Homework
Waves in Evolution Equations
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Due: Wed. June 21, 12:00, V3-128, mailbox 128 (Christian Döding)

Tutorial: Tue. 27.06 2017, 14-16, V5-148

Exercise 16: [Existence of travelling waves]
Consider a parabolic equation

$$u_t = u_{xx} + f(u), x \in \mathbb{R}, t \geq 0, \tag{1}$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$ has the following properties (compare Exercise 2 for an example)

a) $f(b_j) = 0, j = 1, \ldots, 5$ for some $b_1 = 0 < b_2 < b_3 < b_4 < b_5 = 1$,

b) $f < 0$ in $(b_1, b_2), (b_3, b_4)$ and $f > 0$ in $(b_2, b_3), (b_4, b_5)$,

c) $f'(b_j) < 0$ for $j = 1, 3, 5$,

d) $\int_{b_j}^{b_{j+2}} f(x)dx > 0$ for $j = 1, 3$.

Determine an interval $[-\tilde{c}, 0]$ such that (1) has a travelling wave solution u_j with velocity $c_j \in [-\tilde{c}, 0]$ connecting b_j to b_{j+2} for $j = 1, 3$. Give an explicit expression for $\tilde{c}$ in terms of f and the data appearing in a)-d).
Hint: Apply the existence theorem from the lecture to suitably modified nonlinearities.

(7 points)

Exercise 17: [Approximate solutions on the unstable manifold]
Consider the ODE which determines a travelling wave of fixed speed $c \in \mathbb{R}$ for a parabolic equation (1) with $f \in C^2(\mathbb{R}, \mathbb{R})$:

$$w' = G(w) = \begin{pmatrix} w_2 \\ -f(w_1) - cw_2 \end{pmatrix}.$$

Assume $f(0) = 0, f'(0) < 0$ and let $\lambda_- < 0 < \lambda_+$ be the eigenvalues of $DG(0)$ with eigenvectors y_-, y_+. Show that the functions $w(t) = \rho \exp(\lambda_+ t) y_+, t \leq 0$ satisfy for some $C > 0$ and for $\rho > 0$ sufficiently small

$$|w'(t) - G(w(t))| \leq C\rho^2 \exp(2\lambda_+ t), \quad \forall t \leq 0.$$

(7 points)