STABILITY OF TRAVELING WAVES : DICHOTOMIES AND EIGENVALUE CONDITIONS ON FINITE INTERVALS

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Abstract

If a traveling wave is stable or unstable depends essentially on the spectrum of a differential operator P obtained by linearization. We investigate how spectral properties of the all–line operator P are related to eigenvalues of finite–interval BVPs $Pu(x) = su(x)$, $x_- \leq x \leq x_+, Ru = 0$. Here R is a linear boundary operator, for which we will derive determinant conditions, and the x–interval is assumed to be sufficiently large.

Under suitable assumptions, we show (a) resolvent estimates for large s; (b) if s is in the resolvent of the all–line operator P, then s is also in the resolvent for finite–interval BVPs; (c) eigenvalues of P lead to approximating eigenvalues on finite intervals. These results allow to study the stability question for traveling waves by investigating eigenvalues of finite–interval problems. We give applications to the FitzHugh–Nagumo system with small diffusion and to the complex Ginzburg–Landau equations.

Key words: Traveling waves, stability, exponential dichotomies, eigenvalues, finite boundary value problems
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1 Introduction

Under some additional assumptions, the stability of a traveling wave can be characterized in terms of the spectral properties of a linear operator, which we call P: If the spectral values s of P satisfy

$$\mathrm{Re}\, s < 0 \quad \text{or} \quad s = 0$$

and $s = 0$ is a semisimple eigenvalue of P, then the wave is stable. On the other hand, if P has a spectral value s with $\mathrm{Re}\, s > 0$, then the wave is unstable. For results in this direction, see [11, 19, 32], for example.

In this paper we consider waves in one space dimension where P is an operator defined on a subspace V of $L_2 = L_2(\mathbb{R}, \mathbb{C}^m)$, i.e., P acts on functions $v(x)$ defined on the real line $-\infty < x < \infty$. Our aim is to investigate how the desired spectral properties of P are related to spectral properties of *finite–interval* problems,

$$Pv = sv \quad \text{in} \quad J = [x_-, x_+], \quad Rv = 0 \ . \tag{1.1}$$

Here J is a large, but compact interval and R is a suitable linear boundary operator specified below. A motivation for relating the spectral properties of the all-line operator P to those of (1.1) is simply that finite–interval problems (1.1) can be investigated with standard numerical methods whereas a direct computation of the spectrum of P is generally difficult.

A main starting point for our work was the paper by Kreiss, Kreiss, and Peterson [19] where finite–interval eigenvalue problems have also been used to obtain spectral information about an operator on the whole real line. In contrast to [19], we extensively use the theory of exponential dichotomies as we believe that this technical tool makes some of the arguments of [19] more transparent and precise.

We not only consider *spectral properties* of finite–interval problems (1.1), but also obtain sharp estimates for *inhomogeneous BVPs*

$$(sI - P)v = f \quad \text{in} \quad J = [x_-, x_+], \quad Rv = \gamma \ , \tag{1.2}$$

i.e., we estimate the resolvent for finite–interval problems. Our aim is to investigate how the resolvent constant depends on s and J, and what kind of conditions are required for the boundary operator R to obtain such estimates. This is of interest if one replaces a nonlinear evolution problem

$$u_t = f(u, u_x, u_{xx}), \quad x \in \mathbb{R}, \quad t \geq 0 \ , \tag{1.3}$$

on the whole real line by a corresponding problem in a finite interval $x_- \leq x \leq x_+$ with boundary condition $Ru = 0$. For a survey of results using resolvent estimates for nonlinear stability, we refer to [20]. In the present paper we do not discuss the implications of our linear theory for the corresponding nonlinear problems, however.

A traveling wave solution $u(x,t) = U(x - ct)$ of the system (1.3) can be found by solving a second order BVP on the real line for U and by determining the corresponding wave velocity c

simultaneously. In a numerical approximation one usually truncates to a finite interval and then imposes so called asymptotic boundary conditions; see [15], [5], [9]. There is a well developed theory for the error involved in this approximation (cf. [15],[5],[22],[1] and [10], where extensions to cylindrical domains are also discussed). In this paper we assume the wave profile U and the velocity c to be known or to be computed with sufficient accuracy, and we study how the spectra of the linearized finite BVPs depend on the length of the interval and on the boundary conditions. One can show that asymptotic boundary conditions, used for approximating the wave itself, are also well-suited for approximating the zero eigenvalue of the linearization about the wave. (The zero eigenvalue, corresponding to shifts $U(x) \to U(x + x_0)$, is always present.) However, asymptotic boundary conditions generally have no advantage over other types of b.c. when the remaining spectrum, which decides upon stability of the wave, is approximated. In fact, we will derive determinant conditions that the boundary operator should satisfy in order to preserve the stability information. As we will show in Section 5 these conditions are satisfied under reasonable assumptions for Dirichlet, Neumann and periodic boundary conditions but generally may be violated for asymptotic boundary conditions.

We now give a more detailed description of our assumptions and the results of the paper. Let us assume that a formal linearization of (1.3) about a traveling wave u leads to a linear equation

$$v_t = Pv, \quad x \in \mathbb{R}, \quad t \geq 0 , \tag{1.4}$$

where Pv can be written in the form

$$Pv = (A(x)v_x)_x + B(x)v_x + C(x)v . \tag{1.5}$$

We make the following assumptions.

Assumption 1

a) The coefficients $A(x), B(x), C(x)$ take values in $\mathbb{C}^{m\times m}$ and depend continuously on $x \in \mathbb{R}$. Furthermore, the coefficients converge for $x \to \pm\infty$,

$$\begin{aligned} A(x) \to A_+, \quad B(x) \to B_+, \quad C(x) \to C_+ \quad &\text{as} \quad x \to +\infty , \\ A(x) \to A_-, \quad B(x) \to B_-, \quad C(x) \to C_- \quad &\text{as} \quad x \to -\infty . \end{aligned}$$

b) There is $\alpha > 0$ with [1]

$$A(x) + A^*(x) \geq \alpha I > 0 \quad \text{for all} \quad x \in \mathbb{R} . \tag{1.6}$$

[1]For Hermitian matrices $A, B \in \mathbb{C}^{m\times m}$ we write $A \geq B$ if $v^*Av \geq v^*Bv$ for all $v \in \mathbb{C}^m$.

c) There is $\delta > 0$ such that, for all $\omega \in \mathbb{R}$ and all $s \in \mathbb{C}$,

$$det(-\omega^2 A_+ + i\omega B_+ + C_+ - sI) = 0$$

or

$$det(-\omega^2 A_- + i\omega B_- + C_- - sI) = 0$$

implies

$$\mathrm{Re}\, s \leq -\delta < 0 \ .$$

Note that (1.6) is the assumption of parabolicity of the system $v_t = Pv$. Also, Assumption 1c) is referred to as *spectral assumption*, since it implies $\mathrm{Re}\, s \leq -\delta < 0$ for all values of the *essential* spectrum of P. See, for example, [11] for such a result. (Formally, Assumption 1c) can be motivated as follows: If one substitutes $v(x,t) = e^{st+i\omega x}v_0$ into the equation $v_t = A_+v_{xx} + B_+v_x + C_+v$, where $\omega \in \mathbb{R}, s \in \mathbb{C}$, and $v_0 \in \mathbb{C}^m$, then one obtains $v_0 = 0$ unless $\mathrm{Re}\, s \leq -\delta < 0$, i.e., nontrivial solutions of wave type decay exponentially in time.)

Besides the all–line problem (1.4) we also consider finite–interval problems

$$v_t = Pv, \quad x \in J = [x_-, x_+], \quad t \geq 0 \ , \tag{1.7}$$

subject to boundary conditions

$$Rv = 0, \quad t \geq 0 \ . \tag{1.8}$$

Here R is a linear boundary operator of the general form

$$Rv = (P_- \ Q_-) \begin{pmatrix} v(x_-) \\ v_x(x_-) \end{pmatrix} + (P_+ \ Q_+) \begin{pmatrix} v(x_+) \\ v_x(x_+) \end{pmatrix} \tag{1.9}$$

where the coefficient matrices P_-, Q_-, P_+, Q_+ have size $2m \times m$ and

$$rank(P_- \ Q_- \ P_+ \ Q_+) = 2m \ .$$

More specific assumptions for R as well as examples will be discussed below.

Functional Analytical Setting for P. With $H^1 = H^1(\mathbb{R}, \mathbb{C}^m)$ we denote the usual Sobolev space, based on L_2, with norm [2]

$$\|v\|_{H^1} = \Big(\int_{\mathbb{R}} (|v|^2 + |v_x|^2)dx \Big)^{1/2} \ .$$

We consider P as an operator on

$$V = \{ v \ : \ v \in H^1, Av_x \in H^1 \} \subset L_2$$

[2]The Euclidean norm on $\mathbb{C}^m$ is defined by $|v|^2 = \sum_j |v_j|^2$. The corresponding matrix norm will also be denote by $|\cdot|$.

and define the norm on V by

$$\|v\|_{H^2}^2 = \|v\|_{H^1}^2 + \|Av_x\|_{H^1}^2 \ .$$

(If A has a bounded derivative then this norm is equivalent to the usual Sobolev norm $\|\cdot\|_{H^2}$.)

By definition, a complex number s is in the resolvent set of P if for all $f \in L_2$ there is a unique $v \in V$ with

$$(sI - P)v = f \ .$$

The resolvent set of P is denoted by $\rho(P)$, and $\sigma(P) = \mathbb{C} \setminus \rho(P)$ is the spectrum of P. If $s \in \rho(P)$ then, by the inverse–operator theorem, the resolvent

$$(sI - P)^{-1} : L_2 \to V$$

is a *bounded* linear operator. Following the terminology of Henry [11], we write the spectrum of P as a disjoint union,

$$\sigma(P) = \sigma_{eig}(P) \cup \sigma_{ess}(P) \ ,$$

where $\sigma_{eig}(P)$ consists of all spectral values s which are (a) isolated points of $\sigma(P)$ and (b) eigenvalues of finite algebraic multiplicity. Condition (b) means that s is an eigenvalue and the generalized eigenspace

$$E = \{v \in V \ : \ (sI - P)^n v = 0 \quad \text{for some } n = 1, 2, \ldots\} \tag{1.10}$$

has finite dimension. By definition, the essential spectrum of P is

$$\sigma_{ess}(P) = \sigma(P) \setminus \sigma_{eig}(P) \ .$$

Remark: Under Assumption 1 it can be proved that $\mathrm{Re}\, s \le -\delta < 0$ for all $s \in \sigma_{ess}(P)$. See, for example, [11] for the case $A = A^* = const$. The case of variable $A(x)$ is treated in [32].

Notation for Finite–Interval Problems. With $J = [x_-, x_+]$ we denote a compact interval and write

$$J \to \mathbb{R} \quad \text{if} \quad \min\{x_+, -x_-\} \to \infty \ .$$

We set

$$V_J = \{v = v(x) \ : \ v \in H^1(J), \ Av_x \in H^1(J), \ Rv = 0\}$$

suppressing the dependency of V_J on the boundary operator R in our notation. Then

$$P_J : V_J \to L_2(J, \mathbb{C}^m)$$

is the operator defined by $P_J v = Pv, v \in V_J$. With $\sigma(P_J)$ we denote the set of eigenvalues of the eigenvalue problem

$$P_J v = sv, \quad v \in V_J \ .$$

It is well–known that either $\sigma(P_J) = \mathbb{C}$ or $\sigma(P_J)$ consists of a sequence of isolated eigenvalues of finite algebraic multiplicity. Under our assumptions, the case $\sigma(P_J) = \mathbb{C}$ will always be excluded. Then, if

$$s \in \rho(P_J) = \mathbb{C} \setminus \sigma(P_J) \ ,$$

the resolvent $(sI - P_J)^{-1}$ is a bounded linear operator from L_2 onto V_J.

Summary of Results. In Section 2 we will consider the inhomogeneous BVPs

$$sv - Pv = f + g_x \quad \text{in} \quad J = [x_-, x_+], \quad Rv = \gamma \tag{1.11}$$

for large s in a certain sector of the complex plane. (For some applications, e.g. to viscous conservation laws, it is convenient to identify a derivative term, g_x, on the right–hand side; see [19].) The main result of Section 2 is Theorem 2.2, which gives an estimate of the solution v of (1.11) in terms of f, g, and γ. To obtain the estimate it is assumed that

$$s = re^{i\phi} \quad \text{with} \quad |\phi| \le \frac{\pi}{2} + \varepsilon, \quad r \ge c_1 \ ,$$

where $\varepsilon > 0$ is sufficiently small and c_1 is sufficiently large. Furthermore, a determinant condition is required, which only involves the boundary operator R and the limit matrices $A_\pm$. To derive the result, we rewrite the second order equation (1.11) as a first–order system

$$Lz \equiv z_x - M(x,s)z = h \tag{1.12}$$

and apply dichotomy techniques to L. The technical tools concerning dichotomies are collected in the appendix.

In Section 3 we consider complex values s which are in the resolvent of the whole–line operator P and satisfy

$$\mathrm{Re}\, s > -\delta \ . \tag{1.13}$$

(Recall that $\delta > 0$ is the constant appearing in Assumption 1*c*.) The condition $\mathrm{Re}\, s > -\delta$ together with Assumption 1*c* implies the existence of exponential dichotomies for the operator $Lz = z_x - M(x,s)z$ on both half–lines, $-\infty < x \le 0$ and $0 \le x < \infty$. Then the assumption $s \in \rho(P)$ yields a dichotomy on the whole line $\mathbb{R}$. Together with a determinant condition, which now involves the boundary operator R, the limit matrices $A_\pm, B_\pm, C_\pm$, and s, one obtains that $s \in \rho(P_J)$ if $\min\{x_+, -x_-\}$ is large enough. An estimate of the solutions of the corresponding BVP (1.11) is stated in (3.13).

A compactness argument shows that the resulting bound holds uniformly for all large J and $s \in \Omega$ if $\Omega \subset \rho(P)$ is compact and $\mathrm{Re}\, s > -\delta$ for all $s \in \Omega$; see Theorem 3.1. To summarize, if $\mathrm{Re}\, s > -\delta$ and if a determinant condition is fulfilled, then

$$s \in \rho(P) \quad \text{implies} \quad s \in \rho(P_J)$$

provided that J is large enough. The requirement on J is uniform for compact sets of s values.

In Section 4 we assume $\mathrm{Re}\, s_0 > -\delta$ and $s_0 \in \sigma_{eig}(P)$, i.e., s_0 is an isolated point in the spectrum of P and is an eigenvalue of P of finite algebraic multiplicity. Under perturbations, the multiple eigenvalue is expected to break up, leading to the so–called s_0–group of eigenvalues of the perturbed operator [14]. Our aim is to make this precise in the situation where the whole–line operator P is the unperturbed operator and the finite–interval operators P_J are considered as perturbations of P.

For small $\varepsilon > 0$ and large J we define

$$\sigma_J \equiv \sigma_J(\varepsilon) = \sigma(P_J) \cap \{s \in \mathbb{C} : |s - s_0| < \varepsilon\} \ ,$$

i.e., σ_J consists of all eigenvalues of P_J in an ε–neighborhood of s_0. In other words, σ_J is the s_0–group of P_J. Assuming again a determinant condition, we will prove that

$$\max_{s \in \sigma_J} |s - s_0| \to 0 \quad \text{as} \quad J \to \mathbb{R} \ ,$$

i.e., the s_0–group of eigenvalues of P_J approaches s_0 as $J \to \mathbb{R}$. More precisely, we will show an estimate of exponential type in the interval length,

$$\max_{s \in \sigma_J} |s - s_0| \le C \exp\Big(-\beta' \min\{x_+, -x_-\}/k\Big)$$

where $\beta' > 0$ is related to a dichotomy constant and $k = \dim E$ is the algebraic multiplicity of s_0. (See (1.10) for the definition of E.)

If E_J denotes the sum of the generalized eigenspaces of the s_0–group of P_J, then (for large J) the inequality

$$\dim E_J \ge \dim E$$

is rather easy to show using Riesz projectors. However, as one might expect, equality

$$\dim E_J = \dim E$$

holds for large J. To prove this, we apply the power version of the Kreiss Matrix Theorem [17]. Two technical lemmas preparing this application are shown in the appendix.

Let us summarize the results of Sections 2, 3, and 4. We study the spectrum and resolvent estimates for P_J and, for most results, restrict attention to complex s values with

$$\mathrm{Re}\, s > -\delta \ . \tag{1.14}$$

Assumption 1c implies that the essential spectrum of P satisfies $\mathrm{Re}\, \sigma_{ess}(P) \le -\delta$, and therefore our analysis does not show how the essential spectrum of P behaves when P is replaced by P_J. For the stability question raised above, the consideration of the region $\mathrm{Re}\, s > -\delta$ suffices, however. Roughly speaking, if we assume a determinant condition, then Section 2 shows that P_J has no large eigenvalues s with (1.14). Section 3 and 4 consider bounded s and show that points s in the resolvent of P are also in the resolvent of P_J whereas (multiple) eigenvalues s_0 of P break up into groups. The corresponding invariant subspaces have equal dimensions.

In Section 5 we apply the theory to the FitzHugh–Nagumo system with small diffusion and to the complex Ginzburg–Landau equation. We also show numerical results for the resulting finite–interval problems and corresponding finite difference equations.

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2 Resolvent Estimates for Finite–Interval Problems, Large $|s|$

In this section we consider the finite–interval BVP

$$sv - Pv = f + g_x \quad \text{in} \quad J = [x_-, x_+], \quad Rv = \gamma \ . \tag{2.1}$$

Here P is defined in (1.5) and we make Assumption 1 formulated in the introduction. The boundary operator R takes the general form (1.9). The right hand sides $f \in L_2(J), g \in H^1(J)$, and $\gamma \in \mathbb{C}^m$ are given. The main result of this section is Theorem 2.1, which gives an estimate of v in terms of f, g, and γ. An assumption is that

$$s = re^{i\phi} \quad \text{with} \quad |\phi| \le \frac{\pi}{2} + \varepsilon, \quad r \ge c_1 \ , \tag{2.2}$$

where $\varepsilon > 0$ is sufficiently small and c_1 is sufficiently large. Clearly, the condition (2.2) describes all complex numbers s which are sufficiently large in absolute value and lie in a certain sector that contains the right half plane. As is well–known, the fact that ε is allowed to be positive in (2.2), is typical for estimates of parabolic problems and implies analyticity of the associated semigroup.

To arrive at sharp estimates, it will be important to isolate a Dirichlet and a Neumann part of the boundary condition $Rv = \gamma$. To this end, recall the form (1.9) of the boundary operator R and assume that the matrix

$$\begin{pmatrix} Q_- & Q_+ \end{pmatrix}$$

has rank r where $0 \le r \le 2m$. Then, without loss of generality, we may assume that the boundary condition $Rv = \gamma$ reads

$$Rv \equiv \begin{pmatrix} P_-^I & Q_-^I \\ P_-^{II} & 0 \end{pmatrix} \begin{pmatrix} v(x_-) \\ v_x(x_-) \end{pmatrix} + \begin{pmatrix} P_+^I & Q_+^I \\ P_+^{II} & 0 \end{pmatrix} \begin{pmatrix} v(x_+) \\ v_x(x_+) \end{pmatrix} = \begin{pmatrix} \gamma^I \\ \gamma^{II} \end{pmatrix} . \tag{2.3}$$

Here $P_-^I, Q_-^I, P_+^I, Q_+^I$ have size $r \times m$ and P_-^{II}, P_+^{II} have size $(2m-r) \times m$. The vectors γ^I and γ^{II} have size r and $2m-r$, respectively, and

$$\begin{pmatrix} Q_-^I & Q_+^I \end{pmatrix}$$

is a matrix of rank r. The II–part of the boundary conditions does not involve derivatives and will be referred to as the Dirichlet part. Correspondingly, the I–part is called the Neumann part.

Throughout the paper we use the notation

$$|v|_\Gamma^2 = |v(x_-)|^2 + |v(x_+)|^2$$

for measuring a function on the boundary of the interval J and let

$$\rho = |s|^{1/2} .$$

From (1.6) we know that all eigenvalues of $A(x)$ have positive real parts, and with $A^{1/2}(x)$ we always denote the square root whose eigenvalues also have positive real parts.

Theorem 2.1 *Consider the BVP (2.1) for $s = re^{i\phi}, |\phi| \le \frac{\pi}{2} + \varepsilon, r \ge c_1$, and assume that*

$$det \begin{pmatrix} Q_-^I & Q_+^I \\ -P_-^{II} A_-^{1/2} & P_+^{II} A_+^{1/2} \end{pmatrix} \ne 0 . \tag{2.4}$$

There are positive constants c_1, K, and ε so that the BVP (2.1) has a unique solution $v \in V_J$ if the interval J is sufficiently large. This solution satisfies the estimate

$$\rho^2 \|v\|^2 + \|v_x\|^2 + \rho |v|_\Gamma^2 + \frac{1}{\rho} |v_x|_\Gamma^2 \le K \Big(\frac{1}{\rho^2} \|f\|^2 + \|g\|^2 + \frac{1}{\rho} |\gamma^I|^2 + \rho |\gamma^{II}|^2 + \rho |g|_\Gamma^2 \Big) . \tag{2.5}$$

The constants c_1, K, and ε are independent of $f, g, \gamma, s, x_+, x_-$. Moreover, for constant principal part $A(x) = A$, it suffices to have $x_+ - x_- \ge 1$ instead of J sufficiently large.

To prove the theorem, we will use the tools of exponential dichotomies [4]. We introduce this concept here, but refer to the appendix for important properties such as persistence under perturbations (roughness) and extensibility over finite intervals.

Dichotomies. Let $J \subset \mathbb{R}$ denote a finite or infinite interval and let $M(x), x \in J$, denote a continuous matrix function taking values in $\mathbb{C}^{N \times N}$. The solution operator for

$$Lz = z_x - M(x)z, \quad x \in J ,$$

is denoted by $S(x,y)$, i. e., the solution of

$$z_x - M(x)z = h(x), \quad z(x_0) = z_0 \ , \tag{2.6}$$

is

$$z(x) = S(x,x_0)z_0 + \int_{x_0}^{x} S(x,y)h(y)dy \ . \tag{2.7}$$

Definition 2.1 *The operator $Lz = z_x - M(x)z$ has an exponential dichotomy on J if there are positive constants K and β and projectors $\pi(x) : \mathbb{C}^N \to \mathbb{C}^N, x \in J$, such that the following holds:*

$$\begin{array}{rcll} S(x,y)\pi(y) & = & \pi(x)S(x,y), & x,y \in J \ ; \\ |S(x,y)\pi(y)| & \le & Ke^{-\beta(x-y)}, & x \ge y \ ; \\ |S(y,x)(I-\pi(x))| & \le & Ke^{-\beta(x-y)}, & x \ge y \ . \end{array}$$

We call (K,β,π) the (non–unique) dichotomy data. If the matrices $M(x)$ and the projectors $\pi(x)$ depend continuously (in the sup-norm of continuous bounded functions on J) on some parameter $s \in \Omega \subset \mathbb{R}^p$ and if the constants K,β are independent of s then we will call the dichotomy data (K,β,π) continuous with respect to $s \in \Omega$.

Remarks: Intuitively, π is the projection onto the subspace of decaying modes. — The concept of an exponential dichotomy is mostly of interest for infinite intervals. If L has an exponential dichotomy on $\mathbb{R}$, then the projectors $\pi(x)$ are uniquely determined. On the other hand, if L has an exponential dichotomy on the positive half line $\mathbb{R}_+ = [0,\infty)$, then $\pi(x)$ is generally not unique: *range* $\pi(x)$ is unique, but *ker* $\pi(x)$ is not. Correspondingly, if L has an exponential dichotomy on $\mathbb{R}_- = (-\infty,0]$, then *range*$(I-\pi(x))$ = *ker* $\pi(x)$ is unique, but *ker*$(I-\pi(x))$ is not. For half–line operators $L = L(s)$, depending on a parameter s, the requirement of continuity of projectors $\pi(x,s)$ deserves some explanation. For definiteness, consider the case of an interval $\mathbb{R}_+$, where *range* $\pi(x,s)$ is unique, but *ker* $\pi(x,s)$ is not. Then, suppose that $\pi(x,s_0)$ has been constructed for some fixed s_0. For $|s-s_0|$ small, one can construct a *unique* $\pi(x,s)$ by requiring *ker* $\pi(x,s)$ = *ker* $\pi(x,s_0)$, i.e., by fixing the non–unique part. Under suitable assumptions, this will then lead to continuity in s. While this construction works locally, global results can be obtained via the contraction mapping theorem with continuous dependence on parameters. (See our proof of Theorem A.3 in the Appendix.) In any case, the statements of continuity of projectors given below mean that *properly chosen* projectors depend continuously on the parameter.

To prove Theorem 2.1, we transform the BVP (2.1) to a first order system by using the variables

$$w = Av_x + g, \quad z = \begin{pmatrix} v \\ \frac{1}{\rho}w \end{pmatrix}, \quad \rho = |s|^{1/2} \ . \tag{2.8}$$

One obtains

$$z_x = M(x,s)z + h, \quad R_1 z = \eta \ , \tag{2.9}$$

where

$$M(x,s) = \rho \begin{pmatrix} 0 & A^{-1} \\ \frac{s}{\rho^2}I - \frac{1}{\rho^2}C & -\frac{1}{\rho}BA^{-1} \end{pmatrix} , \quad h(x,s) = \begin{pmatrix} -A^{-1}g \\ \frac{1}{\rho}(BA^{-1}g - f) \end{pmatrix} , \tag{2.10}$$

$$R_1 z = \Big(P_- \quad \rho Q_- A(x_-)^{-1}\Big) z(x_-) + \Big(P_+ \quad \rho Q_+ A(x_+)^{-1}\Big) z(x_+) \ , \tag{2.11}$$

and

$$\eta = \gamma + Q_- A(x_-)^{-1} g(x_-) + Q_+ A(x_+)^{-1} g(x_+) \ . \tag{2.12}$$

Remark: If one uses the variable (v, w) instead of the variable z introduced in (2.8), the system matrix becomes

$$\begin{pmatrix} 0 & A^{-1} \\ sI - C & -BA^{-1} \end{pmatrix} .$$

For large $|s|$ this matrix is strongly unbalanced. The purpose of the scaling of the second component of z by $1/\rho$ is to obtain a balanced matrix $M(x,s)$ in (2.10).

Neglecting the small terms C/ρ^2 and BA^{-1}/ρ in $M(x,s)$, one obtains the system

$$z_x = \rho M_\theta z + h, \quad M_\theta(x) = \begin{pmatrix} 0 & A^{-1}(x) \\ e^{2i\theta} I & 0 \end{pmatrix} \quad \text{where} \quad \frac{s}{\rho^2} = e^{2i\theta} \ .$$

For $x \to \pm\infty$, the matrices $M_\theta(x)$ converge to

$$M_\theta^- = \begin{pmatrix} 0 & A_-^{-1} \\ e^{2i\theta} I & 0 \end{pmatrix}, \quad M_\theta^+ = \begin{pmatrix} 0 & A_+^{-1} \\ e^{2i\theta} I & 0 \end{pmatrix} .$$

We now **outline the proof of Theorem 2.1.** First, in Lemma 2.1, the constant coefficient operators

$$L_\theta^- z = z_x - M_\theta^- z, \quad L_\theta^+ z = z_x - M_\theta^+ z \tag{2.13}$$

are shown to have an exponential dichotomy on $\mathbb{R}$ for $|\theta| \le \frac{\pi}{2} - \theta_\pm$, where $0 \le \theta_\pm < \frac{\pi}{4}$ are angles determined by the spectra of $A_\pm$. Using energy and perturbation arguments, the dichotomy is then extended to the variable coefficient operators

$$L_\theta z = z_x - M_\theta(x) z \tag{2.14}$$

for $|\theta| \le \frac{\pi}{4}+\varepsilon$; see Lemma 2.2. In a third step we use that $|s|$ is large and obtain an exponential dichotomy for the family of operators $Lz = z_x - M(x,s)z$, where $M(x,s)$ is given in (2.10); see Lemma 2.3. This allows us to construct a special solution, $z_{sp}(x)$, of the inhomogeneous differential equation $z_x - M(x,s)z = h$ which satisfies the appropriate estimates. Finally, using the determinant condition (2.4), we can also treat inhomogeneous boundary data, which then completes the proof of Theorem 2.1.

The assumption $A_- + A_-^* \ge \alpha I > 0$ implies all eigenvalues of A_- to have positive real parts and therefore we can choose θ_- with

$$0 \le \theta_- < \frac{\pi}{4} \quad \text{and} \quad |\arg(\mu)| < 2\theta_- \tag{2.15}$$

for all eigenvalues μ of A_-.

Lemma 2.1 *Each operator $L_\theta^- z = z_x - M_\theta^- z$, where $|\theta| \le \frac{\pi}{2} - \theta_-$, has an exponential dichotomy on $\mathbb{R}$ with data $(K_-, \beta_-, \pi_\theta^-)$. Here K_- and β_- are independent of θ and the projector is*

$$\pi_\theta^- = \frac{1}{2}\begin{pmatrix} I & -e^{i\theta}A_-^{-1/2} \\ -e^{i\theta}A_-^{1/2} & I \end{pmatrix} . \tag{2.16}$$

Proof: The transformation

$$T = \begin{pmatrix} A_-^{-1/2} & A_-^{-1/2} \\ -e^{i\theta}I & e^{i\theta}I \end{pmatrix}, \quad T^{-1} = \frac{1}{2}\begin{pmatrix} A_-^{1/2} & -e^{-i\theta}I \\ A_-^{1/2} & e^{-i\theta}I \end{pmatrix},$$

transforms M_θ^- to block–diagonal form,

$$T^{-1}M_\theta^- T = \begin{pmatrix} -e^{i\theta}A_-^{-1/2} & 0 \\ 0 & e^{i\theta}A_-^{-1/2} \end{pmatrix} =: M_\theta^{bl} .$$

The eigenvalues μ_j of $A_-^{1/2}$ satisfy $|\arg(\mu_j)| \le \theta_- - \varepsilon$ for some $\varepsilon > 0$ and hence the eigenvalues λ_j of $e^{i\theta}A_-^{-1/2}$ satisfy

$$\operatorname{Re}\lambda_j \ge c > 0$$

with c independent of θ. The projector π^{bl} for M_θ^{bl} is

$$\pi^{bl} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

and the formula $\pi_\theta^- = T\pi^{bl}T^{-1}$ yields (2.16). $\square$

The same arguments apply to $L_\theta^+ z = z_x - M_\theta^+ z$ and yield dichotomy data $(K_+, \beta_+, \pi_\theta^+)$ for L_θ^+ on $\mathbb{R}$ if $|\theta| \le \frac{\pi}{2} - \theta_+$; here $0 \le \theta_+ < \frac{\pi}{4}$.

We define $\varepsilon_0 > 0$ by

$$\frac{\pi}{4} + \varepsilon_0 = \frac{\pi}{2} - \max\{\theta_-, \theta_+\}$$

and consider the variable coefficient operator

$$L_\theta z = z_x - M_\theta(x) z \quad \text{for} \quad |\theta| \leq \frac{\pi}{4} + \varepsilon_0 \ . \tag{2.17}$$

In the following lemma, the number ε satisfies $0 < \varepsilon \leq \varepsilon_0$, and π_θ^- and π_θ^+ denote the projectors for the constant coefficient operators L_θ^- and L_θ^+, respectively.

Lemma 2.2 *There is $\varepsilon > 0$ so that the operators L_θ have exponential dichotomies on $\mathbb{R}$ with data (K, β, π_θ) continuous in θ for $|\theta| \leq \frac{\pi}{4} + \varepsilon$. The projectors $\pi_\theta(x)$ satisfy, uniformly in θ,*

$$|\pi_\theta(x) - \pi_\theta^+| \to 0 \ \text{as } x \to \infty, \quad |\pi_\theta(x) - \pi_\theta^-| \to 0 \ \text{as } x \to -\infty. \tag{2.18}$$

Proof: The lemma is proved in three steps. First, it is shown that the operators (2.17) have exponential dichotomies on both half–lines, $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0, \infty)$. Second, using an energy argument, we show an exponential dichotomy on $\mathbb{R}$ if $|\theta| \leq \frac{\pi}{4}$. Third, a perturbation argument allows for $|\theta| \leq \frac{\pi}{4} + \varepsilon$.

Let $|\theta| \leq \frac{\pi}{4} + \varepsilon_0$ and choose $0 < \beta < \min\{\beta_-, \beta_+\}$. We consider L_θ as a perturbation of L_θ^- on $\mathbb{R}_-$ and of L_θ^+ on $\mathbb{R}_+$. Due to Lemma 2.1 we can apply Theorem A.4 and obtain that L_θ has exponential dichotomies on $\mathbb{R}_-$ and $\mathbb{R}_+$ with data $(\tilde{K}_-, \beta, \tilde{\pi}_-)$ and $(\tilde{K}_+, \beta, \tilde{\pi}_+)$ that are continuous in θ. Moreover, the projectors satisfy, uniformly in θ,

$$|\tilde{\pi}_+(\theta, x) - \pi_\theta^+| \to 0 \text{ as } x \to \infty, \quad |\tilde{\pi}_-(\theta, x) - \pi_\theta^-| \to 0 \text{ as } x \to -\infty.$$

Now we restrict to $|\theta| \leq \frac{\pi}{4}$ and use an energy estimate to show that

$$\text{range}(\tilde{\pi}_+(\theta, 0)) \oplus \text{range}(I - \tilde{\pi}_-(\theta, 0)) = \mathbb{C}^{2m}. \tag{2.19}$$

Since both projectors have the same rank m it is sufficient to prove that the intersection of both ranges is trivial. If z_0 is in the intersection, then $z(x) = S(\theta, x, 0) z_0$ decays exponentially in both directions and satisfies $L_\theta z = 0$. We write $z = (v, w)$ and obtain that $v \in V$ as well as

$$(A(x) v_x)_x = e^{2i\theta} v \ .$$

We multiply by $\bar{v}$ and integrate by parts to obtain,

$$e^{2i\theta} (v, v)_{L_2} = -(v_x, A v_x)_{L_2} \ .$$

Taking real parts we find that

$$0 \leq 2\cos(2\theta)(v, v)_{L_2} = -(v_x, (A + A^*) v_x)_{L_2} \leq -\alpha (v_x, v_x)_{L_2} \leq 0 \ .$$

Therefore, $v_x = 0$, which implies $v = 0$ since $v \in L_2$. Finally, $w = A(x) v_x = 0$ yields $z_0 = 0$, and (2.19) is proved for $|\theta| \leq \frac{\pi}{4}$.

In fact, from the continuity of the projectors in θ for $|\theta| \leq \frac{\pi}{4} + \varepsilon_0$, we conclude that (2.19) holds for $|\theta| \leq \frac{\pi}{4} + \varepsilon$ and some $\varepsilon > 0$. Then Theorem A.5 yields that the operators L_θ have exponential dichotomies with data (K, β, π_θ) continuous in $|\theta| \leq \frac{\pi}{4} + \varepsilon$ and such that the convergence in (2.18) is uniform. $\square$

By rescaling x it is clear that the operators

$$\bar{L}_\theta z = z_x - \rho M_\theta z, \quad |\theta| \leq \frac{\pi}{4} + \varepsilon \ ,$$

have exponential dichotomies on $\mathbb{R}$ with data $(K, \rho\beta, \pi_\theta)$ continuous in θ. Furthermore, the matrix $M(x,s)$ differs from ρM_θ by a term $\Delta(x,s)$ with

$$|\Delta(x,s)| \leq c_2, \quad x \in \mathbb{R}, \quad |s| \geq 1 \ .$$

Choosing $c_1 > 0$ sufficiently large and recalling our assumption

$$|s| \geq c_1, \quad \rho = |s|^{1/2} \ ,$$

we can apply Theorem A.3 (the Roughness Theorem) to the operators

$$Lz = z_x - M(x,s)z \ .$$

In the following result, $\varepsilon > 0$ is the value constructed in the previous lemma.

Lemma 2.3 *There are positive constants $c_1, K_1, K_2, \tilde{\beta}$, and ε so that the operators $Lz = z_x - M(x,s)z$ have an exponential dichotomy on $\mathbb{R}$ if*

$$s = re^{2i\theta}, \quad r \geq c_1, \quad |\theta| \leq \frac{\pi}{4} + \varepsilon \ . \tag{2.20}$$

The dichotomy data are

$$(K_1, \rho\tilde{\beta}, \pi(x,s)) \ ,$$

where π is continuous in s and

$$|\pi(x,s) - \pi_\theta(x)| \leq \frac{K_2}{\rho}, \quad \frac{s}{\rho^2} = e^{2i\theta} \ .$$

Proof: Choose c_1 so large that $4c_2 K \leq \sqrt{c_1}\beta$. Then, since $\rho \geq \sqrt{c_1}$, the assertion follows from Theorem A.3 with $K_1 = 5K, \tilde{\beta} = \frac{\beta}{2}$, and

$$\begin{aligned} |\pi(x,s) - \pi_\theta(x)| &\leq KK_1 \int_{\mathbb{R}} e^{-\rho(\beta+\tilde{\beta})|x-y|} |\Delta(y,s)| dy \\ &\leq \frac{K_2}{\rho} \ . \end{aligned}$$

□

In the remaining parts of this section, s is restricted by (2.20) where c_1 is sufficiently large and $\varepsilon > 0$ is sufficiently small. This ensures that the dichotomy result of Lemma 2.3 holds. We will first use the dichotomy result to construct a special solution of the differential equation $z_x - M(x,s)z = h$ in J. Recall that R_1 denotes the boundary operator in (2.9).

Lemma 2.4 *Let s be restricted by (2.20), thus Lemma 2.3 holds, and let $h \in L_2(J)$. Then the differential equation $z_x - M(x,s)z = h(x)$ has a special solution, $z_{sp}(x)$, in J with*

$$\rho^2 \|z_{sp}\|^2 + \rho |z_{sp}|_\Gamma^2 \le K \|h\|^2 \ . \tag{2.21}$$

If one defines $\eta_{sp} := R_1 z_{sp}$, then

$$\frac{1}{\rho} |\eta_{sp}^I|^2 + \rho |\eta_{sp}^{II}|^2 \le K \|h\|^2 \ . \tag{2.22}$$

The constant K is independent of h, s, and J.

Proof: Because of Lemma 2.3 we can apply Theorem A.1 with $\beta = \rho\tilde{\beta}$ and obtain a special solution z_{sp} in J which satisfies

$$(\rho\tilde{\beta})^2 \|z_{sp}\|^2 + \rho\tilde{\beta} |z_{sp}|_\Gamma^2 \le 5K_1^2 \|h\|^2 \ .$$

This proves (2.21). Furthermore, if $\eta_{sp} := R_1 z_{sp}$, then (2.11) and

$$Q_- = \begin{pmatrix} Q_-^I \\ 0 \end{pmatrix}, \quad Q_+ = \begin{pmatrix} Q_+^I \\ 0 \end{pmatrix}$$

yield

$$|\eta_{sp}^I| \le K\rho |z_{sp}|_\Gamma, \quad |\eta_{sp}^{II}| \le K |z_{sp}|_\Gamma \ .$$

Together with $\rho |z_{sp}|_\Gamma^2 \le K\|h\|^2$ the estimate (2.22) follows. □

Next we solve the homogeneous differential equation $Lz = z_x - M(x,s)z = 0$ with boundary condition $R_1 z = \eta$. The determinant condition (2.4) will be important. We will also require that J is sufficiently large, because we use the convergence of projectors as stated in Lemma 2.2. Requiring J to be large is not necessary in case of a constant matrix $A(x)$.

Lemma 2.5 *Let s be restricted by (2.20), thus Lemma 2.3 holds, and assume (2.4). If J is sufficiently large, then the BVP*

$$Lz = z_x - M(x,s)z = 0, \quad R_1 z = \eta \ , \tag{2.23}$$

has a unique solution $z = z_{hom}(x)$, and the following estimate holds

$$\rho \|z_{hom}\|^2 + |z_{hom}|_\Gamma^2 \le K \Big(\frac{1}{\rho^2} |\eta^I|^2 + |\eta^{II}|^2 \Big) \ . \tag{2.24}$$

The constant K is independent of η, s, and J.

Proof: Denote the dichotomy data of $L = L(s)$ by $(K_1, \rho\beta, \pi(x,s))$; see Lemma 2.3. We will write $\pi(x) = \pi(x,s)$ and $S(x,y) = S(x,y,s)$, where $S(x,y,s)$ is the solution operator, thus suppressing the s–dependence in our notation. Let

$$\alpha_- \in \mathrm{range}(\pi(x_-)), \quad \alpha_+ \in \mathrm{range}(I - \pi(x_+)) \tag{2.25}$$

be arbitrary and consider the function

$$z_{hom}(x) = S(x,x_-)\alpha_- + S(x,x_+)\alpha_+ \ , \tag{2.26}$$

which satisfies $Lz_{hom} = 0$. If $\beta_-, \beta_+ \in \mathbb{C}^m$ are arbitrary and we set

$$\alpha_- = \pi(x_-)\begin{pmatrix} A_-^{-1/2} \\ -e^{i\theta} I \end{pmatrix}\beta_-, \quad \alpha_+ = (I - \pi(x_+))\begin{pmatrix} A_+^{-1/2} \\ e^{i\theta} I \end{pmatrix}\beta_+ \ , \tag{2.27}$$

then α_- and α_+ satisfy (2.25). Here $s = \rho^2 e^{2i\theta}$, as before. Thus, using (2.27), we obtain a solution $z_{hom}(x) = z_{hom}(x, \beta_-, \beta_+)$ depending on $2m$ parameters. (The transformation (2.27) is motivated by the form of T in the proof of Lemma 2.1.) In the following we use the notation $\mathcal{O}(1/\rho, J)$ as an abbreviation for matrices that can be estimated by

$$\frac{C_1}{\rho} + C_2(J)$$

where C_1 is independent of ρ and J, and $C_2(J) \to 0$ as $J \to \mathbb{R}$, uniformly in ρ.

Substitution of (2.26), (2.27) into the boundary condition $R_1 z_h = \eta$ leads to a linear system for (β_-, β_+), which we now derive. First note that

$$\begin{aligned} z_{hom}(x_-) &= \alpha_- + \mathcal{O}(e^{-\rho\tilde{\beta}(x_+ - x_-)})\alpha_+ \\ &= \alpha_- + \mathcal{O}(1/\rho, J)\alpha_+ \ , \end{aligned}$$

and, similarly,

$$z_{hom}(x_+) = \alpha_+ + \mathcal{O}(1/\rho, J)\alpha_- \ .$$

Furthermore, by Lemma 2.3 and Lemma 2.2 we have

$$\pi(x_-) = \pi_\theta(x_-) + \mathcal{O}(1/\rho) = \pi_\theta^- + \mathcal{O}(1/\rho, J) \ ,$$

thus

$$\begin{aligned} \alpha_- &= \pi_\theta^- \begin{pmatrix} A_-^{-1/2} \\ -e^{i\theta} I \end{pmatrix}\beta_- + \mathcal{O}(1/\rho, J)\beta_- \\ &= \begin{pmatrix} A_-^{-1/2} \\ -e^{i\theta} I \end{pmatrix}\beta_- + \mathcal{O}(1/\rho, J)\beta_- \ , \end{aligned}$$

The last equation follows from $\pi_\theta^- T = T\pi^{bl}$; see the proof of Lemma 2.1. Similarly,

$$\alpha_+ = \begin{pmatrix} A_+^{-1/2} \\ e^{i\theta} I \end{pmatrix} \beta_+ + \mathcal{O}(1/\rho, J)\beta_+ \ .$$

The boundary condition $\eta = R_1 z_h$ is equivalent to

$$\begin{aligned} \begin{pmatrix} \frac{1}{\rho}\eta^I \\ \eta^{II} \end{pmatrix} &= \Big\{ \begin{pmatrix} 0 & Q_-^I A_-^{-1} \\ P_-^{II} & 0 \end{pmatrix} + \mathcal{O}(1/\rho, J) \Big\} z_h(x_-) + \Big\{ \begin{pmatrix} 0 & Q_+^I A_+^{-1} \\ P_+^{II} & 0 \end{pmatrix} + \mathcal{O}(1/\rho, J) \Big\} z_h(x_+) \\ &= \Big(\mathcal{M} + \mathcal{O}(1/\rho, J) \Big) \begin{pmatrix} \beta_- \\ \beta_+ \end{pmatrix} \end{aligned}$$

with $\mathcal{M} = (\mathcal{M}_- \ \mathcal{M}_+)$ where

$$\mathcal{M}_- = \begin{pmatrix} 0 & Q_-^I A_-^{-1} \\ P_-^{II} & 0 \end{pmatrix} \begin{pmatrix} A_-^{-1/2} \\ -e^{i\theta} I \end{pmatrix}, \quad \mathcal{M}_+ = \begin{pmatrix} 0 & Q_+^I A_+^{-1} \\ P_+^{II} & 0 \end{pmatrix} \begin{pmatrix} A_+^{-1/2} \\ e^{i\theta} I \end{pmatrix} \ .$$

One obtains

$$\mathcal{M} = \begin{pmatrix} e^{i\theta} I_r & 0 \\ 0 & I_{2m-r} \end{pmatrix} \begin{pmatrix} Q_-^I & Q_+^I \\ -P_-^{II} A_-^{1/2} & P_+^{II} A_+^{1/2} \end{pmatrix} \begin{pmatrix} -A_-^{-1} & 0 \\ 0 & A_+^{-1} \end{pmatrix} \ . \tag{2.28}$$

By assumption (2.4) the matrix $\mathcal{M}$ is nonsingular, and one finds that

$$|\beta_-|^2 + |\beta_+|^2 \le K_1 \Big(\frac{1}{\rho^2} |\eta^I|^2 + |\eta^{II}|^2 \Big)$$

if ρ and J are sufficiently large. The estimate

$$\rho \|z_h\|^2 + |z|_\Gamma^2 \le K_2 (|\beta_-|^2 + |\beta_+|^2)$$

follows from the definition of z_{hom} and the properties of $S(x,y)$ and $\pi(x)$. The lemma is proved. $\square$

It is now easy to complete the **proof of Theorem 2.1**. Write the solution z of

$$Lz = z_x - M(x,s)z = h, \quad R_1 z = \eta \ ,$$

as $z = z_{sp} + z_{hom}$ where

$$\begin{aligned} Lz_{sp} &= h, \quad R_1 z_{sp} =: \eta_{sp} \ , \\ Lz_{hom} &= 0, \quad R_1 z_{hom} = \eta - \eta_{sp} \ . \end{aligned}$$

By Lemma 2.4 we have

$$\rho^2\|z_{sp}\|^2 + \rho|z_{sp}|_\Gamma^2 \le K_1\|h\|^2$$

and

$$\frac{1}{\rho}|\eta^I_{sp}|^2 + \rho|\eta^{II}_{sp}|^2 \le K\|h\|^2 \ .$$

Therefore, Lemma 2.5 yields

$$\rho^2\|z_{hom}\|^2 + \rho|z_{hom}|_\Gamma^2 \le K_2\Big(\|h\|^2 + \frac{1}{\rho}|\eta^I|^2 + \rho|\eta^{II}|^2\Big) \ ,$$

and we obtain

$$\rho^2\|z\|^2 + \rho|z|_\Gamma^2 \le K_3\Big(\|h\|^2 + \frac{1}{\rho}|\eta^I|^2 + \rho|\eta^{II}|^2\Big) \ . \tag{2.29}$$

Furthermore, from the definition of z (see (2.8)) we find that

$$\rho^2\|v\|^2 + \|v_x\|^2 \le K_4\rho^2\|z\|^2 \ , \tag{2.30}$$

$$\rho^2|v|_\Gamma^2 + |v_x|_\Gamma^2 \le K_4\rho^2|z|_\Gamma^2 \ . \tag{2.31}$$

Also, the definition of h (see (2.10)) yields

$$\|h\|^2 \le K_5\Big(\frac{1}{\rho^2}\|f\|^2 + \|g\|^2\Big) \tag{2.32}$$

and the definition of η (see (2.12)) implies

$$|\eta^I|^2 \le K_6(|\gamma^I|^2 + |g|_\Gamma^2), \quad |\eta^{II}|^2 \le K_6(|\gamma^{II}|^2 + |g|_\Gamma^2) \ . \tag{2.33}$$

Collecting the estimates (2.29) to (2.33), the bound (2.5) follows.

3 Resolvent Estimates for Finite–Interval Problems, Bounded $|s|$

In this section we continue to investigate the finite–interval BVP

$$sv - Pv = f + g_x \quad \text{in} \quad J = [x_-, x_+], \quad Rv = \gamma \ . \tag{3.1}$$

The assumptions are as in Section 2, but now we consider s–values in a compact subset Ω of the half plane $\{s \in \mathbb{C} : \operatorname{Re} s > -\delta\}$. By Assumption 1c) we know that P, the all–line operator, has no essential spectrum in this half plane. In addition, we assume in this section that Ω is in the resolvent set $\rho(P)$. The analysis of bounded s is somewhat simpler than the analysis of large s in Section 2 since we do not have to scale by $|s|^{1/2}$.

Using the variables $z = (v, w), w = Av_x + g$, the BVP (3.1) transforms to

$$z_x = M(x, s)z + h, \quad R_1 z = \eta \ , \tag{3.2}$$

where

$$M(x,s)=\left(\begin{array}{cc}0 & A^{-1}\\ sI-C & -BA^{-1}\end{array}\right)\ ,\quad h(x,s)=\left(\begin{array}{c}-A^{-1}g\\ BA^{-1}g-f\end{array}\right)\ ,\tag{3.3}$$

$$R_1 z=\Big(P_- \quad Q_-A(x_-)^{-1}\Big)z(x_-)+\Big(P_+ \quad Q_+A(x_+)^{-1}\Big)z(x_+)\ ,\tag{3.4}$$

and

$$\eta=\gamma+Q_-A(x_-)^{-1}g(x_-)+Q_+A(x_+)^{-1}g(x_+)\ .\tag{3.5}$$

For bounded s the lower–order terms in P are important for the analysis, but we will still obtain an exponential dichotomy for $Lz=z_x-M(x,s)z$ on $\mathbb{R}$ by using the limit matrices for $x\to\pm\infty$,

$$M_-(s)=\left(\begin{array}{cc}0 & A_-^{-1}\\ sI-C_- & -B_-A_-^{-1}\end{array}\right),\quad M_+(s)=\left(\begin{array}{cc}0 & A_+^{-1}\\ sI-C_+ & -B_+A_+^{-1}\end{array}\right)\ .\tag{3.6}$$

For any s, the eigenvalues μ of these matrices are those of the two quadratic eigenvalue problems

$$\det(\mu^2A_\pm+\mu B_\pm+C_\pm-sI)=0\ .\tag{3.7}$$

By Assumption 1c), the spectral assumption, we obtain that for each s with $\operatorname{Re}s>-\delta$ there is $\beta=\beta(s)>0$ so that (3.7) implies $|\operatorname{Re}\mu|\geq\beta(s)$. In fact, if we restrict s to some half plane $\operatorname{Re}s\geq-\delta_0>-\delta$, then there is a uniform bound $|\operatorname{Re}\mu|\geq\beta_0>0$. This holds since for large $|s|$ equation (3.7) has solutions μ close to $\pm\sqrt{s/\lambda_j}$, where the λ_j are the eigenvalues of $A_\pm$. (See [19] for such a reasoning.) Using a continuity argument, this also shows that the $2m\times 2m$–matrices $M_+(s)$ and $M_-(s)$, $\operatorname{Re}s>-\delta$, both have m eigenvalues (counting algebraic multiplicities) in the open right half plane and m eigenvalues in the open left half plane.

As a consequence, for $\operatorname{Re}s>-\delta$, we can choose bases, each with m elements, of the stable subspace of $M_-(s)$ and of the unstable subspace of $M_+(s)$. In matrix notation,

$$M_-(s)\left(\begin{array}{c}Y_-(s)\\ Z_-(s)\end{array}\right)=\left(\begin{array}{c}Y_-(s)\\ Z_-(s)\end{array}\right)\Lambda_-(s),\quad M_+(s)\left(\begin{array}{c}Y_+(s)\\ Z_+(s)\end{array}\right)=\left(\begin{array}{c}Y_+(s)\\ Z_+(s)\end{array}\right)\Lambda_+(s)\ .\tag{3.8}$$

Here all matrices $Y_\pm,Z_\pm,\Lambda_\pm$ have size $m\times m$, the matrices $\left(\begin{array}{c}Y_-\\ Z_-\end{array}\right)$ and $\left(\begin{array}{c}Y_+\\ Z_+\end{array}\right)$ are of rank m and Λ_- and Λ_+ have eigenvalues with Re <0 and Re >0, respectively. Using the form (3.6) of $M_\pm$, the equations (3.8) yield

$$Z_-=A_-Y_-\Lambda_-,\quad Z_+=A_+Y_+\Lambda_+\ .\tag{3.9}$$

The functions

$$z(x)=\left(\begin{array}{c}Y_-\\ Z_-\end{array}\right)e^{\Lambda_-x}\phi\quad\text{and}\quad z(x)=\left(\begin{array}{c}Y_+\\ Z_+\end{array}\right)e^{\Lambda_+x}\psi,\quad \phi,\psi\in\mathbb{C}^m\ ,\tag{3.10}$$

are decaying solutions of $z_x = M_- z$ and growing solutions of $z_x = M_+ z$, respectively, and must be controlled by the boundary conditions. If we insert into (3.4) and replace $A(x_\pm)$ by $A_\pm$ we arrive at a $2m \times 2m$–matrix with determinant

$$D(s) = \det \left(\begin{pmatrix} P_- & Q_- \end{pmatrix} \begin{pmatrix} Y_-(s) \\ Y_-(s)\Lambda_-(s) \end{pmatrix} \quad \begin{pmatrix} P_+ & Q_+ \end{pmatrix} \begin{pmatrix} Y_+(s) \\ Y_+(s)\Lambda_+(s) \end{pmatrix} \right) . \tag{3.11}$$

Theorem 3.1 *Let Ω be a compact set in $\{s \in \mathbb{C} : \operatorname{Re} s > -\delta\}$ which contains no eigenvalues of the operator P and assume*

$$D(s) \neq 0 \quad \text{for all } s \in \Omega . \tag{3.12}$$

Then, for J sufficiently large, there are no eigenvalues in Ω of the finite–interval BVP (3.1), and we have a resolvent estimate uniformly in J and $s \in \Omega$,

$$\|v\|_{H^2} + |v|_\Gamma + |v_x|_\Gamma \leq K\Big(\|f\| + \|g\| + |\gamma| + |g|_\Gamma\Big) . \tag{3.13}$$

The proof proceeds in several steps. It is sufficient to prove the assertion for every fixed $s \in \Omega$. Then the resolvent estimate will give the same result in some $\varepsilon-$neighborhood of s, where ε is independent of J, and a compactness argument proves the theorem.

Since the matrices $M_\pm(s)$ have no eigenvalues on the imaginary axis, the constant–coefficient operators $z_x - M_\pm(s)z$ have exponential dichotomies on $\mathbb{R}$, and we denote their projectors by $\pi_\pm(s)$. For the variable coefficient operator $L(s)z = z_x - M(x,s)z$ we obtain the following result.

Lemma 3.1 *For any $s \in \mathbb{C}$ with $\operatorname{Re} s > -\delta$ the operator $L(s)z = z_x - M(x,s)z$ has exponential dichotomies on both half lines, $\mathbb{R}_-$ and $\mathbb{R}_+$, with data $(K_-, \beta_-, \pi_-(x,s))$ and $(K_+, \beta_+, \pi_+(x,s))$, respectively. The projectors satisfy*

$$|\pi_-(x,s) - \pi_-(s)| \to 0 \quad as \quad x \to -\infty, \quad |\pi_+(x,s) - \pi_+(s)| \to 0 \quad as \quad x \to \infty . \tag{3.14}$$

In addition, if s is also in the resolvent set of P, then $L(s)$ has an exponential dichotomy on $\mathbb{R}$.

Proof: The first part follows directly from Theorem A.4. The second part follows from Theorem A.5, because any non–zero vector in the intersection

$$\operatorname{range}(\pi_+(0,s)) \cap \operatorname{range}(I - \pi_-(0,s))$$

leads to a non–trivial solution of the homogeneous equation $L(s)z = 0$ which decays exponentially in both directions. This contradicts $s \in \rho(P)$. $\square$

The next lemma follows from Theorem A.1 in the same way as Lemma 2.4.

Lemma 3.2 *For any $s \in \rho(P)$ with $\operatorname{Re} s > -\delta$ and any $h \in L_2(J)$, the differential equation $z_x - M(x,s)z = h(x)$ has a solution $z_{sp}(x)$ in J with*

$$\|z_{sp}\| + |z_{sp}|_\Gamma \leq K\|h\| . \tag{3.15}$$

If one defines $\eta_{sp} := R_1 z_{sp}$, then

$$|\eta_{sp}| \leq K\|h\| \ . \tag{3.16}$$

The constant K is independent of h and J.

Furthermore, the analogue of Lemma 2.5 is the following result.

Lemma 3.3 *Consider the BVP*

$$z_x - M(x,s)z = 0 \quad in \quad J, \quad R_1 z = \eta \ , \tag{3.17}$$

under the assumptions

$$\operatorname{Re} s > -\delta, \ s \in \rho(P) \ ,$$

and assume (3.12). If J is sufficiently large, then (3.17) has a unique solution $z = z_{hom}(x)$, and the following estimate holds

$$\|z_{hom}\|_{H^1} + |z_{hom}|_\Gamma \leq K|\eta| \ . \tag{3.18}$$

The constant K is independent of η and J.

Proof : The proof of Lemma 2.5 can be adapted to the current situation. The formula (2.27) becomes

$$\alpha_- = \pi(x_-,s)\begin{pmatrix} Y_- \\ Z_- \end{pmatrix}\beta_-, \quad \alpha_+ = (I - \pi(x_+,s))\begin{pmatrix} Y_+ \\ Z_+ \end{pmatrix}\beta_+ \ , \tag{3.19}$$

and, by using (3.9), the evaluation of the boundary conditions leads to the $2m \times 2m$–matrix which appears in the determinant (3.11). Finally, notice that $\|z_{hom,x}\|$ can be bounded in terms of $\|z_{hom}\|$ by using (3.17) and the boundedness of $M(\cdot,s)$. □

Combining the estimates of the last two lemmas yields the assertion of Theorem 3.1.

In our next result, which is needed for the analysis of eigenvalues in Section 4, we compare the solution v of the infinite–line problem

$$(sI - P)v = f + g_x, \quad v \in V \ , \tag{3.20}$$

with the solution $v_J \in V_J$ of the finite–interval BVP (3.1), assuming homogeneous boundary data $\gamma = 0$. The restriction of v to J is denoted by $v_{|J}$.

Theorem 3.2 *Let the assumptions of Theorem 3.1 hold. Furthermore, let $\gamma = 0$ and let J be sufficiently large. If $|g|_\Gamma \to 0$ as $J \to \mathbb{R}$, then the solutions v_J converge to the solution v of (3.20) in the sense that the error $e_J = v_{|J} - v_J$ satisfies*

$$\|e_J\|_{H^2} + |e_J|_\Gamma + |e_{J,x}|_\Gamma \to 0 \quad as \quad J \to \mathbb{R}. \tag{3.21}$$

If, in addition, the functions f and g satisfy

$$\tilde{f} = e^{\kappa|x|} f \in L_2, \quad \tilde{g} = e^{\kappa|x|} g \in L_2 \tag{3.22}$$

for some $\kappa > 0$, then the following improved estimate holds

$$\|e_J\|_{H^2} + |e_J|_\Gamma + |e_{J,x}|_\Gamma \le C(\|\tilde{f}\| + \|\tilde{g}\| + |\tilde{g}|_\Gamma) e^{-\alpha \min\{x_+, -x_-\}} \ . \tag{3.23}$$

Here the constant C is independent of f, g, J, and the positive exponent α is less than (but can be taken arbitrarily close to) $\min\{\kappa, \beta\}$, where β is the dichotomy exponent for $sI - P$ on $\mathbb{R}$.

Proof : Let z and z_J denote the solutions of the corresponding first order systems (see (3.2)–(3.5)). We set $y = z_{|J} - z_J$, suppressing the dependence of y on J in our notation. The difference y solves the system

$$y_x - M(x,s)y = 0, \quad x \in J, \quad R_1 y = R_1 z - R_1 z_J \ .$$

Here $|R_1 z_J| \le K_0 |g|_\Gamma$ by (3.5) since $\gamma = 0$. By Lemma 3.3 we have the bound

$$\|y\|_{H^1} + |y|_\Gamma \le K_1(|R_1 z| + |g|_\Gamma) \le K_2(|z|_\Gamma + |g|_\Gamma) \tag{3.24}$$

and therefore it suffices to estimate $|z|_\Gamma$. Let us first assume (3.22) and define $\tilde{h} = e^{\kappa|x|} h$ with h given in (3.3). Using the Green's function $G(x,y)$ of the linear operator $Lz = z_x - M(x,s)z$ on $\mathbb{R}$ (see Theorem A.1), we can write $z(x) = \int_{\mathbb{R}} G(x,y)h(y)dy$ and obtain

$$\begin{aligned}
|z(x_+)|^2 &= \Big| \int_{\mathbb{R}} G(x_+,\xi) e^{-\kappa|\xi|} \tilde{h}(\xi) d\xi \Big|^2 \le K \int_{\mathbb{R}} e^{-2(\beta|x_+ - \xi| + \kappa|\xi|)} d\xi \ \|\tilde{h}\|^2 \\
&= K\Big(\frac{1}{2(\beta+\kappa)} (e^{-2\beta x_+} + e^{-2\kappa x_+}) + \int_0^{x_+} e^{-2(\beta(x_+ - \xi) + \kappa\xi)} d\xi \Big) \|\tilde{h}\|^2 \\
&\le C(e^{-2\beta x_+} + e^{-2\kappa x_+} + x_+ e^{-2\min(\beta,\kappa) x_+})(\|\tilde{f}\|^2 + \|\tilde{g}\|^2) \ .
\end{aligned}$$

A similar estimate holds for $|z(x_-)|^2$, and this proves (3.23).

The general case $h \in L_2$ is handled by approximation. Given $\varepsilon > 0$ we choose $h_\varepsilon \in C_0^\infty(\mathbb{R})$ such that $\|h - h_\varepsilon\| \le \varepsilon$. Then $\tilde{h}_\varepsilon = e^{\beta|x|} h_\varepsilon$ is in L_2 and we can apply the previous estimate. Thus we find

$$\begin{aligned}
|z(x_+)| &\le \Big| \int_{\mathbb{R}} G(x_+,\xi)(h(\xi) - h_\varepsilon(\xi)) d\xi \Big| + \Big| \int_{\mathbb{R}} G(x_+,\xi) h_\varepsilon(\xi) d\xi \Big| \\
&\le C_1 e^{-\alpha x_+} \|\tilde{h}_\varepsilon\| + C_2 \varepsilon \ .
\end{aligned}$$

We can bound the right hand side by $(C_2 + 1)\varepsilon$ by making x_+ large, and this completes the proof since C_2 is independent of ε. □

Remark: From the proof we see that the convergence (3.21) holds for any fixed f and g, but is uniform for $s \in \Omega$. Moreover, the condition $|g|_\Gamma \to 0$ vanishes if we solve the BVP (3.1) with a suitable γ that achieves $R_1 z_J = \eta = 0$ in (3.5).

4 Eigenvalues of Finite Multiplicity

In this section s_0 denotes an eigenvalue of P with $\mathrm{Re}\, s_0 > -\delta$. The assumption $\mathrm{Re}\, s_0 > -\delta$ together with Assumption 1 imply that s_0 is an isolated point in $\sigma(P)$ and that the generalized eigenspace E to s_0, defined in (1.10), has finite dimension, which we denote by dim $E = k$. We recall from Section 3 that the differential equation $sv - Pv = f + g_x$ is written as a first order system,

$$L(s)z \equiv z_x - M(x,s)z = h(x) \ ,$$

(see (3.3)) and that $L(s)$ has exponential dichotomies on both half–lines, $\mathbb{R}_+$ and $\mathbb{R}_-$, for $\mathrm{Re}\, s > -\delta$; see Lemma 3.1. Furthermore, we recall the determinant $D(s)$ in (3.11), which involves the coefficients $P_\pm$ and $Q_\pm$ of the boundary operator R (see (1.9)), the limit matrices $A_\pm$, and the matrices $Y_\pm(s)$ and $\Lambda_\pm(s)$ introduced in (3.8). The m columns of

$$\begin{pmatrix} Y_-(s) \\ Z_-(s) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Y_+(s) \\ Z_+(s) \end{pmatrix}$$

span the stable subspace of $M_-(s)$ and the unstable subspace of $M_+(s)$, respectively. The notations for the finite interval $J = [x_-, x_+]$, the space V_J, and the finite–interval operator $P_J : V_J \to L_2(J, \mathbb{C}^m)$ are set in the introduction.

In our perturbation theorem, Theorem 4.1, we will require the determinant condition

$$D(s_0) \neq 0 \ ,$$

and will fix a small ε_0 with $0 < \varepsilon_0 \leq 1$ and the following properties:

(a) the point s_0 is the only spectral point of P in

$$K_{\varepsilon_0} = \{ s \in \mathbb{C} \ : \ |s - s_0| \leq \varepsilon_0 \} \ ; \tag{4.1}$$

(b) all $s \in K_{\varepsilon_0}$ satisfy $\mathrm{Re}\, s > -\delta$;

(c) $D(s) \neq 0$ for all $s \in K_{\varepsilon_0}$.

Then we define

$$\sigma_J = K_{\varepsilon_0} \cap \sigma(P_J) \ , \tag{4.2}$$

i.e., σ_J consists of all eigenvalues of P_J in the ball (4.1). In other words, σ_J is the s_0–group of P_J. Furthermore, E_J denotes the sum of the generalized eigenspaces of P_J to the eigenvalues in σ_J. As $J \to \mathbb{R}$, one expects E_J to approach E and σ_J to approach the one–element set $\{s_0\}$. Such a result is made precise in Theorem 4.1. We recall that $u_{|J}$ denotes the restriction of a function $u(x)$ to the interval J. Furthermore, $\beta_+ > 0$ and $\beta_- > 0$ denote dichotomy exponents for $L(s_0)$ on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively.

Theorem 4.1 *Recall Assumption 1 and let s_0 denote an isolated point of $\sigma(P)$ with corresponding generalized eigenspace E of dimension $1 \leq k < \infty$. We assume $\mathrm{Re}\, s_0 > -\delta$ and $D(s_0) \neq 0$; see (3.11). Then there exists $b > 0$ so that the invariant subspace E_J corresponding to the s_0–group of P_J has dimension*

$$\dim\ E_J = \dim\ E = k$$

if $J \supset [-b, b]$. Furthermore, as $J \to \mathbb{R}$, the s_0–group of P_J converges to $\{s_0\}$ as

$$\max_{s \in \sigma_J} |s - s_0| \leq C \exp\Big(-\beta' \min\{x_+, -x_-\}/k\Big)\ , \tag{4.3}$$

and the space E_J approximates E in the following sense: Given any basis $\{u_j\}_{1 \leq j \leq k}$ in E, there is a basis $\{u_{j,J}\}_{1 \leq j \leq k}$ in E_J with

$$\max_j \|u_{j,J} - u_{j|J}\|_{H^2} \leq C \exp\Big(-\beta' \min\{x_+, -x_-\}\Big)\ . \tag{4.4}$$

Here $\beta' > 0$ is any constant with $\beta' < min\{\beta_+, \beta_-\}$, and $C = C(\beta')$ is independent of the interval $J = [x_-, x_+]$ if $J \supset [-b, b]$.

Remark: The results of the theorem are to be expected in view of the spectral perturbation theory for general linear operators [14]. However, notice that passing from an infinite line problem (with the boundary conditions hidden in the underlying space) to a finite interval (with the boundary conditions explicitly specified) is not a standard perturbation problem in the same function space. For such situations Grigorieff [8] and Vainikko [31] have developed general perturbation results for eigenvalues using the theory of so called *discrete approximations.* By invoking several of the lemmas to follow, it seems possible to embed the current problem into this abstract framework (we have not carried out the details of this approach) and then deduce Theorem 4.1 from the general theory in [31], Ch.6. We notice, however, that our approach differs from Vainikko's in several aspects. For example, for the crucial step $\dim E_J = k$ we use an implication of the Kreiss Matrix Theorem rather than estimates of the measure of noncompactness.

Theorem 4.1 is proved in several steps. We first show a simple convergence result for the eigenvalues of P_J in K_{ε_0}.

Lemma 4.1 *Under the assumptions of Theorem 4.1 we have*

$$\max_{s \in \sigma_J} |s - s_0| \to 0 \quad as \quad J \to \mathbb{R}\ .$$

Proof: Fix any $0 < \varepsilon \leq \varepsilon_0$. By our choice of ε_0, the compact set

$$\Omega(\varepsilon, \varepsilon_0) = \{s\ :\ \varepsilon \leq |s - s_0| \leq \varepsilon_0\}$$

is a subset of $\rho(P)$, it lies in the half–plane $\mathrm{Re}\, s > -\delta$, and furthermore $D(s) \neq 0$ for all $s \in \Omega(\varepsilon, \varepsilon_0)$. By Theorem 3.1 we have $\Omega(\varepsilon, \varepsilon_0) \subset \rho(P_J)$ if $J \supset [-b(\varepsilon), b(\varepsilon)]$. This implies $\max_{s \in \sigma_J} |s - s_0| < \varepsilon$, and the lemma is proved. $\square$

We denote the positively oriented boundary curve of the ball K_{ε_0} by $\mathcal{C}$. As noted in the proof of the previous lemma, we have $\mathcal{C} \subset \rho(P)$ and also $\mathcal{C} \subset \rho(P_J)$ for sufficiently large J. Therefore, the Riesz projectors

$$\Pi_0 = \frac{1}{2\pi i} \int_{\mathcal{C}} (sI - P)^{-1} ds \tag{4.5}$$

and

$$\Pi_J = \frac{1}{2\pi i} \int_{\mathcal{C}} (sI - P_J)^{-1} ds \tag{4.6}$$

are well–defined. The operator Π_0 projects $L_2 = L_2(\mathbb{R}, \mathbb{C}^m)$ onto the generalized eigenspace $E \subset V$, and Π_J projects $L_2(J) = L_2(J, \mathbb{C}^m)$ onto the invariant subspace E_J belonging to the s_0–group of eigenvalues of P_J. (See, for example, [14].)

We will always assume J to be so large that the projector Π_J is well–defined by (4.6). In the following auxiliary result, we let $u \in E$ be arbitrary and compare its restriction $u_{|J}$ with the projection $\Pi_J(u_{|J})$.

Lemma 4.2 *For any $u \in E$ we have*

$$\|u_{|J} - \Pi_J(u_{|J})\|_{H^2} \to 0 \quad as \quad J \to \mathbb{R} \; .$$

Proof: Since $u \in E$ we may write

$$u = \frac{1}{2\pi i} \int_{\mathcal{C}} (sI - P)^{-1} u ds \; , \tag{4.7}$$

and we have

$$\Pi_J(u_{|J}) = \frac{1}{2\pi i} \int_{\mathcal{C}} (sI - P_J)^{-1} u_{|J} ds \; . \tag{4.8}$$

If we fix $s \in \mathcal{C}$ and set

$$v = v(s) = (sI - P)^{-1} u, \quad v_J = v_J(s) = (sI - P_J)^{-1} u_{|J} \; ,$$

then

$$sv - Pv = u \quad \text{on} \quad \mathbb{R}$$

and

$$sv_J - Pv_J = u \quad \text{on} \quad J, \quad Rv_J = 0 \; .$$

By Theorem 3.2 (with $g = 0$) we have

$$\|v_{|J} - v_J\|_{H^2} \to 0 \quad \text{as} \quad J \to \mathbb{R} \; ,$$

and the convergence is uniform for $s \in \mathcal{C}$. Therefore, the lemma follows from (4.7) and (4.8). $\square$

Recall that $\{u_j\}_{1 \le j \le k}$ is any fixed basis of E. We show next that the functions

$$u_{j,J} := \Pi_J(u_{j|J}), \quad j = 1, \ldots, k \ , \tag{4.9}$$

are linearly independent for large J. More quantitatively, we have the following result.

Lemma 4.3 *There exists a constant $\kappa > 0$ such that*

$$\kappa \sum_{j=1}^{k} |\alpha_j| \le \| \sum_{j=1}^{k} \alpha_j u_{j,J} \|_{L_2(J)}$$

for all $\alpha \in \mathbb{C}^k$ and all sufficiently large J. Consequently, for large J the functions (4.9) are linearly independent and $E_J =$ range Π_J has dimension $\ge k$.

Proof: First note that there is $\kappa_1 > 0$ with

$$\kappa_1 \sum_{j=1}^{k} |\alpha_j| \ \le \ \| \sum_{j=1}^{k} \alpha_j u_j \|_{L_2(\mathbb{R})} \tag{4.10}$$

since the functions u_j are linearly independent in $L_2(\mathbb{R})$. Using the previous lemma, we have

$$\begin{aligned} \Big| \, \| \sum_j \alpha_j u_{j|J} \|_{L_2(J)} - \| \sum_j \alpha_j u_{j,J} \|_{L_2(J)} \Big| \ &\le \ \sum_j |\alpha_j| \| u_{j|J} - u_{j,J} \|_{L_2(J)} \\ &\le \ \mu_J \sum_j |\alpha_j| \end{aligned}$$

where $\mu_J \to 0$ as $J \to \mathbb{R}$. Furthermore,

$$\begin{aligned} \| \sum_j \alpha_j u_j \|_{L_2(\mathbb{R})}^2 - \| \sum_j \alpha_j u_{j|J} \|_{L_2(J)}^2 \ &= \ \int_{\mathbb{R} \setminus J} | \sum_j \alpha_j u_j(x) |^2 \ dx \\ &\le \ \mu_J' \Big(\sum_j |\alpha_j| \Big)^2 \end{aligned}$$

where $\mu_J' \to 0$ as $J \to \mathbb{R}$. These estimates yield

$$\Big| \, \| \sum_j \alpha_j u_{j,J} \|_{L_2(J)} - \| \sum_j \alpha_j u_j \|_{L_2(\mathbb{R})} \Big| \le \mu_J'' \sum_j |\alpha_j|$$

where $\mu_J'' \to 0$ as $J \to \mathbb{R}$. Now the lemma follows from (4.10). $\square$

We show next that all functions in E decay exponentially as $|x| \to \infty$.

Lemma 4.4 *Let $u \in E$ be arbitrary and let*

$$0 < \beta' < \beta_0 = \min\{\beta_+, \beta_-\} \ ,$$

where $\beta_\pm$ are dichotomy exponents of $L(s_0)$ on $\mathbb{R}_\pm$. Then we have

$$|u(x)| \le Ce^{-\beta'|x|}, \quad x \in \mathbb{R} \ ,$$

with $C = C(\beta', u)$.

Proof: If $Pu = s_0 u$ and $z = (u, Au_x)$, then we have $L(s_0)z = 0$, and exponential decay, $|u(x)| \le Ce^{-\beta_0|x|}$, follows from Theorem A.2. If $(s_0 I - P)^2 w = 0$, then $(s_0 I - P)w = u$ where $Pu = s_0 u$. Transforming the equation $(s_0 I - P)w = u$ to a first order system and applying the second part of Theorem A.2, we obtain $|w(x)| \le Ce^{-\beta'|x|}$. An induction argument yields the general result. □.

We can now sharpen Lemma 4.2 and show an estimate of exponential type as claimed in (4.4).

Lemma 4.5 *Let $u \in E$ be arbitrary. Then we have*

$$\|u_{|J} - \Pi_J(u_{|J})\|_{H^2} \le C \exp(-\beta' \min\{x_+, -x_-\})$$

with $C = C(\beta', u)$ independent of J. Here β' with $0 < \beta' < \beta_0 = \min\{\beta_+, \beta_-\}$ is arbitrary.

Proof: We use the same notation as in the proof of Lemma 4.2. In particular, for $s \in \mathcal{C} = \partial K_{\varepsilon_0}$, we consider the functions $v \in V$ and $v_J \in V_J$ determined by

$$sv - Pv = u \quad \text{on} \quad \mathbb{R}$$

and

$$sv_J - Pv_J = u \quad \text{on} \quad J, \quad Rv_J = 0 \ .$$

If $\beta' < \beta'' < \beta_0$ then we have

$$|u(x)| \le C_1 e^{-\beta''|x|}$$

by the previous lemma. To apply Theorem 3.2 (with $g = 0$) and estimate $\|v_{|J} - v_J\|_{H^2}$, we note that $L(s)$ has an exponential dichotomy on $\mathbb{R}$ since $\operatorname{Re} s > -\delta$ and $s \in \rho(P)$. Furthermore, if ε_0 is small enough, we may choose β'' as dichotomy exponent for $L(s)$. (See Theorem A.5.) Thus, Theorem 3.2 yields

$$\|v_{|J} - v_J\|_{H^2} \le C \exp(-\beta' \min\{x_+, -x_-\}) \ . \tag{4.11}$$

Now the assertion follows from (4.7) and (4.8). □

To prove Theorem 4.1, it remains to show the exponential estimate (4.3) for the eigenvalues and that dim E_J does not exceed $k = \dim E$ for large J. We will use the following auxiliary result.

Lemma 4.6 *Let $(F_n, \|\cdot\|_n)$ denote a sequence of normed linear spaces and let* $\dim\ F_n = k+1$ *for all n, where k is a fixed integer. Furthermore, let $Q_n : F_n \to F_n$ denote linear operators which are uniformly power bounded, i.e.,*

$$\|Q_n^\nu\|_n \le C$$

for all $n = 1, 2, \dots$ and all $\nu = 1, 2, \dots$ Then there is a constant $\tilde{C}$ with

$$\|Q_n^{k+1}\|_n \le \tilde{C} r(Q_n), \quad n = 1, 2, \dots\ ,$$

where $r(Q_n)$ denotes the spectral radius.

Proof: This follows directly from the matrix version, formulated in Lemma A.1, and Lemma A.2. □

By Lemma 4.3 we have $\dim\ E_J \ge k$ for large J. To show that $\dim\ E_J = k$, we argue by contradiction. Suppose there is a sequence $J_n \to \mathbb{R}$ with $\dim\ E_{J_n} \ge k+1$. Then we can choose subspaces $F_n \subset E_{J_n}$ with $\dim\ F_n = k+1$ and $(P_{J_n} - s_0 I)(F_n) \subset F_n$. We want to apply the previous lemma to the restrictions

$$Q_n := (P_{J_n} - s_0 I)_{|F_n} : F_n \to F_n\ ,$$

where the norm on F_n is the H^2–norm.

Let us first show that the Q_n are uniformly power bounded. For all $v \in E_J$ we have

$$v = \Pi_J v = \frac{1}{2\pi i} \int_{\mathcal{C}} (sI - P_J)^{-1} v\ ds\ ,$$

thus, using the Dunford–Taylor calculus,

$$(P_J - s_0 I)^\nu v = \frac{1}{2\pi i} \int_{\mathcal{C}} (s - s_0)^\nu (sI - P_J)^{-1} v\ ds\ .$$

Since the radius of the curve $\mathcal{C}$ is $\varepsilon_0 \le 1$, it follows that

$$\|(P_J - s_0 I)^\nu v\|_{H^2} \le \max_{s \in \mathcal{C}} \|(sI - P_J)^{-1}\|_{(L_2 \to H^2)}\ \|v\|\ .$$

Using Theorem 3.1 we obtain that, as $J \to \mathbb{R}$, the maximum in the previous formula is bounded uniformly for large J, and we have shown

$$\|(P_J - s_0 I)^\nu v\|_{H^2} \le const \|v\|, \quad v \in E_J\ .$$

Clearly, this implies $\|Q_n^\nu\|_{H^2} \le const.$

We further note that the eigenvalues of Q_n have the form $\lambda = s - s_0$ with $s \in \sigma_{J_n}$, and therefore $r(Q_n) \to 0$ as $n \to \infty$ by Lemma 4.1. Consequently, Lemma 4.6 yields

$$\|Q_n^{k+1}\|_{H^2} \to 0 \quad \text{as} \quad n \to \infty\ . \tag{4.12}$$

The following lemma is a step in obtaining a contradiction to (4.12), which will then show dim $E_J = k$ for all large J. We recall that $\{u_j\}_{1\le j\le k}$ is an arbitrary, but fixed, basis of E and that $u_{j,J} = \Pi_J(u_{j|J})$ are linearly independent. We set

$$G_n = \mathrm{span}\{u_{j,J_n} \ : \ 1 \le j \le k\} \quad \subset \quad E_{J_n}$$

and

$$\mathrm{dist}(v, G_n) = \min\{\|u - v\|_{H^2} \ : \ u \in G_n\} \ .$$

Lemma 4.7 *For a subsequence $n \in \mathbb{N}_1 \subset \mathbb{N}$ there exist $v_n \in F_n$ with*

$$\|v_n\|_{H^2} = 1, \quad \mathit{dist}(v_n, G_n) \ge c > 0 \ ,$$

but

$$\mathit{dist}(Q_n v_n, G_n) \to 0 \quad \mathit{as} \quad n \to \infty, \quad n \in \mathbb{N}_1 \ .$$

Proof: Since dim $G_n = k <$ dim $F_n = k+1$ we can choose $u_n \in F_n$ with

$$\|u_n\|_{H^2} = 1, \quad \mathrm{dist}(u_n, G_n) \ge \frac{1}{2} \ .$$

Consider the sequences

$$\mathrm{dist}(Q_n^\nu u_n, G_n), \quad n = 1, 2, \ldots \tag{4.13}$$

for $\nu = 1, \ldots, k+1$. For $\nu = k+1$ we have $\|Q_n^{k+1} u_n\|_{H^2} \to 0$ by (4.12) and thus (4.13) goes to zero as $n \to \infty$ if $\nu = k+1$. Define l as the *smallest* exponent $\nu \in \{1, 2, \ldots, k+1\}$ for which (4.13) goes to zero as $n \to \infty$. Then we have

$$\|Q_n^{l-1} u_n\|_{H^2} \ge c_1 > 0, \quad n \in \mathbb{N}_1 \subset \mathbb{N}$$

and set

$$v_n = \frac{Q_n^{l-1} u_n}{\|Q_n^{l-1} u_n\|_{H^2}} \ , \quad n \in \mathbb{N}_1 \ .$$

These functions v_n have the required properties. □

The crucial construction leading to a contradiction is carried out next.

Lemma 4.8 *Let $v_n \in F_n, n \in \mathbb{N}_1$, be constructed as in the previous lemma. There exist $v \in V$ and $u \in E$ with $(P - s_0 I)v = u$ and*

$$\|v_n - v_{|J_n}\|_{H^2} \to 0 \quad \mathit{as} \quad n \to \infty, \quad n \in \mathbb{N}_2 \subset \mathbb{N}_1 \ . \tag{4.14}$$

Let us assume the lemma is shown. Then, by the definition of E, it follows that $v \in E$, thus $\Pi_{J_n}(v_{|J_n}) \in G_n$. This yields the estimate

$$\mathrm{dist}(v_n, G_n) \le \|v_n - v_{|J_n}\|_{H^2} + \|v_{|J_n} - \Pi_{J_n}(v_{|J_n})\|_{H^2}, \quad n \in \mathbb{N}_2 \ .$$

By (4.14), the first term on the right hand side goes to zero as $n \to \infty$, and by Lemma 4.2 the second term also goes to zero. However, we have $\mathrm{dist}(v_n, G_n) \ge c > 0$ by Lemma 4.7. This contradiction proves the equality dim $E_J = k$ for all large J, as claimed in Theorem 4.1.

Proof of Lemma 4.8. Since $\mathrm{dist}(Q_n v_n, G_n) \to 0$, there are elements

$$w_n = \sum_{j=1}^{k} \alpha_{jn} u_{j,J_n} \ \in \ G_n$$

with $\|Q_n v_n - w_n\|_{H^2} \to 0$. Also, $\|Q_n v_n\|_{H^2}$ is a bounded sequence, which implies that $\|w_n\|_{H^2}$ is also bounded. By Lemma 4.3 we obtain boundedness of the coefficients, $|\alpha_{jn}| \le const$, and may assume

$$\alpha_{jn} \to \alpha_j \quad \text{as} \quad n \to \infty, \quad n \in \mathbb{N}_2 \subset \mathbb{N}_1 \ ,$$

for $1 \le j \le k$. We set

$$u = \sum_{j=1}^{k} \alpha_j u_j \in E$$

and write

$$Q_n v_n - u_{|J_n} = \Big(Q_n v_n - w_n\Big) \ + \ \Big(w_n - \Pi_{J_n}(u_{|J_n})\Big) \ + \ \Big(\Pi_{J_n}(u_{|J_n}) - u_{|J_n}\Big) \ .$$

The same arguments as above show that

$$\|Q_n v_n - u_{|J_n}\|_{H^2} \to 0 \quad \text{as} \quad n \to \infty, \quad n \in \mathbb{N}_2 \ . \tag{4.15}$$

Set

$$z_n = \begin{pmatrix} v_n \\ A v_{nx} \end{pmatrix}, \quad r_n = \begin{pmatrix} 0 \\ Q_n v_n \end{pmatrix}$$

and recall that

$$(P_{J_n} - s_0 I) v_n = Q_n v_n \quad \text{in} \quad J_n, \quad R v_n = 0 \ . \tag{4.16}$$

Writing (4.16) as a first order system, we have (see (3.2) – (3.4))

$$L(s_0) z_n \equiv z_{nx} - M(x, s_0) z_n = r_n \quad \text{in} \quad J_n, \quad R_1 z_n = 0 \ .$$

Recall that $L(s_0)$ has exponential dichotomies on $\mathbb{R}_+$ and on $\mathbb{R}_-$ with data $(K_\pm, \beta_\pm, \pi_\pm)$. Let us denote the Green's functions of $L(s_0)$ on $\mathbb{R}_\pm$ by $G_\pm(x,y)$; see Theorem A.1. For example,

$$G_+(x,y) = \begin{cases} S(x,y)\pi_+(y), & 0 \le y \le x \ , \\ \\ S(x,y)(\pi_+(y) - I), & 0 \le x < y \ , \end{cases}$$

where $S(x,y)$ is the solution operator of $L(s_0)$. Then, since $L(s_0)z_n = r_n$, Theorem A.1 yields the following representation of $z_n(x)$, where we assume $J_n = [a_n, b_n]$:

$$z_n(x) = \begin{cases} S(x,a_n)\pi_-(a_n)z_n(a_n) + S(x,0)(I - \pi(0))z_n(0) + \rho_{-n}(x), & a_n \le x \le 0 \ , \\ \\ S(x,b_n)(I - \pi_+(b_n))z_n(b_n) + S(x,0)\pi_+(0)z_n(0) + \rho_{+n}(x), & 0 \le x \le b_n \ , \end{cases}$$

with

$$\begin{aligned} \rho_{-n}(x) &= \int_{a_n}^{0} G_-(x,y)r_n(y)\ dy \ , \\ \rho_{+n}(x) &= \int_0^{b_n} G_+(x,y)r_n(y)\ dy \ . \end{aligned}$$

We will determine the limit $z(x)$ of the sequence $z_n(x)$. First, if we define

$$\rho_-(x) = \int_{-\infty}^{0} G_-(x,y)\begin{pmatrix} 0 \\ u(y) \end{pmatrix} dy \ , \quad \rho_+(x) = \int_0^{\infty} G_+(x,y)\begin{pmatrix} 0 \\ u(y) \end{pmatrix} dy \ ,$$

use the estimates $|G_\pm(x,y)| \le K_\pm \exp(-\beta_\pm|x-y|)$, use the exponential decay of u, $|u(y)| \le Ce^{-\beta'|y|}$, and apply (4.15), then it follows that

$$\|\rho_{-n} - \rho_-\|_{L_2[a_n,0]} \to 0, \quad \|\rho_{+n} - \rho_+\|_{L_2[0,b_n]} \to 0 \ .$$

Second, by construction we have $\|v_n\|_{H^2} = 1$, which implies $|z_n(0)| \le$ *const* by Sobolev's inequality. Thus we may assume that

$$z_n(0) \to \eta \in \mathbb{C}^{2m} \quad \text{as} \quad n \to \infty, \quad n \in \mathbb{N}_2 \ .$$

Finally, if we substitute the above representation of $z_n(x)$ into the boundary condition $R_1 z_n = 0$ and use the determinant condition $D(s_0) \ne 0$, then we obtain

$$|\pi_-(a_n)z_n(a_n)| \to 0, \quad |(I - \pi_+(a_n))z_n(a_n)| \to 0 \ .$$

This proves the L_2–convergence

$$\|z_n - z_{|J_n}\| \to 0 \quad \text{as} \quad n \to \infty, \quad n \in \mathbb{N}_2 \ , \tag{4.17}$$

where $z(x)$ is the L_2–function

$$z(x) = \begin{cases} S(x,0)(I - \pi_-(0))\eta + \rho_-(x), & x < 0\ , \\ \\ S(x,0)\pi_+(0)\eta + \rho_+(x), & x > 0\ . \end{cases}$$

Next we will show that $z \in H^1$ and that the convergence (4.17) holds in $\|\cdot\|_{H^1}$. In fact, set

$$w(x) := M(x, s_0)z(x) + \begin{pmatrix} 0 \\ u(x) \end{pmatrix}\ .$$

If $\phi \in C_0^\infty(\mathbb{R}, \mathbb{C}^{2m})$ and if n is so large that $J_n \supset supp(\phi)$, then we have

$$\begin{aligned} |(w,\phi)_{L_2(\mathbb{R})} + (z,\phi')_{L_2(\mathbb{R})}| &= |(w,\phi)_{L_2(J_n)} + (z,\phi')_{L_2(J_n)}| \\ &\le |(w - z_n', \phi)_{L_2(J_n)}| + |(z - z_n, \phi')_{L_2(J_n)}| \\ &\le C_\phi\Big(\|w_{|J_n} - z_n'\| + \|z_{|J_n} - z_n\|\Big)\ . \end{aligned}$$

Here

$$w - z_n' = Mz + \begin{pmatrix} 0 \\ u \end{pmatrix} - Mz_n - \begin{pmatrix} 0 \\ Q_n v_n \end{pmatrix}\ ,$$

thus $\|w_{|J_n} - z_n'\| \to 0$ by (4.15) and (4.17). It follows that $(z, \phi') = -(w, \phi)$, which implies that $z \in H^1$ and $z' = w$. Since $\|w_{|J_n} - z_n'\| \to 0$ has been shown, the desired convergence

$$\|z_n - z_{|J_n}\|_{H^1} \to 0 \tag{4.18}$$

follows, too. Define $v(x)$ as the first m components of $z(x)$. Then we have $v \in H^2$ and $(P - s_0)v = u$ since $L(s_0)z = \begin{pmatrix} 0 \\ u \end{pmatrix}$. Convergence of v_n to v, as stated in Lemma 4.8, is implied by (4.18). This completes the proof of Lemma 4.8 and shows that

$$E_J = \text{range } \Pi_J = \text{span}\{u_{j,J}\}_{1 \le j \le k} \quad \text{with} \quad u_{j,J} = \Pi_J(u_{j|J})$$

for sufficiently large J. □

It remains to show the convergence of eigenvalues (4.3). Clearly,

$$\max_{s \in \sigma_J} |s - s_0|^k \le r((P_J - s_0 I)^k) \le \|(P_J - s_0 I)^k\|$$

if the spectral radius and the L_2–operator norm refer to the restriction of $(P_J - s_0 I)^k$ to E_J. For any $u \in E$ let $u_J = \Pi_J(u_{|J})$ and note that

$$\begin{aligned} 0 = (P - s_0 I)^k u &= \frac{1}{2\pi i}\int_{\mathcal{C}} (s - s_0)^k (sI - P)^{-1} u\, ds\ , \\ (P_J - s_0 I)^k u_J &= \frac{1}{2\pi i}\int_{\mathcal{C}} (s - s_0)^k (sI - P_J)^{-1} u_J\, ds\ . \end{aligned}$$

Subtraction yields

$$\begin{aligned} \|(P_J - s_0 I)^k u_J\| &\leq C_1 \max_{s\in\mathcal{C}} \|v_J(s) - v(s)_{|J}\| \\ &\leq C_2 \exp(-\beta' \min(x_+, -x_-)) \end{aligned}$$

where $v(s)$ and $v_J(s)$ are defined in the proof of Lemma 4.2 and where (4.11) has been applied. Using the above estimate for every element in the basis $\{u_{j,J}\}$ of E_J and noting the uniform independence of $\{u_{j,J}\}$ for all large J (see Lemma 4.3), it follows that

$$\|(P_J - s_0 I)^k\| \leq C \exp(-\beta' \min(x_+, -x_-)) \ .$$

This completes the proof of Theorem 4.1.

5 Applications

In this section we discuss the validity of the determinant conditions of Sections 2–4 for Dirichlet, Neumann, and periodic boundary conditions. Furthermore, we give applications of our theory as well as numerical results for the FitzHugh–Nagumo system and for the complex Ginzburg–Landau equation. As a first step we provide a sufficient criterion for Assumption 1c), which will also guarantee that the matrices $Y_\pm$ appearing in the determinant condition (3.11) are nonsingular.

We consider a quadratic matrix polynomial (cf. [7] for the general theory)

$$\hat{P}(\lambda) = \lambda^2 A + \lambda B + C, \quad A, B, C \in \mathbb{C}^{m\times m}. \tag{5.1}$$

under the following condition:

Assumption 2

For all $u, v \in \mathbb{C}^m$ with $(u, v) \neq 0$ it holds that

$$|\mathrm{Re}\, u^*(B^* - B)v| < u^*(A + A^*)u - v^*(C + C^*)v \ .$$

Note that Assumption 2 implies $A+A^* > 0$ and $C+C^* < 0$. Conversely, these two conditions together with $B = B^*$ are sufficient for Assumption 2. Also, an equivalent formulation of Assumption 2 is

$$\Gamma + \Gamma^* > 0 \quad \text{where} \quad \Gamma = \begin{pmatrix} A & -\frac{1}{2}B \\ \frac{1}{2}B & -C \end{pmatrix} \ . \tag{5.2}$$

The following lemma is an extension of [19], Lemma 2.4, to the non–hermitian case.

Lemma 5.1 *If Assumption 2 holds, then there exists $\delta > 0$ such that all eigenvalues $s \in \mathbb{C}$ of $\hat{P}(i\omega)$, $\omega \in \mathbb{R}$, satisfy*

$$\mathrm{Re}\, s \leq -\delta(1 + \omega^2) \ .$$

Moreover, for $\mathrm{Re}\, s > -\delta$, the matrices $Y_-(s), Y_+(s) \in \mathbb{C}^{m\times m}$ corresponding to the m stable and m unstable eigenvalues in the block decomposition (3.8) are nonsingular.

Proof: By (5.2) there is a constant $\delta > 0$ so that, for all $u, v \in \mathbb{C}^m$,

$$-u^*(A+A^*)u \pm \mathrm{Re}\, u^*(B-B^*)v + v^*(C+C^*)v \le -2\delta(u^*u + v^*v). \tag{5.3}$$

For $u = i\omega v$ we obtain

$$v^*(\hat{P}(i\omega) + \hat{P}^*(i\omega))v \le -2\delta(u^*u + v^*v) = -2\delta(\omega^2+1)v^*v\ .$$

This proves the first assertion.

Let $s \in \mathbb{C}$ with $\mathrm{Re}\, s > -\delta$ be arbitrary and assume $Y_-(s)\phi = 0$ for some $\phi \in \mathbb{C}^m$. The function $u(x)$ defined by (see (3.10))

$$z(x) = \begin{pmatrix} u(x) \\ Au_x(x) \end{pmatrix} = \begin{pmatrix} Y_-(s) \\ Z_-(s) \end{pmatrix} e^{\Lambda_-(s)x}\phi$$

is an exponentially decaying solution of

$$Au_{xx} + Bu_x + (C - sI)u = 0 \quad \text{on} \quad [0,\infty), \quad u(0) = 0\ .$$

We multiply by u^*, integrate over $[0,\infty)$, and take real parts to obtain

$$0 = \int_0^\infty \Big\{ u^*Au_{xx} + u_{xx}^*A^*u + u^*Bu_x + u_x^*B^*u + u^*(C + C^* - 2\mathrm{Re}\,(s)I)u \Big\} dx\ .$$

Since $u(0) = 0$, integration by parts yields

$$0 = \int_0^\infty \Big\{ -u_x^*(A+A^*)u_x + u^*(B-B^*)u_x + u^*(C+C^* - 2\mathrm{Re}\,(s)I)u \Big\} dx\ .$$

From (5.3) and $\mathrm{Re}\, s > -\delta$ we obtain that the real part of the integral is negative unless $u \equiv 0$. Therefore,

$$0 = \begin{pmatrix} u(0) \\ Au_x(0) \end{pmatrix} = \begin{pmatrix} Y_-(s) \\ Z_-(s) \end{pmatrix} \phi\ ,$$

which implies $\phi = 0$ since $\begin{pmatrix} Y_-(s) \\ Z_-(s) \end{pmatrix}$ has full rank m. An analogous argument with $[0,\infty)$ replaced by $(-\infty, 0]$ yields the nonsingularity of $Y_+(s)$. □

For later application we note a simple extension of Lemma 5.1. Let $H = H^* > 0$, define

$$\hat{P}_H(\lambda) = H\hat{P}(\lambda) = \lambda^2 HA + \lambda HB + HC\ ,$$

and assume that the coefficients of $\hat{P}_H(\lambda)$ satisfy Assumption 2. Then the statements of Lemma 5.1 about the eigenvalues of $\hat{P}(i\omega)$ and the nonsingularity of $Y_\pm(s)$ are still valid. In fact, if $\hat{P}(i\omega)v = sv$, one obtains

$$2\mathrm{Re}\,(s)v^*Hv \le -2\delta_H(\omega^2+1)v^*v, \quad \delta_H > 0\ .$$

To prove the nonsingularity of $Y_\pm(s)$, one argues as above but replaces u^*Au_{xx} by u^*HAu_{xx}, etc.

Discussion of the det–Conditions. Let D_∞ and $D(s)$ denote the determinants (2.4) and (3.11), respectively. It is not difficult to show that $D_\infty \neq 0$ for Dirichlet, Neumann, and periodic boundary conditions. In fact, we have the following formulae for the matrices $P_\pm, Q_\pm$ determining the boundary operator R (see (1.9) and (2.3)) and for the corresponding determinants.

Dirichlet B.C.s.

$$\begin{pmatrix} P_- & Q_- & P_+ & Q_+ \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix}$$

$$D_\infty = \det \begin{pmatrix} -A_-^{1/2} & 0 \\ 0 & A_+^{1/2} \end{pmatrix}, \quad D(s) = \det \begin{pmatrix} Y_-(s) & 0 \\ 0 & Y_+(s) \end{pmatrix} .$$

Neumann B.C.s.

$$\begin{pmatrix} P_- & Q_- & P_+ & Q_+ \end{pmatrix} = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

$$D_\infty = \det \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = 1, \quad D(s) = \det \begin{pmatrix} Y_-\Lambda_- & 0 \\ 0 & Y_+\Lambda_+ \end{pmatrix}(s) .$$

Periodic B.C.s.

$$\begin{pmatrix} P_- & Q_- & P_+ & Q_+ \end{pmatrix} = \begin{pmatrix} I & 0 & -I & 0 \\ 0 & I & 0 & -I \end{pmatrix}$$

$$D_\infty = \det \begin{pmatrix} -A_-^{1/2} & -A_+^{1/2} \\ I & -I \end{pmatrix} = \det\left(A_-^{1/2} + A_+^{1/2}\right) ,$$

$$D(s) = \det \begin{pmatrix} Y_- & -Y_+ \\ Y_-\Lambda_- & -Y_+\Lambda_+ \end{pmatrix}(s) .$$

From the parabolicity assumption (1.6) it follows that both matrices $W = A_\pm^{1/2}$ satisfy $W + W^* > 0$ (cf. [14], Ch.V,3.11), which implies $D_\infty \neq 0$ in the case of periodic boundary conditions.

According to Section 3, the determinant $D(s)$ is defined for all s with $\operatorname{Re} s > -\delta$. From the above formulae for $D(s)$ we see that $D(s) \neq 0$ in case of Dirichlet or Neumann boundary conditions if the matrices $Y_\pm(s)$ are nonsingular. As Lemma 5.1 shows, this follows from Assumption 2.

Periodic boundary conditions are a natural choice if $A_+ = A_-, B_+ = B_-, C_+ = C_-$. Then the matrix

$$\begin{pmatrix} Y_- & Y_+ \\ Z_- & Z_+ \end{pmatrix}(s) = \begin{pmatrix} Y_- & Y_+ \\ A_-Y_-\Lambda_- & A_+Y_+\Lambda_+ \end{pmatrix}(s)$$

appearing in (3.8) is nonsingular by construction, and $D(s) \neq 0$ follows from $A_- = A_+$ and $\det A_- \neq 0$.

If the matrices A_-, B_-, C_- are different from A_+, B_+, C_+, then one can construct examples where $D(s) = 0$ for some s–values. We have not investigated, however, if a vanishing determinant $D(s)$ leads to practical difficulties for finite–interval computations.

Remark: For arbitrary boundary conditions of the general form (2.3) one can show that $D(s) \neq 0$ for all large s if $D_\infty \neq 0$. In fact, following the proof of Lemma 2.1, one can choose (with $\rho = |s|^{1/2}$)

$$\begin{pmatrix} Y_-(s) \\ Z_-(s) \end{pmatrix} = \begin{pmatrix} A_-^{-1/2} + \mathcal{O}(\frac{1}{\rho}) \\ -\rho e^{i\theta} I + \mathcal{O}(1) \end{pmatrix}, \quad \begin{pmatrix} Y_+(s) \\ Z_+(s) \end{pmatrix} = \begin{pmatrix} A_+^{-1/2} + \mathcal{O}(\frac{1}{\rho}) \\ \rho e^{i\theta} I + \mathcal{O}(1) \end{pmatrix} .$$

With this choice one obtains $D(s) = \rho^r \det \mathcal{M} + \mathcal{O}(\rho^{r-1})$ where $\mathcal{M}$ is defined in (2.28) and where r is the rank of the Neumann part in (2.3). Since $\det \mathcal{M} = cD_\infty, c \neq 0$, the assertion follows.

The FitzHugh–Nagumo System. (See [3, 6, 25, 26], for example.) The system reads

$$v_t = v_{xx} + f_1(v, r), \quad r_t = \varepsilon r_{xx} + f_2(v, r) \ , \tag{5.4}$$

where f_1 and f_2 are given by

$$f_1 = v - \frac{1}{3}v^3 - r, \quad f_2 = \Phi(v + a - br)$$

with nonnegative constants ε, a, b, Φ. Any traveling wave solution

$$u(x,t) = \begin{pmatrix} v \\ r \end{pmatrix}(x,t) = U(x - ct) = \begin{pmatrix} V \\ R \end{pmatrix}(x - ct)$$

is a steady state of the system

$$u_t = Au_{xx} + Bu_x + f(u) \tag{5.5}$$

with

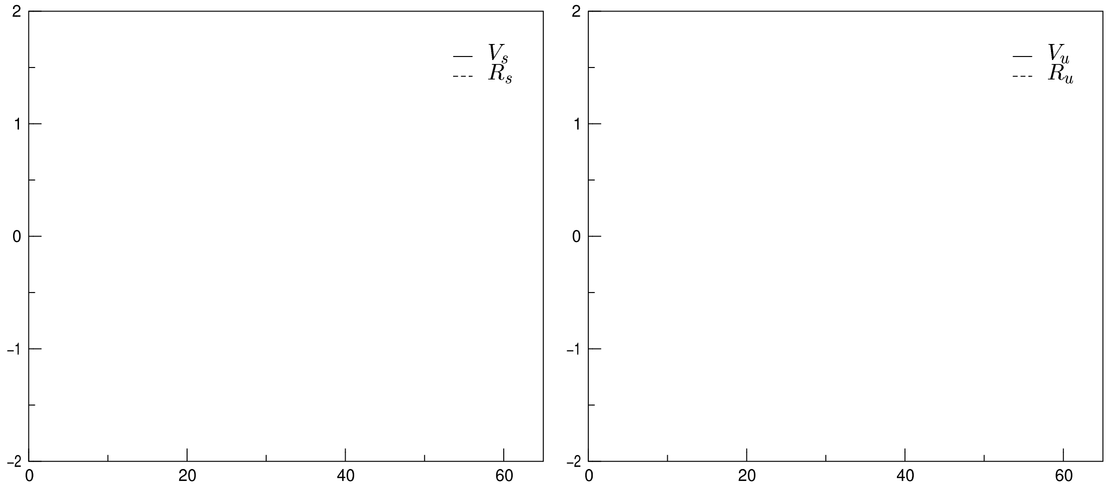
$$A = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad B = cI, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} .$$

In applications to nerve conduction one usually assumes $\varepsilon = 0$, and then (5.4) and (5.5) are parabolic–hyperbolic systems. To make our approximation results applicable, we have chosen a small positive ε, however. (Stability of traveling waves for parabolic–hyperbolic systems is discussed in [3, 19], for example, and an extension of our approximation results to such systems is in preparation.)

For the parameter values (see [25])

$$a = 0.7, \quad b = 0.8, \quad \Phi = 0.08, \quad \varepsilon = 0.1 \tag{5.6}$$

one obtains two traveling waves; one is stable, the other unstable. The velocities are



(a) stable wave, $c_s = -0.789$

(b) unstable wave, $c_u = -0.534$

Figure 1: Wave profiles for the FitzHugh Nagumo equation : interval =[0,65], stepsize $\Delta x = 0.1$, Dirichlet boundary conditions

$$c_s = -0.789 \quad \text{and} \quad c_u = -0.534 \ ,$$

respectively. In Fig.1 we show the corresponding pulse shapes $U_s = (V_s, R_s)$ and $U_u = (V_u, R_u)$.

These profiles and the wave velocities were computed by solving a BVP on the interval $0 \le x \le 65$ using Dirichlet boundary conditions and centered, second–order finite differences with $\Delta x = 0.1$. As usual, a phase fixing condition was applied, accounting for the wave speed c, which is a parameter to be determined. For details, error analysis, and the more accurate projection boundary conditions, we refer to [1].

Let us discuss Assumption 1 for this example. For the parameters (5.6), the nonlinear system

$$f_1(v,r) = 0, \quad f_2(v,r) = 0$$

has a unique solution (v_∞, r_∞), and the matrices $A_\pm$ etc. of Assumption 1b) read

$$A_+ = A_- = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad B_- = B_+ = cI \quad (c = c_s \quad \text{or} \quad c = c_u) \ ,$$

$$C_+ = C_- = \begin{pmatrix} 1 - v_\infty^2 & -1 \\ \Phi & -b\Phi \end{pmatrix} \ .$$

In the following we drop the indices $\pm$. It is clear that Assumption 1b) holds for $\varepsilon > 0$. Moreover, we have $|v_\infty| > 1$ (see Fig. 1), which implies $\operatorname{Re} \sigma(C) < 0$. To show that Assumption 1c) is satisfied, we can not apply Lemma 5.1 directly, because the condition $C + C^* < 0$ does not hold. However, we can apply the simple extension mentioned after Lemma 5.1 with the choice $H = diag(1, \frac{1}{\Phi})$. It follows that Assumption 1c) is satisfied.

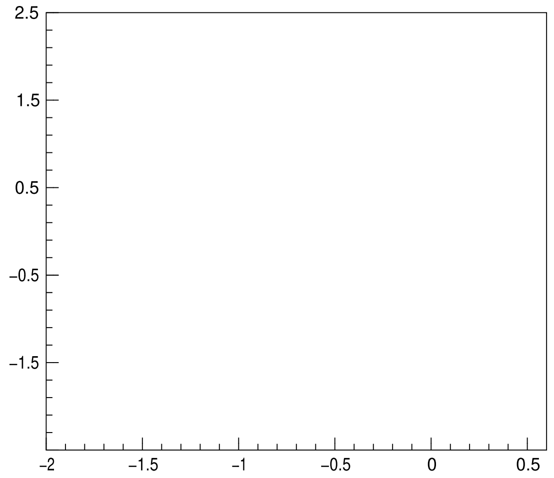


Figure 2: Dispersion relation for the unstable wave

For illustration we consider the algebraic curve S in the complex plane given by the dispersion relation

$$S = \{s \in \mathbb{C} \ : \ \det(-\omega^2 A + i\omega c I + C - sI) = 0 \tag{5.7}$$

Here either $c = c_s$ or $c = c_u$. For the parameter values (5.6) and $c = c_u$ the curve S is shown in Fig.2

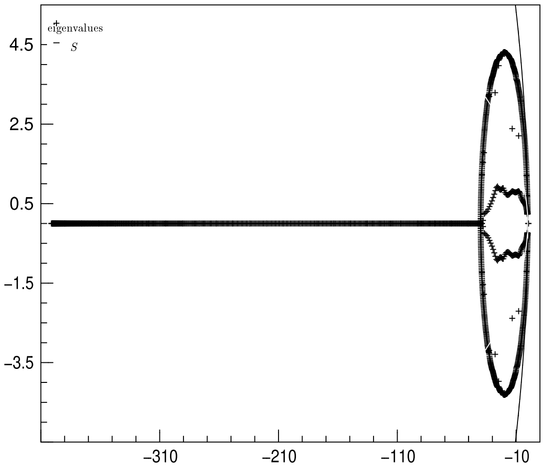
The curve $S = S_\kappa$, $\kappa = s$ or $\kappa = u$, is contained in the essential spectrum $\sigma_{ess}(P)$ of the linearized operator $P = P_\kappa$,

$$Pv = Av_{xx} + c_\kappa v_x + f'(U_\kappa)v, \quad \kappa = s \quad \text{or} \quad \kappa = u \ . \tag{5.8}$$

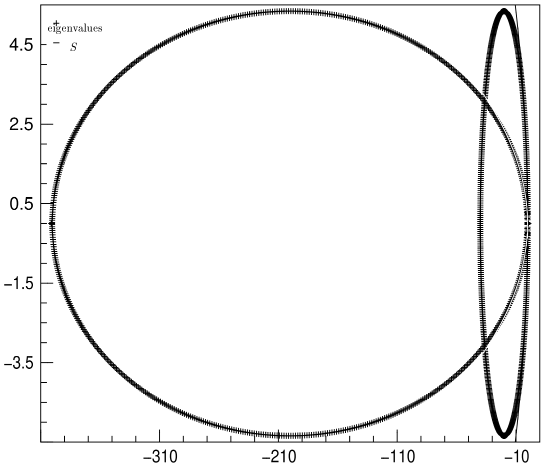
Furthermore, it is known that $\sigma_{ess}(P)$ does not intersect the connected component of $\mathbb{C} \setminus S$ which contains the right half plane; see [11], Chapter 5, Theorem A2. Thus $\sigma_{ess}(P)$ has no points 'to the right' of S, and in particular

$$\mathrm{Re}\, s \le -\delta < 0 \quad \text{for all} \quad s \in \sigma_{ess}(P) \ . \tag{5.9}$$

It is interesting (though irrelevant for the stability question) to study how the essential spectrum $\sigma_{ess}(P)$ breaks up when one restricts the equation $Pv = sv$ to a finite interval (with boundary conditions) and discretizes. Let us emphasize that our theory only says that an essential spectrum with (5.9) cannot produce eigenvalues s for finite and large interval problems with $\mathrm{Re}\, s > -\delta$. It does *not* say, however, if or how the essential spectrum is approximated by eigenvalues. All our results in this direction are numerical. For some analytical results of the effect of discretization we refer to [30].



(a) Dirichlet b.c.



(b) periodic b.c.

Figure 3: Dispersion curve and spectra of matrices obtained by linearizing about the unstable wave

For definiteness, consider the operator P (see (5.8)) where $U = U_u$ is the profile of the unstable wave. If one restricts the equation $Pv = sv$ to a finite interval $J = [x_-, x_+]$, requires a boundary condition $Rv = 0$, and discretizes [3] with a step size $\Delta x = (x_+ - x_-)/M$, then one obtains a matrix eigenvalue problem for a discretization matrix $Q = Q(J, \Delta x, R)$. The eigenvalues of Q were obtained using a standard solver from the NAG–library. They are shown together with the dispersion curve in Fig.3 (a) for $J = [0, 65]$, $\Delta x = 0.1$, and Dirichlet boundary conditions. Fig.3(b) shows the eigenvalues of Q for the same J and Δx, but periodic boundary conditions.

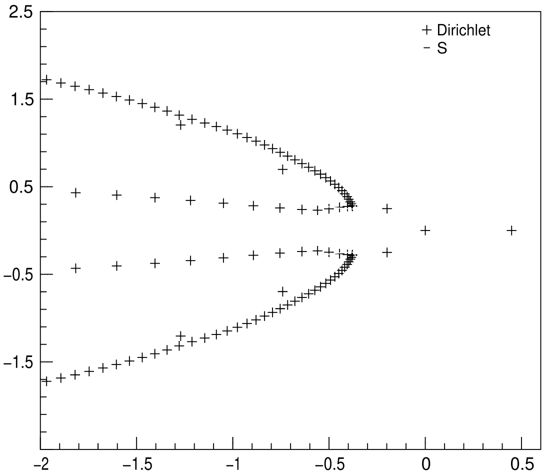
Close–ups of these spectra in a region near $s = 0$ are shown in Figs.4(a) and (b).

We see that the global spectra of Q change dramatically if one replaces Dirichlet by periodic boundary conditions. Near $s = 0$ the change is less dramatic, and both spectra give rough approximations to the dispersion curve S.

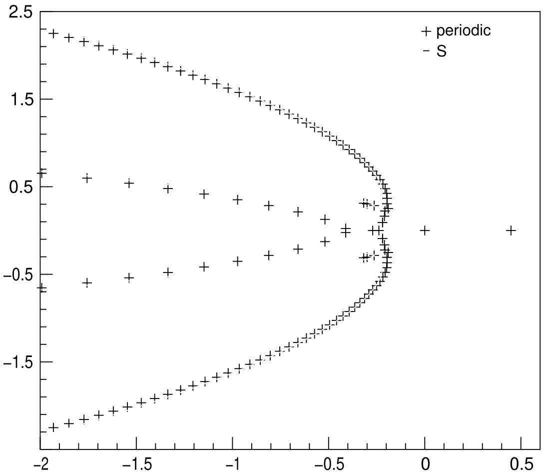
In Figs.4(a) and (b) we also observe an eigenvalue μ_0 near $s = 0$ and an unstable eigenvalue $\mu_1 \approx 0.4485$. We have investigated these eigenvalues, which are covered by our theory, in more detail. Figs. 5(a) and (b) show how $\mu_0 = \mu_0(J, \Delta x, R)$ depends on the interval length and on Δx. As expected, for small Δx the J–error dominates, confirming the exponential estimate (4.3). For large J, the discretization error dominates, indicating an $\mathcal{O}(\Delta x^2)$ behavior.

These considerations hold for Dirichlet and periodic boundary conditions. However, the error plots show additional structure, in particular in the periodic case. Possibly there is a resonance phenomenon, which has not yet been explained.

[3]We have always used centered, second–order differences. The wave profile $U = U(x)$ was only obtained numerically as described above, but we believe the profile error to be negligible.

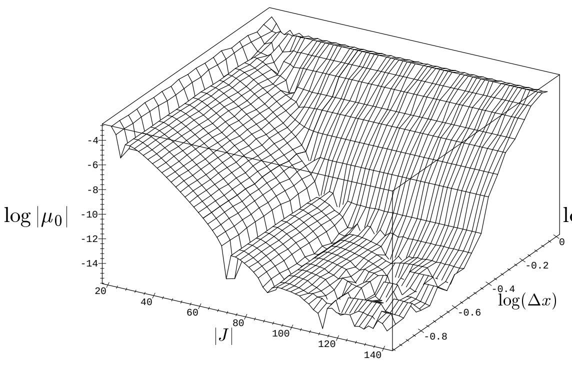


(a) Dirichlet b.c.

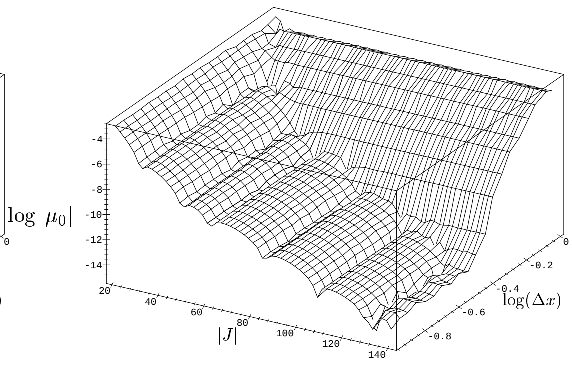


(b) periodic b.c.

Figure 4: Close-ups of spectra from Fig.3 (a),(b)

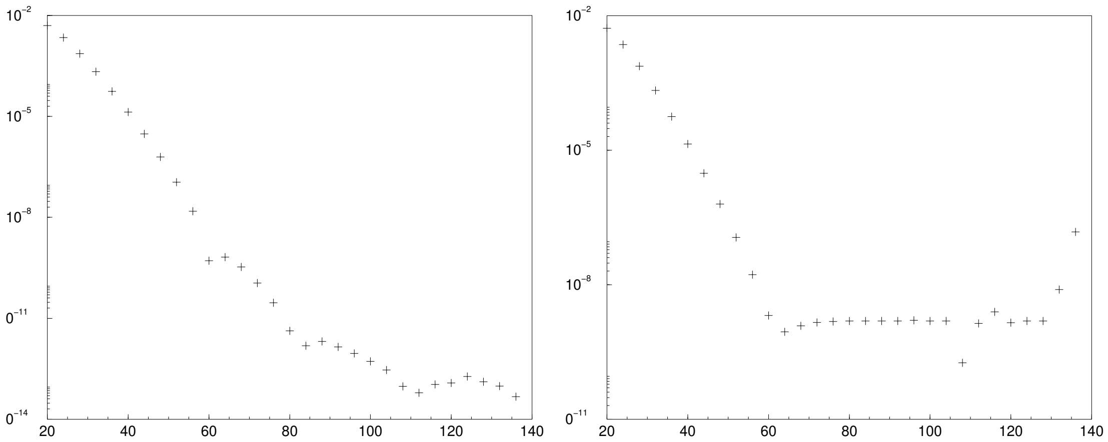


(a) Dirichlet b.c.



(b) periodic b.c.

Figure 5: Unstable wave for the FitzHugh Nagumo system: dependence of the near zero eigenvalue $|\mu_0|$ on the interval length $|J|$ and the stepsize Δx (both Δx and $|\mu_0|$ are shown in logarithmic scale).



(a) Dirichlet b.c. (b) periodic b.c.

Figure 6: Unstable wave for the FitzHugh Nagumo system: the error of the unstable eigenvalue $|\mu_1(J) - \mu_{1,\infty}|$ versus interval length $|J|$

We have also studied the unstable eigenvalue $\mu_1 = \mu_1(J)$ as a function of the interval length for fixed $\Delta x = 0.01$. Fig.6(a) and (b) show the error $\mu_1(J) - \mu_{1,\infty}$, where the reference value $\mu_{1,\infty}$ was obtained for a large interval of length 140.

Similar results, except for the unstable eigenvalue, are obtained when P is the linearization about the stable profile U_s, see [30].

The Complex Ginzburg–Landau Equation. A general reference is [21]. We have used the normalized form (see [23], [24])

$$u_t = (1 + i\alpha)\Big(u_{xx} - (1 + i\beta)^2 u + (1 + i\beta)(2 + i\beta)|u|^2 u\Big) \tag{5.10}$$

with real parameters α, β. Here $u = u(x,t)$ is a complex valued function, and we investigate stability of the stationary solution (see [12])

$$U(x) = (\cosh x)^{-(1+i\beta)} = U_1(x) + iU_2(x) \ . \tag{5.11}$$

Writing the linearization about U in real and imaginary parts, one obtains the system $v_t = Pv$ where

$$Pv = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \Big(v_{xx} - (M_1 + M_2 M_3)v\Big)$$

with

$$M_1 = \begin{pmatrix} 1-\beta^2 & -2\beta \\ 2\beta & 1-\beta^2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2-\beta^2 & -3\beta \\ 3\beta & 2-\beta^2 \end{pmatrix} \ ,$$

$$M_3(x) = \begin{pmatrix} 3U_1^2 & 2U_1U_2 \\ 2U_1U_2 & U_1^2 + 3U_2^2 \end{pmatrix}(x) \ .$$

For any real τ and ϕ, equation (5.10) is invariant under the transformations $u(x) \to u(x+\tau)$ and $u(x) \to e^{i\phi}u(x)$. Therefore, $s=0$ is an eigenvalue of P of geometric multiplicity ≥ 2. The functions U_x and iU (in their real and imaginary parts) are corresponding eigenfunctions.

Assumption 1a) is satisfied for P with

$$A = A_\pm = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}, \quad B_\pm = 0, \quad C = C_\pm = -AM_1 \ .$$

Note that $A + A^* = 2I > 0$; thus the system $v_t = Pv$ is *unsymmetric* (for $\alpha \neq 0$) parabolic, and Assumption 1b) holds.

The algebraic curve S (see (5.7)) consists of two half lines,

$$S = \mathcal{L} \cup \bar{\mathcal{L}}, \quad \mathcal{L} = \{s = (1+i\alpha)(-\omega^2 - (1+i\beta)^2) \text{ for some real } \omega\}.$$

The maximal real part of all $s \in S$ is $-1+\beta^2+2\alpha\beta$. Therefore, Assumption 1c) is satisfied if and only if

$$-\delta := -1 + \beta^2 + 2\alpha\beta < 0 \ . \tag{5.12}$$

In the numerical results shown below we have chosen $\alpha = -2$. For this choice, condition (5.12) holds if

$$2 - \sqrt{5} < \beta < 2 + \sqrt{5} =: \beta_{crit} \ .$$

On the Spectrum of P. It is difficult to obtain rigorous results on the eigenvalues of the all–line operator P. Numerical results can only suggest or indicate certain properties of the eigenvalues.

We describe first how we believe the eigenvalues of P to behave as functions of β (with $\alpha = -2$ fixed). This is mostly inferred from numerics, but we have also used a new analytical result of Mielke [24]. In Fig.7 we show the suggested motion of the eigenvalues schematically.

For $0 \leq \beta < \beta_{crit} = 2+\sqrt{5}$ the operator P has $\mu_0 = 0$ as a geometrically double eigenvalue. Furthermore, there is a real positive eigenvalue μ_1 and a real negative eigenvalue μ_2. There are no other eigenvalues in the right half plane or near zero.

As β increases from 0 to β_{crit}, the negative eigenvalue μ_2 increases and becomes zero for $\beta = \beta_{crit}$, leading to a triple zero eigenvalue. (In fact, it has been proved rigorously [24] that the zero eigenvalue of P has algebraic multiplicity ≥ 3 and geometric multiplicity ≥ 2 for $\beta = \beta_{crit}$.) If β increases beyond β_{crit}, the eigenvalue μ_2 becomes positive and meets with μ_1 for $\beta \approx 4.255$. If β increases further, the positive double eigenvalue $\mu_2 = \mu_1$ splits into a complex conjugate pair, which then moves into the left half plane. At $\beta = \beta_0 \approx 4.62$ there are four eigenvalue with zero real parts (the double eigenvalue zero and the complex conjugate pair), and no unstable eigenvalues.

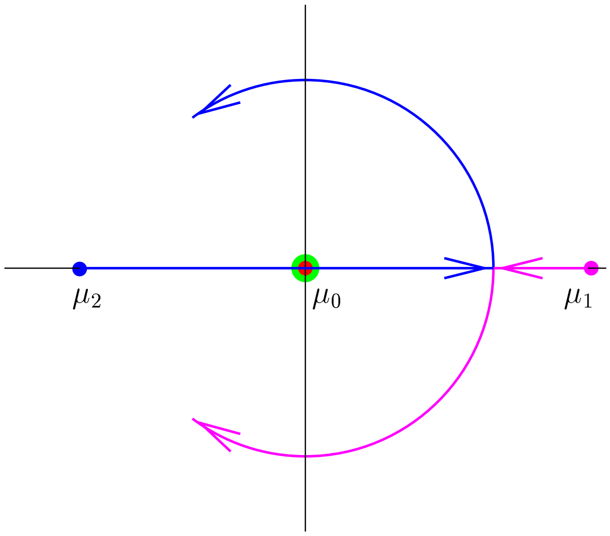


Figure 7: Behavior of complex eigenvalues when β increases from 4 to 5

Though P has no unstable eigenvalues for $\beta > \beta_0$, the stationary solution $U(x)$ remains unstable since the essential spectrum reaches into the right half plane for $\beta > \beta_{crit}$. The instability of $U(x)$ was already inferred from numerical simulations in [12].

Numerical Results.

(a) Motion of Numerical Eigenvalues. A 3D-picture of the four numerical eigenvalues involved is shown in Fig.8 for $J = [-10, 10], \Delta x = 0.004$.

However, it is difficult to extract from this picture the precise motion of the numerical eigenvalues near $\beta = \beta_{crit}$. Therefore, we show the motion for $4 \leq \beta \leq 5$ in a schematic drawing in Fig. 9. In the numerical approximations with $J = [-10, 10], \Delta x = 0.02$, for example, the double eigenvalue $\mu_0 = 0$ of P splits into two real eigenvalues, μ_0^e and μ_0^o. Here μ_0^e corresponds to the even eigenfunction $iU(x)$ and μ_0^o corresponds to $U_x(x)$, which is odd. As β increases from 0 to $\approx 4.16 = \beta_{crit} - \varepsilon_1$, the numerical approximation to the eigenvalue μ_2 meets μ_0^e on the negative real axis. Increasing β further leads to a complex conjugate pair, which moves into the right half plane. At $\beta \approx 4.25 = \beta_{crit} + \varepsilon_2$ the pair again forms a double real eigenvalue, which is now positive. As β increases, this double eigenvalue splits: μ_0^e moves towards zero and μ_2 moves towards μ_1. After forming a positive double eigenvalue $\mu_2 = \mu_1$, a complex conjugate pair forms, which then moves into the left half plane.

Roughly speaking, in numerical approximations the triple zero eigenvalue of P does not occur, but otherwise the four numerical eigenvalues behave essentially like the eigenvalues of P. In fact, our numerical computations first suggested the occurence of a triple zero eigenvalue. Its existence has then been shown by Mielke [24].

(b) Convergence of Eigenvalues as $J \to \infty$. For $\beta = 3, \Delta x = 0.04$ we show in Fig.10 how the eigenvalues μ_0^e, μ_0^o, and μ_1 behave as J increases. In a logarithmic scale we plot $|\mu_0^e|, |\mu_0^o|$

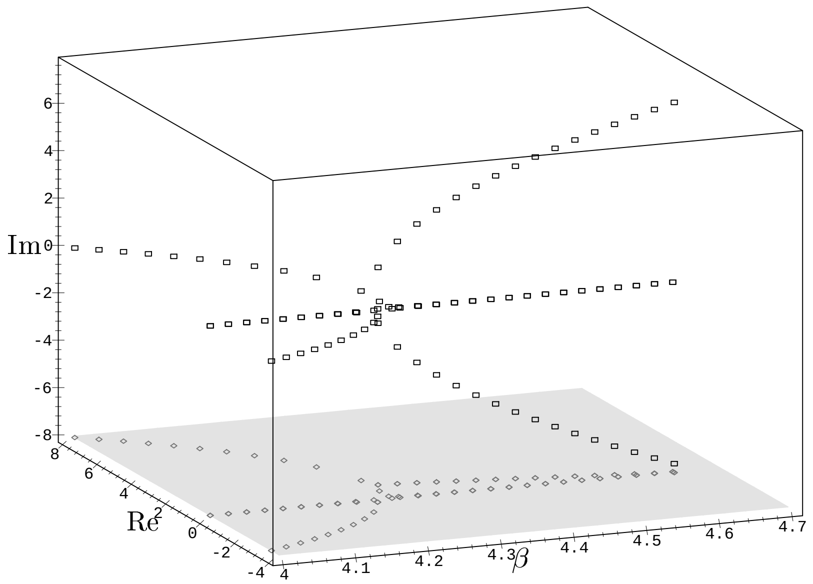
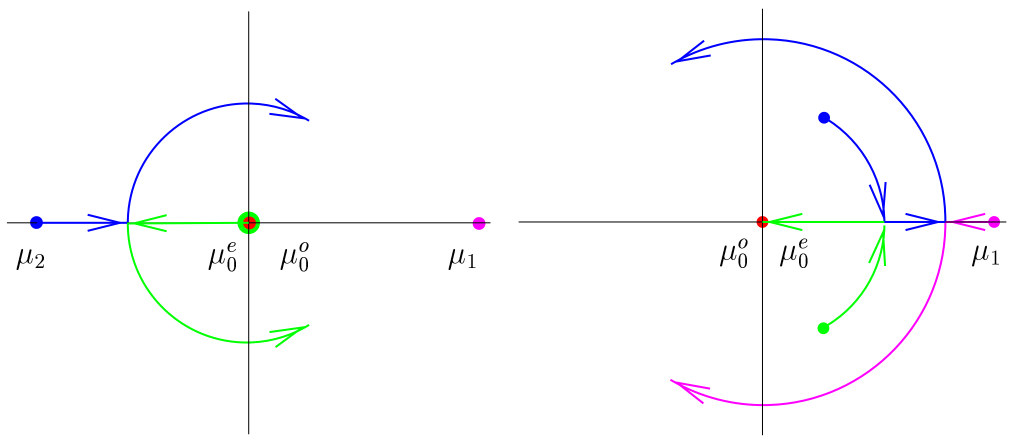


Figure 8: Ginzburg–Landau: four eigenvalues as functions of β

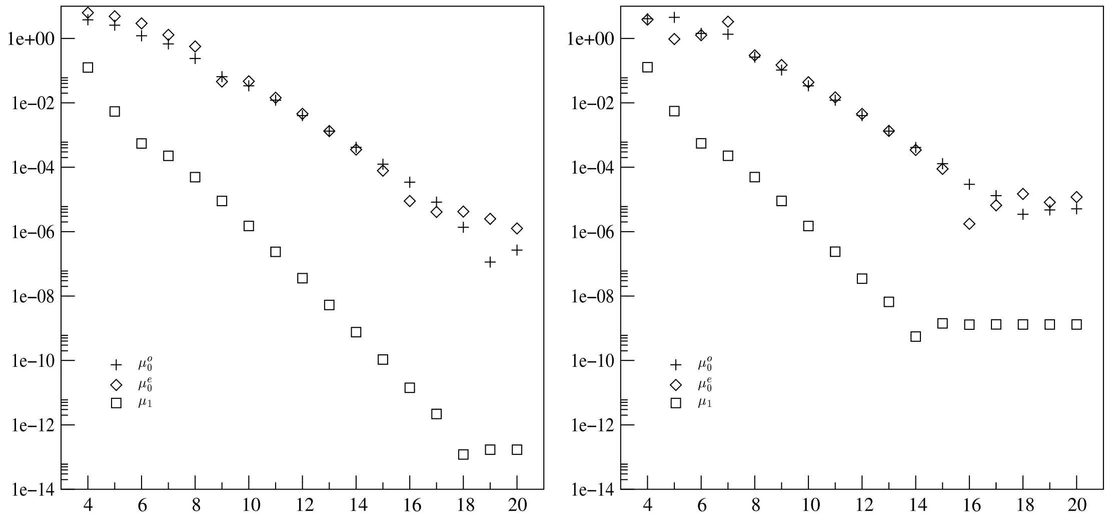


(a) $4 \leq \beta \lesssim \beta_{crit}$

(b) $\beta_{crit} \lesssim \beta \leq 5$

Figure 9: Behavior of numerical eigenvalues for $4 \leq \beta \leq 5$

and $|\mu_1(J) - \mu_{1,\infty}|$ where $\mu_{1,\infty}$ was taken from $J = [-40, 40]$. Note that the rate of convergence for the eigenvalues $|\mu_0^e|, |\mu_0^o|$ is about half the rate for μ_1, which is in agreement with Theorem 4.1. The fourth eigenvalue μ_2 could not be separated safely from the remaining spectrum for all intervals (compare (c) below) and is not shown.



(a) Dirichlet

(b) periodic

Figure 10: Ginzburg–Landau ($\alpha = -2$, $\beta = 3$, $\Delta x = 0.04$) : error of eigenvalues $|\mu_0^e|, |\mu_0^o|$ and $|\mu_1(J) - \mu_{1,\infty}|$ as a function of the length of the interval

(c) Close–Ups of Complete Spectra. In Fig.11 we compare close–ups (near zero) of the complete spectra with the dispersion curve S.

The computations were done for $J = [-20, 20]$ and $\Delta x = 0.04$ for Dirichlet and periodic boundary conditions. Note that the four isolated eigenvalues μ_0^e, μ_0^o, μ_1, and μ_2 are well–represented and are (within graphing accuracy) independent of the boundary conditions. The remaining clusters of eigenvalues of the discrete problems form rough approximations to the dispersion curve S. These clusters show much stronger dependency on the boundary conditions than the isolated eigenvalues. Also, the clusters change considerably with the grid, which is not shown here, however.

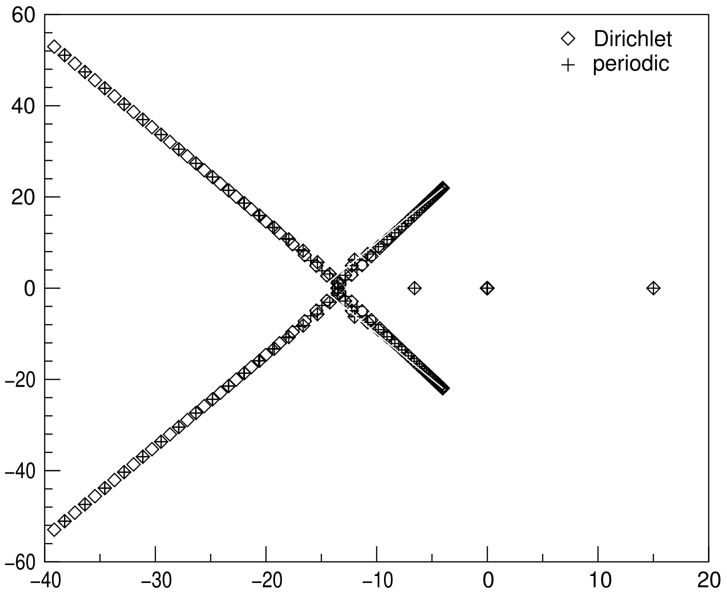


Figure 11: Ginzburg–Landau: Complete spectra for parameter values $\alpha = -2$, $\beta = 3$, $J = -[20, 20]$, $\Delta x = 0.04$ and comparison with dispersion curve

Extension of Theory. Formally our approximation results (for example, Theorem 4.1) are not applicable to study the eigenvalues of P for $\beta \geq \beta_{crit}$ since the essential spectrum of P reaches into the right half plane if $\beta \geq \beta_{crit}$. Then Assumption 1c) is violated. However, the region in the complex plane where $\sigma_{ess}(P)$ attains positive real parts is bounded away from the eigenvalues of P, and it is not difficult to show that the conclusions of Theorem 4.1 remain valid.

A Appendix: Dichotomies and a Matrix Lemma

Consider a subinterval $J = [x_-, x_+] \subset \mathbb{R}$, where both endpoints x_- and x_+ may be finite or infinite, with the understanding that $-\infty$ and $+\infty$ are not contained in J. Let $Lz = z_x - M(x)z, x \in J$, be a linear operator with a bounded continuous $N \times N$–matrix function $M(x)$, and let $S(x, y)$ denote the solution operator of L.

Assuming an exponential dichotomy for L, the following basic theorem gives an estimate for solutions of $Lz = h$ which satisfy suitable boundary conditions. The result is obtained by integrating the appropriate projections in decaying directions.

Theorem A.1 *Let L have an exponential dichotomy on J with data (K, β, π) and define the Green's function with respect to π by*

$$G(x, y) = \begin{cases} S(x, y)\pi(y), & x_- \leq y \leq x \ , \\ S(x, y)(\pi(y) - I), & x < y \leq x_+ \ . \end{cases} \tag{A.1}$$

Then, for any $h \in L_2(J)$ and any $\gamma_- \in range(\pi(x_-)), \gamma_+ \in range(I - \pi_+(x_+))$, the boundary value problem

$$Lz = h, \quad x \in J \ , \tag{A.2}$$

$$(I - \pi(x_+))z(x_+) = \gamma_+, \quad \pi(x_-)z(x_-) = \gamma_- \ , \tag{A.3}$$

has a unique solution $z \in H^1(J)$, namely $z = z_{sp} + z_{hom}$, where

$$z_{sp}(x) = \int_J G(x,y)h(y)dy \ , \tag{A.4}$$

$$z_{hom}(x) = S(x,x_-)\gamma_- + S(x,x_+)\gamma_+ \ . \tag{A.5}$$

The following estimates hold

$$\beta^2\|z_{sp}\|^2 + \beta|z_{sp}|_\Gamma^2 \le 5K^2\|h\|^2 \ , \tag{A.6}$$

$$\beta\|z_{hom}\|^2 + |z_{hom}|_\Gamma^2 \le (2 + 3K^2)(|\gamma_-|^2 + |\gamma_+|^2) \ . \tag{A.7}$$

Remark: In case $x_+ = \infty$ the boundary condition at x_+ in (A.3) is empty and the term $S(x,x_+)\gamma_+$ in (A.5) is zero. A similar remark applies to the case $x_- = -\infty$.

Proof: Defining z_{sp} by (A.4), it is clear that $Lz_{sp} = h$. Furthermore, by the dichotomy assumption,

$$|G(x,y)| \le Ke^{-\beta|x-y|} \ ,$$

and therefore

$$\begin{aligned} |z_{sp}(x)|^2 &\le K^2\Big(\int_J e^{-\beta|x-y|}|h(y)|dy\Big)^2 \\ &\le K^2\Big(\int_J e^{-\beta|x-y|}dy\Big)\Big(\int_J e^{-\beta|x-y|}|h(y)|^2dy\Big) \\ &\le \frac{2K^2}{\beta}\int_J e^{-\beta|x-y|}|h(y)|^2dy \ . \end{aligned}$$

Integration over J yields by Fubini's theorem

$$\|z_{sp}\|^2 \le \frac{4K^2}{\beta^2}\|h\|^2 \ .$$

Also, for finite x_+, we find from the definition of z_{sp},

$$\begin{aligned} |z_{sp}(x_+)|^2 &\le K^2\Big(\int_J e^{-\beta(x_+-y)}|h(y)|dy\Big)^2 \\ &\le \frac{K^2}{2\beta}\|h\|^2 \ , \end{aligned}$$

and a similar estimate holds for $|z_{sp}(x_-)|^2$. This proves (A.6). (If $x_+ = \infty$ then $z_{sp}(x_+) = \lim_{x\to\infty} z_{sp}(x) = 0$.)

Next define z_{hom} by (A.5). Then the dichotomy assumption yields the estimates

$$\|z_{hom}\|^2 \le \frac{K^2}{\beta}(|\gamma_-|^2 + |\gamma_+|^2) \ ,$$

$$|z_{hom}(x_+)|^2 \le 2(K^2 e^{-2\beta(x_+-x_-)}|\gamma_-|^2 + |\gamma_+|^2),$$

A similar estimate holds for $|z_{hom}(x_-)|^2$, and (A.7) is shown. We also note that a bound for z_{sp} in H^1 follows from the L_2–estimate and the boundedness of $M(x)$. The function $z = z_{sp} + z_{hom}$ solves the BVP (A.2),(A.3).

In order to show uniqueness, let z solve the BVP with homogeneous data. In case $-\infty < x_-, x_+ < \infty$ we obtain $z = 0$ from the equality

$$z(x_+) = \pi(x_+)z(x_+) = \pi(x_+)S(x_+, x_-)z(x_-) = S(x_+, x_-)\pi(x_-)z(x_-) = 0 \ .$$

In case $-\infty < x_-$ and $x_+ = \infty$ we use

$$z(x_-) = (I - \pi(x_-))z(x_-) = S(x_-, x)(I - \pi(x))z(x)$$

and the dichotomy assumption to obtain

$$|z(x_-)|^2 \le K^2 e^{-2\beta(x-x_-)}|z(x)|^2 \ .$$

Since the right hand side of this equation has a finite integral over $x_- \le x < \infty$, we conclude $z(x_-) = 0$, and hence $z = 0$. Similar arguments apply to the remaining cases. $\square$

The next result shows that the solution in Theorem A.1 decays exponentially if the inhomogeneous term decays exponentially.

Theorem A.2 *Let the assumptions of Theorem A.1 hold for $J = [0, \infty)$. Then any solution $z \in H^1$ of $Lz = 0$ satisfies*

$$z(0) \in range(\pi(0)), \quad |z(x)| \le Ke^{-\beta x}|z(0)|.$$

Moreover, assume that $h \in L^2(\mathbb{R}_+)$ satisfies

$$|h(x)| \le C_1 e^{-\beta_1 x}$$

for some $C_1, \beta_1 > 0$. If $z \in H^1$ is a solution of $Lz = h$ and if $0 < \beta' < \min\{\beta, \beta_1\}$ then

$$|z(x)| \le C(|\pi(0)z(0)|e^{-\beta x} + e^{-\beta' x}), \quad x \ge 0\,,$$

where $C = C(z, \beta')$. A corresponding result holds on $\mathbb{R}_-$.

Proof: For both parts we apply Theorem A.1 with $\gamma_- = \pi(0)z(0)$ and notice that z solves (A.2),(A.3). Since the solution is unique we obtain $z = z_{hom} + z_{sp}$ with z_{hom}, z_{sp} given by (A.4),(A.5). In the first case, $h = 0$, we find

$$z(x) = z_{hom}(x) = S(x,0)\pi(0)z(0).$$

This implies $z(0) = \pi(0)z(0)$ as well as the exponential estimate. In the second case it suffices to estimate $z_{sp}(x)$. By the exponential dichotomy we have

$$|z_{sp}(x)| \le K \int_0^\infty e^{-\beta|x-y|-\beta_1 y} dy \ .$$

If $\beta_0 = \min\{\beta, \beta_1\}$ then the integral is bounded by

$$\int_0^x e^{-\beta_0 x} dy + \int_x^\infty e^{-2\beta_0 y + \beta_0 x} dy = e^{-\beta_0 x}\Big(x + \frac{1}{2\beta_0}\Big) \ .$$

Since $\beta' < \beta_0$ the estimate

$$|z_{sp}(x)| \le C e^{-\beta' x}$$

follows. $\square$

An important property of exponential dichotomies is persistence under perturbations. The following **Roughness Theorem** is an improved quantitative version of Coppel's result [4]. We will use Definition 2.1 of dichotomy data that are continuous in some parameter.

Theorem A.3 *Let $Lz = z_x - M(x)z$ have an exponential dichotomy on J with data (K, β, π), and let $\Delta \in C(J; \mathbb{C}^{N\times N})$ satisfy*

$$3K|\Delta|_\infty < \beta, \quad |\Delta|_\infty = \sup_{x\in J} |\Delta(x)| \ . \tag{A.8}$$

Then the perturbed operator $\tilde{L}z = z_x - (M + \Delta)z$ has an exponential dichotomy on J with data $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$, where

$$\tilde{K} = K\Big(1 + \frac{4|\Delta|_\infty K}{\beta - 3|\Delta|_\infty K}\Big), \quad \tilde{\beta} = \beta - 2|\Delta|_\infty K \ , \tag{A.9}$$

$$|\pi(x) - \tilde{\pi}(x)| \le K\tilde{K} \int_J e^{-(\beta+\tilde{\beta})|x-y|} |\Delta(y)| dy \ . \tag{A.10}$$

If the unperturbed dichotomy data (K, β, π) and the matrices $\Delta(x)$ are continuous in some parameter and satisfy (A.8) for all parameter values, then the perturbed data $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$ can be chosen continuously in this parameter.

Remark: The original condition in [4] is $4K^2|\Delta|_\infty < \beta$ with $\tilde{K} = \frac{5}{2}K^2$ in (A.9) and the same $\tilde{\beta}$. Our version shows that the perturbed data $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$ can be taken arbitrarily close to the original ones if $|\Delta|_\infty$ is sufficiently small. Moreover, we added the integral estimate of the projector, which is crucial for our applications (see [2] in the semiinfinite case). Various generalizations of the Roughness Theorem have been developed, notably to an infinite dimensional setting in [29],[28]. The outline of proof given below follows the basic idea of Sandstede [29] and employs a further simplification due to [16] for discrete dichotomies.

Outline of proof: Instead of the dichotomy data $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$, we first determine a suitable Green's function $\tilde{G}$ of the perturbed operator $\tilde{L}$. By the variation of constants formula $\tilde{G}$ should satisfy the equation

$$\tilde{G}(x,y) = G(x,y) + \int_J G(x,\xi)\Delta(\xi)\tilde{G}(\xi,y)d\xi \ .$$

Therefore, we set $H = \tilde{G} - G$ and solve the fixed point equation

$$H = F(H) + F(G), \quad F(H)(x,y) = \int_J G(x,\xi)\Delta(\xi)H(\xi,y)d\xi \ . \tag{A.11}$$

Fix some $0 < \tilde{\beta} < \beta$ and consider this equation in the space

$$X = \{H \in C(J \times J, \mathbb{C}^{N\times N}) : \|H\|_* = \sup_{x,y\in J} e^{\tilde{\beta}|x-y|}|H(x,y)| < \infty\} \ .$$

It can be shown that F has a Lipschitz constant $q = |\Delta|_\infty K(\frac{2}{\beta+\tilde{\beta}} + \frac{1}{\beta-\tilde{\beta}})$. If $q < 1$ then (A.11) has a unique solution H in X, and we have

$$\|H\|_* \le \frac{q}{1-q}K \ .$$

Choosing $\tilde{\beta} = \beta - 2|\Delta|_\infty K$ and assuming, tentatively, that $\tilde{G} = G + H$ has the form (A.1) one obtains the estimates (A.9), (A.10). Moreover, the statement about the continuity of the dichotomy data follows directly by using the contraction mapping theorem with continuous dependence on parameters (see e.g. [13]).

In the next step one verifies the relations

$$\tilde{G}(x,\xi)\tilde{G}(\xi,y) = \begin{cases} \tilde{G}(x,y) & \text{for } x \ge \xi \ge y, \\ -\tilde{G}(x,y) & \text{for } x < \xi < y, \\ 0 & \text{otherwise} \ . \end{cases} \tag{A.12}$$

It is then easy to see that this implies (A.1) with $\tilde{\pi}(x) = \tilde{G}(x,x)$. Equation (A.12) follows from the fact that for fixed $\xi \ge y$ both functions, $Y_0(x) = \tilde{G}(x,\xi)\tilde{G}(\xi,y)$ and

$$Y_1(x) = \begin{cases} 0 & \text{for } x < \xi \\ \tilde{G}(x,y) & \text{for } x \ge \xi \end{cases} ,$$

satisfy the same equation,

$$Y(x) = G(x,\xi)\tilde{G}(\xi,y) + \int_J G(x,\eta)\Delta(\eta)Y(\eta)d\eta \ .$$

This equation can be solved by contraction in the norm $\sup_{x\in J} e^{\tilde{\beta}|x-\xi|}|Y(x)|$. A similar argument is used for fixed $\xi < y$. $\square$

If we have a semiinfinite interval $J = [x_-, \infty)$ in TheoremA.3 and, in addition,

$$|\Delta(x)| \to 0 \quad \text{as } x \to \infty \ , \tag{A.13}$$

then the estimate (A.10) immediately leads to

$$|\tilde{\pi}(x) - \pi(x)| \to 0 \quad \text{as } x \to \infty. \tag{A.14}$$

In fact, for $J = [x_-, \infty)$, the smallness assumption (A.8) can be replaced by (A.13) (see [27]). Such a result is proved by applying the roughness theorem on an interval $[x_+, \infty)$ for sufficiently large x_+ and then extending the dichotomy over the finite interval $[x_-, x_+]$. We formulate a corresponding result in the next theorem.

Theorem A.4 *Let $Lz = z_x - M(x)z$ have an exponential dichotomy on a semiinfinite interval $J = [x_-, \infty)$ with data (K, β, π), and let $\Delta \in C(J, \mathbb{C}^{N\times N})$ satisfy (A.13). Then, for any fixed $0 < \tilde{\beta} < \beta$, the perturbed operator $\tilde{L}z = z_x - (M + \Delta)z$ has an exponential dichotomy on J with data $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$. If the data (K, β, π) and the matrices Δ are continuous and the convergence (A.13) is uniform in some compact parameter set, then the perturbed data $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$ are continuous and the convergence (A.14) is uniform in the parameter.*

Finally, a very useful result states that exponential dichotomies for $Lz = z_x - M(x)z$ on both half-lines, $\mathbb{R}_-$ and $\mathbb{R}_+$, carry over to the whole real line if and only if the equation $Lz = 0$ has no non–trivial solution which decays in both x–directions [4], [27].

Theorem A.5 *Let $Lz = z_x - M(x)z$ have exponential dichotomies on $\mathbb{R}_-$ and on $\mathbb{R}_+$, with data (K_-, β_-, π_-) and (K_+, β_+, π_+), respectively. Then L has an exponential dichotomy on $\mathbb{R}$ if and only if*

$$\mathit{range}(\pi_+(0)) \oplus \mathit{range}(I - \pi_-(0)) = \mathbb{C}^N \ . \tag{A.15}$$

If (A.15) holds then the data (K, β, π) on $\mathbb{R}$ can be chosen so that $\beta = \min\{\beta_-, \beta_+\}$ and

$$|\pi(x) - \pi_\pm(x)| \leq C e^{-2\beta|x|} \quad \textit{for } x \in \mathbb{R}_\pm \ . \tag{A.16}$$

If the data $(K_\pm, \beta_\pm, \pi_\pm)$ are continuous in some compact parameter set and if (A.15) holds for all parameter values, then the data (K, β, π) are again continuous in the parameter and the estimate (A.16) holds with a uniform constant C.

Remarks: 1. The projector $\pi(x)$ in (A.16) is given as $\pi(x) = S(x,0)\pi(0)$ where $\pi(0)$ is determined by the conditions

$$\text{range}(\pi(0)) = \text{range}(\pi_+(0)), \quad \text{range}(I - \pi(0)) = \text{range}(I - \pi_-(0)).$$

The constant C in (A.16) depends on the data $K_\pm, \beta_\pm$ and on the norms $|\pi(0) - \pi_+(0)|$, $|\pi(0) - \pi_-(0)|$. It will tend to infinity if the subspaces $\text{range}(\pi_+(0))$ and $\text{range}(I - \pi_-(0))$ of forward and backward decaying modes assume a common direction. On a compact parameter domain, however, these terms can be estimated uniformly.

2. All the results in the above theorems carry over to matrices and projectors which are of type $C^k, 1 \le k \le \infty$, or analytic in some parameter. This follows by using the contraction mapping principle with smooth dependence on parameters. As has been shown in [29] one even has dichotomy estimates for the derivatives with respect to the parameter. For our purposes continuity in the parameter is sufficient to keep track of the uniformity of the data.

Our next auxiliary result is a matrix lemma which follows from the power version of the **Kreiss Matrix Theorem** [17]: For a power–bounded family of $k \times k$ matrices one can estimate the norm of the k–th powers in terms of the spectral radii. In the lemma, $|\cdot|$ denotes an arbitrary norm on $\mathbb{C}^{k\times k}$ though, in the proof, we assume without loss of generality that $|\cdot|$ is the matrix norm corresponding to the (vector) maximum norm. The spectral radius is denoted by $r(A)$.

Lemma A.1 *Let $\mathcal{F} \subset \mathbb{C}^{k\times k}$ denote a set of matrices and assume that there is a constant C with*

$$|A^\nu| \le C \quad \textit{for all } \nu = 1, 2, \dots \quad \textit{and all} \quad A \in \mathcal{F} \ . \tag{A.17}$$

Then there exists a constant $\tilde{C}$ with

$$|A^k| \le \tilde{C} r(A) \quad \textit{for all } A \in F \ . \tag{A.18}$$

Proof: By the Kreiss Matrix Theorem there are constants C_1, C_2 and transformations $T = T(A) \in \mathbb{C}^{k\times k}$ with $|T| + |T^{-1}| \le C_1$ so that $T^{-1}AT$ is upper triangular,

$$T^{-1}AT = \begin{pmatrix} b_{11} & \dots & b_{1k} \\ 0 & \ddots & \vdots \\ 0 & 0 & b_{kk} \end{pmatrix} = B \ , \tag{A.19}$$

and

$$|b_{ij}| \le C_2(1 - |b_{ii}|) \quad \text{for } 1 \le i < j \le k \ . \tag{A.20}$$

We write $B = diag(b_{ii}) + N = D + N$ and obtain

$$|B^k| \le C_3 \sum_{j=0}^{k-1} |D|^{k-j} |N|^j$$

since $N^k = 0$. Clearly,

$$|D|^{k-j} \le r(A) \le 1 \quad \text{for} \quad 0 \le j \le k-1$$

and

$$|N| \le C_2(k-1)$$

by (A.20). Combining these estimates, the bound (A.18) follows. □

Finally, we note that the Kreiss Matrix Theorem as well as the above lemma not only hold for matrices, but also for families of linear operators in normed linear spaces of *fixed* dimension

k. For our application, the important point is that space and norm may change with the operator. To obtain the result, it suffices to prove that in any normed space of dimension k one can choose a basis $\{u_i\}$ so that $\|\sum_i \alpha_i u_i\|$ is equivalent to $\sum_i |\alpha_i|$, where the equivalence constants only depend on k.

Lemma A.2 *In any normed linear space $(X, \| \ \|)$ of dimension k there is a basis $\{u_1, \ldots, u_k\}$ with $\|u_i\| = 1 (1 \le i \le k)$ and*

$$\frac{1}{2^k - 1} \sum_{i=1}^{k} |\alpha_i| \le \| \sum_{i=1}^{k} \alpha_i u_i \| \le \sum_{i=1}^{k} |\alpha_i| \quad \textit{for all} \quad \alpha \in \mathbb{C}^k \ . \tag{A.21}$$

Proof: The upper estimate is trivial, and so is the case $k = 1$. We proceed by induction in k. Let X have dimension k and choose any subspace Y of dimension $k-1$. Then there is a basis $\{u_1, \ldots, u_{k-1}\}$ in Y satisfying (A.21) for $k-1$, and one can choose a vector $u_k \in X$ with $\|u_k\| = 1$ and $\mathrm{dist}(u_k, Y) = 1$. Set $\rho_k = \frac{1}{2^k - 1}$ and consider an arbitrary $u = \sum_{i=1}^{k} \alpha_i u_i$ in X. If $2|\alpha_k| \le \rho_{k-1} \sum_{i=1}^{k-1} |\alpha_i|$, then we obtain

$$\sum_{i=1}^{k} |\alpha_i| \le (1 + \frac{\rho_{k-1}}{2}) \sum_{i=1}^{k-1} |\alpha_i|$$

and, using the induction hypothesis,

$$\begin{aligned} \| \sum_{i=1}^{k} \alpha_i u_i \| \ge \| \sum_{i=1}^{k-1} \alpha_i u_i \| - |\alpha_k| \quad &\ge \quad (\rho_{k-1} - \frac{\rho_{k-1}}{2}) \sum_{i=1}^{k-1} |\alpha_i| \\ \ge \frac{\rho_{k-1}}{2 + \rho_{k-1}} \sum_{i=1}^{k} |\alpha_i| \quad &= \quad \rho_k \sum_{i=1}^{k} |\alpha_i| \ . \end{aligned}$$

If $2|\alpha_k| > \rho_{k-1} \sum_{i=1}^{k-1} |\alpha_i|$, then we note that $\mathrm{dist}(u_k, Y) = 1$ and therefore

$$\begin{aligned} \| \sum_{i=1}^{k} \alpha_i u_i \| = |\alpha_k| \| u_k + \sum_{i=1}^{k-1} \frac{\alpha_i}{\alpha_k} u_i \| \quad &\ge \quad |\alpha_k| \\ > \frac{\rho_{k-1}}{2 + \rho_{k-1}} \sum_{i=1}^{k} |\alpha_i| \quad &= \quad \rho_k \sum_{i=1}^{k} |\alpha_i| \ . \end{aligned}$$

□

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