$, $i=1,2$, $\tau\in\mathbb{R}$, $\theta,\sigma\in\mathbb{R}$ and $u,v \in \mathbb{R}^{m-1}$ with $|u|,|v| < \delta$

$$|H_i(\varepsilon,\tau,\theta,u)| \le C\varepsilon(\delta\omega(\delta) + \varepsilon^2), \tag{4.17}$$

$$|H_i(\varepsilon,\tau,\theta,u) - H_i(\varepsilon,\tau,\sigma,v)| \le C\varepsilon(\omega(\delta)+\varepsilon^2)(|\theta-\sigma|+|u-v|). \tag{4.18}$$

Moreover, by (C3) there exist constants $\beta, c>0$ such that for $s\le 0, u\in\mathbb{R}^{m-1}$

$$|e^{-\varepsilon s B}u| \le c e^{\varepsilon\beta s}|u|. \tag{4.19}$$

In the following we assume $p \le \delta$ so that we can use (4.17) and (4.18) to estimate terms involving

$$H_i(\varepsilon,s,\theta(s,\tau,\varepsilon,\theta_0,h),h(\varepsilon,s,\theta(s,\tau,\varepsilon,\theta_0,h))).$$

In the following we collect various conditions on $p, p_1, \varepsilon_0, \delta$ and finally show that they can be satisfied simultaneously.

For $h \in X$ we obtain from (4.17), (4.19)

$$|Th(\varepsilon,\tau,\theta_0)| \le \int_{-\infty}^{\tau} e^{\varepsilon\beta(s-\tau)}\varepsilon(\delta\omega(\delta)+\varepsilon^2)ds = \frac{c}{\beta}(\delta\omega(\delta)+\varepsilon^2) \le p.$$

Hence we require

$$\frac{c}{\beta}(\delta\omega(\delta) + \varepsilon^2) \le p. \tag{4.20}$$

Next for $\theta_1, \theta_2 \in \mathbb{R}$ and $s \le \tau$ we have from (4.14), (4.18)

$$\begin{aligned}&|\theta(s,\tau,\varepsilon,\theta_1,h) - \theta(s,\tau,\varepsilon,\theta_2,h)|\\ \le\ & |\theta_1-\theta_2| + c\int_{s}^{\tau} \varepsilon(\omega(\delta)+\varepsilon^2)(1+p_1)|\theta(t,\tau,\varepsilon,\theta_1,h) - \theta(t,\tau,\varepsilon,\theta_2,h)|dt\end{aligned}$$

and therefore by Gronwall's inequality

$$|\theta(s,\tau,\varepsilon,\theta_1,h) - \theta(s,\tau,\varepsilon,\theta_2,h)| \le |\theta_1-\theta_2| e^{-\varepsilon\gamma(s-\tau)} \tag{4.21}$$

where $\gamma = c(1+p_1)(\omega(\delta)+\varepsilon^2)$.

Using (4.15), (4.18) and (4.21) we obtain

$$\begin{aligned} |(Th)(\varepsilon,\tau,\theta_1)-(Th)(\varepsilon,\tau,\theta_2)| &\leq c\gamma\varepsilon|\theta_1-\theta_2|\int_{-\infty}^{\tau} e^{\varepsilon(\beta-\gamma)(s-\tau)}ds \\ &= \frac{c\gamma}{\beta-\gamma}|\theta_1-\theta_2| \end{aligned}$$

which leads to the conditions

$$\gamma = c(1+p_1)(\omega(\delta)+\varepsilon^2) < \beta, \quad \frac{c\gamma}{\beta-\gamma} \leq p_1. \tag{4.22}$$

Consider now $h_1, h_2 \in X$ and $\tau, \theta_0 \in \mathbb{R}$. For $s \leq \tau$ we obtain from (4.14), (4.18) (omitting the arguments $\tau, \varepsilon, \theta_0$)

$$\begin{aligned} &|\theta(s,h_1)-\theta(s,h_2)| \\ \leq &\int_s^{\tau} |H_1(t,\theta(t,h_1),h_1(t,\theta(t,h_1)))-H_1(t,\theta(t,h_1),h_2(t,\theta(t,h_1)))| \\ &+|H_1(t,\theta(t,h_1),h_2(t,\theta(t,h_1)))-H_1(t,\theta(t,h_2),h_2(t,\theta(t,h_2)))|dt \\ \leq &\int_s^{\tau} c\varepsilon(\omega(\delta)+\varepsilon^2)\|h_1-h_2\|+c\varepsilon(\omega(\delta)+\varepsilon^2)(1+p_1)|\theta(t,h_1)-\theta(t,h_2)|dt \end{aligned}$$

and again by Gronwall's inequality

$$|\theta(s,h_1)-\theta(s,h_2)| \leq c\varepsilon(\omega(\delta)+\varepsilon^2)e^{\varepsilon\gamma(\tau-s)}\|h_1-h_2\|, \quad s \leq \tau.$$

Combining this with (4.15) and (4.18) yields

$$|(Th_1)(\varepsilon,\tau,\theta_0)-(Th_2)(\varepsilon,\tau,\theta_0)| \leq \kappa_\varepsilon\|h_1-h_2\|$$

where $\kappa_\varepsilon = \frac{c}{\beta}(\omega(\delta)+\varepsilon^2)+c\varepsilon(\omega(\delta)+\varepsilon^2)^2\frac{1+p_1}{\beta-\gamma}$. For contraction we require

$$\kappa_\varepsilon \leq \kappa < 1 \text{ for all } 0<\varepsilon\leq\varepsilon_0. \tag{4.23}$$

It is then easy to see that with $p=\delta$, $p_1=1$ and δ, ε_0 sufficiently small all the conditions (4.20), (4.22), (4.23) can be satisfied. Finally, it is readily seen that $h(\varepsilon,\tau,0)=0$ implies $\theta(s,\tau,\varepsilon,0,h)=0$ and thus $(Th)(\varepsilon,\tau,0)=0$.

Periodicity of h in τ follows by showing that

$$\hat{h}(\varepsilon,\tau,\theta) := h(\varepsilon,\tau+1,\theta)$$

is a fixed point of T in X and hence coincides with h. In the main step one uses the τ-periodicity of H_1, H_2 and shows

$$\theta(s,\tau,\varepsilon,\theta_0,\hat{h}) = \theta(s+1,\tau+1,\varepsilon,\theta_0,h).$$

In order to prove (iii) and (iv) we notice that the contraction argument above can be repeated for any fixed $\varepsilon \in (0, \varepsilon_0)$ with X replaced by

$$\begin{aligned}\tilde{X} &= \Big\{ h \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}^{m-1}) : \ h(\tau, 0) = 0, \ |h(\tau, \theta)| \le p, \\ &\qquad |h(\tau, \theta_1) - h(\tau, \theta_2)| \le p_1 |\theta_1 - \theta_2| \Big\}\end{aligned}$$

and with $T_\varepsilon : \tilde{X} \to \tilde{X}$ defined as in (4.15). Then $h(\varepsilon, \cdot, \cdot)$ is the unique fixed point of T_ε in $\tilde{X}$ and with κ from (4.23) we have the estimate

$$\|h_1 - h_2\| \le \frac{1}{1-\kappa} \|(I - T_\varepsilon)(h_1) - (I - T_\varepsilon)(h_2)\|, \ \forall h_1, h_2 \in \tilde{X}. \tag{4.24}$$

We now consider the autonomous case when the g_i in (4.11) is identically zero. With $H_i(\varepsilon, \theta, u) = \varepsilon f_i(\theta \psi(\theta/\delta), u)$, $i = 1, 2$ we then define the operator $\tilde{T}_\varepsilon$ as in (4.15) and obtain contraction on $\tilde{X}$ with the same constant κ as before. The corresponding fixed points $\tilde{h}_\varepsilon(\tau, \theta)$ in $\tilde{X}$ are independent of ε and τ as may be seen from the fact that $\tilde{h}_\varepsilon(\tau + \Delta, \theta)$, $\Delta \in \mathbb{R}$ and $\tilde{h}_{\varepsilon_0}(\tau, \theta)$ are also fixed points of $\tilde{T}_\varepsilon$. Hence we have

$$\tilde{h}_\varepsilon(\tau, \theta) = \tilde{h}(\theta), \ \theta \in \mathbb{R}, \ 0 < \varepsilon < \varepsilon_0, \tau \in \mathbb{R}$$

for some function $\tilde{h} \in C(\mathbb{R}, \mathbb{R}^{m-1})$ with $\tilde{h}(0) = 0$, $|\tilde{h}(\theta)| \le p$, $|\tilde{h}(\theta_1) - \tilde{h}(\theta_2)| \le p_1 |\theta_1 - \theta_2|$.

We then use $h_1 = h(\varepsilon, \cdot, \cdot)$ and $h_2 = \tilde{h}$ in (4.24) and find

$$\|\tilde{h} - h(\varepsilon, \cdot, \cdot)\| \le \frac{1}{1-\kappa} \|\tilde{T}_\varepsilon \tilde{h} - T_\varepsilon \tilde{h}\|.$$

The estimate (4.10) then follows from

$$\begin{aligned} & |(\tilde{T}_\varepsilon \tilde{h} - T_\varepsilon \tilde{h})(\tau, \theta_0)| \\ = \ & \Big| \int_{-\infty}^{\tau} e^{-\varepsilon B(s-\tau)} (\tilde{H}_2 - H_2)(\varepsilon, \theta(s, \tau, \varepsilon_0, \tilde{h}), \tilde{h}(\theta(s, \tau, \varepsilon_0, \tilde{h}))) ds \Big| \\ \le \ & \int_{-\infty}^{\tau} e^{\varepsilon \beta (s-\tau)} \varepsilon^3 \|g_2\| ds = C \varepsilon^2. \end{aligned}$$

The $C^{l-1, Lip}$-smoothness of the function $h(\varepsilon, \tau, \cdot)$ follows in the standard way by proving contraction of the operator T in the following set Y with respect to th C^{l-1} norm:

$$\begin{aligned} Y = \Big\{ & h \in X : h(\varepsilon, \tau, \cdot) \in C^{l-1}((-\delta, \delta), \mathbb{R}^{m-1}), |D_\theta^j h(\varepsilon, \tau, \theta)| \le q \\ & \text{for } j = 0, \cdots, l-1 \text{ and } |D_\theta^{l-1} h(\varepsilon, \tau, \theta_1) - D_\theta^{l-1} h(\varepsilon, \tau, \theta_2)| \le q_1 |\theta_1 - \theta_2| \Big\} \end{aligned}$$

Actually, the functions $h(\varepsilon, \tau, \cdot)$ are of class C^l in θ which can be proved by well known techniques, cf. [3]. ■

Combining Theorem 2.3 and 4.1 we obtain C^{k-1}-manifolds

$$W_\varepsilon = \{(\theta, u) \in \mathbb{R}^m : u = h(\varepsilon, 0, \theta), |\theta| < \delta\} \tag{4.25}$$

which are locally invariant under the one-step mapping $\Phi(\varepsilon, \cdot)$. Suppose that $(\theta, u) = (\theta, h(\varepsilon, 0, \theta)) \in W_\varepsilon$ and that $(\theta_1, u_1) := \Phi(\varepsilon, \theta, u)$ satisfies $|\theta_1| < \delta$. Then from $G(\varepsilon, \varepsilon, \theta, u) = \Phi(\varepsilon, \theta, u)$ and the invariance of $\tilde{M}_\varepsilon$ from (4.5) we obtain

$$u_1 = h(\varepsilon, 1, \theta_1) = h(\varepsilon, 0, \theta_1), \text{ i.e. } (\theta_1, u_1) \in W_\varepsilon.$$

Summarizing we arrive at the following corollary.

Corollary 4.2 *Let the condition* (H1) – (H3) *hold. Then there exists an* $\varepsilon_0 > 0$ *and invariant* C^{k-1}*-manifolds for the one-step method which are of the form* (4.25) *and satisfy*

$$h(\varepsilon, 0, 0) = 0, \quad D_\theta h(\varepsilon, 0, \theta) = 0, \tag{4.26}$$

$$\sup_{|\theta|<\delta} |h(\varepsilon, 0, \theta) - \tilde{h}(\theta)| \leq C\varepsilon^2 \tag{4.27}$$

where $\tilde{h}$ *defines a suitable locally invariant* C^k*-center manifold of the system* (1.1).

Remark 4.3 As in Theorem 2.3 we can allow in the corollary a general set of eigenvalues on the imaginary axis. Similarly the results can be extended to general one-step schemes. However, the relation $D_\theta h(\varepsilon, 0, 0) = 0$ will only hold if the growth function of the method has the property mentioned in Remark 2.4. In general, the invariant manifold W_ε need not be a center manifold for $\Phi(\varepsilon, \cdot)$ and this is discussed in [5]. Such a result is also compatible with the more general discretization theorems for 'pseudo - hyperbolic' manifolds in [14]. Finally, we notice that we can extend the estimate in (4.10) to the θ-derivatives and obtain

$$|D_\theta^j h(\varepsilon, \tau, \theta) - D_\theta^j \tilde{h}(\theta)| \leq c\varepsilon^{\min(2, k-1-j)}, \; j = 1, \cdots, k-1.$$

This will then yield corresponding estimates in (4.27) as in [14].

5 An example

For an illustration of the results from section 3 and section 4 we consider as an example the simplest system with a saddle node

$$\begin{aligned} \theta' &= \theta^2 \\ u' &= -u. \end{aligned} \tag{5.1}$$

It is easy to see that the exact flow is

$$F(t,x) = \begin{pmatrix} \theta/(1-t\theta) \\ u\exp(-t) \end{pmatrix} \tag{5.2}$$

where $x = (\theta, u)$. This system does not satisfy the global condition (H1). But we will only consider it in a bounded domain $|\theta| \le 1$, $|u| \le 1$ and assume that it has been cut off outside.

Consider the discretization of (5.1) by the centered Euler method

$$\begin{aligned} \frac{\theta_{n+1} - \theta_n}{\varepsilon} &= (\frac{\theta_{n+1} + \theta_n}{2})^2 \\ \frac{u_{n+1} - u_n}{\varepsilon} &= -\frac{u_{n+1} + u_n}{2}. \end{aligned} \tag{5.3}$$

This does not have a unique solution θ_{n+1} globally. But if we consider the bounded domain above and add the ε-restriction

$$\Delta = 1 - 2\varepsilon\theta \ge 0$$

we obtain the one-step map

$$\Phi(\varepsilon, x) = \left(\frac{2\theta + \frac{1}{2}\varepsilon\theta^2}{1 - \frac{1}{2}\varepsilon\theta + \sqrt{\Delta}}, \ \frac{1 - \frac{1}{2}\varepsilon}{1 + \frac{1}{2}\varepsilon} u \right)^T. \tag{5.4}$$

For simplicity we choose a cut-off function in C^3 rather than in C^∞ for constructing the interpolated equation (2.2) as follows

$$\chi_0(\tau) = \begin{cases} 1 & , \ \tau \le 0 \\ (1-\tau^3)^3 & , \ 0 < \tau < 1 \\ 0 & , \ \tau \ge 1 \end{cases}$$

and define $\chi_1(\tau) = 1 - \chi_0(\tau)$. Using the same procedure as described in Section 2, we get the flow $G(t, \varepsilon, x)$ of (2.2) from equation (2.10).

In the following discussion, we fix the step-size at $\varepsilon = 0.1$.

In Figure 5.1, we show how the exact solution $F(t, x_0)$, the numerical solution $\Phi^n(\varepsilon, x_0)$ and the interpolated solution $G(t, \varepsilon, x_0)$ fit to each other. At time $t = 0$, we choose the initial value $x_0 = (-1, -1)$.

For $\theta > 0$ the center manifold of the equation (5.1) (resp. (2.2)) is unique, and given by $u = 0$. But for $\theta < 0$ it is nonunique and can be any of the curves

$$h_c(\theta) = ce^{1/\theta}, \quad c \in \mathbb{R}.$$

We assume that the cut-off has been done in such a way that the point $x_0 = (-1, -1)$ is on the manifold, i.e.

$$\tilde{h}(\theta) = -e^{1+1/\theta}, \quad \theta < 0.$$

In Figure 5.2 we show the invariant manifold M_ε of the interpolated equation (2.2) with t restricted to $[0, \varepsilon]$.

As we know, the invariant manifold M_ε of the interpolated equation (2.2) is ε-periodic in t. Therefore, in Figure 5.3 we redraw the invariant manifold using cylindrical coordinates with minimal radius 0.2. The coordinates in Figure 5.3 can be expressed in terms of those from Figure 5.2 as follows

$$\begin{aligned}
&\tilde{t} = (u - 0.2)\cos(2\pi t/\varepsilon) \\
&\tilde{\theta} = \theta \\
&\tilde{u} = -(u - 0.2)\sin(2\pi t/\varepsilon)
\end{aligned}$$

In Figure 5.4, we plot the difference between the center manifold of (5.1) and the invariant manifold of the corresponding interpolated equation (2.2), given by $\tilde{h}(\theta) - h(\varepsilon, t/\varepsilon, \theta)$.

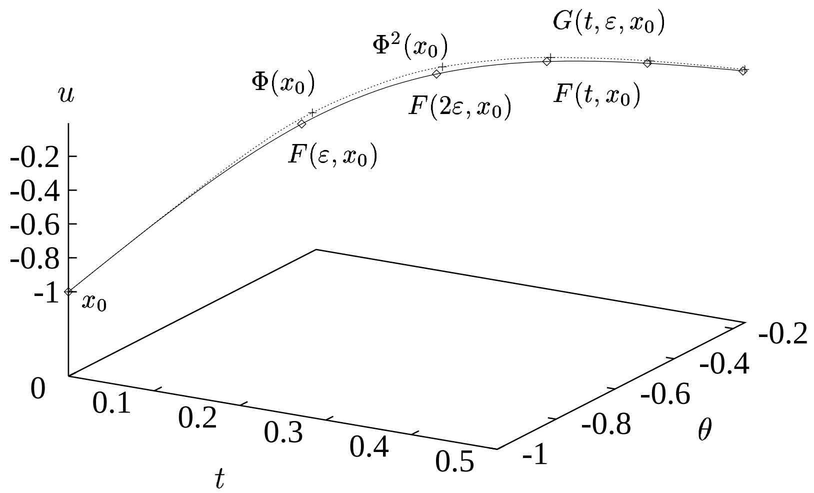

Figure 5.1 Exact versus interpolated flow.

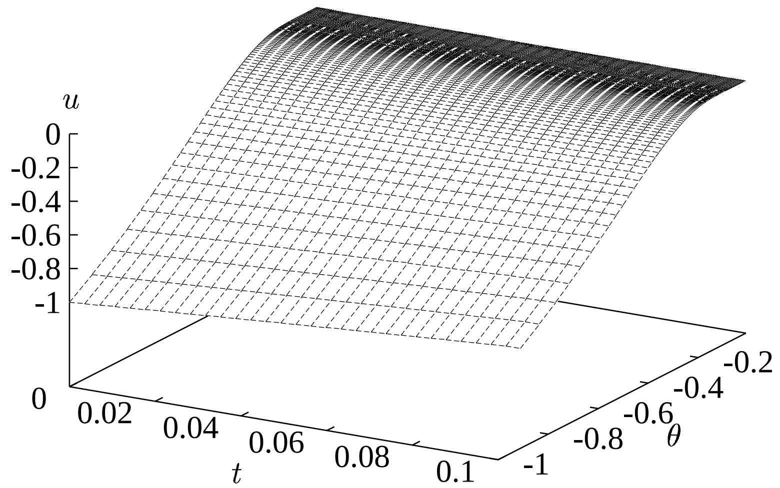

Figure 5.2 The invariant manifold of the interpolated system.

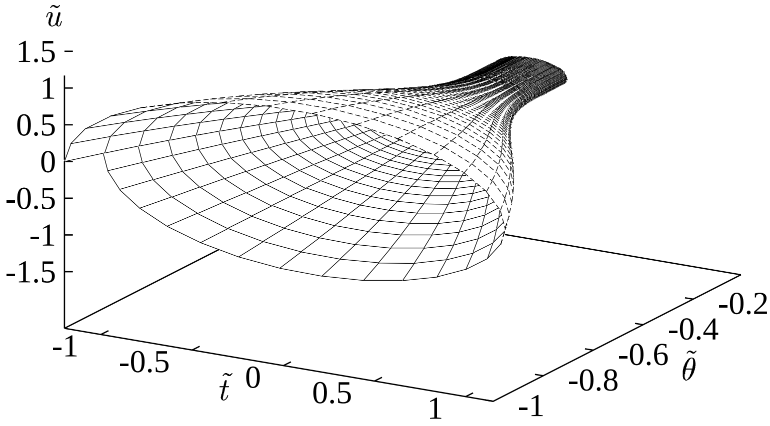

Figure 5.3 The invariant manifold in cylindrical coordinates.

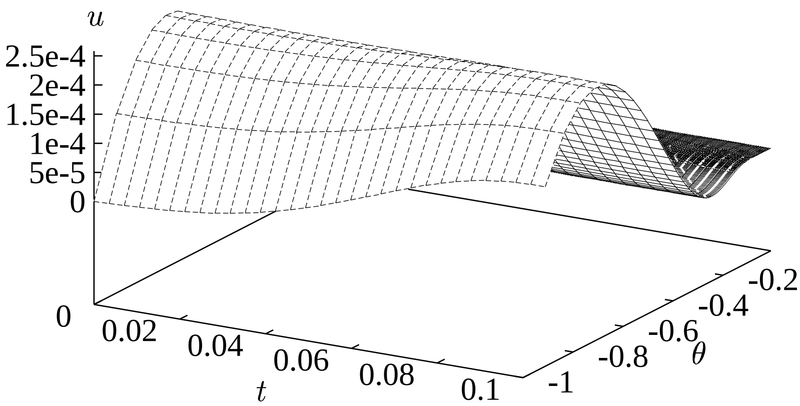

Figure 5.4 Difference of center manifolds.

6 Dependence on parameters

In this final section we discuss the smoothness of the rapidly oscillating manifolds in Theorem 4.1 with respect to the parameter ε and with respect to further parameters in the given system (1.1).

We were not able to prove smoothness of the function h in (4.3) with respect to ε at $\varepsilon = 0$. This is plausible from the fact that the ε-derivative of the perturbation term in (2.2) is

$$\varepsilon^2 D_\varepsilon g(\varepsilon, t/\varepsilon, x) - t D_\tau g(\varepsilon, t/\varepsilon, x)$$

which does not converge as $\varepsilon \to 0$ and is bounded as $\varepsilon \to 0$ only on finite time intervals. It is possible however to prove smoothness in ε for $\varepsilon > 0$ and to derive bounds on $D_\varepsilon h$ as $\varepsilon \to 0$.

Proposition 6.1 *Under the assumptions of Theorem* 4.1 *with* $l \geq 2$, *the derivative* $D_\varepsilon h(\varepsilon, \tau, \theta)$ *exists, is continuous in* $(0, \varepsilon_0) \times \mathbb{R} \times (-\delta, \delta)$ *and satisfies an estimate*

$$|D_\varepsilon h(\varepsilon, \tau, \theta)| \leq \frac{c}{\varepsilon}, \; 0 < \varepsilon < \varepsilon_0, \; \tau \in \mathbb{R}, \; |\theta| < \delta. \tag{6.1}$$

Proof. We do not give all the details of the proof which is quite similar to the proof of the C^1-smoothness of $h(\varepsilon, \tau, \cdot)$ in Theorem 4.1. The basic idea is to use the norm

$$\|h\|_\varepsilon = \|h\| + \|D_\theta h\| + \varepsilon \|D_\varepsilon h\|$$

and to prove contraction of the operator T from (4.17) on the set

$$Z = \Big\{ h \in X : \|h\|_\varepsilon \leq q \text{ and } |D_\theta h(\varepsilon, \tau, \theta_1) - D_\theta h(\varepsilon, \tau, \theta_2)| \leq q_1 |\theta_1 - \theta_2| \\ \text{for } 0 < \varepsilon < \varepsilon_0, \; \theta_1, \theta_2 \in \mathbb{R} \Big\}.$$

In addition to (4.19), (4.20) we have bounds on the terms $|D_\varepsilon H_i|$, $|D_\theta H_i|$ and $|D_u H_i|$ of the type $c[\delta\omega(\delta) + \delta^2 + \varepsilon^2\delta]$, $\varepsilon\omega(\delta)$ and $\varepsilon(\omega(\delta) + \varepsilon^2)$ and on their Lipschitz contants of type $\omega(\delta) + c\varepsilon^2$, $c\varepsilon$ and $c\varepsilon$ respectively. The crucial step is to bound $\varepsilon D_\varepsilon(Th)$ and to compute its Lipschitz constant with respect to h. The bound turns out to be of the form $c(\delta\omega(\delta) + \delta^2 + \varepsilon) + q(\omega(\delta) + \varepsilon^2)$ and the Lipschitz constant to be $c(\omega(\delta) + \varepsilon) + c\varepsilon q_1$. Contraction on Z is then obtained by a suitable choice of ε_0, δ, q and q_1. ■

There is no problem in dealing with further parameters in the given system, such as

$$\dot{x} = f(x, \mu), \quad (x, \mu) \in \mathbb{R}^m \times \mathbb{R}. \tag{6.2}$$

Similar to section 2 we assume for some small $\mu_0 > 0$

(HP1) f and its derivatives up to order $k \geq 4$ are continuous and uniformly bounded for $x \in \mathbb{R}^m$, $|\mu| < \mu_0$.

(HP2) $f(0, \mu) = 0$ for all $|\mu| < \mu_0$.

(HP3) $D_x f(0, \mu)$ has a simple zero eigenvalue 0 with right eigenvector $e_r(\mu)$ which is C^{k-1}-smooth for $|\mu| < \mu_0$. The remaining eigenvalues of $D_x f(0, \mu)$ have nonzero real parts.

Condition (HP3) seems rather artificial at first sight, because in generic systems the saddle node will be destroyed by an additional parameter. However, in the application to the saddle node homoclinic orbit we will have a two dimensional parameter plane in which a curve of saddle nodes exists (see [8]). Condition (HP3) is then satisfied if μ is used for parametrizing the saddle node curve.

The centered Euler scheme for (6.2) reads

$$\frac{x_{n+1} - x_n}{\varepsilon} = f(\frac{x_{n+1} + x_n}{2}, \mu) \tag{6.3}$$

and it determines a family of C^k-maps

$$x_{n+1} = \Phi(\mu, \varepsilon, x_n), \quad n = 0, 1, 2, \ldots . \tag{6.4}$$

The interpolated systems are now of the form

$$\dot{x}(t) = f(x(t), \mu) + \varepsilon^2 g(\mu, \varepsilon, t/\varepsilon, x(t)). \tag{6.5}$$

By a straightforward extension of Theorem 2.3 we can prove the following results.

Theorem 6.2 *Let the conditions* (HP1) – (HP3) *hold. Then there exist* $\varepsilon_0 > 0$, $\mu_1 > 0$ *and a vector field* $g \in C^{k-3}((-\mu_1, \mu_1) \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{R} \times \mathbb{R}^m, \mathbb{R}^m)$ *such that the following properties hold for all* $|\mu| < \mu_1$, $|\varepsilon| < \varepsilon_0$, $\tau \in \mathbb{R}$ *and* $x \in \mathbb{R}^m$.

(i) *the map* $(\mu, \varepsilon, \tau, x) \to \varepsilon^2 g(\mu, \varepsilon, \tau, x)$ *is of class* C^{k-1} *and* 1*-periodic in* τ,

(ii) $|g(\mu, \varepsilon, \tau, x)| + |D_x g(\mu, \varepsilon, \tau, x)| + |D_\mu g(\mu, \varepsilon, \tau, x)| \leq C$ *for some* $C > 0$,

(iii) $g(\mu, \varepsilon, \tau, 0) = 0$ *and* $D_x g(\mu, \varepsilon, \tau, 0)$ *has the simple eigenvalue* 0 *with right eigenvector* $e_r(\mu)$ *and no other eigenvalues on the imaginary axis,*

(iv) *For* $\varepsilon \neq 0$ *we have* $G(\mu, \varepsilon, \varepsilon, x) = \Phi(\mu, \varepsilon, x)$ *where* $G(\mu, t, \varepsilon, x)$ *is the* t*-flow of the system* (6.5) *with* $G(\mu, 0, \varepsilon, x) = x$.

For completeness we will also state the μ-dependent analogue of Theorem 4.1 and Corollary 4.2.

Theorem 6.3 *Assume that the conditions* (HP1)-(HP3) *are satisfied. Then there exist constants* $\mu_2, \varepsilon_1, \delta, c > 0$ *and functions*

$$h \in C((-\mu_2, \mu_2) \times (0, \varepsilon_1) \times \mathbb{R} \times (-\delta, \delta), \mathbb{R}^{m-1}),$$
$$\tilde{h} \in C^{k-1}((-\mu_2, \mu_2) \times (-\delta, \delta), \mathbb{R}^{m-1})$$

with the following properties.

(i) $h(\mu, \varepsilon, \tau, \theta)$ *is of class* C^1 *in the variables* (μ, τ, θ) *and of class* C^{k-1} *in the variables* (μ, θ). *Moreover*

$$h(\mu, \varepsilon, \tau, 0) = 0, \;\; D_\theta h(\mu, \varepsilon, \tau, 0) = 0, \;\; h(\mu, \varepsilon, \tau+1, \theta) = h(\mu, \varepsilon, \tau, \theta),$$

(ii) *The manifold*

$$\{(\mu, \varepsilon, t, \theta, h(\mu, \varepsilon, t/\varepsilon, \theta)) : |\mu| < \mu_2, \;\; 0 < \varepsilon < \varepsilon_1, \;\; t \in \mathbb{R}, \;\; |\theta| < \delta\}$$

is locally invariant for the system (6.5),

(iii) $\tilde{h}(\mu, 0) = 0$, $D_\theta \tilde{h}(\mu, 0) = 0$ *and the manifold*

$$\{(\mu, \theta, \tilde{h}(\mu, \theta)) : |\mu| < \mu_2, \;\; |\theta| < \delta\}$$

is locally invariant for the system (6.2),

(iv)

$$|h(\mu, \varepsilon, \tau, \theta) - \tilde{h}(\mu, \theta)| + |D_\theta h(\mu, \varepsilon, \tau, \theta) - D_\theta \tilde{h}(\mu, \theta)|$$
$$+ |D_\mu h(\mu, \varepsilon, \tau, \theta) - D_\mu \tilde{h}(\mu, \theta)| \leq c\varepsilon^2,$$

(v) *the manifolds*

$$M_{\mu,\varepsilon} = \{(\theta, h(\mu, \varepsilon, 0, \theta)) : |\mu| < \mu_1, 0 < \varepsilon < \varepsilon_1, |\theta| < \delta\}$$

are locally invariant for the one step map $\Phi(\mu, \varepsilon)$.

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