A direct approach to homoclinic orbits in the fast dynamics of singularly perturbed systems

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Abstract

Homoclinic orbits in the fast dynamics of singular perturbation problems are usually analyzed by a combination of Fenichel's invariant manifold theory with general transversality arguments (see [26] and the Exchange Lemma in [14]). In this paper an alternative direct approach is developed which uses a two–time scaling and a contraction argument in exponentially weighted spaces. Homoclinic orbits with one fast transition are treated and it is shown how ϵ–expansions can be extracted rigorously from this approach. The result is applied to a singularly perturbed Bogdanov point in the FitzHugh–Nagumo system.

1 Introduction

In this paper a constructive approach is developed for proving the existence of homoclinic orbits in singularly perturbed systems of the form

$$\begin{aligned} \dot{x} &= f(x,y,\lambda,\epsilon) \in \mathbb{R}^n, \qquad n \geq 2 \\ \dot{y} &= \epsilon \cdot g(x,y,\lambda,\epsilon) \in \mathbb{R}\ . \end{aligned} \tag{1.1}$$

More precisely, we assume a homoclinic orbit within the fast $\epsilon = 0$ system $\dot{x} = f(x,y,\lambda,0)$ which is continued for $\epsilon \neq 0$ along two parameter curves $\lambda^+(\epsilon)$ and $\lambda^-(\epsilon)$. A typical unfolding diagram for $x \in \mathbb{R}^2$ is depicted in Fig.1 below.

The indicated existence result is essentially known and can be proved in a rather geometric way with a general transversality argument [26] which depends heavily on the complete machinery of invariant manifold theory [11]. An extension of that approach leads to the Exchange Lemma [14] which allows to treat certain homoclinic orbits with multiple fast and slow transitions. However it seems difficult to extract from this geometric approach the precise quantitative behavior of the solutions, for example the ϵ-expansions or the asymptotic

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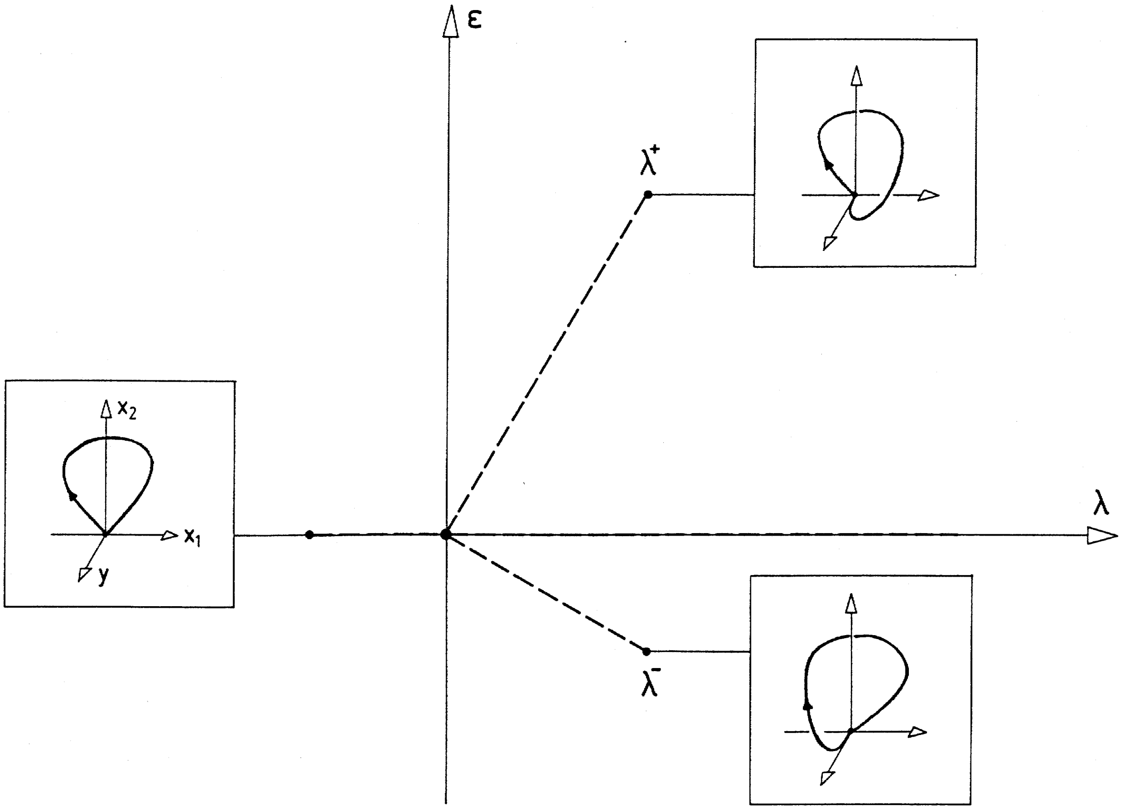

Figure 1: (λ, ϵ)–unfolding diagram of homoclinic orbits.

time estimates. In [16] this was taken as a motivation in order to develop an extended and formula oriented version of the Exchange Lemma for $x \in \mathbb{R}^2$ which allows a bifurcation analysis with the implicit function theorem.

In this paper we use a completely different approach for handling the homoclinic orbits of the system (1.1) with one fast transition. Our approach is based on a two time scaling for the slow y-variable of the form

$$y(t, \epsilon t, \epsilon) \; = \; \epsilon \cdot \mathrm{e}^{-\epsilon t} \cdot z(t, \epsilon) \tag{1.2}$$

and it uses the contraction mapping principle in appropriate spaces of exponentially decaying functions. The corresponding abstract framework and the details of the proof are given in sections 3 and 4. In section 7 we show how the approach can be extended to several slow variables $y \in \mathbb{R}^m$.

A particular outcome of our construction is that one directly obtains the leading order approximations of the solution branches without appealing to general invariant manifold theory [11]. Only a pretransformation of the system is performed which employs a classical local center manifold. As originally proposed by Chow, Hale, Mallet–Paret [6] we solve boundary value problems in suitable function spaces and our techniques are close to those developed by Lin [17], [18] though the specific situation considered here is not analyzed in his papers.

Another motivation for our approach is provided by the numerical results in [2], [3]. These suggest to develop formulations of boundary value problems that are well posed uniformly in ϵ and hence are amenable to numerical computations.

We apply the results to the classical example of the FitzHugh–Nagumo equation. In particular we examine a highly degenerate situation obtained from a singularly perturbed Bogdanov point [22] which occurs in the FitzHugh–Nagumo equation when a certain parameter tends to zero.

2 The setting and main results

Consider the system

$$\begin{aligned} \dot{u} &= F(u,v,\lambda,\epsilon) \in \mathbb{R}^n, \qquad n \geq 2 \\ \dot{v} &= \epsilon \cdot G(u,v,\lambda,\epsilon) \in \mathbb{R} \end{aligned} \tag{2.1}$$

where $(F,G) \in C^k(\mathbb{R}^{n+3}, \mathbb{R}^{n+1})$ and k is sufficiently large. We assume the steady state conditions

$$\begin{gathered} (F,G)^0 := (F,G)(0,0,0,0) = 0, \qquad F_u^0 \quad \text{hyperbolic}, \\ \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}^0 \quad \text{regular} \end{gathered} \tag{2.2}$$

implying the existence of a unique function $(u,v)^s(\lambda,\epsilon)$ with

$$\begin{gathered} (F,G)((u,v)^s(\lambda,\epsilon),\lambda,\epsilon) = 0, \qquad (u,v)^s(0,0) = 0, \\ \begin{pmatrix} u_\lambda^s \\ v_\lambda^s \end{pmatrix}(0,0) = -\begin{pmatrix} F_u^0 & F_v^0 \\ G_u^0 & G_v^0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} F_\lambda^0 \\ G_\lambda^0 \end{pmatrix}. \end{gathered} \tag{2.3}$$

Concerning the dynamical properties we assume that the system $\dot{u} = F(u,0,0,0)$ has a homoclinic orbit $\bar{u}(\tau)$ with base point $u = 0$ such that the only bounded solutions (u,λ) of the variational equation

$$\dot{u} = F_u(\bar{u}(\tau),0,0,0) \cdot u + [F_u, F_v, F_\lambda](\bar{u}(\tau),0,0,0) \cdot \begin{pmatrix} u_\lambda^s(0,0) \\ v_\lambda^s(0,0) \\ 1 \end{pmatrix} \cdot \lambda \tag{2.4}$$

are $u = c\dot{\bar{u}}$, $c \in \mathbb{R}$ and $\lambda = 0$. As explained below this can be interpreted as a transversality or Melnikov condition.

Under these assumptions homoclinic orbits of the system (2.1) typically appear at the (λ,ϵ)–curves indicated in Fig.1; i.e. there exists a trivial curve of homoclinic orbits on the λ-axis and two nontrivial curves of homoclinic orbits for $\epsilon \neq 0$ which meet at $(\lambda,\epsilon) = (0,0)$ in the central homoclinic orbit $(u,v) = (\bar{u}(\tau),0)$. This unfolding picture of homoclinic orbits is derived after some standard transformations and the crucial scaling transformation (1.2) have been applied. The purpose of this section is to introduce these transformations and to formulate the precise properties of the emanating homoclinic orbits.

At first, the (u,v)-linearization at the base point of the unperturbed homoclinic orbit satisfies

$$N[\begin{pmatrix} F_u^0 & F_v^0 \\ 0 & 0 \end{pmatrix}] = \mathbb{R} \cdot \begin{pmatrix} -F_u^{0^{-1}} \cdot F_v^0 \\ 1 \end{pmatrix}$$

where N denotes the kernel of a matrix. Thus there exists a locally defined center manifold $u^c(v,\lambda,\epsilon)$ of class C^k [21] such that

$$u^c(0,0,0) = 0, \qquad u_v^c(0,0,0) = -F_u^{0^{-1}} \cdot F_v^0 \tag{2.5}$$

and the following invariance property holds

$$F[u^c(v,\lambda,\epsilon), v, \lambda, \epsilon] - u_v^c(v,\lambda,\epsilon) \cdot \epsilon \cdot G[u^c(v,\lambda,\epsilon), v, \lambda, \epsilon] = 0. \tag{2.6}$$

Moreover, the center manifold contains the stationary points of the system implying

$$u^c[v^s(\lambda,\epsilon), \lambda, \epsilon] = u^s(\lambda,\epsilon), \qquad F[u^c(v,\lambda,0), v, \lambda, 0] = 0 \tag{2.7}$$

(use (2.6) with $\epsilon = 0$). Then with the setting

$$d(\lambda,\epsilon) := (G_u \cdot F_u^{-1} \cdot F_v - G_v)^{-1}[(u,v)^s(\lambda,\epsilon), \lambda, \epsilon] \neq 0 \tag{2.8}$$

we perform a time scaling and a regular transformation of the coordinates according to

$$\begin{aligned} x(t) &= u(t \cdot d(\lambda,\epsilon)) - u^c[v(t \cdot d(\lambda,\epsilon)), \lambda, \epsilon] \\ y(t) &= v(t \cdot d(\lambda,\epsilon)) - v^s(\lambda,\epsilon) \ . \end{aligned} \tag{2.9}$$

This leads to the system

$$\begin{aligned} \dot{x} &= d(\lambda,\epsilon) \cdot F(u^c[v^s(\lambda,\epsilon) + y, \lambda, \epsilon] + x, v^s(\lambda,\epsilon) + y, \lambda, \epsilon) \\ &\quad -\epsilon \cdot d(\lambda,\epsilon) \cdot u_v^c[v^s(\lambda,\epsilon) + y, \lambda, \epsilon] \cdot G(u^c[v^s(\lambda,\epsilon) + y, \lambda, \epsilon] + x, v^s(\lambda,\epsilon) + y, \lambda, \epsilon) \\ &=: f(x,y,\lambda,\epsilon) \\ \dot{y} &= \epsilon \cdot d(\lambda,\epsilon) \cdot G(u^c[v^s(\lambda,\epsilon) + y, \lambda, \epsilon] + x, v^s(\lambda,\epsilon) + y, \lambda, \epsilon) \\ &=: \epsilon \cdot g(x,y,\lambda,\epsilon). \end{aligned} \tag{2.10}$$

Setting $f_x^0 := f_x(0,0,0,0)$ and using (2.2), (2.3), (2.5)–(2.8) this system satisfies

$$f(0,y,\lambda,\epsilon) \ \equiv \ 0, \qquad |\ \mathrm{Re}\ \sigma[f_x^0]\ | > \ \gamma \ > \ 0, \tag{2.11}$$

$$g(0,0,\lambda,\epsilon) \ \equiv \ 0, \qquad g_y(0,0,\lambda,\epsilon) \ \equiv \ -1 \tag{2.12}$$

where $\sigma[\cdot]$ denotes the spectrum of a matrix. In order to translate the transversality condition (2.4) to the transformed system (2.10) we notice that (2.4) implies that the homogenous equation $\dot{u} - F_u(\bar{u}(\tau),0,0,0)u = 0$ has the only bounded solutions $u = c\dot{\bar{u}}$, $c \in \mathbb{R}$ and that the inhomogenous equation

$$\dot{u} - F_u(\bar{u}(\tau),0,0,0)u = r, \qquad r \quad \text{bounded}$$

has a bounded solution if and only if $\int_{-\infty}^{\infty} \Psi^T(\tau) r(\tau) d\tau = 0$. Here $\Psi(\tau)$ denotes the unique bounded solution (up to a constant multiple) of the adjoint equation $\dot{u} + F_u^T(\bar{u}(\tau), 0, 0, 0) u = 0$ (cf. [19]). Since $\Psi^T(\tau)\dot{\bar{u}}(\tau) \equiv 0$ this applies to $r = \dot{\bar{u}}$ and taking λ–derivatives in (2.10) and using (2.7) we find from (2.4) that the only bounded solutions (x, λ) of the transformed variational equation

$$\dot{x} = f_x(\bar{x}(t), 0, 0, 0) \cdot x + f_\lambda(\bar{x}(t), 0, 0, 0) \cdot \lambda \tag{2.13}$$

are $x = c\dot{\bar{x}}$, $c \in \mathbb{R}$ and $\lambda = 0$. Here $\bar{x}(t) := \bar{u}(td(0,0))$ is the homoclinic solution of the transformed system $\dot{x} = f(x, 0, 0, 0)$. Further note that (2.13) implies

$$M \ := \ \int_{-\infty}^{\infty} \bar{\Psi}^T(t) \cdot f_\lambda(\bar{x}(t), 0, 0, 0) \ dt \ \neq \ 0 \tag{2.14}$$

where $c\bar{\Psi}(t) = c\Psi(td(0,0))$, $c \in \mathbb{R}$ represent the only bounded solutions of the adjoint equation $\dot{x} + f_x^T(\bar{x}(t), 0, 0, 0)x = 0$. Finally we define

$$\bar{z}(t) := \int_{-\infty}^{t} g(\bar{x}(\tau), 0, 0, 0) \ d\tau \tag{2.15}$$

and with $|\cdot|$ denoting any norm in $\mathbb{R}^n$, $n \geq 1$, the main results are now stated in terms of the normalized system (2.10).

Theorem 2.1: *Let the assumptions (2.11) and (2.12) hold and let $\bar{x}(t)$ be a homoclinic orbit of the sytem $\dot{x} = f(x, 0, 0, 0)$ that satisfies condition (2.13). Then there exists a differentiable function $\lambda^+(\epsilon)$, $\epsilon \geq 0$ such that the system (2.10) has homoclinic orbits $[x(t,\epsilon), y(t,\epsilon)]$ at $\lambda = \lambda^+(\epsilon)$. More precisely, given $0 < \alpha < \beta < \gamma$ there exists an $\epsilon_0 > 0$ and a constant $c > 0$ such that the following properties hold for $0 \leq \epsilon \leq \epsilon_0$:*

$$\begin{aligned} x(t,\epsilon) &= \bar{x}(t) + \epsilon \cdot r_1(t,\epsilon) \\ y(t,\epsilon) &= \epsilon \cdot \mathrm{e}^{-\epsilon t} \cdot [\ \bar{z}(t) + \epsilon \cdot r_2(t,\epsilon)\ ] \\ \lambda^+(\epsilon) &= \lambda_1^+ \cdot \epsilon + \epsilon^2 \cdot r_3(\epsilon) \end{aligned} \tag{2.16}$$

with

$$\begin{aligned} |r_1(t,\epsilon)| &\leq c \cdot \mathrm{e}^{-\beta|t|} \quad , t \in \mathbb{R} \\ |r_2(t,\epsilon)| &\leq c \cdot \begin{cases} \mathrm{e}^{-\alpha|t|} & ,t \leq 0 \\ 1 & ,t \geq 0 \end{cases} \\ |r_3(\epsilon)| &\leq c \end{aligned} \tag{2.17}$$

and

$$\lambda_1^+ = -\frac{1}{M} \cdot \int_{-\infty}^{\infty} \bar{\Psi}^T(t) \cdot [f_y(\bar{x}(t), 0, 0, 0) \cdot \bar{z}(t) \ + \ f_\epsilon(\bar{x}(t), 0, 0, 0)] \ dt \tag{2.18}$$

where M and $\bar{z}$ are defined in (2.14) and (2.15). Moreover the homoclinic solutions are unique (up to a phase shift) within the class of solutions satisfying these relations.

Theorem 2.1 proves the existence of the homoclinic orbits depicted in Fig.1 for $\epsilon \geq 0$. The homoclinic orbits for $\epsilon \leq 0$ follow with $(t,\epsilon) \to (-t,-\epsilon)$ and applying Theorem 2.1. After retransforming this leads to a curve

$$\lambda^-(\epsilon) \;=\; \lambda_1^- \cdot \epsilon \;+\; O(\epsilon^2), \qquad -\epsilon_0 < \epsilon \leq 0$$

with

$$\lambda_1^- = -\frac{1}{M} \cdot \int\limits_{-\infty}^{\infty} \bar{\Psi}^T(t) \cdot \left[-f_y(\bar{x}(t),0,0,0) \cdot \int\limits_{t}^{\infty} g(\bar{x}(\tau),0,0,0)\; d\tau \;+\; f_\epsilon(\bar{x}(t),0,0,0) \right] \; dt \; .$$

Thus, by (2.18) the two curves $\lambda^+(\epsilon)$ and $\lambda^-(\epsilon)$ form a smooth curve at $\epsilon = 0$ if and only if

$$\int_{-\infty}^{\infty} \bar{\Psi}^T(t) \cdot f_y(\bar{x}(t),0,0,0)\; dt = 0 \qquad \text{or} \qquad \int_{-\infty}^{\infty} g(\bar{x}(\tau),0,0,0)\; d\tau = 0 \; . \tag{2.19}$$

It should be noted that the passage from $\epsilon = 0$ to $\epsilon \neq 0$ within a singularly perturbed system of the form (2.1), (2.10) almost always involves bifurcation problems. This aspect is stressed in [23], [24] where limit points and Hopf points are investigated which occur in the fast dynamics of singularly perturbed systems. We refer also to section 6 where the limit points and the Hopf points coalesce to form a singularly perturbed Bogdanov point. In the setting of Theorem 2.1 the current bifurcation structure can be described as follows.

First note that the (y,λ)–dependent x–system $\dot{x} = f(x,y,\lambda,0)$ which results at $\epsilon = 0$ has the surface of stationary points $(x,y,\lambda) = (0,y,\lambda)$ (cf. (2.11), (2.12)). ¿From the special curve $(x,y,\lambda) = (0,0,\lambda)$ within this 2–dimensional surface S_1 another surface S_2 of stationary points $(x,y,\lambda,\epsilon) = (0,0,\lambda,\epsilon)$ branches off for the full (x,y)–system (2.10) with $\epsilon \neq 0$. Hence the λ–axis represents a curve of stationary bifurcation points with respect to ϵ. This simple but basic constellation is depicted in the (y,λ,ϵ)–diagram in Fig.2.

The constant M from (2.14) represents a Melnikov coefficient of the system $\dot{x} = f(x,y,\lambda,0)$ and $M \neq 0$ ensures that the homoclinic orbit $\bar{x}(\tau)$ results from the transversal intersection of the stable and unstable manifold along the curve of stationary bifurcation points. Moreover, this regularity condition implies a unique curve of homoclinic orbits with base points of the form $(x,y,\lambda) = (0,y,\lambda^0(y))$ (cf. Fig.2). Hence within the 2–dimensional surface S_1 we obtain a curve of homoclinic base points and a curve of stationary bifurcation points which intersect transversally in the central point $(x,y,\lambda) = (0,0,0)$. The case $\frac{d}{dy}\lambda^0(0) < 0$ is shown in Fig.2.

¿From the central point two new nontrivial curves of homoclinic base points $(x,y,\lambda,\epsilon) = (0,0,\lambda^\pm(\epsilon),\epsilon)$ branch off in S_2 for $\epsilon \neq 0$; i.e. the central homoclinic orbit represents a bifurcation point of three branches of homoclinic orbits. The projection of the dashed lines of homoclinic base points onto the (λ,ϵ)–plane yields Fig.1. In case of $\frac{d}{dy}\lambda^0(0) \neq 0$ the trivial homoclinic orbits $\lambda^0(y)$ fill the λ–axis in Fig.1. However also the case $\lambda^0(y) \equiv 0$ occurs rather often, for example if the equation $\dot{x} = f(x,y,0,0)$ represents a y–dependent family of Hamiltonian systems (see section 6). Then the projection onto the (λ,ϵ)–plane shrinks to a point.

It is worth transforming the results of Theorem 2.1 back to the original system (2.1). We set $\delta(\epsilon) := d(\lambda^+(\epsilon),\epsilon)^{-1}$ (cf. (2.8)) and assume

$$\delta(0) \;=\; (G_u \cdot F_u^{-1} \cdot F_v - G_v)^0 \;>\; 0 \tag{2.20}$$

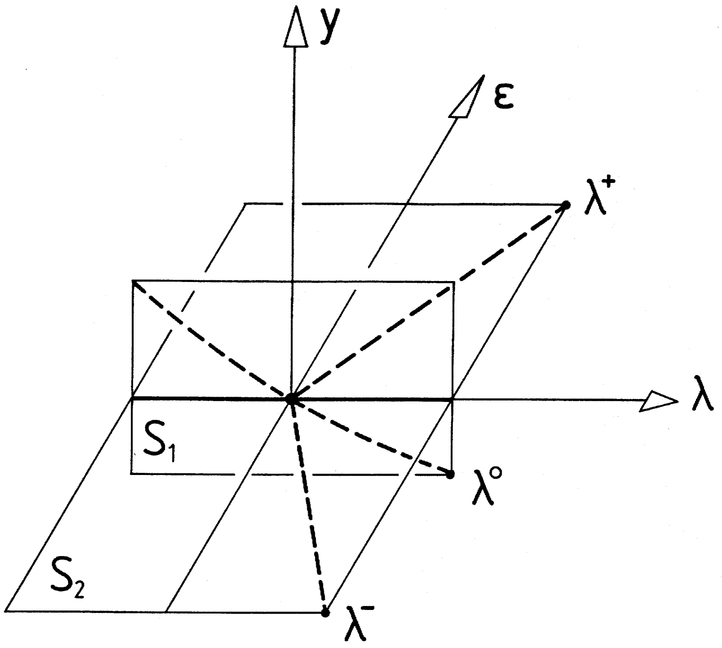

Figure 2: (y, λ, ϵ)–bifurcation diagram.

which can always be achieved by time reversal. From (2.9) and (2.15), (2.16) we find homoclinic orbits of the system (2.1) in the form

$$\begin{array}{rcll} u(\tau,\epsilon) &=& u^c(v(\tau,\epsilon),\lambda^+(\epsilon),\epsilon) \;+ & \bar{u}\left(\tau\frac{\delta(\epsilon)}{\delta(0)}\right) \;+\; \epsilon\cdot r_1\left(\tau\delta(\epsilon),\epsilon\right) \\ v(\tau,\epsilon) &=& v^s(\lambda^+(\epsilon),\epsilon) & +\;\epsilon\cdot \mathrm{e}^{-\epsilon\tau\delta(\epsilon)}\cdot\left[\;\bar{z}\left(\tau\delta(\epsilon)\right) \;+\; \epsilon\cdot r_2\left(\tau\delta(\epsilon),\epsilon\right)\;\right]\;. \end{array}$$

Using the estimates (2.17) we arrive at the following result.

Corollary 2.2: *Let the system (2.1) satisfy the assumptions (2.2)–(2.4) and (2.20). Define*

$$\bar{v}(\tau) := \int_{-\infty}^{\tau} G(\bar{u}(s),0,0,0)\;ds$$

and let $0 < \tilde{\alpha} < \tilde{\beta} < \tilde{\gamma} < |Re\;\sigma[F_u(0,0,0,0)]|$. *Then there exists an* $\epsilon_0 > 0$ *and a constant* $c > 0$, *such that the system (2.1) has homoclinic orbits* $(u(\cdot,\epsilon), v(\cdot,\epsilon))$ *at* $\lambda = \lambda^+(\epsilon)$, $0 \leq \epsilon \leq \epsilon_0$ *with the properties*

$$\begin{array}{rcll} u(\tau,\epsilon) &=& u^c(v(\tau,\epsilon),\lambda^+(\epsilon),\epsilon) \;+ & \bar{u}(\tau) \;+\; \epsilon\cdot\tilde{r}_1(\tau,\epsilon) \\ v(\tau,\epsilon) &=& v^s(\lambda^+(\epsilon),\epsilon) & +\;\epsilon\cdot\mathrm{e}^{-\epsilon\tau\delta(\epsilon)}\cdot[\;\bar{v}(\tau) \;+\; \epsilon\cdot\tilde{r}_2(\tau,\epsilon)\;] \\ \lambda^+(\epsilon) &=& & \lambda_1^+\cdot\epsilon \;+\; \epsilon^2\cdot r_3(\epsilon) \\ \delta(\epsilon) &=& & \delta(0) \;+\; \epsilon\cdot r_4(\epsilon) \end{array}$$

and

$$\begin{array}{lcl} |\tilde{r}_1(\tau,\epsilon)| & \leq & c\cdot \mathrm{e}^{-\tilde{\beta}|\tau|} \quad ,\tau\in\mathbb{R} \\ |\tilde{r}_2(\tau,\epsilon)| & \leq & c\cdot\begin{cases} \mathrm{e}^{-\tilde{\alpha}|\tau|} & ,\tau\leq 0 \\ 1 & ,\tau\geq 0 \end{cases} \\ |r_3(\epsilon)|,\ |r_4(\epsilon)| & \leq & c\ . \end{array}$$

Here $u = u^c(v,\lambda,\epsilon)$ *is a local center manifold of the* x*–equation in (2.1). Further the following representations of the exponent* $\delta(0)$*, the Melnikov coefficient* M *and the derivative* λ_1^+ *hold*

$$\delta(0) \;=\; (G_u\cdot F_u^{-1}\cdot F_v - G_v)^0\ ,$$

$$M \;=\; \int\limits_{-\infty}^{\infty} \Psi^T(\tau)\cdot[F_u,F_v,F_\lambda](\bar{u}(\tau),0,0,0)\cdot\begin{pmatrix} u_\lambda^s(0,0) \\ v_\lambda^s(0,0) \\ 1 \end{pmatrix} d\tau\ ,$$

$$\begin{aligned} \lambda_1^+ \;=\; -\frac{1}{M}\cdot\int\limits_{-\infty}^{\infty}\Psi^T(\tau)\cdot\Bigg\{ & [F_u,F_v](\bar{u}(\tau),0,0,0)\cdot\begin{pmatrix} u_v^c(0,0,0) \\ 1\end{pmatrix}\cdot\bar{v}(\tau) \\ & + [F_u,F_v,F_\epsilon](\bar{u}(\tau),0,0,0)\cdot\begin{pmatrix} u_\epsilon^s(0,0) \\ v_\epsilon^s(0,0) \\ 1\end{pmatrix} - u_v^c(0,0,0)\cdot G(\bar{u}(\tau),0,0,0)\Bigg\}\, d\tau \end{aligned} \tag{2.21}$$

with

$$\begin{array}{lcl} u_v^c(0,0,0) & = & -(F_u^{-1}\cdot F_v)^0\ , \\ \begin{pmatrix} u_\lambda^s & u_\epsilon^s \\ v_\lambda^s & v_\epsilon^s\end{pmatrix}(0,0) & = & -\begin{pmatrix} F_u^0 & F_v^0 \\ G_u^0 & G_v^0\end{pmatrix}^{-1}\cdot\begin{pmatrix} F_\lambda^0 & F_\epsilon^0 \\ G_\lambda^0 & G_\epsilon^0\end{pmatrix}. \end{array}$$

3 A blow–up procedure and the abstract framework

At first note that the stationary properties (2.11), (2.12) are only valid in a neighborhood of $(y,\lambda,\epsilon)=0$. However, these conditions can be made to hold globally after cutting off and extending the system appropriately. In the next step the singular scaling transformation of the slow y–variable

$$y \;=\; \epsilon\cdot\mathrm{e}^{-\epsilon t}\cdot z \tag{3.1}$$

is performed. Note that this setting can be interpreted as a two time scaling with fast time t and slow time ϵt (cf. (1.2)) as well as a blow–up transformation of the y–variable as ϵ tends to zero. After cancelling ϵ in the z–variable we obtain

$$\begin{array}{lclcl} \dot{x} & = & f(x,\epsilon\mathrm{e}^{-\epsilon t}z,\lambda,\epsilon) & =: & f_1(t,x,z,\lambda,\epsilon) \\ \dot{z} & = & \mathrm{e}^{\epsilon t}\cdot g(x,\epsilon\mathrm{e}^{-\epsilon t}z,\lambda,\epsilon)\ +\ \epsilon\cdot z & =: & f_2(t,x,z,\lambda,\epsilon)\ . \end{array} \tag{3.2}$$

yielding at $(\lambda,\epsilon) = (0,0)$ the blown–up system

$$\begin{aligned} \dot{x} &= f_1(t,x,z,0,0) &= f(x,0,0,0) \\ \dot{z} &= f_2(t,x,z,0,0) &= g(x,0,0,0) \end{aligned} \tag{3.3}$$

which turns out to be independent of the z–variable on the right hand side. This system has the continuum of heteroclinic orbits

$$(x,z) = (\bar{x}(t), \bar{z}(t) + c), \quad c \in \mathbb{R} \tag{3.4}$$

where $\bar{z}(t)$ is defined in (2.15). Hence the scaling procedure blows up the unperturbed homoclinic orbit $(x,y,\lambda,\epsilon) = (\bar{x}(t),0,0,0)$ of the system (2.10) to a continuum of heteroclinic orbits of the system (3.2). From these heteroclinic orbits at $(\lambda,\epsilon) = (0,0)$ we will select the orbit $(x,z) = (\bar{x},\bar{z})$, i.e. $c = 0$, as an approximation to the heteroclinic orbits which exist for $\epsilon > 0$ for the transformed system (3.2). Via the scaling (3.1) one finally obtains the homoclinic orbits of the original system (2.10).

The equation (3.2) is now rewritten as an operator equation in suitable function spaces

$$\begin{pmatrix} \dot{x} - f_1(t,x,z,\lambda,\epsilon) \\ \dot{z} - f_2(t,x,z,\lambda,\epsilon) \end{pmatrix} = 0\ . \tag{3.5}$$

Our aim is to solve this equation with the contraction mapping principle for fixed but small $\epsilon > 0$. For this purpose the equation is split into a linear and a nonlinear part

$$\begin{pmatrix} \dot{x} - f_1(t,x,z,\lambda,\epsilon) \\ \dot{z} - f_2(t,x,z,\lambda,\epsilon) \end{pmatrix} = L(x,z,\lambda) - F_\epsilon(x,z,\lambda) = 0 \tag{3.6}$$

where

$$\begin{aligned} L(x,z,\lambda) &:= \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} - \begin{pmatrix} f_{1x} & f_{1z} & f_{1\lambda} \\ f_{2x} & f_{2z} & f_{2\lambda} \end{pmatrix} (t,\bar{x}(t),\bar{z}(t),0,0) \cdot \begin{pmatrix} x \\ z \\ \lambda \end{pmatrix} \\ &= \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} - \begin{pmatrix} f_x & f_\lambda \\ g_x & g_\lambda \end{pmatrix} (\bar{x}(t),0,0,0) \cdot \begin{pmatrix} x \\ \lambda \end{pmatrix} \end{aligned} \tag{3.7}$$

and

$$F_\epsilon(x,z,\lambda) := \begin{pmatrix} f_1(t,x,z,\lambda,\epsilon) \\ f_2(t,x,z,\lambda,\epsilon) \end{pmatrix} - \begin{pmatrix} f_x & f_\lambda \\ g_x & g_\lambda \end{pmatrix} (\bar{x}(t),0,0,0) \cdot \begin{pmatrix} x \\ \lambda \end{pmatrix}. \tag{3.8}$$

Note that the linear part $L(x,z,\lambda)$ can be interpreted as the linearisation with respect to x, z and λ at the heteroclinic orbit $(x,z) = (\bar{x},\bar{z})$. In particular, the (x,λ)–part of $L(x,z,\lambda)$ decouples completely from the z–part whereas the z–part remains as a direct integration. These properties simplify the linear analysis considerably and indicate that there is only a weak coupling between the x– and the z–equation. Nevertheless, this weak dynamical coupling which is preserved up to $\epsilon = 0$ (cf. (3.3)) suffices for solving the problem in a constructive way. Note that the unscaled system (2.10) inherits at $\epsilon = 0$ merely a static dependency of the x–components on the y–component which acts as a parameter due to $\dot{y} = 0$.

It will be convenient to work with generalized norms; i.e. assume a linear space X and a mapping $|\cdot|^*$: $X \to \mathbb{R}^m$ which satisfies

$$|x|^* \geq 0, \quad |x|^* = 0 \Leftrightarrow x = 0, \quad |x_1 + x_2|^* \leq |x_1|^* + |x_2|^*, \quad |\kappa \cdot x|^* = |\kappa| \cdot |x|^*, \quad \kappa \in \mathbb{R}$$

with the natural ordering in $\mathbb{R}^m$. Then the contraction mapping principle can be formulated in the following way.

Theorem 3.1: *Let X_0, X_1 be Banach spaces with respect to the generalized norms $|\cdot|_0^*, |\cdot|_1^*$ in $\mathbb{R}^m$ and let $D(L) \subset X_0$ be a linear subspace with associated linear operator $L : D(L) \to X_1$. Further assume $F : X_0 \to X_1$ to be an (nonlinear) operator such that for suitable $u_0 \in D(L)$ and $Q,\ P \in \mathbb{R}^{m,m},\ \varrho \in \mathbb{R}^m$ the following properties hold :*

Q has a nonnegative inverse $Q^{-1} \geq 0$ and for any $v \in X_1$ the equation $Lu = v$ has a unique solution $u \in D(L)$ with

$$|u|_0^* \;\leq\; Q^{-1} \cdot |Lu|_1^* \,. \tag{3.9}$$

$P \geq 0,\ \varrho > 0$ and for any $u, v \in B_\varrho(u_0) = \{u \in X_0 : |u - u_0|_0^* \leq \varrho\}$ the Lipschitz condition

$$|F(u) - F(v)|_1^* \;\leq\; P \cdot |u - v|_0^* \tag{3.10}$$

holds. Finally

$$|Lu_0 - F(u_0)|_1^* \;<\; (Q - P)\varrho \,. \tag{3.11}$$

Then the equation $Lu - F(u) = 0,\ u \in D(L)$ has a unique solution $\bar{u}$ in $B_\varrho(u_0) \cap D(L)$, the matrix $Q - P$ has a nonnegative inverse and we have the estimate

$$|u - v|_0^* \;\leq\; (Q - P)^{-1} \cdot |(Lu - F(u)) \;-\; (Lv - F(v))|_1^* \tag{3.12}$$

for all $u, v \in B_\varrho(u_0) \cap D(L)$.

In [4] somewhat more general theorems of this type are proved and the importance of the stability inequality (3.12) is stressed. We indicate here a proof of Theorem 3.1 for completeness (see also [3]).

By (3.9)–(3.11) we have $Q^{-1}P\varrho < \varrho$ and $0 \leq Q^{-1}P$. Hence $I_m - Q^{-1}P$ has a nonnegative inverse and $(Q - P)^{-1} = (I_m - Q^{-1}P)^{-1}Q^{-1} \geq 0$. With these properties it is easy to see that $T(u) = L^{-1}F(u)$ maps $B_\varrho(u_0)$ into itself and is an ordinary contraction with respect to the norm

$$\|u\|_\varrho \;=\; \max\left\{ \frac{(|u|_0^*)_i}{\varrho_i} : i = 1, ..., m \right\}. \tag{3.13}$$

Moreover (3.12) follows from the inequality

$$|u - v|_0^* \;\leq\; Q^{-1} \cdot |Lu - Lv|_1^* \;\leq\; Q^{-1} \cdot (\; |(Lu - F(u)) - (Lv - F(v))|_1^* + P|u - v|_0^* \;) \,.$$

Next the spaces and generalized norms suitable for our original problem (3.6) are defined. The effect of Theorem 3.1 will be to measure fast and slow components in (3.2) separately and to reduce their interplay to matrix computations. First set up a phase condition

$$\Pi(x) := \dot{\bar{x}}(0)^T \cdot [x(0) - \bar{x}(0)] = 0 \ .$$

Then introduce for $\epsilon > 0$ the weight functions

$$w_1(t) := \mathrm{e}^{-\beta|t|} \quad ,t \in \mathbb{R} \tag{3.14}$$

$$w_2(t) := \begin{cases} \mathrm{e}^{-\alpha|t|} & ,t \leq 0 \\ 1 & ,t \geq 0 \end{cases} \tag{3.15}$$

$$w_3(t) := \begin{cases} \mathrm{e}^{-\alpha|t|} & ,t \leq 0 \\ \mathrm{e}^{-\alpha t} + \epsilon \mathrm{e}^{-\epsilon t} & ,t \geq 0 \end{cases} \tag{3.16}$$

with fixed $0 < \alpha < \beta < \gamma$ and for sufficiently small $\epsilon > 0$ satisfying

$$\epsilon < \alpha \qquad \text{and} \qquad \alpha + \epsilon < \beta < \gamma - \epsilon \ . \tag{3.17}$$

Further with the setting

$$\|x\|_w \ := \ \sup\left\{ \frac{|x(t)|}{w(t)} : t \in \mathbb{R} \right\}$$

define the spaces

$$\begin{aligned} X_0 \ &:= \ \{u = (x, z, \lambda) \in \mathrm{C}(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}) \times \mathbb{R} \ : \ \Pi(x) = 0, \\ &\quad\ \|x\|_{w_1} + \|z\|_{w_2} < \infty\} \end{aligned}$$

$$\begin{aligned} D(L) \ &:= \ \{u = (x, z, \lambda) \in \mathrm{C}^1(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}) \times \mathbb{R} \ : \ \Pi(x) = 0, \\ &\quad\ \|x\|_{w_1} + \|z\|_{w_2} < \infty, \ \|\dot{x}\|_{w_1} + \|\dot{z}\|_{w_3} < \infty\} \ \subset \ X_0 \end{aligned}$$

$$X_1 \ := \ \{v = (\xi, \zeta) \in \mathrm{C}(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}) \ : \ \|\xi\|_{w_1} + \|\zeta\|_{w_3} < \infty\}$$

and the generalized norms

$$|u|_0^* \ = \ |(x, z, \lambda)|_0^* \ := \ \begin{pmatrix} \|x\|_{w_1} + |\lambda| \\ \|z\|_{w_2} \end{pmatrix} \in \mathbb{R}^2 \tag{3.18}$$

$$|v|_1^* \ = \ |(\xi, \zeta)|_1^* \ := \ \begin{pmatrix} \|\xi\|_{w_1} \\ \|\zeta\|_{w_3} \end{pmatrix} \in \mathbb{R}^2 \ . \tag{3.19}$$

Then for fixed $\epsilon > 0$ the assumptions of Theorem 3.1 are verified for the decomposition (3.6)–(3.8) and for $u_0 = (\bar{x}(t), \bar{z}(t), 0)$ in the following sequence :

Step 1 : prove the linear stability inequality (3.9) with $c > 0$ independent of ϵ and

$$Q^{-1} = c \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{3.20}$$

Step 2 : prove the Lipschitz condition (3.10) with $\varrho = (\varrho_1, \varrho_2)$ and

$$P = c \begin{pmatrix} \varrho_1 + \epsilon & \epsilon \\ \varrho_1 + \epsilon & \epsilon \end{pmatrix}. \tag{3.21}$$

Step 3 : prove the residual condition

$$|Lu_0 - F_\epsilon(u_0)|_1^* \ \leq \ c \begin{pmatrix} \epsilon \\ \epsilon \end{pmatrix}. \tag{3.22}$$

Then a direct calculation yields

$$c \begin{pmatrix} \epsilon \\ \epsilon \end{pmatrix} \ < \ (Q - P) \begin{pmatrix} \varrho_1 \\ \varrho_2 \end{pmatrix}$$

provided we choose $0 < \epsilon \ll \varrho_1 < \varrho_2 \ll 1$. This proves the desired estimate (3.11). Hence, recalling the scaling transformation (3.1) and the generalized norms (3.18), (3.19), the existence and the uniqueness of the homoclinic orbits satisfying the relations (2.16), (2.17) are shown. The differentiability of $\lambda^+(\epsilon)$ will follow from the nonlinear stability inequality (3.12) and from a parameter dependent version of Theorem 3.1. This will be shown in section 5.

4 Details of the proof

At first a few basic estimates are shown and the properties $L(D(L)) \subset X_1$ and $F_\epsilon(X_0) \subset X_1$ are verified for the operators from (3.7) and (3.8). Subsequently the claims in step 1 to 3 are proved. The equations (2.11), (2.12), (3.17) and (4.1)–(4.3) will be frequently used without explicit reference.

In the following $c > 0$ denotes a generic constant independent of $t \in \mathbb{R}$ and $\epsilon \geq 0$ sufficiently small. From (2.11), (2.12), (2.15) and (3.15) one obtains

$$|\bar{x}(t)| \ \leq \ c\mathrm{e}^{-\gamma|t|}, \quad t \in \mathbb{R} \tag{4.1}$$

$$|g(\bar{x}(t), 0, 0, 0)| \ \leq \ c\mathrm{e}^{-\gamma|t|}, \quad t \in \mathbb{R} \tag{4.2}$$

$$\bar{z}(t) \ \leq \ c \left\{ \begin{array}{ll} \mathrm{e}^{-\gamma|t|} & , t \leq 0 \\ 1 & , t \geq 0 \end{array} \right\} \ \leq \ c w_2(t), \quad t \in \mathbb{R}\ . \tag{4.3}$$

Further a Taylor expansion to the second order with respect to $y = \epsilon \mathrm{e}^{-\epsilon t} z$ yields

$$f_2(t, x, z, \lambda, \epsilon) \ = \ \mathrm{e}^{\epsilon t} g(x, 0, \lambda, \epsilon) + \epsilon (g_y(x, 0, \lambda, \epsilon) + 1) z + \epsilon^2 \mathrm{e}^{-\epsilon t} g_2(t, x, z, \lambda, \epsilon) z^2 \tag{4.4}$$

where

$$g_2(t,x,z,\lambda,\epsilon) \;=\; \int_0^1 (1-s) g_{yy}(x, s\epsilon \mathrm{e}^{-\epsilon t} z, \lambda, \epsilon)\; ds\;. \tag{4.5}$$

Hence

$$f_{2,\lambda}(t,\bar{x},\bar{z},0,\epsilon) \;=\; \mathrm{e}^{\epsilon t} g_\lambda(\bar{x},0,0,\epsilon) + \epsilon g_{y\lambda}(\bar{x},0,0,\epsilon)\bar{z} + \epsilon^2 \mathrm{e}^{-\epsilon t} g_{2,\lambda}(t,\bar{x},\bar{z},0,\epsilon)\bar{z}^2 \tag{4.6}$$

and

$$\begin{aligned} |f_{2,\lambda}(t,\bar{x},\bar{z},0,\epsilon)| \;&\le\; c\;(\mathrm{e}^{\epsilon t}|\bar{x}(t)| + \epsilon|\bar{x}(t)||\bar{z}(t)| + \epsilon^2 \mathrm{e}^{-\epsilon t}|\bar{z}(t)|^2) \\ &\le\; c\;(\mathrm{e}^{-(\gamma-\epsilon)|t|} + \epsilon \mathrm{e}^{-\gamma|t|} + \epsilon^2 \left\{ \begin{array}{ll} \mathrm{e}^{-\gamma|t|} & ,t\le 0 \\ \mathrm{e}^{-\epsilon t} & ,t \ge 0 \end{array} \right\}) \;\le\; c w_3(t). \end{aligned} \tag{4.7}$$

Now we show $L(D(L)) \subset X_1$ and $F_\epsilon(X_0) \subset X_1$. For $(x,z,\lambda) \in D(L)$ we have $(\dot{x},\dot{z}) \in X_1$ by definition. Then with $\bar{c}$ denoting a generic constant which depends on (x,z,λ)

$$\begin{aligned} |f_x(\bar{x},0,0,0)\cdot x + f_\lambda(\bar{x},0,0,0)\cdot\lambda| \;&\le\; c\;(|x(t)| + |\bar{x}(t)||\lambda|) \;\le\; \bar{c} w_1(t) \\ |g_x(\bar{x},0,0,0)\cdot x + g_\lambda(\bar{x},0,0,0)\cdot\lambda| \;&\le\; c\;(|x(t)| + |\bar{x}(t)||\lambda|) \;\le\; \bar{c} w_3(t). \end{aligned} \tag{4.8}$$

Hence $L(D(L)) \subset X_1$. Furthermore for $(x,z,\lambda) \in X_0$ we obtain by (2.11) and (4.4)

$$\begin{aligned} |f_1(t,x,z,\lambda,\epsilon)| \;&=\; |f(x,\epsilon \mathrm{e}^{-\epsilon t} z,\lambda,\epsilon)| \;\le\; \bar{c}|x(t)| \;\le\; \bar{c} w_1(t) \\ |f_2(t,x,z,\lambda,\epsilon)| \;&\le\; \bar{c}\;(\mathrm{e}^{\epsilon t}|x(t)| + \epsilon|x(t)||z(t)| + \epsilon^2 \mathrm{e}^{-\epsilon t}|z(t)|^2) \\ &\le\; \bar{c}\;(\mathrm{e}^{-(\beta-\epsilon)|t|} + \epsilon \mathrm{e}^{-\beta|t|} + \epsilon^2 \mathrm{e}^{-\epsilon t} \left\{ \begin{array}{ll} \mathrm{e}^{-2\alpha|t|} & ,t\le 0 \\ 1 & ,t\ge 0 \end{array} \right\}) \;\le\; \bar{c} w_3(t) \end{aligned} \tag{4.9}$$

implying $F_\epsilon(X_0) \subset X_1$.

4.1 Linear stability inequality (step 1)

In this section we prove that for any $(\xi,\zeta) \in X_1$ the equation $L(x,z,\lambda) = (\xi,\zeta)$ has a unique solution $(x,z,\lambda) \in D(L)$ which satisfies

$$|(x,z,\lambda)|_0^* \;=\; \begin{pmatrix} \|x\|_{w_1} + |\lambda| \\ \|z\|_{w_2} \end{pmatrix} \;\le\; c \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \|\xi\|_{w_1} \\ \|\zeta\|_{w_3} \end{pmatrix} \;=\; Q^{-1}\cdot|L(x,z,\lambda)|_1^* \tag{4.10}$$

with c independent of $0 < \epsilon \ll 1$. First consider the (x,λ)–part of $L(x,z,\lambda) = (\xi,\zeta)$; i.e.

$$M_\beta(x,\lambda) := \dot{x} - f_x(\bar{x},0,0,0)\cdot x - f_\lambda(\bar{x},0,0,0)\cdot\lambda = \xi \tag{4.11}$$

which decouples completely from the z–component. Then recalling $w_1(t) = \mathrm{e}^{-\beta|t|}$ define the spaces

$$\begin{aligned} D(M_\beta) \;&:=\; \{(x,\lambda) \in \mathrm{C}^1(\mathbb{R},\mathbb{R}^n) \times \mathbb{R} \;:\; \Pi(x) = 0,\; \|x\|_{w_1} + \|\dot{x}\|_{w_1} + |\lambda| < \infty\} \\ Y_\beta \;&:=\; \{\xi \in \mathrm{C}(\mathbb{R},\mathbb{R}^n) \;:\; \|\xi\|_{w_1} < \infty\}\;. \end{aligned}$$

Due to the assumptions $|\mathrm{Re}\ \sigma[f_x(0)]| > \gamma > 0$ and (2.13) the operator $M_\beta : D(M_\beta) \to Y_\beta$ is a linear homeomorphism for any $\beta \in [0, \gamma]$. In [2] this is stated for bounded functions ($\beta = 0$) which converge as $t \to \pm\infty$. For $\beta > 0$ the proof remains essentially the same except that one makes explicit use of the fact that the differential operator in M_β has an exponential dichotomy with exponent $> \gamma$. Related Fredholm properties for operators between exponentially weighted spaces appear also in [7].

We obtain the following linear stability inequality for the (x, λ)–component

$$\|x\|_{w_1} + |\lambda| \ \le\ c\ \|\xi\|_{w_1}\ . \tag{4.12}$$

It remains to estimate the z–component of $L(x, z, \lambda) = (\xi, \zeta)$; i.e.

$$\dot z - g_x(\bar x, 0, 0, 0) \cdot x - g_\lambda(\bar x, 0, 0, 0) \cdot \lambda = \zeta$$

where $(x, \lambda) \in D(M_\beta)$ denotes the solution of (4.11) and $\|\zeta\|_{w_3} < \infty$. The general solution is simply given for $r \in \mathbb{R}$ by integration

$$z_r(t) = r + \int_{-\infty}^{t} [\ g_x(\bar x(s), 0, 0, 0) \cdot x(s) + g_\lambda(\bar x(s), 0, 0, 0) \cdot \lambda + \zeta(s)\ ]\ ds\ .$$

Obviously $z_0(t)$ is the unique solution which converges to zero as $t \to -\infty$. Moreover this solution satisfies $\|z_0\|_{w_2} < \infty$ according to

$$\begin{aligned}
|z_0(t)| &\le c \int_{-\infty}^{t} [\ \|x\|_{w_1} \cdot w_1(s)\ +\ \|\bar x\|_{w_1} \cdot w_1(s) \cdot |\lambda|\ +\ \|\zeta\|_{w_3} \cdot w_3(s)\ ]\ ds \\
&\le c\ (\|x\|_{w_1}\ +\ |\lambda|) \cdot \int_{-\infty}^{t} w_1(s)\ ds\ +\ c\ \|\zeta\|_{w_3} \cdot \int_{-\infty}^{t} w_3(s)\ ds \\
&\le c\ (\|x\|_{w_1}\ +\ |\lambda|) \cdot w_2(t)\ +\ c\ \|\zeta\|_{w_3} \cdot w_2(t) \\
&\le c\ (\|\xi\|_{w_1}\ +\ \|\zeta\|_{w_3}) \cdot w_2(t)
\end{aligned} \tag{4.13}$$

where we used (4.12) and the crucial property

$$\int_0^\infty \epsilon e^{-\epsilon s} ds\ \equiv\ 1 \qquad \text{for} \qquad \epsilon > 0$$

implying $\int_{-\infty}^{\infty} w_3(s) ds < c < \infty$ independent of $0 < \epsilon \ll 1$. Now combining (4.12) and (4.13) gives the desired estimate (4.10).

4.2 Small Lipschitz constant (step 2)

Consider $u_0 = (\bar x, \bar z, 0)$ and $u = (x, z, \lambda),\ v = (\xi, \zeta, \mu) \in X_0$ with $|u - u_0|_0^* \le \varrho,\ \ |v - u_0|_0^* \le \varrho$ or in detail

$$\begin{pmatrix} \|x - \bar x\|_{w_1} + |\lambda| \\ \|z - \bar z\|_{w_2} \end{pmatrix} \le \begin{pmatrix} \varrho_1 \\ \varrho_2 \end{pmatrix}, \quad \begin{pmatrix} \|\xi - \bar x\|_{w_1} + |\mu| \\ \|\zeta - \bar z\|_{w_2} \end{pmatrix} \le \begin{pmatrix} \varrho_1 \\ \varrho_2 \end{pmatrix}. \tag{4.14}$$

Then with $(\diamond) = (t, \xi + \tau(x - \xi), \zeta + \tau(z - \zeta), \mu + \tau(\lambda - \mu), \epsilon) =: (t, \tilde x, \tilde z, \tilde\lambda, \epsilon),\ (\cdot) = (\bar x, 0, 0, 0)$ and

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} := \begin{pmatrix} \int_0^1 f_{1x}(\diamond) - f_x(\cdot)\ d\tau & \int_0^1 f_{1z}(\diamond)\ d\tau & \int_0^1 f_{1\lambda}(\diamond) - f_\lambda(\cdot)\ d\tau \\ \int_0^1 f_{2x}(\diamond) - g_x(\cdot)\ d\tau & \int_0^1 f_{2z}(\diamond)\ d\tau & \int_0^1 f_{2\lambda}(\diamond) - g_\lambda(\cdot)\ d\tau \end{pmatrix}$$

we obtain by the mean value theorem

$$F_\epsilon(x,z,\lambda) \;-\; F_\epsilon(\xi,\zeta,\mu) \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x-\xi \\ z-\zeta \\ \lambda-\mu \end{pmatrix} .$$

Now for proving step 2 (cf. (3.10), (3.21)) it will be shown

$$\left| \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x-\xi \\ z-\zeta \\ \lambda-\mu \end{pmatrix} \right|_1^* = \begin{pmatrix} \|a_{11}(x-\xi)+a_{12}(z-\zeta)+a_{13}(\lambda-\mu)\|_{w_1} \\ \|a_{21}(x-\xi)+a_{22}(z-\zeta)+a_{23}(\lambda-\mu)\|_{w_3} \end{pmatrix}$$

$$\leq c \begin{pmatrix} \varrho_1+\epsilon & \epsilon \\ \varrho_1+\epsilon & \epsilon \end{pmatrix} \begin{pmatrix} \|x-\xi\|_{w_1}+|\lambda-\mu| \\ \|z-\zeta\|_{w_2} \end{pmatrix} \;=\; P \left| \begin{pmatrix} x-\xi \\ z-\zeta \\ \lambda-\mu \end{pmatrix} \right|_0^* . \tag{4.15}$$

In the following we use the assumptions (2.11), (2.12) and the formulas (4.1)–(4.7) without explicit reference. Moreover note that the arguments $(x,\lambda)=(\bar{x},0)$ and $(x,\lambda)=(\tilde{x},\tilde{\lambda})$ are $O(\varrho_1)$–close and that the arguments $y=0$ and $y=\epsilon e^{-\epsilon t}\tilde{z}$ are $O(\epsilon)$–close at least uniformly in $t\in\mathbb{R}$ and $\tau\in[0,1]$ (cf. (4.14)).

At first with $(*)=(\bar{x}+s(\tilde{x}-\bar{x}), s\epsilon e^{-\epsilon t}\tilde{z}, s\tilde{\lambda}, s\epsilon)$ we obtain by the mean value theorem

$$\begin{aligned}
|a_{11}(x-\xi)| \;&\leq\; \sup_{\tau\in[0,1]} \, |f_x(\tilde{x},\epsilon e^{-\epsilon t}\tilde{z},\tilde{\lambda},\epsilon) \;-\; f_x(\bar{x},0,0,0)| \cdot |x-\xi| \\
&\leq\; \sup_{\tau\in[0,1]} \, \sup_{s\in[0,1]} \, \{\, |f_{xx}(*)|\cdot|\tilde{x}-\bar{x}| \;+\; |f_{xy}(*)|\cdot\epsilon e^{-\epsilon t}|\tilde{z}| \\
&\qquad + |f_{x\lambda}(*)|\cdot|\tilde{\lambda}| \;+\; |f_{x\epsilon}(*)|\cdot\epsilon \,\}\cdot|x-\xi| \\
&\leq\; c \, \sup_{\tau\in[0,1]} \, \{\, |\tilde{x}-\bar{x}| \;+\; \epsilon \;+\; |\tilde{\lambda}| \;+\; \epsilon \,\}\cdot|x-\xi| \\
&\leq\; c\,(\varrho_1 \;+\; \epsilon)\cdot\|x-\xi\|_{w_1}\cdot w_1(t) \;.
\end{aligned}$$

Next we compute

$$\begin{aligned}
|a_{12}(z-\zeta)| \;&\leq\; \sup_{\tau\in[0,1]} \, \epsilon e^{-\epsilon t}\cdot|f_y(\tilde{x},\epsilon e^{-\epsilon t}\tilde{z},\tilde{\lambda},\epsilon)|\cdot|z-\zeta| \\
&\leq\; c\,\epsilon e^{-\epsilon t}\cdot\sup_{\tau\in[0,1]}|\tilde{x}|\cdot|z-\zeta| \\
&\leq\; c\,\epsilon e^{-\epsilon t}\cdot w_1(t)\cdot\|z-\zeta\|_{w_2}\cdot w_2(t) \\
&\leq\; c\,\epsilon\cdot\|z-\zeta\|_{w_2}\cdot w_1(t) \;.
\end{aligned}$$

The last term in the first row of (4.15) reads

$$\begin{aligned}
|a_{13}(\lambda-\mu)| \;&\leq\; \sup_{\tau\in[0,1]} \, |f_\lambda(\tilde{x},\epsilon e^{-\epsilon t}\tilde{z},\tilde{\lambda},\epsilon) \;-\; f_\lambda(\bar{x},0,0,0)|\cdot|\lambda-\mu| \\
&\leq\; \sup_{\tau\in[0,1]} \, \sup_{s\in[0,1]} \, \{\, |f_{\lambda x}(*)|\cdot|\tilde{x}-\bar{x}| \;+\; |f_{\lambda y}(*)|\cdot\epsilon e^{-\epsilon t}|\tilde{z}| \\
&\qquad + |f_{\lambda\lambda}(*)|\cdot|\tilde{\lambda}| \;+\; |f_{\lambda\epsilon}(*)|\cdot\epsilon \,\}\cdot|\lambda-\mu| \\
&\leq\; c \, \sup_{\tau\in[0,1]} \, \{\, \|\tilde{x}-\bar{x}\|_{w_1}\cdot w_1(t) \\
&\qquad + w_1(t)\cdot(\epsilon e^{-\epsilon t}|\tilde{z}| \;+\; |\tilde{\lambda}| \;+\; \epsilon) \,\}\cdot|\lambda-\mu| \\
&\leq\; c\,(\varrho_1 \;+\; \epsilon)\cdot|\lambda-\mu|\cdot w_1(t) \;.
\end{aligned}$$

This finishes the Lipschitz estimate (4.15) for the x–component. Next consider

$$\begin{aligned}
|a_{21}(x-\xi)| \;&\leq\; \sup_{\tau\in[0,1]}\ |\mathrm{e}^{\epsilon t}\cdot g_x(\tilde{x},\epsilon\mathrm{e}^{-\epsilon t}\tilde{z},\tilde{\lambda},\epsilon)\ -\ g_x(\bar{x},0,0,0)|\cdot|x-\xi| \\
&=\; \sup_{\tau\in[0,1]}\ |\mathrm{e}^{\epsilon t}\cdot[\ g_x(\bar{x},0,0,0)\ +\ \textstyle\int_0^1\{\ g_{xx}(*)\cdot(\tilde{x}-\bar{x})\ +\ g_{xy}(*)\cdot\epsilon\mathrm{e}^{-\epsilon t}\tilde{z} \\
&\qquad +\ g_{x\lambda}(*)\cdot\tilde{\lambda}\ +\ g_{x\epsilon}(*)\cdot\epsilon\ \}\ ds\ ]\ -\ g_x(\bar{x},0,0,0)|\cdot|x-\xi| \\
&\leq\; c\ \sup_{\tau\in[0,1]}\ \{\ |\mathrm{e}^{\epsilon t}-1|\cdot|g_x(\bar{x},0,0,0)| \\
&\qquad +\ \mathrm{e}^{\epsilon t}\cdot(\ |\tilde{x}-\bar{x}|+\epsilon+|\tilde{\lambda}|+\epsilon\ )\ \}\cdot|x-\xi| \\
&\leq\; c\ \{\ |\mathrm{e}^{\epsilon t}-1|\ +\ \mathrm{e}^{\epsilon t}\cdot(\varrho_1+\epsilon)\ \}\cdot\|x-\xi\|_{w_1}\cdot w_1(t) \\
&\leq\; c\ (\varrho_1+\epsilon)\cdot\|x-\xi\|_{w_1}\cdot w_3(t)\ .
\end{aligned} \tag{4.16}$$

In the last inequality we used

$$|\mathrm{e}^{\epsilon t}-1|\cdot w_1(t) \quad\leq\quad c\ \epsilon\cdot w_3(t) \tag{4.17}$$

which follows from

$$\begin{aligned}
|\mathrm{e}^{\epsilon t}-1|\cdot w_1(t) \;&=\; \epsilon\cdot|\textstyle\int_0^t\mathrm{e}^{\epsilon s}ds|\cdot w_1(t) \quad\leq\quad \epsilon\left\{\begin{array}{ll}|t| & ,t\leq 0\\ t\mathrm{e}^{\epsilon t} & ,t\geq 0\end{array}\right\}\cdot w_1(t) \\
&\leq\; c\ \epsilon\cdot w_3(t)\ .
\end{aligned}$$

Further with $(*)=(s\tilde{x},s\epsilon\mathrm{e}^{-\epsilon t}\tilde{z},\tilde{\lambda},\epsilon)$ we obtain

$$\begin{aligned}
|a_{22}(z-\zeta)| \;&\leq\; \sup_{\tau\in[0,1]}\ \epsilon\cdot|g_y(\tilde{x},\epsilon\mathrm{e}^{-\epsilon t}\tilde{z},\tilde{\lambda},\epsilon)+1|\cdot|z-\zeta| \\
&\leq\; \sup_{\tau\in[0,1]}\ \epsilon\cdot\sup_{s\in[0,1]}\ \{\ |g_{yx}(*)|\cdot|\tilde{x}|\ +\ |g_{yy}(*)|\cdot\epsilon\mathrm{e}^{-\epsilon t}|\tilde{z}|\ \}\cdot|z-\zeta| \\
&\leq\; c\ \epsilon\cdot\{\ |\tilde{x}|+\epsilon\mathrm{e}^{-\epsilon t}|\tilde{z}|\ \}\cdot\|z-\zeta\|_{w_2}\cdot w_2(t) \\
&\leq\; c\ \epsilon\cdot\{\ w_1(t)+\epsilon\mathrm{e}^{-\epsilon t}\ w_2(t)\ \}\cdot\|z-\zeta\|_{w_2}\cdot w_2(t) \\
&\leq\; c\ \epsilon\cdot\|z-\zeta\|_{w_2}\cdot w_3(t)
\end{aligned} \tag{4.18}$$

and finally in view of (4.4), (4.17) and with $(*)=(\bar{x}+s(\tilde{x}-\bar{x}),0,s\tilde{\lambda},s\epsilon)$

$$\begin{aligned}
|a_{23}(\lambda-\mu)| \;&\leq\; \sup_{\tau\in[0,1]}\ |\ \mathrm{e}^{\epsilon t}\cdot g_\lambda(\tilde{x},0,\tilde{\lambda},\epsilon)+\epsilon\cdot g_{y\lambda}(\tilde{x},0,\tilde{\lambda},\epsilon)\cdot\tilde{z} \\
&\qquad +\ \epsilon^2\mathrm{e}^{-\epsilon t}\cdot g_{2\lambda}(t,\tilde{x},\tilde{z},\tilde{\lambda},\epsilon)\cdot\tilde{z}^2-g_\lambda(\bar{x},0,0,0)\ |\cdot|\lambda-\mu| \\
&=\; \sup_{\tau\in[0,1]}\ |\ \mathrm{e}^{\epsilon t}\cdot[\ g_\lambda(\bar{x},0,0,0)\ +\ \textstyle\int_0^1\{\ g_{\lambda x}(*)\cdot(\tilde{x}-\bar{x}) \\
&\qquad +\ g_{\lambda\lambda}(*)\cdot\tilde{\lambda}\ +\ g_{\lambda\epsilon}(*)\cdot\epsilon\ \}\ ds\ ]\ +\ \epsilon\cdot g_{y\lambda}(\tilde{x},0,\tilde{\lambda},\epsilon)\cdot\tilde{z} \\
&\qquad +\ \epsilon^2\mathrm{e}^{-\epsilon t}\cdot g_{2\lambda}(t,\tilde{x},\tilde{z},\tilde{\lambda},\epsilon)\cdot\tilde{z}^2-g_\lambda(\bar{x},0,0,0)\ |\cdot|\lambda-\mu| \\
&\leq\; c\ \sup_{\tau\in[0,1]}\ \{\ |\mathrm{e}^{\epsilon t}-1|\cdot w_1(t)|+\mathrm{e}^{\epsilon t}\cdot w_1(t)\cdot(\ \|\tilde{x}-\bar{x}\|_{w_1}+|\tilde{\lambda}|+\epsilon\ ) \\
&\qquad +\ \epsilon\cdot w_1(t)\cdot w_2(t)+\epsilon^2\mathrm{e}^{-\epsilon t}\cdot w_2^2(t)\ \}\cdot|\lambda-\mu| \\
&\leq\; c\ (\varrho_1+\epsilon)\cdot|\lambda-\mu|\cdot w_3(t)\ .
\end{aligned} \tag{4.19}$$

From (4.16)–(4.19) we obtain the Lipschitz estimate (4.15) for the z–component and the second step is accomplished.

4.3 Small residual (step 3)

Consider the residual at $u_0 = (\bar{x}, \bar{z}, 0)$, i.e.

$$|Lu_0 - F_\epsilon(u_0)|_1^* \quad = \quad \begin{pmatrix} \|\dot{\bar{x}} - f_1(t, \bar{x}, \bar{z}, 0, \epsilon)\|_{w_1} \\ \|\dot{\bar{z}} - f_2(t, \bar{x}, \bar{z}, 0, \epsilon)\|_{w_3} \end{pmatrix} . \tag{4.20}$$

With $(*) = (\bar{x}, s\epsilon e^{-\epsilon t}\bar{z}, 0, s\epsilon)$ we obtain for the first component

$$\begin{aligned} |f_1(t, \bar{x}, \bar{z}, 0, \epsilon) - \dot{\bar{x}}| &= |f(\bar{x}, \epsilon e^{-\epsilon t}\bar{z}, 0, \epsilon) - f(\bar{x}, 0, 0, 0)| \\ &\leq \sup_{s\in[0,1]} \{ \ |f_y(*)| \cdot \epsilon e^{-\epsilon t}|\bar{z}| + |f_\epsilon(*)| \cdot \epsilon \ \} \\ &\leq c \ (|\bar{x}(t)| \cdot \epsilon e^{-\epsilon t}|\bar{z}| + |\bar{x}(t)| \cdot \epsilon) \\ &\leq c \ \epsilon \cdot w_1(t) \ . \end{aligned} \tag{4.21}$$

For the second row we use (4.4) and (4.17)

$$\begin{aligned} |f_2(t, \bar{x}, \bar{z}, 0, \epsilon) - \dot{\bar{z}}| \ &\leq \ |e^{\epsilon t} \cdot g(\bar{x}, 0, 0, \epsilon) - g(\bar{x}, 0, 0, 0)| + \epsilon \cdot |g_y(\bar{x}, 0, 0, \epsilon) + 1| \cdot |\bar{z}| \\ &\quad + \epsilon^2 e^{-\epsilon t} \cdot |g_2(t, \bar{x}, \bar{z}, 0, \epsilon)| \cdot |\bar{z}|^2 \\ &= \ |e^{\epsilon t} \cdot [\ g(\bar{x}, 0, 0, 0) \ + \ \textstyle\int_0^1 g_\epsilon(\bar{x}, 0, 0, s\epsilon) \cdot \epsilon \ ds \ ] - g(\bar{x}, 0, 0, 0)| \\ &\quad + \epsilon \cdot |g_y(\bar{x}, 0, 0, \epsilon) - g_y(0, 0, 0, \epsilon)| \cdot |\bar{z}| + \epsilon^2 e^{-\epsilon t} \cdot |g_2(t, \bar{x}, \bar{z}, 0, \epsilon)| \cdot |\bar{z}|^2 \\ &\leq \ c \ \{ \ |e^{\epsilon t} - 1| \cdot w_1(t) + e^{\epsilon t} \cdot w_1(t) \cdot \epsilon + \epsilon \cdot w_1(t) \cdot w_2(t) + \epsilon^2 e^{-\epsilon t} \cdot w_2^2(t) \ \} \\ &\leq \ c \ \epsilon \cdot w_3(t) \ . \end{aligned} \tag{4.22}$$

Thus by (4.20)–(4.22) step 3 is finished.

5 Smoothness in ϵ

First the differentiability of $\lambda^+(\epsilon)$ is proved at $\epsilon = 0$ with the help of the stability inequality (3.12) and a higher order approximation of the solution branch. More precisely, if we find $x_1(t)$ and λ_1 with $u_\epsilon := (\bar{x} + \epsilon x_1, \bar{z}, \lambda_1\epsilon) \in B_\varrho(u_0) \cap D(L)$ and residual

$$|Lu_\epsilon - F_\epsilon(u_\epsilon)|_1^* \ = \ \begin{pmatrix} \| \ \dot{\bar{x}} \ + \ \epsilon\dot{x}_1 \ \ -f_1(t, \bar{x} + \epsilon x_1, \bar{z}, \lambda_1\epsilon, \epsilon) \ \|_{w_1} \\ \| \ \dot{\bar{z}} \ \ -f_2(t, \bar{x} + \epsilon x_1, \bar{z}, \lambda_1\epsilon, \epsilon) \ \|_{w_3} \end{pmatrix} \ \leq \ c \begin{pmatrix} \epsilon^2 \\ \epsilon \end{pmatrix} \tag{5.1}$$

then we obtain from the first row in (3.12)

$$\|x(t, \epsilon) - \bar{x}(t) - \epsilon x_1(t)\|_{w_1} \ + \ |\lambda^+(\epsilon) - \lambda_1\epsilon| \quad \leq \quad c \ (1, \epsilon) \cdot \begin{pmatrix} \epsilon^2 \\ \epsilon \end{pmatrix} \quad \leq \quad c \ \epsilon^2$$

with a generic constant $c > 0$. In particular $\lambda^+(\epsilon)$ is differentiable at $\epsilon = 0$ with derivative λ_1. Note that it suffices to determine a higher order approximation of the (x, λ)–components of the solution branch. This is possible because we used generalized norms in the contraction mapping Theorem 3.1.

Now for finding $x_1(t)$ and λ_1 which satisfy (5.1), insert the ansatz $(x,\lambda) = (\bar{x}(t)+\epsilon x_1(t), \lambda_1\epsilon)$ into the x–equation of (3.5), perform a Taylor expansion with respect to ϵ, solve the equation of the first nontrivial coefficient and finally estimate the remainder term with respect to the w_1–norm.

With the setting $[\epsilon] = (\bar{x}+\epsilon x_1, \epsilon \mathrm{e}^{-\epsilon t}\bar{z}, \lambda_1\epsilon, \epsilon)$ the Taylor expansion reads

$$
\begin{aligned}
&\dot{\bar{x}} + \epsilon\dot{x}_1 - f_1(t,\bar{x}+\epsilon x_1,\bar{z},\lambda_1\epsilon,\epsilon) \;=\; \dot{\bar{x}} + \epsilon\dot{x}_1 - f[\epsilon] \;=\; \dot{\bar{x}} + \epsilon\dot{x}_1 - f(\bar{x},0,0,0)\\
&- \epsilon\cdot\{\; f_x[\epsilon]x_1 + f_\lambda[\epsilon]\lambda_1 + f_\epsilon[\epsilon] + f_y[\epsilon](1-\epsilon t)\mathrm{e}^{-\epsilon t}\bar{z} \;\}_{|\epsilon=0} \;-\; \epsilon^2\cdot\textstyle\int_0^1(1-s)\\
&\cdot\{\; f_{xx}[s\epsilon]x_1^2 + f_{\lambda\lambda}[s\epsilon]\lambda_1^2 + f_{\epsilon\epsilon}[s\epsilon] + 2f_{x\lambda}[s\epsilon]x_1\lambda_1 + 2f_{x\epsilon}[s\epsilon]x_1 + 2f_{\lambda\epsilon}[s\epsilon]\lambda_1\\
&+ 2(f_{xy}[s\epsilon]x_1 + f_{\lambda y}[s\epsilon]\lambda_1 + f_{\epsilon y}[s\epsilon])(1-s\epsilon t)\mathrm{e}^{-s\epsilon t}\bar{z}\\
&+ f_{yy}[s\epsilon][(1-s\epsilon t)\mathrm{e}^{-s\epsilon t}\bar{z}]^2 \;+\; f_y[s\epsilon](s\epsilon t-2)t\mathrm{e}^{-s\epsilon t}\bar{z}\;\}\;ds \;=\; 0\;.
\end{aligned} \tag{5.2}
$$

The constant term vanishes and the linear ϵ–term implies the condition

$$
\begin{aligned}
\dot{x}_1 \;&=\; f_x(\bar{x},0,0,0)\cdot x_1 \;+\; f_y(\bar{x},0,0,0)\bar{z} \;+\; f_\lambda(\bar{x},0,0,0)\lambda_1 \;+\; f_\epsilon(\bar{x},0,0,0)\\
&=:\; f_x(\bar{x},0,0,0)\cdot x_1 \;+\; r(t,\lambda_1)
\end{aligned}
$$

where the inhomogeneity $r(t,\lambda_1)$ is bounded by $c(\lambda_1)\mathrm{e}^{-\gamma|t|}$. Hence for fixed $\beta' \in (\beta,\gamma)$ there exists a constant $c > 0$ and a solution $x_1(t)$ with $|x_1(t)| \le c\mathrm{e}^{-\beta'|t|}$ if and only if $\int_{-\infty}^{\infty}\bar{\Psi}^T(t)r(t,\lambda_1)dt = 0$ or equivalently $\lambda_1 = \lambda_1^+$ according to (2.14), (2.15) and (2.18). Note that the phase condition $\Pi(x_1) = 0$ can be enforced by adding a suitable multiple of $\dot{\bar{x}}$.

Now chose $\lambda_1 = \lambda_1^+$, calculate the corresponding solution $x_1(t)$ and show that the ϵ^2–remainder term in (5.2) is uniformly bounded with respect to the w_1–norm. Using $f(0,y,\lambda,\epsilon) \equiv 0$ this follows by inspection where the inequality $\beta < \beta' < \gamma$ is needed for estimating the f_y–term for $t \ge 0$. This leads to the upper inequality in (5.1). The lower inequality follows directly along the lines of (4.22).

Now we turn to the differentiability of $\lambda^+(\epsilon)$ for $\epsilon > 0$. Fix $\epsilon^+ \in (0,\epsilon_0)$ and introduce the weight function

$$
w_3^+(t) := \begin{cases} \mathrm{e}^{-\alpha|t|} & ,t \le 0 \\ \mathrm{e}^{-\alpha t} \;+\; \frac{\epsilon^+}{2}\,\mathrm{e}^{-\frac{\epsilon^+}{2}t} & ,t \ge 0 \end{cases}
$$

with associated spaces $D^+(L) \subset X_0$ and X_1^+ obtained from $D(L)$ and X_1 by replacing w_3 by w_3^+. The generalized norm in X_1^+ reads $|(\xi,\zeta)|_1^+ = (\|\xi\|_{w_1}, \|\zeta\|_{w_3^+})^T$. Our aim is to apply Theorem 3.1 uniformly for

$$
\epsilon \in (\epsilon^+ - \delta,\; \epsilon^+ + \delta) =: I, \qquad \delta < \epsilon^+/2, \qquad \epsilon^+ + \delta < \epsilon_0
$$

with respect to the spaces $(X_0, |\cdot|_0^*)$, $(X_1^+, |\cdot|_1^+)$ and $D^+(L) \subset X_0$ which are independent of $\epsilon \in I$. Moreover we obtain

$$
w_3(t) \;\le\; cw_3^+(t), \qquad \|\zeta\|_{w_3^+} \;\le\; c\|\zeta\|_{w_3}, \qquad D(L) \;\subset\; D^+(L), \qquad X_1 \;\subset\; X_1^+ \tag{5.3}
$$

with c independent of $\epsilon \in I$. Note that X_1 is continously embedded into X_1^+ with respect to the norms $\|\xi\|_{w_1} + \|\zeta\|_{w_3}$ and $\|\xi\|_{w_1} + \|\zeta\|_{w_3^+}$.

Next the assumptions of Theorem 3.1 are verified for $\epsilon \in I$. We obtain $L(D^+(L)) \subset X_1^+$ and $F_\epsilon(X_0) \subset X_1^+$ according to (4.8), (4.9) and (5.3). Concerning the linear stability inequality (3.9), we notice that the (x, λ)–components remain unchanged and that we can simply replace w_3 by w_3^+ in (4.13). This yields Q^{-1} as in (3.20). Concerning the Lipschitz condition (3.10) we obtain from (5.3) and (3.21)

$$\begin{aligned} |F_\epsilon(u) - F_\epsilon(v)|_1^+ &\leq c\,|F_\epsilon(u) - F_\epsilon(v)|_1^* \leq c \begin{pmatrix} \varrho_1 + \epsilon & \epsilon \\ \varrho_1 + \epsilon & \epsilon \end{pmatrix} |u - v|_0^* \\ &\leq c \begin{pmatrix} \varrho_1 + \epsilon^+ & \epsilon^+ \\ \varrho_1 + \epsilon^+ & \epsilon^+ \end{pmatrix} |u - v|_0^* =: P^+ |u - v|_0^* \end{aligned}$$

with a generic constant c independent of $\epsilon \in I$. The residual condition (3.22) follows in the same way according to

$$|Lu_0 - F_\epsilon(u_0)|_1^+ \leq c\,|Lu_0 - F_\epsilon(u_0)|_1^* \leq c \begin{pmatrix} \epsilon \\ \epsilon \end{pmatrix} \leq c \begin{pmatrix} \epsilon^+ \\ \epsilon^+ \end{pmatrix}$$

which leads to the desired inequality $|Lu_0 - F_\epsilon(u_0)|_1^+ < (Q - P^+)\varrho$ (cf. (3.11)) if we choose $0 < \epsilon^+ \ll \varrho_1 < \varrho_2 \ll 1$. Thus, by Theorem 3.1 the iteration

$$u_{n+1} = L^{-1} F_\epsilon(u_n)\,, \qquad L^{-1} F_\epsilon(\cdot) : X_0 \to X_1^+ \to X_0$$

is a contraction in $B_\varrho(u_0) \subset X_0$ with respect to the norm $\|u\|_\varrho$ from (3.13) and we obtain a function $u(\epsilon) : I \to B_\varrho(u_0)$ which represents all solutions of $Lu - F_\epsilon(u) = 0$ for $(u, \epsilon) \in I \times B_\varrho(u_0)$. Recalling $X_1 \subset X_1^+$ these solutions agree with the solutions of the last section.

Finally it is well known that we obtain $u(\epsilon) \in C^1(I, X_0)$ if $L^{-1}F_\epsilon(u) \in C^1(I \times X_0, X_0)$ with respect to the norms $|\epsilon| + \|u\|_\varrho$ in $I \times X_0$ and $\|u\|_\varrho$ in X_0. If we choose the norm $\|\xi\|_{w_1} + \|\zeta\|_{w_3^+}$ in X_1^+ this differentiability can be seen from the linear stability inequality $|u|_0^* \leq Q^{-1}|Lu|_1^+$ which implies the boundedness of $L^{-1} : X_1^+ \to X_0$ and from a straightforward calculation concerning the differentiablity of $F_\epsilon(u) : X_0 \to X_1^+$. Hence the solutions satisfy $u(\epsilon) = [x(t, \epsilon), z(t, \epsilon), \lambda^+(\epsilon)] \in C^1(I, X_0)$ and together with the differentiability at $\epsilon = 0$ we have that $\lambda^+(\epsilon)$ is differentiable in $[0, \epsilon_0)$.

It is clear that these arguments can be continued to give higher order smoothness for $\epsilon > 0$. Moreover, we could equally obtain higher derivatives at $\epsilon = 0$ by extending the Taylor expansion (5.2) including correction terms for $\bar{z}$ and by applying the same procedure to the second row in (5.1). We have not carried out the details of such an approach.

6 Applications

In this section we investigate slow travelling waves of the FitzHugh–Nagumo equation. The results of the first part are well known and they are mainly derived for interpreting geometrically the assumptions of Theorem 2.1. In the second part we examine more closely the degenerate situation of a singularly perturbed Bogdanov point that occurs in the FitzHugh–Nagumo system when the parameter a tends to zero. Our results become applicable to this system after an appropriate scaling transformation is performed.

6.1 Slow waves of the FitzHugh–Nagumo equation

The FitzHugh–Nagumo equation in one space dimension

$$\begin{aligned} \dot{u} &= -f_a(u) - w + \Delta u\,, \qquad & u \in \mathbb{R} \\ \dot{w} &= \delta\,(u - \gamma w)\,, & w \in \mathbb{R} \end{aligned}$$

is a typical example of a reaction–diffusion equation with diffusion restricted to the fast u–variable. Travelling waves of this system are solutions $(u,w)(s,t)$ which can be expressed as $(u,w)(s,t) = (u_1,v)(s+\lambda t)$ where $|\lambda| > 0$ represents the speed of the wave. The sign of λ determines the direction along which the wave travels, whereas the shape $(u_1,v)(\cdot)$ of the wave is governed by the ordinary differential equation

$$\begin{aligned} \dot{u_1} &= u_2 \\ \dot{u_2} &= \lambda u_2 + f_a(u_1) + v \\ \dot{v} &= \frac{\delta}{\lambda}\,(u_1 - \gamma v) \end{aligned} \tag{6.1}$$

where $f_a(u_1) = u_1(u_1 - a)(u_1 - 1)$ and $a \in (0, \frac{1}{2})$, $\gamma \geq 0$ are fixed constants. Then with $\epsilon = \delta/\lambda$ we obtain a system of the form

$$\begin{aligned} \dot{u} &= F(u,v,\lambda,\epsilon) \in \mathbb{R}^2 \\ \dot{v} &= \epsilon \cdot G(u,v,\lambda,\epsilon) \in \mathbb{R}\,. \end{aligned} \tag{6.2}$$

The corresponding (v,λ)–dependent u–system at $\epsilon = 0$ reads

$$\begin{aligned} \dot{u}_1 &= u_2 \\ \dot{u}_2 &= \lambda u_2 + f_a(u_1) + v\,. \end{aligned} \tag{6.3}$$

This unperturbed system has the 2–dimensional surface S_1 of stationary points

$$(u_1,u_2,v,\lambda) = (u_1, 0, -f_a(u_1), \lambda)$$

parametrized by (u_1,λ). The second surface S_2 of stationary points of the full (u,v)–system (6.2) is given by $(u_1,u_2,v,\lambda,\epsilon) = (0,0,0,\lambda,\epsilon)$, $\epsilon \neq 0$. It intersects the first surface at the curve $(u_1,u_2,v,\lambda,\epsilon) = (0,0,0,\lambda,0)$ which therefore represents the curve of stationary bifurcation points with respect to ϵ (cf. Fig.2 with y replaced by v). Assumption (2.3) holds with $(u,v)^s(\lambda,\epsilon) \equiv 0$.

Next we turn to the dynamical condition (2.4) of the system (6.3). First note that we obtain at $\lambda = 0$ a v–dependent family of Hamiltonian systems $\dot{u}_1 = u_2$, $\dot{u}_2 = f_a(u_1) + v$. As is well known each of these systems contains for $|v| \ll 1$ a Hopf point and a homoclinic orbit which are connected by a continuum of periodic orbits. The base points of the homoclinic orbits are positioned within the 2–dimensional surface S_1 of stationary points at $(u_1,u_2,v,\lambda) = (u_1, 0, -f_a(u_1), 0)$; i.e. along the $y = v$–axis.

Now from a geometrical point of view the Melnikov condition (2.4) requires that the central homoclinic orbit $\bar{u}(\tau)$ results from the transversal intersection of the stable and unstable manifold along the curve of stationary bifurcation points. Analytically this transversality condition is shown if the only bounded solutions of $\dot{u} - F_u(\bar{u}(\tau), 0, 0, 0)u = 0$ are $u = c\dot{\bar{u}}$, $c \in \mathbb{R}$ and $M = \int_{-\infty}^{\infty} \Psi^T(\tau) F_\lambda(\bar{u}(\tau), 0, 0, 0) d\tau \neq 0$. These two conditions are fulfilled as can be seen from the behavior of the Wronski–determinant as τ tends to infinity and from $\Psi^T(\tau) = [-\dot{\bar{u}}_2(\tau), \dot{\bar{u}}_1(\tau)]$ which leads to

$$M \; = \; \int_{-\infty}^{\infty} \bar{u}_2(\tau)^2 \; d\tau \; > \; 0 \; .$$

Moreover we obtain $\delta(0) = a/(\gamma a + 1) > 0$ (cf. (2.20)) and hence we can apply Corollary 2.2. In particular the derivative of $\lambda^+(\epsilon)$ at $\epsilon = 0$ reads after performing some integration by parts

$$\lambda_1^+ \; = \; \int_{-\infty}^{\infty} \bar{u}_1(\tau)^2 \; d\tau \; / \int_{-\infty}^{\infty} \bar{u}_2(\tau)^2 \; d\tau \; > \; 0 \; .$$

For an early reference of this simple formula see [5], where it is derived by formal asymptotic expansions.

Summarizing we proved homoclinic orbits of the system (6.2) along $\lambda = \lambda_1^+ \epsilon + O(\epsilon^2)$, $0 \leq \epsilon \ll 1$, as depicted in Fig.1 and Fig.2. Concerning the homoclinic orbits at $\lambda = \lambda_1^- \epsilon + O(\epsilon^2)$, $-1 \ll \epsilon \leq 0$, a simple calculation yields the left identity in (2.19) which implies $\lambda_1^- = \lambda_1^+$ and we obtain a smooth nontrivial curve of homoclinic orbits for $-1 \ll \epsilon \ll 1$. Further note that the trivial homoclinic orbits of the (v, λ)–dependent system $\dot{u} = F(u, v, \lambda, 0)$ all appear at $\lambda = 0$; i.e. in the v–dependent family of Hamiltonian systems $\dot{u} = F(u, v, 0, 0)$. Hence we meet the case that the trivial branch is perpendicular to the (λ, ϵ)–plane in Fig.2. Nevertheless the central homoclinic orbit $(u, v, \lambda, \epsilon) = (\bar{u}, 0, 0, 0)$ represents a bifurcation point of branches of homoclinic orbits.

Finally, recalling $\epsilon = \delta/\lambda$ the nontrivial orbits along $\lambda = \lambda_1^+ \epsilon + O(\epsilon^2)$, $|\epsilon| \ll 1$ transfer to homoclinic orbits of our original travelling wave system (6.1) at

$$\delta \; = \; \frac{1}{\lambda_1^+} \, \lambda^2 \; + \; O(\lambda^3) \; , \qquad 0 \; < \; |\lambda| \; \ll \; 1$$

leading to slow travelling waves in both directions of the s–axis.

6.2 An example of a singularly perturbed Bogdanov point

The calculations in the last section are valid for every $a \in (0, \frac{1}{2})$. This yields an a–dependent function $\lambda(\epsilon, a)$, $|\epsilon| \leq \epsilon_0(a)$ of homoclinic orbits of the system (6.2) with $\epsilon_0(a) \to 0$ as $a \to \frac{1}{2}$ or $a \to 0$. Note that the central homoclinic orbit $\bar{u}(\tau)$ blows up to a heteroclinic loop as $a \to \frac{1}{2}$, whereas the homoclinic orbit shrinks to a stationary point as $a \to 0$ (cf. [5], [13], [16]). Qualitatively we obtain the left diagram within Fig.3 below where the behavior of $\epsilon_0(a)$ is drawn in an arbitrary fashion as a tends to $\frac{1}{2}$ or as a tends to zero.

Now in [16] the behavior of the homoclinic orbits is completely investigated in a neighborhood of $a = \frac{1}{2}$, whereas near $a = 0$ no properties seem to be known. In the following we examine analytically the interval of existence of the homoclinic orbits as a tends to zero. Our aim is to extend the surface $\lambda(\epsilon, a)$ near $a = 0$ as depicted in the middle diagram of Fig.3; i.e. we shall show that the border line of the surface satisfies $a/\lambda = o(1)$ if we approach the origin.

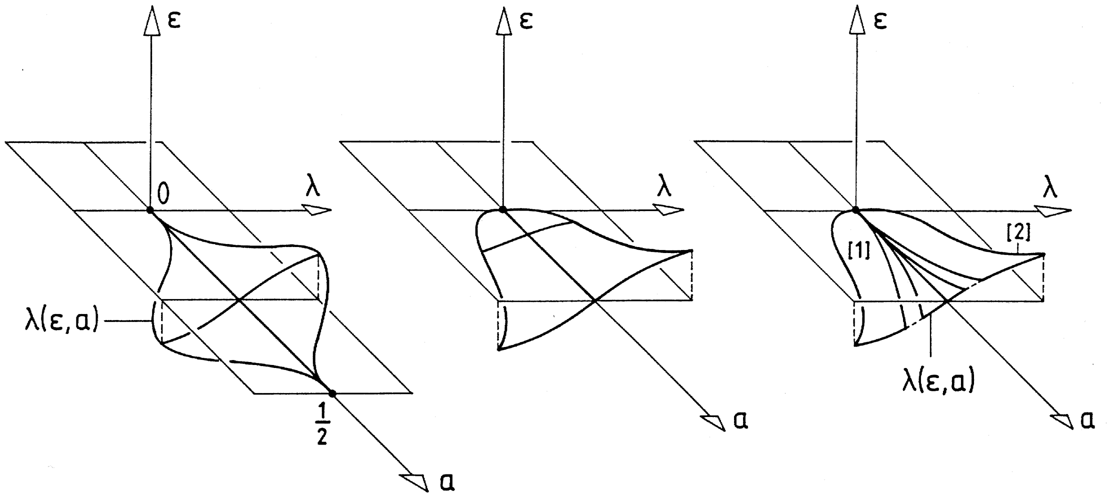

Figure 3: (λ, a, ϵ)–unfolding diagram of homoclinic orbits.

Unfortunately we can only prove the right diagram; i.e. in addition to the known surface $\lambda(\epsilon, a)$ from the left diagram we obtain the two outer surfaces [1] and [2] all of which are presumably connected by the surface depicted in the middle diagram.

With $\epsilon = \delta/\lambda$ and $(x, y) = (u, v)$ the travelling wave system (6.1) reads

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \lambda x_2 + f_a(x_1) + y \\ \dot{y} &= \epsilon \cdot (x_1 - \gamma y) \ . \end{aligned} \tag{6.4}$$

At $\epsilon = 0$ we obtain a (y, λ, a)–dependent x–system which assumes the following form at $(y, \lambda, a) = 0$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 \cdot (x_1 - 1) \ . \end{aligned}$$

Hence the stationary point $(x_1, x_2) = 0$ of the central system has the algebraically double and geometrically simple eigenvalue zero and represents a Bogdanov point. In this sense the system (6.4) can be interpreted as a singular perturbation of a Bogdanov point which occurs in the fast x–dynamics.

In a general framework it is shown in [22] and in the forthcoming paper [25] that a singularly perturbed Bogdanov point of this kind creates curves of singularly perturbed saddle–node points, Hopf points and homoclinic orbits. Thus the typical phenomena near a regularly perturbed Bogdanov point [12] carry over to Bogdanov points viewed in the context of singular perturbation. Singularly perturbed saddle–nodes are examined in many papers, see for example [8], [10], [1], [9], [23], whereas singularly perturbed Hopf points are investigated in [24]. Moreover, it turns out that the singularly perturbed homoclinic orbits can be investigated with the help of Theorem 2.1 and Corollary 2.2. In the following we perform this analysis in case of the FitzHugh–Nagumo equation. Other examples of singularly perturbed Bogdanov points appear in [15] and [22] where Van der Pol–Duffing oscillators and cell communication problems are studied.

The singularly perturbed Bogdanov point of the FitzHugh–Nagumo equation (6.4) reads $(x, y, \lambda, a, \epsilon) = 0$. First we perform a scaling transformation which blows up this point appropriately. This singular transformation between the original variables $(x_1, x_2, y, \lambda, a, \epsilon, t)$ and the new variables $(u_1, u_2, v, \bar{\lambda}, \bar{k}, \bar{\epsilon}, \tau)$ reads

$$
\begin{array}{llcl}
\text{coordinates :} & (x_1,\ x_2,\ y) & = & (\ a/2 \ - \ \bar{\lambda}^2 \cdot u_1, \quad \bar{\lambda}^3 \cdot u_2, \quad -f_a(a/2) \ + \ \bar{\lambda}^4 \cdot v\ ) \\
\text{parameters :} & (\lambda,\ a,\ \epsilon) & = & (\ -\bar{\lambda}^2 \cdot \bar{k}, \quad 4\bar{\lambda}^2, \quad \bar{\lambda}^3 \cdot \bar{\epsilon}\ ) \\
\text{time :} & t & = & -\tau/\bar{\lambda}\ .
\end{array}
\tag{6.5}
$$

In principle this scaling is a direct extension of the well known scaling near a regularly perturbed Bogdanov point (cf. [12]). However note that we maintain a linear dependency between the variables y and v. This seems appropriate because v represents a dynamic variable for $\bar{\epsilon} \neq 0$ and it is convenient to decouple the parameter transformation between (λ, a, ϵ) and $(\bar{\lambda}, \bar{k}, \bar{\epsilon})$ from the coordinate transformation between (x_1, x_2, y) and (u_1, u_2, v). Then a straightforward calculation yields the transformed system

$$
\begin{array}{rclcl}
\dot{u}_1 & = & u_2 & =: & F_1(u_1, u_2, v, \bar{\lambda}, \bar{k}, \bar{\epsilon}) \\
\dot{u}_2 & = & u_1^2 \ - \ v \ + \ \bar{\lambda}\bar{k}u_2 \ + \ O(\bar{\lambda}^2) & =: & F_2(u_1, u_2, v, \bar{\lambda}, \bar{k}, \bar{\epsilon}) \\
\dot{v} & = & \bar{\epsilon} \cdot [\ u_1 \ - \ 2 \ + \ O(\bar{\lambda}^2)\ ] & =: & \bar{\epsilon} \cdot G(u_1, u_2, v, \bar{\lambda}, \bar{k}, \bar{\epsilon})\ .
\end{array}
\tag{6.6}
$$

Our aim is to apply Corollary 2.2 for fixed $\bar{k} \in \mathbb{R}$ with respect to $(\bar{\epsilon}, \bar{\lambda})$. First we are looking for trivial homoclinic orbits at $\bar{\epsilon} = 0$ whose base points satisfy additionally $G = 0$. Secondly we verify the Melnikov condition (2.4).

At $\bar{\epsilon} = 0$, $\bar{\lambda} = 0$ and $\bar{k} \in \mathbb{R}$ the u–system $\dot{u}_1 = u_2$, $\dot{u}_2 = u_1^2 - v$ represents a well known v–dependent family of Hamiltonian systems [12] with corresponding family of homoclinic orbits. The homoclinic base points are positioned at $(u_1, u_2, v) = (\sqrt{v}, 0, v)$, $v > 0$ and consequently the unique homoclinic base point with $G = 0$ reads $(u_1, u_2, v) = (2, 0, 4)$. The associated homoclinic orbit is given by

$$
\bar{u}_1(\tau) \ = \ 2\,[\ 1 \ - \ 3\ \mathrm{sech}^2(\tau)\ ]\ , \qquad \bar{u}_2(\tau) \ = \ 12\ \mathrm{sech}^2(\tau) \cdot \tanh(\tau)
$$

(cf. [3]). The assumption (2.2) is obviously satisfied (cf. (6.6)). Hence there exists a unique function $(u, v)^s(\bar{\lambda}, \bar{k}, \bar{\epsilon})$ with (2.3) and $(u, v)^s_{\bar{\lambda}}$ vanishes at $(\bar{\lambda}, \bar{k}, \bar{\epsilon}) = (0, \bar{k}, 0)$. For verifying (2.4) with respect to $\bar{\lambda}$ it therefore suffices again to show that the only bounded solutions of $\dot{u} - F_u(\bar{u}(\tau), 4, 0, \bar{k}, 0)u = 0$ are $u = c\dot{\bar{u}}$, $c \in \mathbb{R}$ and that

$$
\begin{aligned}
0 \ \neq \ & \int_{-\infty}^{\infty} [-\dot{\bar{u}}_2,\ \dot{\bar{u}}_1] \cdot [F_u, F_v, F_{\bar{\lambda}}](\bar{u}(\tau), 4, 0, \bar{k}, 0) \cdot \begin{pmatrix} u^s_{\bar{\lambda}}(0, \bar{k}, 0) \\ v^s_{\bar{\lambda}}(0, \bar{k}, 0) \\ 1 \end{pmatrix} d\tau \\
= \ & \int_{-\infty}^{\infty} [-\dot{\bar{u}}_2,\ \dot{\bar{u}}_1] \cdot F_{\bar{\lambda}}(\bar{u}(\tau), 4, 0, \bar{k}, 0)\ d\tau \quad = \quad \bar{k} \cdot \int_{-\infty}^{\infty} \dot{\bar{u}}_1^2\ d\tau \ =: \ -M(\bar{k})\ .
\end{aligned}
$$

Now the first condition follows again from the behavior of the Wronski–determinant as τ tends to infinity, whereas the second condition is satisfied in case of $\bar{k} \neq 0$; i.e. we can apply Corollary 2.2 for every fixed $\bar{k}$ different from zero. Within the $(\bar{\lambda}, \bar{k}, \bar{\epsilon})$–parameter space of the system (6.6) we obtain surfaces

$$
\bar{\lambda} \ = \ \bar{\lambda}^+(\bar{\epsilon}, \bar{k}), \quad 0 \leq \bar{\epsilon} \leq \bar{\epsilon}_0(\bar{k}) \qquad \text{and} \qquad \bar{\lambda} \ = \ \bar{\lambda}^-(\bar{\epsilon}, \bar{k}), \quad -\bar{\epsilon}_0(\bar{k}) \leq \bar{\epsilon} \leq 0
$$

of homoclinic orbits which branch off from the $\bar{k}$–axis for $\bar{k} \neq 0$. Further it is not difficult to see the antisymmetry $\bar{\lambda}^+(\bar{\epsilon}, \bar{k}) = -\bar{\lambda}^-(-\bar{\epsilon}, \bar{k})$ which results from the symmetry within the scaling transformation (6.5) and a uniqueness argument (cf. [3]). In particular the surface is smooth along the $\bar{k}$–axis and the homoclinic orbits which correspond to $\bar{\lambda} = \bar{\lambda}^+(\bar{\epsilon}, \bar{k})$ and $\bar{\lambda} = -\bar{\lambda}^-(-\bar{\epsilon}, \bar{k})$ represent the same homoclinic orbit of the unscaled system (6.4). Hence we can restrict to

$$\bar{\lambda}^+(\bar{\epsilon}, \bar{k}) \;=\; \bar{\lambda}_1^+(\bar{k}) \cdot \bar{\epsilon} \;+\; O(\bar{\epsilon}^2), \quad 0 \leq \bar{\epsilon} \leq \bar{\epsilon}_0(\bar{k})$$

with $\bar{\epsilon}_0(\bar{k}) \to 0$ as $\bar{k} \to \pm\infty$ or $\bar{k} \to 0$. After some calculations equation (2.21) leads to the following formula for the derivative $\bar{\lambda}_1^+(\bar{k})$ at $\bar{\epsilon} = 0$

$$\bar{\lambda}_1^+(\bar{k}) \;=\; -\frac{1}{M(\bar{k})} \cdot \int_{-\infty}^{\infty} (\bar{u}_1 \;-\; 2)^2 \, d\tau \;=\; -\frac{1}{28\bar{k}} \;. \tag{6.7}$$

Next we transform the results back to the system (6.4) using the scaling transformation (6.5). According to (6.7) the inverse function of $\bar{\lambda}^+(\cdot, \bar{k})$ reads

$$\bar{\epsilon}^+(\bar{\lambda}, \bar{k}) \;=\; -28\bar{k} \cdot \bar{\lambda} \;+\; O(\bar{\lambda}^2) \;, \qquad \bar{\lambda} \;\in\; \left\{ \begin{array}{ll} [0, \; \bar{\lambda}_0(\bar{k})] & \text{in case of} \;\; \bar{k} < 0 \\ [-\bar{\lambda}_0(\bar{k}), \; 0] & \text{in case of} \;\; \bar{k} > 0 \end{array} \right\} \tag{6.8}$$

with $\bar{\lambda}_0(\bar{k}) \to 0$ as $\bar{k} \to \pm\infty$ or $\bar{k} \to 0$. Then recalling (6.5) the system (6.4) has homoclinic orbits at parameter values

$$(\lambda, \; a, \; \epsilon) \;=\; (\; -\bar{\lambda}^2 \cdot \bar{k}, \;\; 4\bar{\lambda}^2, \;\; -28\bar{k} \cdot \bar{\lambda}^4 \;+\; O(\bar{\lambda}^5) \;)$$

with $\bar{k} \neq 0$ and $\bar{\lambda}$ from (6.8). Hence for fixed $\bar{k}$ we obtain a curve within the (λ, a, ϵ)–space. The projection of this curve onto the (λ, a)–plane forms the straight line

$$\lambda = -\tfrac{1}{4} \; \bar{k} \cdot a \;, \qquad a \in [0, \; 4\bar{\lambda}_0(\bar{k})^2] \;.$$

On contrary the projection of the curve onto the (a, ϵ)–plane yields the parabola

$$\epsilon = -\tfrac{7}{4} \; \bar{k} \cdot a^2 \;+\; O(a^2\sqrt{a}) \;.$$

During variation of $\bar{k} \in (-\infty, 0)$ and $\bar{k} \in (0, \infty)$ this yields the two surfaces [1] and [2] in the right diagram of Fig.3. Note that along the border line

$$(\lambda, \; a) \;=\; (-\bar{\lambda}_0(\bar{k})^2 \cdot \bar{k}, \; 4\bar{\lambda}_0(\bar{k})^2)$$

the surface shows the limiting behavior

$$a/\lambda \;=\; -4/\bar{k} \;\to\; 0 \qquad \text{as} \qquad \bar{k} \;\to\; \pm\infty \;;$$

i.e. we have $a/\lambda = o(1)$ if we approach the origin. Further, according to numerical results, we obtain homoclinic orbits of Shilnikov type if we continue the surfaces [1] and [2] towards $\epsilon \to 0$ (cf. [22] for a similar situation). These orbits are not covered by our approach and they vanish presumably by a Duck like blow–up within a small ϵ interval.

7 Several slow stable variables

In this section we extend our approach to several slow stable variables, i.e. equation (2.1) now has the form

$$
\begin{aligned}
\dot{u} &= F(u,v,\lambda,\epsilon) \in \mathbb{R}^n\,, \quad n \geq 2 \\
\dot{v} &= \epsilon\cdot G(u,v,\lambda,\epsilon) \in \mathbb{R}^m\,, \quad m \geq 1\ .
\end{aligned}
$$

This situation is also considered in [14] within the framework of general invariant manifold theory. As before we require the conditions (2.2)–(2.4) for the steady states and the unperturbed homoclinic orbit. The main deviation is that we can no longer make $g_y(0,0,\lambda,\epsilon) \in \mathbb{R}^{m,m}$ constant as in (2.12) by a simple time scaling. Rather we perform the transformation (2.9) without time scaling and find a system

$$
\begin{aligned}
\dot{x} &= f(x,y,\lambda,\epsilon) \in \mathbb{R}^n \\
\dot{y} &= \epsilon\cdot g(x,y,\lambda,\epsilon) \in \mathbb{R}^m
\end{aligned}
\tag{7.1}
$$

which satisfies

$$
f(0,y,\lambda,\epsilon) \equiv 0, \qquad |\ \mathrm{Re}\ \sigma[f_x^0]\ | > \gamma > 0, \tag{7.2}
$$

$$
g(0,0,\lambda,\epsilon) \equiv 0, \tag{7.3}
$$

with nonsingular matrices

$$
A(\lambda,\epsilon) := g_y(0,0,\lambda,\epsilon) = (G_u\cdot F_u^{-1}\cdot F_v - G_v)[(u,v)^s(\lambda,\epsilon),\lambda,\epsilon] \in \mathbb{R}^{m,m} \tag{7.4}
$$

(cf. (2.11), (2.12), (2.8)). In addition to (7.4) we assume the slow variables to be stable; i.e.

$$
\mathrm{Re}\ \sigma[A(0,0)] < 0\ . \tag{7.5}
$$

It turns out that some nonresonance conditions on the eigenvalues of $A(0,0)$ are needed for our method of proof to work. This seems to be related to the fact that our approach allows to expand the slow and fast parts in terms of ϵ and to discuss the smoothness of the parameters. No such conditions are needed for the transversality arguments in [14] which are independent of the precise slow behavior of the homoclinic orbits.

In addition to (7.2), (7.3) and (7.5) we assume

$$
D_y^j\, g(0,0,\lambda,\epsilon) \equiv 0 \qquad \text{for} \qquad j = 2,...,k-1 \tag{7.6}
$$

where k is the smallest integer ≥ 2 satisfying

$$
\mathrm{Max}\{\ |\mathrm{Re}\ \mu|\ :\ \mu \in \sigma[A(0,0)]\ \} < k\cdot \mathrm{Min}\{\ |\mathrm{Re}\ \mu|\ :\ \mu \in \sigma[A(0,0)]\ \}\ . \tag{7.7}
$$

We notice that (7.6) is trivially satisfied if (7.7) holds for $k = 2$ and this includes the case $m = 1$ treated previously. In general it is well known that (7.6) can be achieved by a smooth normal form transformation [12] provided $A(0,0)$ satisfies the nonresonance condition

$$
\sum_{i=1}^{j} \lambda_i \notin \sigma[A(0,0)] \qquad \text{for all} \qquad j = 2,...,k-1 \qquad \text{and} \qquad \lambda_i \in \sigma[A(0,0)]\ .
$$

Moreover, under an additional nonresonance condition on the real parts of sums of eigenvalues for $j = k$ the y–system admits a C^1–linearization (see the strong Sternberg conditions in [20]).

The singular transformation (3.1) now has the form

$$y \;=\; \epsilon \cdot \mathrm{e}^{\epsilon A(\lambda,\epsilon) t} \cdot z$$

and (7.1) becomes

$$\begin{aligned} \dot{x} &= f(x, \epsilon \mathrm{e}^{\epsilon A(\lambda,\epsilon)t} z, \lambda, \epsilon) & &=: f_1(t,x,z,\lambda,\epsilon) \\ \dot{z} &= \mathrm{e}^{-\epsilon A(\lambda,\epsilon)t} \cdot g(x, \epsilon \mathrm{e}^{\epsilon A(\lambda,\epsilon)t} z, \lambda, \epsilon) \;-\; \epsilon A(\lambda,\epsilon)\cdot z & &=: f_2(t,x,z,\lambda,\epsilon) \;. \end{aligned}$$

This leads to the same blown–up system (3.3) with the same heteroclinic orbit $(\bar{x}, \bar{z})$ as before (see (2.15), (3.4)). Theorem 2.1 then generalizes as follows.

Theorem 7.1: *Let the system (7.1) satisfy assumptions (7.2)–(7.7) and let $\bar{x}$ be a homoclinic orbit of the x–equation (at $y = \lambda = \epsilon = 0$) which satisfies (2.13). Then there exists a differentiable function $\lambda^+(\epsilon)$, $0 \le \epsilon \le \epsilon_0$ such that the system (7.1) has homoclinic orbits $(x,y)(t,\epsilon)$ at $\lambda = \lambda^+(\epsilon)$. The further assertions (2.16)–(2.18) hold verbatim with $y(t,\epsilon)$ written as*

$$y(t,\epsilon) \;=\; \epsilon \cdot e^{\epsilon A(\lambda^+(\epsilon),\epsilon)t} \cdot [\; \bar{z}(t) \;+\; \epsilon \cdot r_2(t,\epsilon) \;] \;.$$

We indicate the major changes in the proof of Theorem 2.1 that are needed for the proof of Theorem 7.1. First we use the same decomposition as in (3.6) with L and F_ϵ given by (3.7) and (3.8). The weight functions w_1, w_2 in (3.14), (3.15) remain the same, but w_3 is changed to

$$w_3(t) := \begin{cases} \mathrm{e}^{-\alpha|t|} & , t \le 0 \\ \mathrm{e}^{-\alpha t} + \epsilon \mathrm{e}^{-\epsilon\kappa t} & , t \ge 0 \end{cases}$$

with a suitable constant $\kappa > 0$. According to (7.7) we find constants δ_1, $\delta_2 > 0$ such that for all λ, ϵ small

$$-\,(k - \delta_2)\delta_1 \;<\; \mathrm{Re}\;\sigma[A(\lambda,\epsilon)] \;<\; -\delta_1 \;. \tag{7.8}$$

The restrictions on κ, ϵ are

$$\kappa \;<\; \delta_1\delta_2 \qquad \text{and} \qquad 2\epsilon\kappa\delta_1 \;<\; \mathrm{Min}\{\alpha,\; \beta - \alpha\} \;. \tag{7.9}$$

We introduce the abbreviations

$$Y(t,\lambda,\epsilon) \;:=\; \mathrm{e}^{\epsilon A(\lambda,\epsilon)t} \;, \qquad Z(t,\lambda,\epsilon) \;:=\; \frac{d}{d\lambda}\, Y(t,\lambda,\epsilon) \tag{7.10}$$

and suppress the arguments (t,λ,ϵ) whenever appropriate. The estimates (4.1)–(4.3) and (4.8) remain valid and (4.4) is replaced by the Taylor expansion up to order k. Using (7.6) we obtain

$$\begin{aligned} f_2(t,x,z,\lambda,\epsilon) \;&=\; \sum_{j=0}^{k-1} \frac{1}{j!}\, \epsilon^j\, Y^{-1} \cdot [\; D_y^j\, g(x,0,\lambda,\epsilon) \;-\; D_y^j\, g(0,0,\lambda,\epsilon) \;] \cdot (Yz)^j \\ &\quad +\; \epsilon^k\, Y^{-1} \cdot g_k(t,x,z,\lambda,\epsilon) \cdot (Yz)^k \end{aligned} \tag{7.11}$$

where

$$g_k(t,x,z,\lambda,\epsilon) \;=\; \frac{1}{(k-1)!} \int_0^1 (1-s)^{k-1}\, D_y^k\, g(x, s\epsilon Y z, \lambda, \epsilon)\; ds\;. \tag{7.12}$$

From (7.8) and (7.10) we have the estimates

$$|Y(t,\lambda,\epsilon)| \;\le\; c\, \mathrm{e}^{\epsilon(k-\delta_2)\delta_1|t|}\,, \qquad |Y(t,\lambda,\epsilon)^{-1}| \;\le\; c\, \mathrm{e}^{-\epsilon\delta_1|t|} \qquad ,t \le 0 \tag{7.13}$$

$$|Y(t,\lambda,\epsilon)| \;\le\; c\, \mathrm{e}^{-\epsilon\delta_1 t}\,, \qquad |Y(t,\lambda,\epsilon)^{-1}| \;\le\; c\, \mathrm{e}^{\epsilon(k-\delta_2)\delta_1 t} \qquad ,t \;\ge\; 0\,. \tag{7.14}$$

Using this we find for the λ–derivative

$$\begin{aligned}
|Z(t,\lambda,\epsilon)| \;&=\; \epsilon\;|\; \textstyle\int_0^t\; Y(t-s,\lambda,\epsilon)\cdot A_\lambda(\lambda,\epsilon)\cdot Y(s,\lambda,\epsilon)\; ds\;| \\
&\le\; c\,\epsilon\cdot \begin{cases} |\; \int_0^t\; \mathrm{e}^{\epsilon(k-\delta_2)\delta_1(s-t)}\cdot \mathrm{e}^{-\epsilon(k-\delta_2)\delta_1 s}\; ds\;| & ,t\;\le\;0 \\ \int_0^t\; \mathrm{e}^{-\epsilon\delta_1(t-s)}\cdot \mathrm{e}^{-\epsilon\delta_1 s}\; ds & ,t\;\ge\;0 \end{cases}
\end{aligned}$$

and hence

$$|Z(t,\lambda,\epsilon)| \;\le\; c\,\epsilon\;|t|\cdot \begin{cases} \mathrm{e}^{\epsilon(k-\delta_2)\delta_1|t|} & ,t\;\le\;0 \\ \mathrm{e}^{-\epsilon\delta_1 t} & ,t\;\ge\;0\,. \end{cases} \tag{7.15}$$

In particular, Z is bounded for $t \ge 0$ uniformly in ϵ and λ. The first inequality in (4.9) follows as before, while the second is a consequence of (7.11)–(7.14) and the crucial conditions (7.3), (7.6), (7.8). We obtain

$$\begin{aligned}
|f_2(t,x,z,\lambda,\epsilon)| \;&\le\; c\;|x(t)|\cdot \left\{\begin{array}{ll} \mathrm{e}^{-\epsilon\delta_1|t|} & ,t\;\le\;0 \\ \mathrm{e}^{\epsilon(k-\delta_2)\delta_1 t} & ,t\;\ge\;0 \end{array}\right\}\cdot \sum_{j=0}^{k-1} \epsilon^j\; |Y|^j\; |z|^j \\
&\quad +\; c\,\epsilon^k\cdot \left\{\begin{array}{ll} \mathrm{e}^{[\epsilon\delta_1(-1+k(k-\delta_2))-k\alpha]|t|} & ,t\;\le\;0 \\ \mathrm{e}^{\epsilon[(k-\delta_2)\delta_1-k\delta_1]t} & ,t\;\ge\;0 \end{array}\right\} \\
&\le\; c\;\sum_{j=0}^{k-1} \epsilon^j\cdot \left\{\begin{array}{ll} \mathrm{e}^{[-\beta+j\epsilon(k-\delta_2)\delta_1-j\alpha]|t|} & ,t\;\le\;0 \\ \mathrm{e}^{[-\beta+\epsilon(k-\delta_2-j)\delta_1]t} & ,t\;\ge\;0 \end{array}\right\} \\
&\quad +\; c\,\epsilon^k\cdot \left\{\begin{array}{ll} \mathrm{e}^{k[\epsilon\delta_1 k-\alpha]|t|} & ,t\;\le\;0 \\ \mathrm{e}^{-\epsilon\delta_1\delta_2 t} & ,t\;\ge\;0 \end{array}\right\} \;\;\le\;\; \bar c\; w_3(t)\;.
\end{aligned}$$

In the last step the restrictions (7.9) are used. Hence we showed again $L(D(L)) \subset X_1$ and $F_\epsilon(X_0) \subset X_1$.

In principle we now proceed with the steps 1–3 as in 4.1–4.3. The linear stability inequality is the same as in section 4.1 with the only exception that now

$$\int_0^\infty \epsilon \mathrm{e}^{-\epsilon\kappa s}\; ds \;\le\; \frac{1}{\kappa} \qquad \text{for all} \qquad \epsilon > 0\;.$$

The small Lipschitz constant and the small residual are essentially shown with the help of the expansion (7.11) and the equations (7.2)–(7.6), (7.13)–(7.15). The key point is to replace

(4.17) in section 4 by

$$| \, Y^{-1}(t,\tilde\lambda,\epsilon) - I \, | \cdot w_1(t) \;=\; | \, \epsilon A(\tilde\lambda,\epsilon) \cdot \int_0^t Y(-s,\tilde\lambda,\epsilon) \; ds \, | \cdot w_1(t)$$

$$\le \; c \; \epsilon \; |t| \cdot \left\{ \begin{array}{ll} 1 & , t \le 0 \\ e^{\epsilon(k-\delta_2)\delta_1 t} & , t \ge 0 \end{array} \right\} \cdot e^{-\beta|t|} \quad \le \quad c \; \epsilon \cdot w_3(t) \; .$$

Using this and $k \ge 2$ allows to estimate the second component of the residual as follows

$$\begin{aligned}
|f_2(t,\bar x,\bar z,0,\epsilon) \; - \; \dot{\bar z}| \;\; \le \;\; & c \; \{ \; |Y(t,0,\epsilon)^{-1} \cdot g(\bar x,0,0,\epsilon) \; - \; g(\bar x,0,0,0)| \\
& + \sum_{j=1}^{k-1} \epsilon^j \; |Y(t,0,\epsilon)^{-1}| \cdot |\bar x(t)| \; |Y(t,0,\epsilon)|^j \; |\bar z|^j \\
& + \epsilon^k \; |Y(t,0,\epsilon)^{-1}| \cdot |Y(t,0,\epsilon)|^k \; |\bar z|^k \; \} \\
\le \;\; & c \; \{ \; |Y(t,0,\epsilon)^{-1} \; - \; I| \cdot |g(\bar x,0,0,0)| \\
& + |Y(t,0,\epsilon)^{-1}| \; \epsilon \int_0^1 |g_\epsilon(\bar x,0,0,s\epsilon) - g_\epsilon(0,0,0,s\epsilon)| \; ds \\
& + \sum_{j=1}^{k-1} \epsilon^j \cdot \left\{ \begin{array}{ll} e^{[\epsilon(-\delta_1+j(k-\delta_2)\delta_1)-(j+1)\gamma]|t|} & , t \; \le \; 0 \\ e^{[\epsilon((k-\delta_2)\delta_1-j\delta_1)-\gamma]t} & , t \; \ge \; 0 \end{array} \right\} \\
& + \epsilon^k \cdot \left\{ \begin{array}{ll} e^{[\epsilon(-\delta_1+k(k-\delta_2)\delta_1)-k\gamma]|t|} & , t \; \le \; 0 \\ e^{-\epsilon\delta_1\delta_2 t} & , t \; \ge \; 0 \end{array} \right\} \; \} \\
\le \;\; & c \; \epsilon \; w_3(t) \; .
\end{aligned}$$

This finishes the proof of Theorem 7.1.

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