Homework
Waves in Evolution Equations
Summer term 2017

Wolf-Jürgen Beyn
Christian Döding

Due: Wed. May 24, 12:00, V3-128, mailbox 128 (Christian Döding)

Tutorial: Tue. 30.05. 2017, 14-16, V5-148

Exercise 8: [Euclidean transformations and matrix exponential]
The special Euclidean group $\mathrm{SE}(d)$ and its Lie algebra $\mathrm{se}(d)$ may be represented in $\mathbb{R}^{d+1,d+1}$ as

$$\mathrm{SE}(d) = \left\{ \begin{pmatrix} Q & b \\ 0 & 1 \end{pmatrix} : Q \in \mathbb{R}^{d,d}, Q^\top Q = I_d, \det(Q) = 1, b \in \mathbb{R}^d \right\},$$
$$\mathrm{se}(d) = \left\{ \begin{pmatrix} S & a \\ 0 & 0 \end{pmatrix} : S \in \mathbb{R}^{d,d}, S^\top = -S, a \in \mathbb{R}^d \right\}.$$

Compute the matrix exponential $\exp\left(\begin{pmatrix} S & a \\ 0 & 0 \end{pmatrix} \right)$ for an element in $\mathrm{se}(d)$ and show that it belongs to $\mathrm{SE}(d)$. Conversely, show that every element of $\mathrm{SE}(d)$ can be written as the matrix exponential of an element from $\mathrm{se}(d)$.
Hint: Solve the linear homogeneous differential equation

$$Y' = \begin{pmatrix} S & a \\ 0 & 0 \end{pmatrix} Y, \quad Y(0) = I_{d+1},$$

in block form and evaluate the solution at $t = 1$.

(7 points)

Exercise 9: [Equivariance of the Navier Stokes equation]
The three-dimensional Navier-Stokes equation describes the incompressible viscous flow in a domain (here $\mathbb{R}^3$) through the evolution equation

$$\begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}_t = \begin{pmatrix} \frac{1}{R}\Delta_x u - u_x u - p_x^T \\ \mathrm{tr}(u_x) \end{pmatrix}, \quad x \in \mathbb{R}^3, t \geq 0. \tag{NSE}$$

Here $u : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ is the velocity field and $p : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$ is the pressure field. The constant $R > 0$ is called the Reynolds number, by $u_x(x,t) \in \mathbb{R}^{3,3}$ we denote the total derivative, and its trace $\mathrm{tr}(u_x) = \mathrm{div} u$ is called the divergence of u. Show that this evolution equation is equivariant with respect to the following action of the special Euclidean group SE(3):

$$\left[a(\gamma) \begin{pmatrix} u \\ p \end{pmatrix} \right](x,t) = \begin{pmatrix} Qu \\ p \end{pmatrix} (Q^T(x-b), t) \quad \text{for} \quad \gamma = \begin{pmatrix} Q & b \\ 0 & 1 \end{pmatrix} \in \mathrm{SE}(3).$$

(7 points)