Homework
Waves in Evolution Equations
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Due: Wed. May 31, 12:00, V3-128, mailbox 128 (Christian Döding)

Tutorial: Tue. 06.06 2017, 14-16, V5-148

Exercise 10: [Characterization of a Sobolev space]
From the lecture we know that that the shift map

$$a(\cdot)v : \mathbb{R} \to L^2(\mathbb{R}, \mathbb{R}^m), \quad [a(\gamma)v](x) = v(x-\gamma), x \in \mathbb{R}$$

is continuously differentiable for a fixed $v \in H^1(\mathbb{R}, \mathbb{R}^m)$ with derivative at $\gamma = 0$ given by

$$d_\gamma[a(0)v] = -v', \quad v' = \text{ weak derivative.}$$

Show the converse, i.e. if $v \in L^2(\mathbb{R}, \mathbb{R}^m)$ and the limit

$$w := \lim_{h \to 0} \frac{1}{h}(v(\cdot + h) - v(\cdot))$$

exists in $L^2(\mathbb{R}, \mathbb{R}^m)$ then $v \in H^1(\mathbb{R}, \mathbb{R}^m)$ and the weak derivative v' of v agrees with w.
Remark: This shows that the infinitesimal generator of the shift semigroup

$$\varphi^\gamma = a(\gamma) : L^2(\mathbb{R}, \mathbb{R}^m) \to L^2(\mathbb{R}, \mathbb{R}^m), \gamma \in \mathbb{R}$$

has domain $H^1(\mathbb{R}, \mathbb{R}^m)$ and agrees with $-\frac{d}{dx}$.

(7 points)

Exercise 11: [Travelling waves of a viscous conservation law]
For a given $f \in C^1(\mathbb{R}, \mathbb{R})$ consider a scalar parabolic equation in so-called conservation form

$$u_t = u_{xx} - [f(u)]_x, \quad x \in \mathbb{R}, \quad t \geq 0, \tag{1}$$

(a) Set up the ODE that is satisfied by the profile of a travelling wave.

(b) Let $v_+ < v_-$ be given in $\mathbb{R}$ such that the line $\ell(v), v \in \mathbb{R}$ determined by $\ell(v_\pm) = f(v_\pm)$ satisfies the following properties (sketch !)

1. $f(v) < \ell(v)$ for all $v_+ < v < v_-$,
2. $f'(v_+) < \ell'(v_+) = \ell'(v_-) < f'(v_-)$.

Show that (1) has a travelling wave with a profile $v_\star$ which satisfies a first order ODE as well as $\lim_{\xi \to \pm\infty} v_\star(\xi) = v_\pm$. Determine the speed of the wave.

Hint: use (and prove!) the following simple fact. Given $g \in C^1(\mathbb{R}, \mathbb{R})$ and two consecutive zeroes $v_+ < v_-$ of g such that $g(v) \neq 0$ for all $v_+ < v < v_-$. Then any solution of $v' = g(v)$ with $v_+ < v(0) < v_-$ exists for all times and satisfies either $\lim_{\xi \to \pm\infty} v(\xi) = v_\pm$ or $\lim_{\xi \to \pm\infty} v(\xi) = v_\mp$.

(7 points)