# Homework
# Waves in Evolution Equations
## Summer term 2017

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**Due: Wed. July 19, 12:00, V3-128, mailbox 128 (Christian Döding)**

Tutorial: Tue. 25.07.2017, 14-16, V5-148

**Exercise 24:** [Eigenvalues of a periodic boundary value problem]
Repeat the computation from Exercise 23 for the case of periodic boundary conditions

$$\mathcal{L}v = \lambda v \quad \text{in} \quad J = [x_-, x_+], \quad v(x_-) = v(x_+),\ v'(x_-) = v'(x_+). \tag{1}$$

Set up the discretized eigenvalue problem with $h = \frac{x_+ - x_-}{N}$ for the scalar case $m = 1$

$$L_h v_h = \lambda v_h, \quad v_h \in \mathbb{R}^N,$$

where $v_h(x_i), x_i = x_- + ih, i = 0, \ldots, N-1$ denote approximations of $v(x_i)$ $i = 0, \ldots, N-1$. Use central differences at the grid points $x_i, i = 0, \ldots, N-1$ for all terms of the differential equation. Implement the boundary conditions by setting $v_h(x_{-1}) = v_h(x_{N-1})$ and $v_h(x_N) = v_h(x_0)$. Draw the eigenvalues in the complex plane in two diagrams for the following cases

$$\begin{aligned} h =& 1, \quad -x_- = x_+ = 10, 100, 250, 500, 750, 1000, \\ h =& 2^{-n}, n = 0, \ldots, 5, \quad -x_- = x_+ = 10. \end{aligned}$$

For comparison include in your diagram the two parabolas

$$\lambda = -\omega^2 + i\mu_\star \omega + Df(v_\pm), \quad \omega \in \mathbb{R}.$$

(7 points)

**Exercise 25:** [Eigenvalues of the linearization about a rotating wave]
For $\theta \in \mathbb{R}$ define rotation and reflection matrices by

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad S_\theta = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $u(x,t) = v_\star(R_{-\omega t} x), x \in \mathbb{R}^2, t \in \mathbb{R}$ be a wave solution with profile $v_\star \in C_b^3(\mathbb{R}^2, \mathbb{R}^m)$ and angular velocity $\omega \neq 0$ of the reaction diffusion system

$$u_t = \Delta u + f(u), \quad x \in \mathbb{R}^2, t \geq 0, \quad u(x,t) \in \mathbb{R}^m, \quad f \in C^1(\mathbb{R}^m, \mathbb{R}^m).$$

Then the profile $v_\star$ solves the rotating wave equation equation (see lecture)

$$0 = \Delta v + v_x S_\omega x + f(v), \quad x \in \mathbb{R}^2. \tag{2}$$

Show that the linear differential operator obtained by linearizing about $v_\star$

$$\mathcal{L}v = \Delta v + v_x S_\omega x + Df(v_\star)v \quad v \in C_b^2(\mathbb{R}^2, \mathbb{R}^m).$$

has eigenvalues $0$ and $\pm i\omega$ with corresponding eigenfunctions $v_{\star,x} S_1 x$ and $D_1 v_\star \pm i D_2 v_\star$
**Hint:** Differentiate (2) with respect to $x_1$ and $x_2$.

(7 points)