Homework
Waves in Evolution Equations
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Due: Wed. June 14, 12:00, V3-128, mailbox 128 (Christian Döding)

Tutorial: Tue. 20.06 2017, 14-16, V5-148

Exercise 14: [The space H^1_{Eucl}]
For the analysis of rotating waves we used the function space

$$H^1_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{R}^m) = \{v \in H^1(\mathbb{R}^d,\mathbb{R}^m) : \mathcal{L}_S v \in L^2(\mathbb{R}^d,\mathbb{R}^m)\ \forall S \in \mathrm{so}(\mathbb{R}^d)\}$$

with norm

$$\|v\|^2_{H^1_{\mathrm{Eucl}}} = \|v\|^2_{H^1} + \sup\{\|\mathcal{L}_S v\|^2_{L^2} : S \in \mathrm{so}(\mathbb{R}^d),\ |S|_2 \le 1\}, \tag{1}$$

where $\mathcal{L}_S v(x) = v_x(x)Sx, x \in \mathbb{R}^d, S \in \mathrm{so}(\mathbb{R}^d)$. Using the Hilbert space property of H^1 prove that $H^1_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{R}^m)$ is a Banach space with respect to the norm (1). Can you turn it into a Hilbert space after replacing $\|\cdot\|_{H^1_{\mathrm{Eucl}}}$ by a different but equivalent norm?
Hint: Consider a basis of $\mathrm{so}(\mathbb{R}^d)$.

(6 points)

Exercise 15: [Self-similar solutions of Burgers' equation]

(a) Show that $G = (0,\infty) \ltimes \mathbb{R}$ is a Lie group with respect to the operation

$$g \circ \gamma = (g_1, g_2) \circ (\gamma_1, \gamma_2) = (g_1\gamma_1, g_2 + g_1\gamma_2).$$

(b) As in Exercise 11 and 12 consider Burgers' equation

$$u_t = u_{xx} - \frac{1}{2}[u^2]_x =: F(u), \quad x \in \mathbb{R}, \quad t \ge 0 \tag{2}$$

and the group action

$$[a(g)u](x) = \frac{1}{g_1} u\left(\frac{x - g_2}{g_1}\right), \quad x \in \mathbb{R}, \quad g = (g_1, g_2) \in G$$

on sufficiently smooth functions $u : \mathbb{R} \to \mathbb{R}$. Show that F satisfies the generalized equivariance condition

$$F(a(g)u) = \frac{1}{g_1^2} a(g)F(u), \quad g \in G.$$

(c) For a relative equilibrium of (2) with respect to this group action use the ansatz

$$u(\cdot,t) = a(g(\tau(t)))v_\star, \quad t \in [0,T)$$

for some smooth profile $v_\star : \mathbb{R} \to \mathbb{R}$, some group orbit $g \in C^1([0,\infty), G)$ and some time transformation $\tau \in C^1([0,T),[0,\infty))$ satisfying $\tau(0) = 0$ and $\tau' > 0$ in $[0,T)$. Which differential equation is satisfied by the profile $v_\star$?

Hint: It is enough to solve these problems in a formal sense.

(8 points)