Numerical approximation of connecting orbits with asymptotic rate

Thorsten Pampel*

email: pampel@mathematik.uni-bielefeld.de

Department of Mathematics, University of Bielefeld
P. O. Box 100131, D–33501 Bielefeld, Germany

Abstract

In this paper we set up and analyze a numerical method for so called **connecting orbits with asymptotic rate** in parameterized dynamical systems. A connecting orbit with asymptotic rate has its initial value in a given submanifold of the phase space (or its cross product with parameter space) and it converges with an exponential rate to a given orbit, e. g. a steady state or a periodic orbit. It is well known that orbits with asymptotic rate can be used to foliate stable or strong stable manifolds of invariant sets.

We show that the problem of determining a connecting orbit with asymptotic rate is well-posed if a certain transversality condition is made and a specific relation between the number of stable dimensions and the number of parameters holds. For the proof we employ the implicit function theorem in spaces of exponentially decaying functions.

Using asymptotic boundary conditions we truncate the original problem to a finite interval and show that the error decays exponentially.

Typically the asymptotic boundary conditions by themselves are the result of a boundary value problem, e. g. if the limiting orbit is periodic. Thus it is expensive to calculate them in a parameter dependent way during the approximation procedure. To avoid this we develop a boundary corrector method which turns out to be nearly optimal after very few steps.

Keywords: Numerical approximation, connecting orbits, asymptotic rate, asymptotic boundary condition

AMS subject classification: Primary 58F22, 65L10. Secondary 58F18, 34C37.

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1 Introduction

Consider a parameterized dynamical system

$$\dot{x} = f(x, \lambda), \quad x(t) \in \mathbb{R}^m, \quad \lambda \in \Lambda \subset \mathbb{R}^p \tag{1.1}$$

and a family $\{y(\lambda)(\cdot)\}_{\lambda \in \Lambda}$ of solutions of (1.1) on $\mathbb{R}_+$.

Our aim is to approximate so called **connecting orbits with asymptotic rate γ from $\mathcal{W}$ to y**. These are pairs (x, λ) where x is a solution of (1.1) on $\mathbb{R}_+$ converging asymptotically to a given $y(\lambda)$ with an exponential rate $\gamma < 0$ and with initial value in a given manifold $\mathcal{W} \subset \mathbb{R}^{m+p}$, i. e. we require $\big(x(0), \lambda\big) \in \mathcal{W}$ and

$$||x(t) - y(\lambda)(t)|| \leq C_x e^{\gamma t} \quad \forall \quad t \geq 0 \tag{1.2}$$

for some constant $C_x > 0$. More precisely, we say that $x(t)$ is **γ–asymptotic** with $y(\lambda)(t)$ if (1.2) holds.

For uniqueness of connecting orbits with asymptotic rate we need a transversality condition which will be seen to be equivalent to the regularity of a solution of an operator equation.

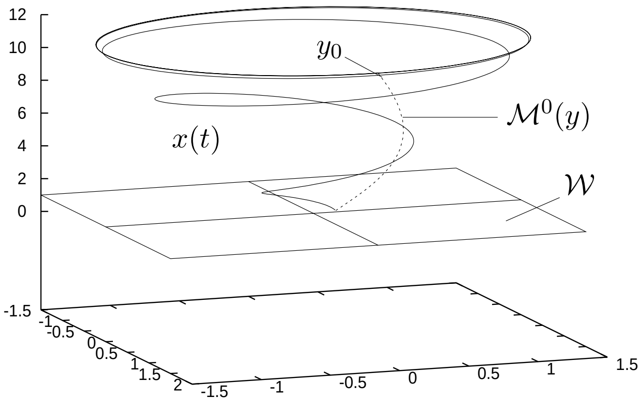

Figure 1.1: The phase portrait of example 5.1. Here $y(\lambda)$ is periodic and the manifold $\mathcal{W}$ is a hyperplane in $\mathbb{R}^3$. The dotted line $\mathcal{M}^0(y)$ is the stable manifold of the periodic solution with initial value $y(0) = y_0$.

We are interested in that part of the stable manifold of an orbit $y(\lambda)$ which intersects the given manifold $\mathcal{W}$. Problems of this type occur as subproblems in the computation of connecting orbits of general type [6], [19] or in solving optimal control problems with infinite horizon [11], [22],[23].

We show how to extend a given γ–asymptotic solution to a smooth manifold of orbits that are γ–asymptotic with the family $y(\lambda)$. We use the concept of shifted exponential dichotomy [15] and we employ the implicit function theorem in spaces of exponentially decaying functions.

In particular, we prove parameter dependent results even in the strong stable case. This includes the smoothness of the manifold in the parameter. The first derivative with respect to the parameter is used for our approximation result in Section 4.

In the parameter independent case a general theory of foliations of stable and unstable manifolds of invariant sets has been developed by Fenichel [12, 13, 14]. There the technique of the Hadamard graph transform is used.

A related result for foliations of generalized stable and unstable manifolds on infinite dimensional spaces and its smoothness is shown in Chow, Lin and Lu [9]. In order to obtain general foliations, additional gap condition are assumed.

For nonautonomous differential equations of Carathéodory type analogous foliations of integral manifolds are proved by Aulbach and Wanner [1, 2] and their smoothness is a recent result from the PhD-Thesis of Siegmund [24].

Smoothness in the parameter only appears in the case of generalized center unstable manifolds in [9] (and not in the (strong) stable case mentioned here).

In Section 4 we use our previous results to study the approximations of connecting orbits with asymptotic rate. We truncate the original problem to a finite interval $[0, T_+]$ and use an asymptotic boundary condition at T_+ as well as the initial condition $\big(x(0), \lambda\big) \in \mathcal{W}$. By a proper use of weighted norms on finite intervals we show that the error caused by truncation decays exponentially.

To get appropriate asymptotic boundary conditions we typically have to solve spectral problems for a linear boundary value problem, e. g. the variational equation in the periodic case, during the approximation. To avoid this we develop the boundary corrector method as follows: A first approximation is used to get a predictor for the unknown parameters. Then we update the asymptotic boundary condition at these parameters and we show that the second approximation has nearly optimal error estimates (Proposition 4.4).

If the $y(\lambda)$ are hyperbolic equilibria we establish a super-convergence result (Corollary 4.3) similar to those in [4], [5], [21].

We will apply our general results to orbits that converge to equilibria or periodic orbits and discuss the errors for an example where the solutions $y(\lambda)$ of (1.1) are known explicitly.

Orbits connecting steady states or periodic orbits to each other can be split into two orbits of the kind treated in this article. This will be used in the Lorenz system to find an initial approximation of a point to periodic connection (see [6]).

A numerical method and its error estimates for orbits connecting steady states or periodic orbits to each other is developed in [19].

2 Preliminaries on linear systems

An important tool for describing linearizations along solutions which converge with an exponential rate to another trajectory is the notion of shifted exponential dichotomy as defined in [15]. In particular, we apply different forms of roughness theorems to obtain "similar" behavior in a neighborhood of such trajectories.

We define **Banach spaces of k times differentiable exponentially weighted functions** from an interval J into a Banach space $(X, |\cdot|)$ by

$$BC^k_\gamma(J,X) := \left\{ x \in C^k(J,X) \Big| \sup_{t\in J}(|x^{(j)}(t)|e^{-\gamma t}) < \infty, \quad j = 0,\dots,k \right\},$$

where $\gamma \in \mathbb{R}$. The norm on $BC^k_\gamma(J,X)$ is

$$||x||_{\gamma,k} = \sum_{j=0}^{k} ||x^{(j)}||_\gamma, \quad \text{where } ||x||_\gamma = \sup_{t\in J}(|x(t)|e^{-\gamma t}).$$

In case $\gamma = 0$ we obtain the **Banach space of k times differentiable and bounded functions** on J

$$BC^k(J,X) := BC^k_0(J,X) \text{ with } ||x||_{0,k} = \sum_{j=0}^{k} ||x^{(j)}||_\infty,\ ||x||_\infty := \sup_{t\in J}(|x(t)|).$$

Consider a linear differential operator

$$Lx = \dot{x} - A(\cdot)x$$

where $A \in BC(J, \mathbb{R}^{m,m})$, $x \in C^1(J,\mathbb{R}^m)$ and $J \subset \mathbb{R}$ is some interval. Denote the solution operator of $Lx = 0$ by $S(\cdot,\cdot)$, i. e. $LS(\cdot, s) = 0$ and $S(s,s) = \mathrm{Id}_{\mathbb{R}^m}$ for all $s \in J$.

We define a shifted exponential dichotomy as in [15] and its smoothness in the parameter as in [20],[7].

Definition 2.1 (Shifted exponential dichotomy) $L := \frac{\mathrm{d}}{\mathrm{dt}} - A(\cdot)$ *has a* **shifted exponential dichotomy** *on J with exponents $\alpha < \beta$ if there exist a constant K and projectors $P_a(t)$, $P_b(t)$, $t \in J$ with $P_b(t) = \mathrm{Id}_{\mathbb{R}^m} - P_a(t)$ such that*

$$S(t,s)P_\kappa(s) = P_\kappa(t)S(t,s), \qquad \text{for } \kappa = a,b \text{ and all } t,s \in J \tag{2.1}$$

and

$$||P_a(t)S(t,s)|| \le Ke^{\alpha(t-s)}, \quad ||P_b(s)S(s,t)|| \le Ke^{-\beta(t-s)} \quad \text{for } t \ge s. \tag{2.2}$$

We call (K,α,β,P_a,P_b) the **dichotomy data**. *The dichotomy data are* **of type C^l**, *$l \ge 0$ with respect to $\lambda \in \Lambda$ if $A \in BC^l\big(\Lambda, BC(J,\mathbb{R}^{m,m})\big)$ and $L(\lambda) = \frac{\mathrm{d}}{\mathrm{dt}} - A(\lambda)(\cdot)$ has a shifted exponential dichotomy with dichotomy data $\big(K,\alpha,\beta,P_a(\lambda),P_b(\lambda)\big)$ for all $\lambda \in \Lambda$, where $P_a(\cdot)(t), P_b(\cdot)(t) \in C^l(\Lambda, \mathbb{R}^{m,m})$, and the data K, α, β are independent of λ.*

Using this notion we define Green's function on some interval J by

$$G(t,s) := \begin{cases} S(t,s)P_a(s), & s \le t \\ -S(t,s)P_b(s), & t < s \end{cases} \quad \text{for } s,t \in J \tag{2.3}$$

and see that

$$s_J(v)(t) := \int_J G(t,s)v(s)\mathrm{d}s, \tag{2.4}$$

solves $Lx = v$ for $v \in C(J, \mathbb{R}^m)$, if the integral is well defined for each $t \in J$.

Thus the basic solution theorem in [7, Theorem A1] can easily be extended to shifted exponential dichotomies as follows

THEOREM 2.2 *Let $L = \frac{\mathrm{d}}{\mathrm{dt}} - A(\cdot)$ have a shifted exponential dichotomy on $J = [t_-, \infty)$ with data $(K, \alpha, \beta, P_a, P_b)$ and assume $\gamma \in (\alpha, \beta)$.*

Then any solution $z \in BC^1_\gamma\big([t_-,\infty), \mathbb{R}^m\big)$ of $Lz = 0$ satisfies $z(t_-) \in \mathcal{R}\big(P_a(t_-)\big)$.

For any $v \in BC^0_\gamma(J, \mathbb{R}^m)$ and $\xi_a \in \mathcal{R}\big(P_a(t_-)\big)$ the boundary value problem

$$Lz = v, \quad t \in J \tag{2.5}$$

$$P_a(t_-)z(t_-) = \xi_a \tag{2.6}$$

has a unique solution $z \in BC^1_\gamma(J, \mathbb{R}^m)$ namely $z = s_J(v) + S(\cdot, t_-)\xi_a$. Furthermore the following estimates hold with $C_\gamma = \frac{1}{\gamma - \alpha} + \frac{1}{\beta - \gamma}$:

$$||s_J(v)||_\gamma \le KC_\gamma ||v||_\gamma, \tag{2.7}$$

$$||S(\cdot, t_-)\xi_a||_\gamma \le Ke^{-\gamma t_-}|\xi_a|, \tag{2.8}$$

$$||s_J(v)||_{\gamma,1} \le (K(1 + ||A||_\infty)C_\gamma + 1)||v||_\gamma, \tag{2.9}$$

$$||S(\cdot, t_-)\xi_a||_{\gamma,1} \le K(1 + ||A||_\infty)e^{-\gamma t_-}|\xi_a|. \tag{2.10}$$

Proof: Let $z \in BC^1_\gamma\big([t_-,\infty), \mathbb{R}^m\big)$ be a solution of the homogeneous equation $Lz = 0$ on J then $z(t) = S(t, t_-)z(t_-)$, $z(t_-) \in \mathbb{R}^m$ and $|z(t)|e^{-\gamma t} \le ||z||_\gamma$. Thus for any $t \in [t_-, \infty)$ we see

$$\begin{aligned} |P_b(t_-)z(t_-)| &= |S(t_-,t)P_b(t)S(t,t_-)z(t_-)| \le ||S(t_-,t)P_b(t)|| \cdot |z(t)| \\ &\le Ke^{-\beta(t-t_-)} \cdot e^{\gamma t}||z||_\gamma \le Ke^{\beta t_-}||z||_\gamma e^{(\gamma-\beta)t} \overset{t\to\infty}{\longrightarrow} 0. \end{aligned}$$

This implies $z(t_-) \in \mathcal{R}\big(P_a(t_-)\big)$. On the other hand any solution z of $Lz = 0$ on J with $z(t_-) \in \mathcal{R}\big(P_a(t_-)\big)$ is in $BC^1_\gamma\big([t_-,\infty), \mathbb{R}^m\big)$, because from $\dot{z} = A(\cdot)z$ we see that $||z||_{\gamma,1} \le K(1 + ||A(\cdot)||_\infty)e^{-\gamma t_-}|z(t_-)|$ by the shifted exponential dichotomy.

To obtain $z = s_J(v) + S(\cdot, t_-)\xi_a \in BC^1_\gamma\big([t_-,\infty),\mathbb{R}^m\big)$ we prove (2.9), (2.10) by

$$\begin{aligned}
|P_a(t)s_J(v)(t)| &\leq \int_{t_-}^{t} |S(t,s)P_a(s)v(s)|\mathrm{ds} \\
&\leq K\int_{t_-}^{t} e^{\alpha(t-s)}\cdot e^{\gamma s}|v(s)|e^{-\gamma s}\mathrm{ds} \qquad (2.11)\\
&\leq Ke^{\alpha t}\frac{1}{\gamma-\alpha}\big(e^{(\gamma-\alpha)t} - e^{(\gamma-\alpha)t_-}\big)||v||_\gamma \\
&\leq K\frac{1}{\gamma-\alpha}e^{\gamma t}||v||_\gamma
\end{aligned}$$

and analogously

$$\begin{aligned}
|P_b(t)s_J(v)(t)| &\leq \int_{t}^{\infty} |S(t,s)P_b(s)v(s)|\mathrm{ds} \\
&\leq K\int_{t}^{\infty} e^{\beta(t-s)}e^{\gamma s}|v(s)|e^{-\gamma s}\mathrm{ds} \qquad (2.12)\\
&\leq Ke^{\beta t}\frac{1}{\beta-\gamma}e^{(\gamma-\beta)t}||v||_\gamma \\
&\leq K\frac{1}{\beta-\gamma}e^{\gamma t}||v||_\gamma
\end{aligned}$$

and therefore (2.7). The dichotomy properties imply (2.8) directly. By (2.5) we see that $|\dot{z}(t)| \leq ||A(t)||\ |z(t)| + |v(t)|$ holds and hence $||z||_{\gamma,1} \leq (1+||A(\cdot)||_\infty)\ ||z||_\gamma + ||v||_\gamma$. This implies (2.9) and (2.10). Thus $\big\{s_J(v) + S(\cdot,t_-)\xi_a | \xi_a \in \mathcal{R}\big(P_a(t_-)\big)\big\}$ is the set of all solutions of (2.5) in $BC^1_\gamma\big([t_-,\infty),\mathbb{R}^m\big)$ and by (2.6) z is unique. ■

For our later approximations we need corresponding estimates on a finite interval $[t_-,t_+]$.

THEOREM 2.3 *Let $L = \frac{\mathrm{d}}{\mathrm{dt}} - A(\cdot)$ have a shifted exponential dichotomy on $J = [t_-,t_+]$ with data (K,α,β,P_a,P_b).*

Then for any $v \in C(J,\mathbb{R}^m)$, $\xi_a \in \mathcal{R}\big(P_a(t_-)\big)$ and $\xi_b \in \mathcal{R}\big(P_b(t_+)\big)$ the boundary value problem

$$Lz = v, \quad t \in J \qquad (2.13)$$

$$P_a(t_-)z(t_-) = \xi_a, \qquad P_b(t_+)z(t_+) = \xi_b \qquad (2.14)$$

has a unique solution $z \in C^1(J,\mathbb{R}^m)$ namely $z = s_J(v) + S(t,t_-)\xi_a + S(t,t_+)\xi_b$.

For any $\gamma \in (\alpha,\beta)$ the following estimates hold with $C_\gamma = \frac{1}{\gamma-\alpha} + \frac{1}{\beta-\gamma}$

$$||s_J(v)||_\gamma \leq KC_\gamma||v||_\gamma, \qquad (2.15)$$

and for $\gamma \in [\alpha,\beta]$ we can estimate

$$\begin{aligned}
||s_J(v)||_\gamma &\leq K\int_J |v(s)|e^{-\gamma s}\mathrm{ds} \leq K|t_+ - t_-|\ ||v||_\gamma \qquad (2.16)\\
||S(t,t_-)\xi_a + S(t,t_+)\xi_b||_\gamma &\leq K\big(e^{-\gamma t_-}|\xi_a| + e^{-\gamma t_+}|\xi_b|\big). \qquad (2.17)
\end{aligned}$$

Proof: Defining $s_J(v)$ as in (2.3) and (2.4) implies, on a finite interval, that $s_J(v)$ solves(2.13) with $P_a(t_-)s_J(v)(t_-) = 0$ and $P_b(t_+)s_J(v)(t_+) = 0$. Let x be any solution of the homogeneous equation $Lz = 0$ with (2.14), i. e. $x(t) = S(t,t_-)x(t_-)$, then

$$\begin{aligned} x(t) &= P_a(t)x(t) + P_b(t)x(t) \\ &= S(t,t_-)P_a(t_-)x(t_-) + S(t,t_+)P_b(t_+)S(t_+,t_-)x(t_-) \\ &= S(t,t_-)\xi_a + S(t,t_+)\xi_b. \end{aligned}$$

Thus $z = s_J(v) + S(t,t_-)\xi_a + S(t,t_+)\xi_b$ is the unique solution of (2.13), (2.14). The estimates (2.15) and (2.17) are proved analogously to (2.7), (2.8) in Theorem 2.2 and (2.16) is obtained from

$$\begin{aligned} |s_J(v)(t)|e^{-\gamma t} &\leq K\left(\int_{t_-}^{t} \underbrace{e^{\alpha(t-s)}}_{\leq e^{\gamma(t-s)}} \cdot |v(s)|\mathrm{ds} + \int_{t}^{t_+} \underbrace{e^{\beta(t-s)}}_{\leq e^{\gamma(t-s)}} \cdot |v(s)|\mathrm{ds}\right) e^{-\gamma t} \\ &\leq K \int_{t_-}^{t_+} |v(s)|e^{-\gamma s}\mathrm{ds} \leq K|t_+ - t_-| \, ||v||_\gamma. \end{aligned}$$

■

Another important tool is a quantitative version of the roughness theorem on finite and infinite intervals J ([10],[18],[20],[6],[7]).

PROPOSITION 2.4 (Roughness theorem) *Let $\bar{L}$ have a shifted exponential dichotomy on J with dichotomy data $(\bar{K}, \bar{\alpha}, \bar{\beta}, \bar{P}_a, \bar{P}_b)$ and let $B \in C(J, \mathbb{R}^{m,m})$ satisfy*

$$6\bar{K}^2 b_\infty < \bar{\beta} - \bar{\alpha} \text{ where } b_\infty = \sup_{t\in J} |B(t)|. \tag{2.18}$$

Then the perturbed operator $L = \bar{L} - B(\cdot)$ has a shifted exponential dichotomy on J with data $(K, \alpha, \beta, P_a, P_b)$, where

$$\begin{gathered} K = \bar{K}\left(1 + \frac{8b_\infty \bar{K}}{\bar{\beta} - \bar{\alpha} - 6b_\infty \bar{K}}\right), \\ \alpha = \bar{\alpha} + 2b_\infty \bar{K} < \beta = \bar{\beta} - 2b_\infty \bar{K}. \end{gathered}$$

and the projectors $P_a(t)$, $P_b(t)$, $t \in J$ satisfy with $\kappa = a, b$

$$\begin{aligned} ||P_\kappa(t) - \bar{P}_\kappa(t)|| &\leq \bar{K}K \int_J e^{-(\beta-\bar{\alpha})|t-s|}||B(s)||\mathrm{ds} \leq 2\bar{K}K\frac{b_\infty}{\beta - \bar{\alpha}}, && (2.19) \\ ||\bar{P}_a(t)P_b(t)|| &\leq \bar{K}K \int_{t_-}^{t} e^{-(\beta-\bar{\alpha})|t-s|}||B(s)||\mathrm{ds} \leq \bar{K}K\frac{b_\infty}{\beta - \bar{\alpha}}, && (2.20) \\ ||\bar{P}_b(t)P_a(t)|| &\leq \bar{K}K \int_{t}^{t_+} e^{-(\beta-\bar{\alpha})|t-s|}||B(s)||\mathrm{ds} \leq \bar{K}K\frac{b_\infty}{\beta - \bar{\alpha}}. && (2.21) \end{aligned}$$

If $B \in BC^l(\Lambda, BC(J, \mathbb{R}^{m,m}))$ and (2.18) holds uniformly in Λ, then the dichotomy data are of type C^l and the estimates (2.19)–(2.21) hold uniformly in Λ.

Proof: In the case of an exponential dichotomy, i. e. $\alpha = -\beta$ ([10]), this is the same result as Theorem A.3 in [7]. Therefore we shift the exponents with $\frac{1}{2}(\alpha+\beta)$ using Lemma 2.2 in [6], apply Theorem A.3 in [7] and shift the exponents back with $-\frac{1}{2}(\alpha+\beta)$. ■

A result for exponentially decaying functions can be shown as in Proposition 2.4 in [6].

PROPOSITION 2.5 (Asymptotic roughness theorem) *Let $\bar{L}$ have a shifted exponential dichotomy on $J = [0,\infty)$ with dichotomy data $(\bar{K}, \bar{\alpha}, \bar{\beta}, \bar{P}_a, \bar{P}_b)$ and let $B \in C(J, \mathbb{R}^{m,m})$ satisfy $|B(t)| \le C_B e^{-\epsilon t}$ for all $t \ge 0$ and some $\epsilon \in (0, \bar{\beta} - \bar{\alpha})$.*

Then the perturbed operator $L = \bar{L} - B(\cdot)$ has a shifted exponential dichotomy on J with exponents $\alpha < \beta$, which may be choosen arbitrarily close to $\bar{\alpha} < \bar{\beta}$, a constant K and projectors P_a, P_b satisfying for some constant $C > 0$

$$\begin{aligned} ||P_\kappa(t) - \bar{P}_\kappa(t)|| &\le Ce^{-\epsilon t}, \quad \kappa = a, b, \\ ||\bar{P}_a(t) P_b(t)|| &\le Ce^{-\epsilon t}, \\ ||\bar{P}_b(t) P_a(t)|| &\le Ce^{-\epsilon t}. \end{aligned}$$

3 Connecting orbits with asymptotic rate

Throughout this article we deal with solutions of (1.1) which are "γ–asymptotic" with a parameterized family $\{y(\lambda)\}_{\lambda\in\Lambda}$ of solutions of (1.1).

For the function f and the family $\{y(\lambda)\}_{\lambda\in\Lambda}$ we make the following assumptions:

A1 (System properties) $f \in C^k(\mathbb{R}^{m+p}, \mathbb{R}^m)$, $k \ge 2$ and $f^{(k)}$ is locally Lipschitz with respect to x.

A2 (Solution properties) $y \in C^k\big(\Lambda, BC^1(\mathbb{R}_+, \mathbb{R}^m)\big)$ where $\Lambda \subset \mathbb{R}^p$ is an open, bounded set. Moreover, $y(\lambda)$ is a solution of $\dot{x} = f(x, \lambda)$ for each $\lambda \in \Lambda$ and y is uniformly bounded in Λ.

A3 (Dichotomy condition) $\bar{L}(\lambda) := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f\big(y(\lambda), \lambda\big)$, $\lambda \in \Lambda$ has a shifted exponential dichotomy with data $\big(\bar{K}, \bar{\alpha}, \bar{\beta}, \bar{P}_a(\lambda), \bar{P}_b(\lambda)\big)$, $\bar{\alpha} < 0$ which are continuous with respect to λ. The ranks of the projectors are independent of λ and given by $m_a := \dim \mathcal{R}\big(\bar{P}_a(t)(\lambda)\big)$ and $m_b := \dim \mathcal{R}\big(\bar{P}_b(t)(\lambda)\big)$.

Remark: The assumptions**A1** and **A2** imply, that $y(\lambda) \in C^{k+1}(\mathbb{R}_+, \mathbb{R}^m)$ and that $y^{(j)}(\lambda)$ and $f^{(j)}\big(y(\lambda)(\cdot), \lambda\big)$ are uniformly bounded in Λ for $0 \le j \le k$.

The dichotomy condition **A3** can be weakened by the assumption that **A3** holds for some $\bar{\lambda} \in \Lambda$. Then Proposition 2.4 implies the existence of some $\Lambda_0 \subset \Lambda$ such that **A3** holds on Λ_0.

In 4.1 we prove that these assumptions hold, if the $y(\lambda)$ are either equilibria or scaled periodic orbits.

3.1 Asymptotic solutions

In this section we define "γ–asymptotic" solutions and derive a local parametrization of these solutions. Using this parametrization, we introduce the notion of a "local γ–stable manifold".

In the parameter independent case results on stable and unstable manifolds have been developed by Fenichel [12, 13, 14]. Those results hold for overflowing invariant sets (and not only for the manifold of solutions converging to a **single** trajectory, discussed here). The proofs are based on the Hadamard graph transform and allow under additional assumptions to describe the dynamic on the (possibly higher-dimensional) overflowing invariant set.

A foliation of generalized stable and unstable manifolds of a steady state and its smoothness is also proved by Chow, Lin and Lu [9]. Their results are valid in infinite dimensional spaces but they need an additional gap condition on α, β to achieve smoothness. Moreover they need a global Lipschitz constant. Smoothness in the parameter is only achieved for the case where the center part belongs to the manifold and not in the strong stable case, handled here.

Related results on existence and smoothness of integral manifolds for nonautonomous differential equations of Carathéodory type are proved by Aulbach and Wanner [1, 2] (existence) and Siegmund [24] (smoothness).

We restrict to single bounded orbits. This allows to achieve smoothness with respect to parameters even in case of a strong stable manifold without additional global assumptions or conditions on α, β.

In [19] a C^{k-1}–structure on the set of solutions which are γ–asymptotic with $y(\lambda)$ and hence a "global γ–stable manifold evaluated at a point" is defined. Moreover, smoothness with respect to time is obtained in [19] and hence a "global γ–stable manifold along a solution", which can be foliated, is achieved there.

Definition 3.1 *Two solutions x, y of $\dot{x} = f(x, \lambda)$ on $\mathbb{R}_+$ are called* **asymptotic** *if*

$$\lim_{t\to\infty} ||x(t) - y(t)|| = 0 \tag{3.1}$$

and $\boldsymbol{\gamma}$**–asymptotic** ($\gamma < 0$) *if in addition there exists some $C_x > 0$ such that*

$$||x(t) - y(t)|| \leq C_x e^{\gamma t} \quad \forall\, t \geq 0. \tag{3.2}$$

We call two points x and $y \in \mathbb{R}^m$ $\boldsymbol{\gamma}$**–asymptotic** *if the corresponding solutions $u(\cdot, 0, x, \lambda)$ and $u(\cdot, 0, y, \lambda)$ of $\dot{x} = f(x, \lambda)$ with $x(0) = x$, $y(0) = y$ are γ–asymptotic.*

A local parametrization of γ–asymptotic solutions is established by our next proposition.

PROPOSITION 3.2 (γ–asymptotic solutions) *Let* **A1**–**A2** *hold and let* $\bar{x}$ *and* $\bar{y} = y(\bar{\lambda})$ *be* $\bar{\gamma}$*–asymptotic solutions* ($\bar{\gamma} < 0$) *of* $\dot{x} = f(x, \bar{\lambda})$ *for some* $\bar{\lambda} \in \Lambda$.

Furthermore assume that $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(\bar{x}(t), \bar{\lambda})$ *has a shifted exponential dichotomy on* $[0, \infty)$ *with dichotomy data* $(K, \alpha, \beta, P_a, P_b)$ *such that* $\bar{\gamma} \in [\alpha, \beta)$. *Then for any* $\gamma \in (\bar{\gamma}, \beta)$ *there exist open neighborhoods* $V \times \Omega \subset \mathcal{R}\big(P_a(0)\big) \times \Lambda$ *of* $(0, \bar{\lambda})$ *and* $U \subset BC^1(\mathbb{R}_+, \mathbb{R}^m)$ *of* $\bar{x}$ *and a function* $x \in C^{k-1}(V \times \Omega, U)$, *such that*

P1 $x(0, \bar{\lambda}) = \bar{x}$

P2 $x(\xi, \lambda) \in U$ *for all* $(\xi, \lambda) \in V \times \Omega$

P3 *For* $(\xi, \lambda) \in V \times \Omega$ *the function* $x(\xi, \lambda)$ *is the unique solution in* U *of the problem*

$$\begin{cases} x \text{ and } y(\lambda) \text{ are } \gamma\text{–asymptotic solutions of } \dot{x} = f(x, \lambda), \\ P_a(0)\big(x(0) - \bar{x}(0)\big) = \xi. \end{cases} \qquad (A_\gamma(\bar{x}, \bar{\lambda}))$$

P4 *The partial derivatives of* x *at* $(0, \bar{\lambda})$ *are*

$$\frac{\partial}{\partial \xi} x(0, \bar{\lambda}) = S(\cdot, 0) P_a(0) \qquad (3.3)$$

$$\frac{\partial}{\partial \lambda} x(0, \bar{\lambda}) = s_{[0,\infty)}\big(\Delta(\cdot)\big) - S(\cdot, 0) P_a(0) \frac{\partial}{\partial \lambda} y(\bar{\lambda})(0) + \frac{\partial}{\partial \lambda} y(\bar{\lambda})(\cdot) \qquad (3.4)$$

with $s_{[0,\infty)}(w)(t) = \int_0^t S(t,s) P_a(s) w(s) \mathrm{d}s - \int_t^\infty S(t,s) P_b(s) w(s) \mathrm{d}s$ *and*

$$\begin{aligned} \Delta(s) := & \left(\frac{\partial}{\partial x} f\big(\bar{x}(s), \bar{\lambda}\big) - \frac{\partial}{\partial x} f\big(\bar{y}(s), \bar{\lambda}\big) \right) \frac{\partial}{\partial \lambda} y(\bar{\lambda})(s) \\ & + \left(\frac{\partial}{\partial \lambda} f\big(\bar{x}(s), \bar{\lambda}\big) - \frac{\partial}{\partial \lambda} f\big(\bar{y}(s), \bar{\lambda}\big) \right). \end{aligned} \qquad (3.5)$$

Notations: For abbreviation let $BC^\kappa_\gamma := BC^\kappa_\gamma(\mathbb{R}_+, \mathbb{R}^m)$, and $X^\kappa_\gamma := BC^\kappa_\gamma \times \mathcal{R}\big(P_a(0)\big)$, $\kappa = 0, 1$.

We denote the Lipschitz constant of some function g by L_g and for the j–th derivative by L^j_g. Analogously we denote the upper bounds by M_g respectively M^j_g.

Proof: The main idea is to apply the implicit function theorem to an appropriate operator F, such that the zeros of F define the γ–asymptotic solutions. To get smoothness results, in particular with respect to the parameter, we have to take some care and need the Lipschitz continuity of the k–th derivative of f.

Any function x which is γ–asymptotic with $y(\lambda)$ can be written as

$$x = \bar{x} + y(\lambda) - \bar{y} + z \text{ for some } z \in BC^1_\gamma(\mathbb{R}_+, \mathbb{R}^m). \qquad (3.6)$$

We find appropriate $z \in BC^1_\gamma(\mathbb{R}_+, \mathbb{R}^m)$ and define neighborhoods V, Ω, U and a function $x(\xi, \lambda)$ by the following steps:

Step: 1 We define suitable neighborhoods and an operator F such that any zero z of F defines a solution of $(A_\gamma(\bar{x}, \bar{\lambda}))$ by (3.6) (see Lemma 3.3).

Step: 2 In Lemma 3.4 we prove smoothness properties of the operator F which allows to apply the implicit function theorem and to get $z(\xi,\lambda)$ with $F(z(\xi,\lambda),\xi,\lambda)=0$.

Step: 3 We use (3.6) and Lemma 3.3 to define sets V, Ω, U and the function $x(\xi,\lambda)$ with the properties of Proposition 3.2. To prove the uniqueness in Proposition 3.2 we apply the contraction mapping theorem.

Step 1: Let (x,ξ,λ) be a solution of $(A_\gamma(\bar{x},\bar{\lambda}))$ and define

$$z = x - y(\lambda) - \bar{x} + \bar{y}. \tag{3.7}$$

Then (z,ξ,λ) is a solution of the operator equation

$$F(z;\xi,\lambda) := \begin{pmatrix} \dot{z} - h(\cdot, z(\cdot), \lambda) \\ P_a(0)\big(z(0) + y(\lambda)(0) - \bar{y}(0)\big) - \xi \end{pmatrix} = 0 \tag{3.8}$$

where $h \in C^k(\mathbb{R}_+ \times \mathbb{R}^{m+p}, \mathbb{R}^m)$ is defined by

$$h(t,v,\lambda) := \\ f\big(v + \bar{x}(t) + y(\lambda)(t) - \bar{y}(t), \lambda\big) - f\big(y(\lambda)(t),\lambda\big) + f\big(\bar{y}(t),\bar{\lambda}\big) - f\big(\bar{x}(t),\bar{\lambda}\big).$$

We relate γ–asymptotic solutions to regular solutions of the operator equation $F=0$ as follows:

Lemma 3.3 *Suppose that the assumptions of Proposition 3.2 hold, let $(\xi,\lambda) \in V \times \Omega$, where $V \times \Omega$ is an open neighborhood of $\big(0,\bar{\lambda}\big)$ and define for $\delta > 0$ the open set $U_\delta := \big\{x \in BC^1(\mathbb{R}_+,\mathbb{R}^m) \big| ||x-\bar{x}||_\infty < \delta\big\}$. Then $(x,\xi,\lambda) \in U_\delta \times V \times \Omega$ solves $(A_\gamma(\bar{x},\bar{\lambda}))$ if and only if $(z,\xi,\lambda) \in W_\delta^{V,\Omega} := \big\{(z,\xi,\lambda) \in BC^1_\gamma \times V \times \Omega \quad \big| \; ||z + y(\lambda) - \bar{y}||_\infty < \delta\big\}$ and (z,ξ,λ) solves the operator equation*

$$F(z;\xi,\lambda) = 0$$

where z is defined as in (3.7).

Proof: "$\Longrightarrow$" Let $(x,\xi,\lambda) \in U_\delta \times V \times \Omega$ be a solution of $(A_\gamma(\bar{x},\bar{\lambda}))$. ¿From (3.7) and $x \in U_\delta$ we see $z \in BC^1(\mathbb{R}_+,\mathbb{R}^m)$ and $||z + y(\lambda) - \bar{y}||_\infty < \delta$. Moreover

$$\begin{aligned} |z(t)|e^{-\gamma t} &\leq |x(t) - y(\lambda)(t)|e^{-\gamma t} + |\bar{x}(t) - \bar{y}(t)|e^{-\gamma t} \leq C_x + C_{\bar{x}} &\quad (3.9)\\ |\dot{z}(t)|e^{-\gamma t} &= |h(t,z(t),\lambda)|e^{-\gamma t} &\quad (3.10)\\ &\leq |f(x(t),\lambda) - f(y(\lambda)(t),\lambda)|e^{-\gamma t} + |f(\bar{x}(t),\bar{\lambda}) - f(\bar{y}(t),\bar{\lambda})|e^{-\gamma t} \\ &\leq L_f|x(t) - y(\lambda)(t)|e^{-\gamma t} + L_f|\bar{x}(t) - \bar{y}(t)|e^{-\gamma t} \leq L_f(C_x + C_{\bar{x}}). \end{aligned}$$

Therefore $z \in BC^1_\gamma(\mathbb{R}_+,\mathbb{R}^m)$ and $(z,\xi,\lambda) \in W_\delta^{V,\Omega} \times V$ follow. Clearly, $F(z;\xi,\lambda)=0$ is clear by definition of z and h.

"$\Longleftarrow$" Let $(z,\xi,\lambda) \in W_\delta^{V,\Omega}$ be a solution of (3.8). Then x defined by (3.6) is in U_δ and solves $\dot{x} = f(x,\lambda)$, $P_a(0)(x(0) - \bar{x}(0)) = \xi$. $\bar{x}$, $\bar{y}$ are γ–asymptotic and $z \in BC^1_\gamma(\mathbb{R}_+, \mathbb{R}^m)$ hence

$$|x(t) - y(\lambda)(t)| = |z(t) + \bar{x}(t) - \bar{y}(t)| \leq |z(t)| + |\bar{x}(t) - \bar{y}(t)| \leq (||z||_\gamma + C_{\bar{x}})e^{\gamma t}$$

and x, $y(\lambda)$ are γ–asymptotic. ■

Step 2: In order to apply the implicit function theorem to F we prove

Lemma 3.4 *The operator F has the following properties:*

1. $F \in C^{k-1}(X^1_\gamma \times \Lambda, X^0_\gamma)$
2. $F(0;0,\bar{\lambda}) = 0$
3. $Q := \frac{\partial}{\partial z}F(0;0,\bar{\lambda}) \in \mathcal{L}\big(BC^1_\gamma, X^0_\gamma\big)$ *is nonsingular with inverse*

$$Q^{-1}(y,\xi) = S(\cdot,0)P_a(0)\xi + s_{[0,\infty)}(y) = S(\cdot,0)P_a(0)\xi + \int_0^\infty G(t,s)y(s)\mathrm{d}s.$$

Proof: 1. Some care is necessary when dealing with differentiability with respect to exponential weighted norms, so we display some of the details.

By **A2** we see that the second component F is in $C^k\big(X^1_\gamma \times \Lambda, \mathcal{R}(P_a(0))\big)$.

The first component of F is independent of ξ, thus we neglect ξ and define $F_1(z,\lambda) = \dot{z} - h(\cdot, z(\cdot), \lambda)$. To see $F_1(z,\lambda) \in BC^0_\gamma$ we notice (similar to (3.9)) that

$$||\dot{z} - h(\cdot, z, \lambda)||_\gamma \leq ||\dot{z}||_\gamma + L_f(||z||_\gamma + 2C_{\bar{x}}).$$

We define a linear operator which will be shown to be the derivative of F_1 at (z,λ) by

$$DF_1(z,\lambda) : \begin{array}{ccc} BC^1_\gamma \times \Lambda & \to & BC^0_\gamma \\ (v,\mu) & \mapsto & \dot{v} - \frac{\partial}{\partial x}f\big(z + \bar{x} + y(\lambda) - \bar{y}, \lambda\big)\, v - \frac{\partial}{\partial \lambda}h\big(\cdot, z, \lambda\big)\, \mu \end{array}$$

To see $DF_1(z,\lambda)(v,\mu) \in BC^0_\gamma$ for any $(v,\mu) \in BC^1_\gamma \times \mathbb{R}^p$ we estimate at $t \geq 0$

$$\begin{array}{c} |\frac{\partial}{\partial \lambda}h(t, z(t), \lambda)\mu|e^{-\gamma t} \leq L^1_f\big(||z||_\gamma + C_{\bar{x}}\big)(1 + M_y)\ |\mu| \\ |\frac{\partial}{\partial x}f\big(z(t) + \bar{x}(t) + y(\lambda)(t) - \bar{y}(t), \lambda\big)\ v(t)|e^{-\gamma t} \leq M^1_f||v||_\gamma \end{array}$$

hence

$$\begin{aligned} ||DF_1(z,\lambda)||_\gamma &\leq \sup_{\substack{(v,\mu)\in BC^1_\gamma\times\Lambda \\ ||(v,\mu)||\leq 1}} ||DF_1(z,\xi,\lambda)[v,\mu]||_\gamma \\ &\leq \sup_{\substack{(v,\mu)\in BC^1_\gamma\times\Lambda \\ ||(v,\mu)||\leq 1}} \big(||\dot{v}||_\gamma + M^1_f||v||_\gamma + L^1_f\big(||z||_\gamma + C_{\bar{x}}\big)(1 + M_y)\ |\mu|\big) \\ &\leq \max\big(1, M^1_f, L^1_f\big(||z||_\gamma + C_{\bar{x}}\big)(1 + M_y)\big). \end{aligned}$$

In fact, $DF_1(z,\lambda)$ is the derivative of F_1 at (z,λ). This is seen by the following estimate where we use $\tilde{x} := z + \bar{x} + y(\lambda) - \bar{y}$ as abbreviation:

$$\begin{aligned}
&|F_1(z+v,\lambda+\mu)(t) - F_1(z,\lambda)(t) - \big(DF_1(z,\lambda)[v,\mu]\big)(t)|e^{-\gamma t} \\
=\ & |F_1(z+v,\lambda+\mu)(t) - F_1(z+v,\lambda)(t) - \frac{\partial}{\partial\lambda}h(t,z(t)+v(t),\lambda)\ \mu \\
& + f\big(v(t)+\tilde{x}(t),\lambda\big) - f\big(\tilde{x}(t),\lambda\big) - \frac{\partial}{\partial x}f\big(\tilde{x}(t),\lambda\big)\ v(t) \\
& + \frac{\partial}{\partial\lambda}h(t,z(t)+v(t),\lambda)\ \mu - \frac{\partial}{\partial\lambda}h(t,z(t),\lambda)\ \mu|e^{-\gamma t} \\
\leq\ & |\mu|^2 L_f^2\big(1+2M_y^1+(M_y^1)^2\big)|\bar{x}(t)-\bar{y}(t)|e^{-\gamma t} \\
& + |v(t)|^2 L_f^1 e^{-\gamma t} + |\mu| L_f^1\big(1+M_y^1\big)|v(t)|e^{-\gamma t} \\
\leq\ & C\big(|\mu| + ||v||_\gamma\big)^2.
\end{aligned}$$

Similarly we can define multi-linear operators $D^jF_1(z,\lambda)$, $j = 2,\dots,k$ and prove that $||D^jF_1(z,\lambda)||_\gamma$ exists and show that $D^jF_1(z,\xi,\lambda)$ is the derivative of $D^{j-1}F_1$ at (z,ξ,λ) if $j = 2,\dots,k-1$. For details see [19].
This implies that the first component of F_1 is in $C^{k-1}(X_\gamma^1 \times \Lambda, BC_\gamma^0)$.

Remark: The main argument for the exponential decay of the derivatives with respect to λ is the Lipschitz continuity of $f^{(k)}$ which implies

$$\left|\frac{\partial^l}{\partial\lambda^l}h(t,z,\lambda)\right| \leq C|z(t)+\bar{x}(t)-\bar{y}(t)| \leq C(||z||_\gamma + C_{\bar{x}})e^{\gamma t}.$$

To differentiate $D^{k-2}F_1$ with respect to λ we use the Lipschitz constant L_f^k of $f^{(k)}$. On the other hand $D^{k-1}F_1$ is once differentiable with respect to z.

2. $F(0;0,\bar{\lambda}) = 0$ is obvious by definition of F.

3. If we differentiate F at $(0,0,\bar{\lambda})$ we get

$$F'(0,0,\bar{\lambda}) = \begin{pmatrix} L & 0 & -\Delta(\cdot) \\ P_a(0)E_0 & -\mathrm{Id}_{\mathcal{R}(P_a(0))} & P_a(0)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0) \end{pmatrix} \tag{3.11}$$

where E_0 is the evaluation operator $E_0x := x(0)$. Note that

$$|\Delta(t)| \leq L_f(1+M_y)C_{\bar{x}}e^{\bar{\gamma}t} \leq L_f(1+M_y)C_{\bar{x}}e^{\gamma t} \tag{3.12}$$

implies $\Delta(\cdot) \in BC_\gamma^0(\mathbb{R}_+,\mathbb{R}^m)$. Thus Theorem 2.2 implies **3.** ■

Choose $\delta > 0$ and a small neighborhood $\tilde{\Omega} \subset \Lambda$ of $\bar{\lambda}$ such that $||y(\lambda) - y(\bar{\lambda})||_\infty < \delta$ for all $\lambda \in \tilde{\Omega}$. Defining an open neighborhood of $0 \in BC_\gamma^1$ by $\tilde{W} = \big\{(z,\xi,\lambda) \in X_\gamma \times \tilde{\Omega}\big|\,||z - y(\lambda) + \bar{y}||_\infty < \delta\big\}$ we can apply the implicit function theorem to F defined on $\tilde{W}$. Thus we get some open neighborhood $V \times \Omega \subset \mathcal{R}(P_a(0)) \times \Omega$ of $(0,\bar{\lambda})$ and a function $z \in C^{k-1}\big(V \times \Omega, BC_\gamma^1(\mathbb{R}_+,\mathbb{R}^m)\big)$ with $z(0,\bar{\lambda}) = 0$ and $\big(z(\xi,\lambda),\xi,\lambda\big) \in \tilde{W}$,

$F(z(\xi,\lambda),\xi,\lambda) = 0$ for all $(\xi,\lambda) \in V\times\Omega$. In particular, $\big(z(\xi,\lambda),\xi,\lambda\big) \in W_\delta^{V,\Omega}$ as defined in Lemma 3.3.

Furthermore the derivatives of z at $(0,\bar\lambda)$ are

$$\begin{aligned}
\frac{\partial}{\partial\xi} z(0,\bar\lambda) = -Q^{-1}\frac{\partial}{\partial\xi}F(0,0,\bar\lambda) &= S(\cdot,0)P_a(0)\\
\frac{\partial}{\partial\lambda} z(0,\bar\lambda) = -Q^{-1}\frac{\partial}{\partial\lambda}F(0,0,\bar\lambda) &= s_{[0,\infty)}(\Delta(\cdot)) - S(\cdot,0)P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0).
\end{aligned}$$

Step 3: Define $U = U_\delta$ as in Lemma 3.3 and set $x(\xi,\lambda) := z(\xi,\lambda) + \bar x + y(\lambda) - \bar y$, then Lemma 3.3 implies that $x \in C^{k-1}(V\times\Omega, U)$ has the properties **P1**–**P4** claimed in Proposition 3.2 except for the uniqueness of **P3**.

For the proof of the uniqueness of **P3** we give the idea and refer for details to [19].

For any solution $(\tilde x,\xi,\lambda)$ of $(A_\gamma(\bar x,\bar\lambda))$ we define $z = \tilde x - x(\xi,\lambda)$ and show that this must be a fixed point of an operator $\mathcal{G}$ defined on $\mathcal{K}_{2\delta}(0) := \big\{z \in BC^0_\gamma \bigm| ||z||_\infty < 2\delta\big\}$ and mapping to BC^0_γ. This operator has a Lipschitz constant of the form $C_1\delta + C_2|\lambda-\bar\lambda|$, thus for sufficiently small δ and Ω it is a contraction and has at most one fixed point in this ball. In particular, $z = \tilde x - x(\xi,\lambda)$ and 0 are fixed points in $\mathcal{K}_{2\delta}(0)$, thus we conclude $\tilde x = x(\xi,\lambda)$. ■

In the later approximation part we only use the evaluation of γ–asymptotic solutions at 0. Hence, we define

$$b(\xi,\lambda) := \big(x(\xi,\lambda)(0),\lambda\big), \quad b \in C^{k-1}(V\times\Omega,\mathbb{R}^{m+p}) \tag{3.13}$$

and its image

$$W^0 := \big\{b(\xi,\lambda) \bigm| (\xi,\lambda)\in V\times\Omega\big\} \subset \mathcal{B}_\delta(\bar x(0))\times\Omega \subset \mathbb{R}^{m+p}, \tag{3.14}$$

where $\delta > 0$ is as in Proposition 3.2. The derivative $B(\xi,\lambda)$ of b is

$$B(\xi,\lambda): \quad \begin{array}{ccc} \mathcal{R}\big(P_a(0)\big)\times\mathbb{R}^p & \to & \mathbb{R}^{m+p} \\ (\eta,\mu) & \mapsto & \begin{pmatrix} \frac{\partial}{\partial\xi}x(\xi,\lambda)(0)\eta + \frac{\partial}{\partial\lambda}x(\xi,\lambda)(0)\mu \\ \mu \end{pmatrix}. \end{array} \tag{3.15}$$

This implies for $(x,\lambda) = \big(x(\xi,\lambda)(0),\lambda\big) \in W^0$ that the tangent space at (x,λ) is

$$T_{(x,\lambda)}W^0 = \mathcal{R}\big(B(\xi,\lambda)\big). \tag{3.16}$$

In particular, with $B := B(0,\bar\lambda)$ we obtain

$$B(\eta,\mu) = \begin{pmatrix} \eta + \frac{\partial}{\partial\lambda}x(0,\bar\lambda)(0)\mu \\ \mu \end{pmatrix}. \tag{3.17}$$

Hence b is immersive at $(0,\bar\lambda)$ and there exists an open neighborhood $\tilde V\times\tilde\Omega \subset V\times\Omega$ of $(0,\bar\lambda)$ such that $\operatorname{im}(b|_{\tilde V\times\tilde\Omega})$ is a C^{k-1} submanifold of $\mathbb{R}^{m+p}$ (see [16]).

The results of Proposition 3.2 also hold on this (perhaps smaller) neighborhood, thus with $V\times\Omega := \tilde V\times\tilde\Omega$ and $x := x|_{\tilde V\times\tilde\Omega}$ we see that $(U\times V\times\Omega, x)$ satisfy **P1**–**P4** and

P5 $W^0 = \mathrm{im}(b|_{V\times\Omega})$ is a C^{k-1} submanifold of $\mathbb{R}^{m+p}$.

This leads to the definition of a "γ–parametrization" and a "local γ–stable manifold" of $\bar{x}$ For the definition of a "global γ–stable manifold" or a "manifold along a solution" see [19].

DEFINITION 3.5 *Suppose that the assumptions of Proposition 3.2 hold, then we call a pair* $(U \times V \times \Omega, x)$ *with* **P1**–**P5** *a* **local γ–parametrization of $(\bar{x}, \bar{\lambda})$ at 0** *and*

$$\begin{aligned} \mathcal{M}^0_\gamma(\bar{x},\bar{\lambda}) \quad := \quad & \big\{(z,\lambda) \in \mathbb{R}^{m+p} \;\big|\; z \text{ and } y(\lambda)(0) \text{ are } \gamma\text{–asymptotic at } \lambda \in \Omega, \\ & \qquad P_a(0)\big(z - \bar{x}(0)\big) \in V \text{ and } u(\cdot,0,z,\lambda) \in U\big\}. \end{aligned}$$

is called a **local γ–stable manifold of $(\bar{x},\bar{\lambda})$ evaluated at 0**.

Proposition 3.2 yields the existence of a local γ–stable manifold for the $y(\lambda)$ as well as for any x converging to it.

COROLLARY 3.6 *Suppose that* **A1**–**A3** *hold and let* $\gamma \in (\bar{\alpha}, \bar{\beta})$. *Then there exists a local γ–stable manifold of* $\big(y(\lambda), \lambda\big)$ *evaluated at* 0 *for all* $\lambda \in \Lambda$.

For any $(\tilde{x}, \tilde{\lambda}) \in BC^1(\mathbb{R}_+, \mathbb{R}^m) \times \Lambda$, *where* $\tilde{x}$ *and* $y(\tilde{\lambda})$ *are γ–asymptotic, there exists a local γ–stable manifold of* $\big(\tilde{x}, \tilde{\lambda}\big)$ *evaluated at* 0.

Proof: Each $y(\lambda)$ is $\bar{\alpha}$–asymptotic with itself and **A3** implies that we can apply Proposition 3.2 with $\bar{x} = y(\lambda)$ and $\bar{\lambda} = \lambda$ to get a local γ–stable manifold of $\big(y(\lambda), \lambda\big)$.

For the general case Proposition 2.5 implies that $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(\tilde{x}(t), \tilde{\lambda})$ has a shifted exponential dichotomy on $[0, \infty)$ with exponents $\alpha < \gamma < \beta$, α, β near $\bar{\alpha}$, $\bar{\beta}$, a constant K and projectors P_a, P_b. Hence Proposition 3.2 yields the existence of a local γ–stable manifold. ■

3.2 Nondegenerate connecting orbits with asymptotic rate

In this section we define "**connecting orbits with asymptotic rate γ**". These are solutions of the dynamical system which are γ–asymptotic with some $y(\lambda)$ and with initial values in a given manifold $\mathcal{W}$.

We define nondegeneracy of a connecting orbit with asymptotic rate and relate this to the non-singularity of a linear map and the regularity of an operator equation.

In addition to **A1**–**A3** we assume that

A4 (dimension condition) $\mathcal{W}$ is an m_b–dimensional manifold in $\mathbb{R}^{m+p}$ with representation $\mathcal{W} = \{(x,\mu) \in \mathbb{R}^{m+p} | g(x,\mu) = 0\}$ where $g \in C^{k-1}(\mathbb{R}^{m+p}, \mathbb{R}^{m_a+p})$ and $g'(x,\mu)$ has full rank for all $(x,\mu) \in \mathcal{W}$.

A5 (transversality condition) $T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} + T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0 = \mathbb{R}^{m+p}$.

Assumption **A4** implies by $m+p = (m_a+p)+m_b$ that **A5** is equivalent to

$$T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} \cap T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0 = \{0\}. \tag{3.18}$$

Remark: The arbitrary choice of parameter space (in particular its dimension p) is hidden in the construction of the m_b–dimensional manifold $\mathcal{W}$.

If **A5** holds and $\mathcal{W}$ is (m_b+q)–dimensional then we add q extra conditions $h(x,\mu)=0$ to $g(x,\mu)=0$ such that the new manifold is m_b–dimensional and the assumptions **A4** and **A5** hold.

To approximate the set of connecting orbits with asymptotic rate in this case we may fix q initial values by using extra conditions $x_{i_j}(0)-\bar{x}_{i_j}$, $j=1,\dots,q$ (such that **A4** and **A5** hold). Then we use the $\bar{x}_{i_j}$ as continuation parameters.

If $\mathcal{W}$ is only (m_b-q)–dimensional and (3.18) holds then we may use q additional parameters $\mu\in\mathbb{R}^q$ which have no influence on $\mathcal{W}$ such that $\mathcal{W}\times\mathbb{R}^q$ is an m_b–dimensional manifold and the assumptions**A4** and **A5** hold.

In the periodic case typically the appropriate phase of the periodic orbit $y(\lambda)$ is unknown and the dimension of $\mathcal{W}$ is m_b-1. Now we introduce the phase τ as additional parameter and define $y(\lambda,\tau) := y(\lambda)(\cdot+\tau)$. This method presumes that $f(\bar{x}(0),\bar{\lambda}) \notin T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W}$.

DEFINITION 3.7 (connecting orbit with asymptotic rate) *Let* **A1**–**A4** *hold.*

We call $(\bar{x},\bar{\lambda})$ *an* **connecting orbit with asymptotic rate γ from $\mathcal{W}$ to y** *if* $\bar{x}$ *and* $y(\bar{\lambda})$ *are* γ*–asymptotic at* $\bar{\lambda}$ *and* $(\bar{x}(0),\lambda)\in\mathcal{W}$.

If in addition the transversality condition $T_{(\bar{x}(0),\bar{\lambda})}\mathcal{W} \cap T_{(\bar{x}(0),\bar{\lambda})}\mathcal{M}_\gamma^0 = \{0\}$ *holds then it is called a* **nondegenerate connecting orbit with asymptotic rate γ from $\mathcal{W}$ to y**.

If we define for $(x,\lambda)\in\mathcal{W}$

$$D(x,\lambda) := \left(\frac{\partial g(x,\lambda)}{\partial x}, \frac{\partial g(x,\lambda)}{\partial \lambda}\right) \in \mathbb{R}^{m_a+p,m+p} \tag{3.19}$$

then $T_{(x,\lambda)}\mathcal{W} = \mathcal{N}\big(D(x,\lambda)\big)$ follows from the definition of $\mathcal{W}$.

This and (3.16) implies $T_{(x(0),\lambda)}\mathcal{W}\cap T_{(x(0),\lambda)}\mathcal{M}_\gamma^0 = \mathcal{N}\big(D(x(0),\lambda)\big)\cap\mathcal{R}\big(B(\xi,\lambda)\big)$ for $(x,\lambda) := (x(\xi,\lambda),\lambda)\in\mathcal{W}\cap W^0$ and hence we obtain the following lemma:

LEMMA 3.8 *Let* **A1**–**A4** *hold, let* $\bar{x}$ *be* γ*–asymptotic with* $y(\bar{\lambda})$ *at* $\bar{\lambda}$ *and let* $(x,\lambda) := (x(\xi,\lambda),\lambda)\in W$ *be a connecting orbit with asymptotic rate* γ *from* $\mathcal{W}$ *to* y*. Then* (x,λ) *is nondegenerate if and only if* $D\big(x(0),\lambda\big)\circ B(\xi,\lambda)$ *is nonsingular.*

Proof: Let (x,λ) be a nondegenerate connecting orbit with asymptotic rate γ from $\mathcal{W}$ to y and let (η,μ) be a solution of $D(x(0),\lambda)\circ B(\xi,\lambda)(\eta,\mu)=0$. Then we see $B(\xi,\lambda)(\eta,\mu)\in \mathcal{N}\big(D(x(0),\lambda)\big)\cap\mathcal{R}\big(B(\xi,\lambda)\big)$ hence $(\eta,\mu)=0$ follows from definition of B (see (3.17)).
Let $D\big(x(0),\lambda\big)\circ B(\xi,\lambda)$ be nonsingular and let $(y,\mu)\in\mathcal{N}\big(D(x(0),\lambda)\big)\cap\mathcal{R}\big(B(\xi,\lambda)\big)$. Then there exists some $(\eta,\mu)\in\mathcal{R}\big(P_a(0)\big)\times\mathbb{R}^p$ such that $(y,\mu)=B(\xi,\lambda)(\eta,\mu)$ and $0=D\big(x(0),\lambda\big)(y,\mu)=D\big(x(0),\lambda\big)\circ B(\xi,\lambda)(\eta,\mu)$ hence $(y,\mu)=0$ and (x,λ) is a nondegenerate connecting orbit with asymptotic rate γ from $\mathcal{W}$ to y. ∎

For a connecting orbit with asymptotic rate $(\bar{x},\bar{\lambda})$ we define $D:=D(\bar{x}(0),\bar{\lambda})$. Similar to Lemma 3.3 we relate connecting orbits with asymptotic rate (x,λ) to solutions of an operator equation and show that regular solutions of this operator correspond to nondegenerate connecting orbits with asymptotic rate.

Theorem 3.9 (Nondegenerate connecting orbit with asymptotic rate) *Suppose that* **A1**–**A4** *hold and let* $(\bar{x},\bar{\lambda})$ *be an connecting orbit with asymptotic rate* γ *from* $\mathcal{W}$ *to* y. *Then the following conditions are equivalent:*

1. $(\bar{x},\bar{\lambda})$ *is a nondegenerate connecting orbit with asymptotic rate* γ *from* $\mathcal{W}$ *to* y.
2. $D\circ B\in\mathcal{L}(\mathcal{R}(P_a(0))\times\mathbb{R}^p,\mathbb{R}^{m_a+p})$, *is nonsingular*
3. $(0,\bar{\lambda})$ *is a regular solution of the operator equation*

$$G_\infty(z,\lambda):=\begin{pmatrix}\dot{z}-h(\cdot,z,\lambda)\\ g(z(0)+\bar{x}(0)+y(\lambda)(0)-\bar{y}(0),\lambda)\end{pmatrix}=0, \tag{3.20}$$

$G_\infty: BC^1_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\Lambda\to BC^0_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^{m_a+p}$.

Proof: "1. $\Longleftrightarrow$ 2." This follows from Lemma 3.8.

"2. $\Rightarrow$ 3." By regular we mean that the operator G_∞ has a total derivative $G'_\infty(0,\bar{\lambda})$ at $(0,\bar{\lambda})$ which is a homeomorphism from $BC^1_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^p$ to $BC^0_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^{m_a+p}$.
As in Lemma 3.3 we see that $G_\infty\in C^{k-1}(BC^1_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\Lambda, BC^0_\gamma(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^{m_a+p}$ and

$$\begin{aligned} G'_\infty(0,\bar{\lambda}) &= \begin{pmatrix} L & -\Delta(\cdot)\\ \frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})E_0 & \frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)+\frac{\partial}{\partial\lambda}g(\bar{x}(0)\bar{\lambda})\end{pmatrix}\\ &= \begin{pmatrix} L & -\Delta(\cdot)\\ R_0E_0 & Q_0+R_0\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)\end{pmatrix}\end{aligned}$$

where $(R_0,Q_0):=D=\big(\frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})\ \frac{\partial}{\partial\lambda}g(\bar{x}(0)\bar{\lambda})\big)$ and L is Fredholm of index m_a. Furthermore Δ, defined in (3.5), is in $\mathcal{L}(\mathbb{R}^p,BC^0_\gamma(\mathbb{R}_+,\mathbb{R}^m))$ and $\frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})E_0\in\mathcal{L}(BC^1_\gamma(\mathbb{R}_+,\mathbb{R}^m),\mathbb{R}^{m_a+p})$ and $\frac{\partial}{\partial x}g(\bar{x}(0),\bar{\lambda})\frac{\partial}{\partial\lambda}y(\bar{\lambda})(0)+\frac{\partial}{\partial\lambda}g(\bar{x}(0)\bar{\lambda})\in\mathcal{L}(\mathbb{R}^p,\mathbb{R}^{m_a+p})$. By the bordering lemma [4] we see that $G'_\infty(0,\bar{\lambda})$ is Fredholm of index 0.

Hence it is sufficient to show that $G'_\infty(0,\bar\lambda)(y,\mu)=0$ has only the trivial solution.

The equations $Ly=\Delta(\cdot)\mu$ and $\mathcal{N}(L)=\{S(\cdot,0)\xi \,|\, \xi\in\mathcal{R}(P_a(0))\}$ imply by Theorem 2.2 that $y=s_{[0,\infty)}(\Delta(\cdot)\mu)+S(\cdot,0)\xi$ for some $\xi\in\mathcal{R}(P_a(0))$. Thus

$$\begin{aligned}
0 &= R_0E_0y+Q_0\mu+R_0\frac{\partial}{\partial\lambda}y(\bar\lambda)(0)\mu\\
&= Q_0\mu+R_0\left(\xi+\frac{\partial}{\partial\lambda}x(0,\bar\lambda)(0)\mu+P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0)\mu\right)\\
&= (R_0\quad Q_0)\begin{pmatrix}\xi+\frac{\partial}{\partial\lambda}x(0,\bar\lambda)(0)\mu+P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0)\mu\\ \mu\end{pmatrix}\\
&= (D\circ B)\left(\xi+P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0)\mu,\mu\right).
\end{aligned}$$

The non-singularity of $D\circ B$ implies $(\xi+P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0)\mu,\mu)=0$ hence $\mu=0$, $\xi=0$ and $y=s_{[0,\infty)}(\Delta(\cdot)\mu)+S(\cdot,0)\xi=0$.

"3. $\Rightarrow$ 2." Let $(\xi,\mu)\in\mathcal{R}(P_a(0))\times\mathbb{R}^p$ be a solution of $(D\circ B)(\xi,\mu)=0$. Define the function $y:=s_{[0,\infty)}(\Delta(\cdot)\mu)+S(\cdot,0)(\xi-P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0)\mu)$, then (y,μ) satisfies $Ly-\Delta(\cdot)\mu=0$ and

$$\begin{aligned}
0 &= (D\circ B)(\xi,\mu)\\
&= Q_0\mu+R_0\left(\xi+\frac{\partial}{\partial\lambda}x(0,\bar\lambda)(0)\mu\right)\\
&= Q_0\mu+R_0\left(s_{[0,\infty)}(\Delta(\cdot)\mu)(0)+S(0,0)\xi+P_b(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0)\mu\right)\\
&= R_0E_0y+\left(Q_0+R_0\frac{\partial}{\partial\lambda}y(\bar\lambda)(0)\right)\mu.
\end{aligned}\tag{3.21}$$

Thus (y,μ) solves $G'_\infty(0,\bar\lambda)(y,\mu)=0$ hence by regularity $(y,\mu)=0$ and $\xi=y(0)=0$.

■

4 Numerical approximation of connecting orbits

In this section we approximate nondegenerate connecting orbits with asymptotic rate $(\bar x,\bar\lambda)$, i. e.

$$(\bar x(0),\bar\lambda)\in\mathcal{W},\qquad (\bar x(T_+),\bar\lambda)\in\mathcal{M}_\gamma^{T_+}\ \text{ for some } T_+\in\mathbb{R}_+.$$

where $\mathcal{M}_\gamma^{T_+}$ is the γ–stable manifold of $y(\lambda)(T_++\cdot)$ evaluated at 0. Thus we approximate $(\bar x,\bar\lambda)$ by some (x_J,λ_J) where x_J is a solution of $\dot x=f(x,\lambda_J)$ on $J=[0,T_+]$ with initial value $(x_J(0),\lambda_J)\in\mathcal{W}$ and an asymptotic boundary condition at T_+, i. e. we assume that $(\bar x(T_+),\bar\lambda)$ is in the tangent space of $\mathcal{M}_\gamma^{T_+}$ at $(\hat y_J,\lambda_J):=(y(\lambda_J)(T_+),\lambda_J)$.

$$\begin{aligned} \dot{x}_J(t) - f(x_J(t), \lambda_J) &= 0, \quad t \in J, & (4.1)\\ g(x_J(0), \lambda_J) &= 0, & (4.2)\\ (x_J(T_+), \lambda_J) &\in \big(y(\lambda_J)(T_+), \lambda_J\big) + T_{(\hat{y}_J, \lambda_J)} \mathcal{M}_\gamma^{T_+}. & (4.3) \end{aligned}$$

Evaluating $\mathcal{M}_\gamma^{T_+}$ at $\big(y(\lambda_J)(T_+ + \cdot), \lambda_J\big)$ yields

$$T_{(\hat{y}_J,\lambda_J)} \mathcal{M}_\gamma^{T_+} = \Big\{ \big(\eta + \bar{P}_b(\lambda_J)(T_+) \frac{\partial}{\partial \lambda} y(\lambda_J)(T_+)\mu, \mu\big) \Big| (\eta, \mu) \in \mathcal{R}\big(\bar{P}_a(\lambda_J)(T_+)\big) \times \mathbb{R}^p \Big\}$$

as denoted in [19]. Thus the asymptotic boundary condition (4.3) is equivalent to

$$\big(x_J(T_+) - y(\lambda_J)(T_+)\big) \in \mathcal{R}\big(\bar{P}_a(\lambda_J)(T_+)\big) = \mathcal{N}\big(\bar{P}_b(\lambda_J)(T_+)\big). \tag{4.4}$$

$\mathcal{N}\big(\bar{P}_b(\lambda_J)(T_+)\big)$ is m_a–dimensional, therefore (4.3) is equivalent to

$$\bar{M}(T_+, \lambda_J)\big(x_J(T_+) - y(\lambda_J)(T_+)\big) = 0, \tag{4.5}$$

where $\bar{M}(T_+, \lambda) \in \mathbb{R}^{m_b, m}$ is a matrix with $\mathcal{N}\big(\bar{M}(T_+, \lambda)\big) = \mathcal{N}\big(\bar{P}_b(\lambda)(T_+)\big)$ for each λ. In the equilibrium case and the periodic case it can be proved that for $k \geq 3$ and $\bar{T}_+$ sufficiently large there exists some constant $C_{\bar{M}} > 0$ such that for all $T_+ \geq \bar{T}_+$

$$|\bar{M}(T_+, \lambda)\big(x(T_+) - y(\lambda)(T_+)\big)| \leq C_{\bar{M}} |x(T_+) - y(\lambda)(T_+)|^2 \tag{4.6}$$

holds for each $(x, \lambda) \in \mathcal{M}_\gamma$ (see [19]).

In general (e. g. the periodic case) it is expensive to compute such $\bar{M}(T_+, \lambda)$ depending on λ, thus for the approximation result in 4.2 we only assume $\hat{M}(T_+, \lambda) \in \mathbb{R}^{m_b, m}$ to be a matrix with $\mathcal{N}\big(\hat{M}(T_+, \lambda)\big) \cap \mathcal{R}\big(\bar{P}_b(\lambda)(T_+)\big) = \big\{0\big\}$ (see **A6** on page 22). In particular, we may choose $\hat{M}(T_+, \lambda_J)$ constant in λ.

We apply our theory to the cases where the $y(\lambda)$ are either equilibria or periodic orbits. In 4.1 we show how to compute $y(\lambda)$ and $\bar{M}(T_+, \lambda_J)$ in these cases.

In 4.2 set up and analyze an approximation method for connecting orbits with asymptotic rate and in 4.3 we develop a boundary corrector method which allows to use constant $\hat{M}(T_+, \lambda_J)$ and get estimates as in the case with $\bar{M}(T_+, \lambda_J)$.

4.1 Equilibria and periodic solutions

For the numerical applications we restrict ourself to the cases where the $y(\lambda)$ are equilibria or periodic orbits. In general it is difficult compute $y(\lambda)$ and to ensure that **A2** and **A3** holds. We always assume $f \in C^k(\mathbb{R}^{m+p}, \mathbb{R}^m)$ like in **A1**.

Equilibria

In this case we compute the intersection of $\mathcal{W}$ with the (strong) stable manifold of an equilibrium.

Consider a C^k-branch $y(\lambda)$ of equilibria on Λ, i. e. $f\big(y(\lambda),\lambda\big)=0$ for all $\lambda\in\Lambda$. Choose $\alpha<0$, $\beta>\alpha$ such that $\frac{\partial}{\partial x}f\big(y(\lambda),\lambda\big)$ has $m_a\geq 1$ eigenvalues μ with $\mathrm{Re}(\mu)<\alpha<0$ and $m_b=m-m_a\geq 1$ eigenvalues μ with $\mathrm{Re}(\mu)>\beta$ for each $\lambda\in\Lambda$.

Then $L:=\frac{\mathrm{d}}{\mathrm{dt}}-\frac{\partial}{\partial x}f\big(y(\lambda),\lambda\big)$ has a shifted exponential dichotomy with exponents $\alpha<\beta$, $\alpha<0$ and **A1**–**A3** hold.

Asymptotic boundary conditions for equilibria

To get a smooth matrix valued function $\bar{M}(\lambda)$ (independent of T_+) in a neighborhood of $\hat{\lambda}$ we choose some $\gamma\in(\alpha,\beta)$ and solve

$$V\frac{\partial}{\partial x}f(y(\lambda),\lambda) \;=\; U(\lambda)V,\quad \mathrm{Re}\big(\sigma(U)\big)>\gamma,\quad U\in\mathbb{R}^{m_b,m_b},\ V\in\mathbb{R}^{m_b,m} \tag{4.7}$$

$$\hat{V}^\top V \;=\; \hat{V}^\top\hat{V} \tag{4.8}$$

where $\hat{V}$, $\hat{U}$ solves (4.7) at $\hat{\lambda}$. Using this and (4.1), (4.2) and (4.5) we solve

$$G_J^e(z,\nu,y,V,U)=\begin{pmatrix} f(y,\nu)\\ \dot{z}-f(z,\nu)\\ g\big(z(0),\nu\big)\\ V\big(z(T_+)-y\big)\\ V\frac{\partial}{\partial x}f(y,\lambda)-UV\\ \hat{V}^\top V-\hat{V}^\top\hat{V}\end{pmatrix}=0. \tag{4.9}$$

As alternative of adding (4.7) and (4.8) to the boundary value problem we may determine $\bar{M}(\lambda)$ at each evaluation of (4.9) by an eigenvalue solver combined with a normalization procedure ([5],[8]) applied to $\frac{\partial}{\partial x}f(y_0(\lambda),\lambda)-\gamma\mathrm{Id}_{\mathbb{R}^m}$ which is hyperbolic.

Periodic Solutions

In the periodic case let $(\hat{y}(\cdot),\bar{\lambda})$ be a $\bar{T}$ periodic orbit of $\dot{x}=\hat{f}(x,\bar{\lambda})$ where 1 is a simple Floquet multiplier.

Furthermore define $\bar{y}(t):=\hat{y}(t\cdot\bar{T})$ and a phase fixing function $\chi\in C^k(BC^1(\mathbb{R}_+,\mathbb{R}^m)\times\Lambda,\mathbb{R})$ satisfying

$$\chi(\bar{y}(\cdot),\bar{\lambda})=0,\quad \frac{\partial}{\partial x}\chi(\bar{y},\bar{\lambda})\bar{y}'\neq 0. \tag{4.10}$$

Then $(\bar{y}(\cdot),\bar{T},\bar{\lambda})$ is a solution of

$$F_p(y,T;\lambda)=\begin{pmatrix}\dot{y}-T\hat{f}(y,\lambda)\\ \chi(y,\lambda)\\ y(0)-y(1)\end{pmatrix}=0, \tag{4.11}$$

where $F_p\in C^k\big(BC^1(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}_+\setminus\{0\}\times\mathbb{R}^p, BC^0(\mathbb{R}_+,\mathbb{R}^m)\times\mathbb{R}^{m+1}\big)$.

The implicit function theorem applied to (4.11) yields the existence of some neighborhood $U \times \Xi \times \Lambda \subset BC^1(\mathbb{R}_+, \mathbb{R}^m) \times \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}^p$ of $(\bar{y}, \bar{T}, \bar{\lambda})$ and a function $(y, T) \in C^k(\Lambda; U, \Xi)$ with $\big(y(\bar{\lambda}), T(\bar{\lambda})\big) = (\bar{y}, \bar{T})$ (see Jepson and Keller [17]).

Therefore, $y(\lambda)$ is a 1–periodic solution of

$$\dot{y} - f(y, \lambda), \qquad T(\lambda)\hat{f}(y, \lambda) =: f(y, \lambda). \tag{4.12}$$

Thus **A1**–**A2** hold.

Furthermore, assume that all $y(\lambda)$ have $m_a \geq 1$ Floquet multipliers μ with $|\mu| < e^{\bar{\alpha}}$, $\bar{\alpha} < 0$ and $m_b = m - m_a \geq 1$ Floquet multipliers with $|\mu| \geq e^{\bar{\beta}}$, $\bar{\alpha} < \bar{\beta} \leq 0$. Then each $L := \frac{\mathrm{d}}{\mathrm{dt}} - \frac{\partial}{\partial x} f(y(\lambda)(\cdot), \lambda)$ has a shifted exponential dichotomy and **A3** holds.

Remark: If the periodic orbit $(\bar{y}(\cdot), \bar{\lambda})$ is hyperbolic then L has an exponential trichotomy ([6]) with center exponent 0. In particular, L has shifted exponential dichotomy with $\alpha < 0 = \beta$.

Asymptotic boundary conditions for periodic orbits

We choose $\hat{M}(T_+, \lambda)$ 1–periodic in T_+ on $\mathcal{D} = \cup_{N \in \mathbb{N}} [N - \tau_-, N + \tau_+]$ for some $\tau_-, \tau_+ \in [0, 1]$, such that $\mathcal{N}\big(\hat{M}(T_+, \bar{\lambda})\big) \cap \mathcal{R}\big(\bar{P}_b(T_+)\big) = \{0\}$ holds on the interval $[\tau_-, \tau_+]$.

A good choice for $\hat{M}(T_+, \lambda) \in \mathbb{R}^{m_b, m}$ is $\hat{M}(T_+, \lambda) = V(0, \lambda)$ where V is a solution of the adjoint variational equation

$$\begin{aligned} \dot{V} &= -V \frac{\partial}{\partial x} f(y(\lambda)(T_+ + \cdot), \lambda), && (4.13) \\ V(1, \lambda) &= U(\lambda) V(0, \lambda) && (4.14) \end{aligned}$$

for some $U(\lambda) \in \mathbb{R}^{m_b, m_b}$ with $|\mu| \leq e^{-\gamma}$ for all eigenvalues μ of $U(\lambda)$ (see [6],[19]).

Adding this equations would blow up the dimensions of the boundary value problem from $2m$ to $(2 + m_b)m$. Computing it smoothly like in the equilibrium case would imply that in each evaluation step we have to solve an $(m_b m)$–dimensional boundary value problem.

To avoid this we develop the **boundary corrector method** (for detail see 4.3),

1. Compute $\hat{M}(T_+, \mu_0)$ as solution of (4.13), (4.14) at μ_0,

2. Calculate $\big(x_J, \mu_J, y(\mu_J), T(\mu_J)\big)$, the solution

$$G_J^p(z, \nu, y, T) = \begin{pmatrix} \dot{y} - T\hat{f}(y, \nu) \\ \chi(y, \nu) \\ y(0) - y(1) \\ \dot{z} - T\hat{f}(z, \nu) \\ g\big(z(0), \nu\big) \\ \hat{M}(T_+, \mu_0)\big(z(T_+) - y(T_+)\big) \end{pmatrix} = 0, \tag{4.15}$$

3. Repeat with **1.** once if $\beta = 0$ or twice if $\beta < 0$ with $\mu_0 = \mu_J$.

We apply the approximation result to the scaled system. If $(\hat{x}, \bar{\lambda})$ is an connecting orbit with asymptotic rate of the unscaled system. Then $(\bar{x}, \bar{\lambda})$, $\bar{x}(t) := \hat{x}(t \cdot \bar{T})$ is a connecting orbit with asymptotic rate of the scaled system. Furthermore let (x_J, λ_J) be a solution of (4.15), i. e. (x_J, λ_J) is an approximation of $(\bar{x}, \bar{\lambda})$ in the scaled system. Then $(\hat{x}_J, \lambda_J)$, $\hat{x}_J(t) := x_J(\frac{t}{T(\lambda_J)})$ is an approximation of $(\hat{x}, \bar{\lambda})$ in the unscaled system and we can estimate

$$||\hat{x}_J - \hat{x}||_\infty \leq ||x_J - \bar{x}||_\infty + C\frac{T'(\bar{\lambda})}{T(\bar{\lambda})} \cdot T_+ \cdot |\lambda_J - \bar{\lambda}| \quad \text{where } C \approx M_f \tag{4.16}$$

(see [19]). Thus the error estimates in the scaled system carried over to the unscaled system except for the factor T_+.

4.2 Approximation theorem

In this section we assume that $y(\lambda)$ is known and that **A1**–**A4** hold. Furthermore, let $(\bar{x}, \bar{\lambda})$ be a nondegenerate connecting orbit with asymptotic rate γ from $\mathcal{W}$ to y, where $\bar{\alpha} < \gamma < \min(0, 2\bar{\beta})$ (see **A3**). Proposition 2.5 implies that $L := \frac{\mathrm{d}}{\mathrm{dt}} - Df_x(\bar{x}, \bar{\lambda})$ has a shifted exponential dichotomy with projectors P_a and P_b and exponents α, β, $(\alpha < \gamma < 2\beta)$ near $\bar{\alpha}$, $\bar{\beta}$.

To set up an appropriate approximation operator (see (4.19)), the boundary condition at T_+ has to be regular in the following sense:

A6 (Regular boundary condition) Let $\hat{M}(T_+, \cdot) \in C^1\big(\Lambda, \mathbb{R}^{m_b, m}\big)$ be a matrix valued function with $\mathcal{N}\big(\hat{M}(T_+, \bar{\lambda})\big) \cap \mathcal{R}\big(\bar{P}_b(T_+)\big) = \{0\}$ for all $T_+ \in \mathcal{D}$ where $\mathcal{D} \subset \mathbb{R}_+$ is a set and there exists a sequence $\{T_i\}_{i \in \mathbb{N}} \subset \mathcal{D}$ with $\lim_{i \to \infty} T_i = \infty$. Furthermore let $\hat{M}(T_+, \lambda)$, $\frac{\partial}{\partial \lambda}\hat{M}(T_+, \lambda)$ be uniformly bounded by some $M_{\hat{M}}$, $M^1_{\hat{M}}$ in $T_+ \in \mathcal{D}$ and $\lambda \in \Lambda$ and Lipschitz continuous w. r. t. $\lambda \in \Lambda$ with constants $L_{\hat{M}}$, $L^1_{\hat{M}}$. Let $(\hat{M}(T_+, \bar{\lambda})|_{\mathcal{R}(\bar{P}_b(T_+))})^{-1}$ be uniformly bounded by $\hat{M}^{-1} > 0$ for all $T_+ \in \mathcal{D}$.

Remark: Uniformity of upper bounds and Lipschitz constants M_M, M^1_M, L_M, L^1_M and $\bar{N}$ in $T_+ \in \mathbb{R}_+$ might be too strong an assumption. Thus we define $\mathcal{D}$ as in **A6** and only require uniformity on $\mathcal{D}$. In particular, in the case where the $y(\lambda)$ are 1–periodic we choose $\mathcal{D} = \{n + [-a, b] \mid n \in \mathbb{N}\}, a, b \in [0, 1]$ and in the case where the $y(\lambda)$ are equilibria we choose $\mathcal{D} = \mathbb{R}_+$.

$\mathcal{N}\big(\hat{M}(T_+, \bar{\lambda})\big) \cap \mathcal{R}\big(\bar{P}_b(T_+)\big) = \{0\}$ implies the non-singularity of $\hat{M}(T_+, \bar{\lambda})|_{\mathcal{R}(\bar{P}_b(T_+))}$. Furthermore $\hat{M}(T_+, \bar{\lambda})|_{\mathcal{R}(P_b(T_+))}$ is a homeomorphism and its inverse is uniformly bounded by some $M^{-1} := 4K\hat{M}^{-1} > 0$ for sufficiently large $T_+ \in \mathcal{D}$ (see [19]).

To get approximation results even for $\beta < 0$, we have to choose functions spaces where the solution properties will be anticipated. Thus, the main trick is to use, even on finite intervals, exponentially weighted norms. In addition the integral in (4.18) avoids to estimate (4.18) by $(1 + T_+)||v||_\beta$.

For some $T_+ \in \mathcal{D}$, $J := [0, T_+]$ we define Banach spaces $\mathcal{Y} = C^1(J, \mathbb{R}^m) \times \mathbb{R}^p$ with

$$||(x,\lambda)||_{\mathcal{Y}} = ||x||^*_{\mathcal{Y}} + |\lambda|, \quad ||x||^*_{\mathcal{Y}} = \sup_{t\in J}\big(|x(t)|q(t)\big), \quad q(t) = \begin{cases} e^{-\beta t} & : \beta \geq 0 \\ 1 & : \beta < 0 \end{cases} \tag{4.17}$$

and $\mathcal{Z} = C^0(J, \mathbb{R}^m) \times \mathbb{R}^{m_a+p} \times \mathbb{R}^{m_b}$ with

$$\begin{aligned} ||(v, r_0, r_+)||_{\mathcal{Z}} &= ||v||^*_{\mathcal{Z}} + |r_0| + e^{-\beta T_+}|r_+|, \\ ||v||^*_{\mathcal{Z}} &= ||v||_\beta + \int_0^{T_+} |v(s)|e^{-\beta s}\mathrm{ds}, \quad ||v||_\beta = \sup(|v(t)|e^{-\beta t}) \end{aligned} \tag{4.18}$$

Theorem 4.1 *Suppose that* **A1**–**A6** *and let $(\bar{x}, \bar{\lambda})$ be a nondegenerate connecting orbit with asymptotic rate γ from $\mathcal{W}$ to y where $L := \frac{\mathrm{d}}{\mathrm{dt}} - Df_x(\bar{x}, \bar{\lambda})$ has a shifted exponential dichotomy with dichotomy data $(K, \alpha, \beta, P_a, P_b)$ and $\alpha < \gamma < \min(0, 2\beta)$.*

Then there exist some $\delta > 0$, $C > 0$ and $\bar{T}_+ > 0$ sufficiently large, such that for $T_+ \in \mathcal{D}$, $T_+ \geq \bar{T}_+$ the operator equation

$$G_J(z,\nu) = \begin{pmatrix} \dot{z} - f(z,\nu) \\ g\big(z(0),\nu\big) \\ \hat{M}(T_+,\nu)\big(z(T_+) - y(\nu)(T_+)\big) \end{pmatrix} = 0, \tag{4.19}$$

has a unique solution (z_J, ν_J) in the ball $\mathcal{B}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J, \bar{\lambda})$ in $\mathcal{Y}$ where

$$r(\beta, T_+) := \sup_{t\in J}\left(e^{|\beta|t}\right) + \int_0^{T_+} e^{|\beta|t}\mathrm{dt} = \begin{cases} e^{|\beta|T_+} + \frac{1}{|\beta|}\left(e^{|\beta|T_+} - 1\right) & : \beta \neq 0 \\ 1 + T_+ & : \beta = 0. \end{cases}$$

Furthermore, the following piecewise estimates hold with $d = 1$

$$|\nu_J - \bar{\lambda}| \leq Ce^{(d\gamma-\beta)T_+} \tag{4.20}$$
$$|z_J(t) - \bar{x}|_J(t)| \leq Ce^{d\gamma T_+} \cdot e^{-\beta(T_+-t)} \quad : \beta \geq 0 \tag{4.21}$$
$$|z_J(t) - \bar{x}|_J(t)| \leq Ce^{(d\gamma-\beta)T_+} \quad : \beta < 0. \tag{4.22}$$

If $\hat{M}(T_+)$ satisfies

$$|\hat{M}(T_+,\bar{\lambda})\big(\bar{x}(T_+) - y(\bar{\lambda})(T_+)\big)| \leq C_{\hat{M}}|\bar{x}(T_+) - y(\bar{\lambda})(T_+)|^2 \tag{4.23}$$

for a constant $C_{\hat{M}} > 0$ and sufficiently large $T_+ \in \mathcal{D}$ then the estimates (4.20)–(4.22) hold for $d = 2$. In this case the assumption $\alpha < \gamma < \min(0, 2\beta)$ can be weakened to $\alpha < \gamma < \min(0, \beta)$.

Proof: For the proof of this theorem we apply the following lemma (see [5], [26]):

Lemma 4.2 (Perturbation lemma) *Let $F : K_\delta(y_0) \to \mathcal{Z}$ be a C^1–function from a ball in $\mathcal{Y}$ into $\mathcal{Z}$ (Banach spaces). Assume that $F'(y_0)$ is a homeomorphism and there exist constants κ and σ, such that*

$$||F'(y) - F'(y_0)|| \leq \kappa < \sigma \leq ||F'(y_0)^{-1}||^{-1} \tag{4.24}$$
$$||F(y_0)|| \leq (\sigma - \kappa)\delta. \tag{4.25}$$

holds for all $y \in K_\delta(y_0)$. *Then* F *has a unique zero* $\bar{y}$ *in* $K_\delta(y_0)$ *and*

$$||\bar{y} - y_0|| \leq (\sigma - \kappa)^{-1}||F(y_0)||, \tag{4.26}$$

$$||y_1 - y_2|| \leq (\sigma - \kappa)^{-1}||F(y_1) - F(y_2)|| \tag{4.27}$$

for all $y_1, y_2 \in K_\delta(y_0)$.

Sketch of the proof of Theorem 4.1

Apply Lemma 4.2 with $F = G_J$ and $y_0 = (\bar{x}|_J, \bar{\lambda})$.

- **S1** Let (v,μ) be a solution of $G'_J(\bar{x}|_J, \bar{\lambda})(v,\mu) = (w, r_0, r_+)$ then we can estimate $||(v,\mu)||_{\mathcal{Y}} \leq C_{lin}||(w, r_0, r_+)||_{\mathcal{Z}}$.
- **S2** $G'_J(\bar{x}|_J, \bar{\lambda})$ is a homeomorphism and $\sigma \leq ||G'_J(\bar{x}|_J, \bar{\lambda})^{-1}||^{-1}$ with $\sigma := \frac{1}{C_{lin}} > 0$.
- **S3** There exists some $C_{Lip} > 0$ such that for each $0 < \delta \leq \frac{\sigma}{2C_{Lip}}$ we have $||G'_J(z,\nu) - G'_J(\bar{x}|_J, \bar{\lambda})|| \leq \kappa := \frac{\sigma}{2}\ \forall (z,\nu) \in \mathcal{B}_{\tilde{\delta}}(\bar{x}|_J, \bar{\lambda})$ with $\tilde{\delta} := \frac{\delta}{r(\beta, T_+)}$
- **S4** $||G_J(\bar{x}|_J, \bar{\lambda})|| \leq C_{tr} e^{(d\gamma - \beta)T_+} \leq (\sigma - \kappa)\frac{\delta}{r(\beta, T_+)} = (\sigma - \kappa)\tilde{\delta}$ for some $C_{tr} > 0$ and $T_+ \in \mathcal{D}$ sufficiently large
- **S5** Lemma 4.2 implies that (4.19) has a unique solution (z_J, ν_J) in $\mathcal{B}_{\tilde{\delta}}(\bar{x}|_J, \bar{\lambda})$ with $C := C_{lin}C_{tr}$

$$||(z_J, \nu_J) - (\bar{x}|_J, \bar{\lambda})||_{\mathcal{Y}} \leq Ce^{(d\gamma - \beta)T_+}$$

S1 Let $(w, r_0, r_+) \in \mathcal{Z}$ and let $(v,\mu) \in \mathcal{Y}$ be a solution of the inhomogeneous equation

$$G'_J(\bar{x}|_J, \bar{\lambda})(v,\mu) = (w, r_0, r_+). \tag{4.28}$$

This is equivalent to the variational equation

$$\dot{v} - A(\cdot)v - V(\cdot)\mu = w \tag{4.29}$$

$$R_0 v(0) + Q_0\mu = r_0 \tag{4.30}$$

$$R_+ v(T_+) + Q_+\mu = r_+ \tag{4.31}$$

where

$$\begin{aligned} A(\cdot) &= \tfrac{\partial}{\partial x} f(\bar{x}(\cdot), \bar{\lambda}), & V(\cdot) &= \tfrac{\partial}{\partial \lambda} f(\bar{x}(\cdot), \bar{\lambda}), \\ R_0 &= \tfrac{\partial}{\partial x} g(\bar{x}(0), \bar{\lambda}), & Q_0 &= \tfrac{\partial}{\partial \lambda} g(\bar{x}(0), \bar{\lambda}), \\ R_+ &= \hat{M}(T_+, \bar{\lambda}), & Q_+ &= \tfrac{\partial}{\partial \lambda}\hat{M}(T_+, \bar{\lambda})\big(\bar{x}(T_+) - y(\bar{\lambda})(T_+)\big) \\ & & & \quad -\hat{M}(T_+, \bar{\lambda})\tfrac{\partial}{\partial \lambda} y(\bar{\lambda})(T_+). \end{aligned}$$

If we define $\xi_a = P_a(0)v(0) \in \mathcal{R}(P_a(0))$ and $\xi_b = P_b(T_+)\big(v(T_+) - \frac{\partial}{\partial \lambda} y(\bar{\lambda})(T_+)\mu\big) \in \mathcal{R}\big(P_b(T_+)\big)$ then v is the unique solution of

$$Lv = \big(V(\cdot)\mu + w\big), \quad t \in J$$
$$P_a(0)v(0) = \xi_a, \quad P_b(T_+)v(T_+) = \xi_b + P_b(T_+)\tfrac{\partial}{\partial \lambda} y(\bar{\lambda})(T_+)\mu \quad .$$

Using Theorem 2.3 we obtain

$$\begin{aligned} v(t) &= S(t,0)\xi_a + S(t,T_+)\xi_b + s_J(w)(t) + s_J(V(\cdot)\mu)(t) \\ &\quad + S(t,T_+)P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+)\mu. \end{aligned} \tag{4.32}$$

Now we derive another representation of $s_J(V(\cdot)\mu)(t)$ and hence v. ¿From (3.4) and $\frac{\partial}{\partial s}\big(S(t,s)\frac{\partial}{\partial\lambda}y(\bar\lambda)(s)\big) = S(t,s)\big(V(s) - \Delta(s)\big)$ we obtain by integration

$$\begin{aligned} & s_J\big(V(\cdot)\big)(t) - s_J\big(\Delta(\cdot)\big)(t) \\ &= \frac{\partial}{\partial\lambda}y(\bar\lambda)(t) - S(t,0)P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0) - S(t,T_+)P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+) && (4.33) \\ &= \frac{\partial}{\partial\lambda}x(0,\bar\lambda)(t) - s_{[0,\infty)}(\Delta(\cdot))(t) - S(t,T_+)P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+) && (4.34) \\ &= \frac{\partial}{\partial\lambda}x(0,\bar\lambda)(t) - S(t,T_+)P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+) && (4.35) \\ &\quad - s_J\big(\Delta(\cdot)\big)(t) + S(t,T_+)\int_{T_+}^{\infty} S(T_+,s)P_b(s)\Delta(s)\mathrm{d}s. \end{aligned}$$

This and (4.32) imply further representations of v. Forthcoming (4.36) follows from (4.33) and (4.37) follows from (4.35).

$$\begin{aligned} v(t) &= S(t,0)\xi_a + S(t,T_+)\xi_b + s_J(w)(t) && (4.36) \\ &\quad + \frac{\partial}{\partial\lambda}y(\bar\lambda)(t)\mu - S(t,0)P_a(0)\frac{\partial}{\partial\lambda}y(\bar\lambda)(0)\mu + s_J\big(\Delta(\cdot)\big)(t)\mu \\ &= S(t,0)\xi_a + S(t,T_+)\xi_b + s_J(w)(t) + \frac{\partial}{\partial\lambda}x(0,\bar\lambda)(t)\mu && (4.37) \\ &\quad + S(t,T_+)\int_{T_+}^{\infty} S(T_+,s)P_b(s)\Delta(s)\mu\mathrm{d}s. \end{aligned}$$

We use $D\circ B(\xi_a,\mu) = R_0\xi_a + R_0\frac{\partial}{\partial\lambda}x(0,\bar\lambda)(0)\mu + Q_0\mu$ (see (3.17) and (3.19)) to get forthcoming (4.38). By $R_0 v(0) + Q_0\mu = r_0$ and (4.37) evaluated at 0 we arrive at (4.39).

$$\begin{aligned} & \begin{pmatrix} D\circ B & R_0 S(0,T_+)|_{\mathcal{R}(P_b(T_+))} \end{pmatrix} \begin{pmatrix} (\xi_a,\mu) \\ \xi_b \end{pmatrix} \\ &= D\circ B(\xi_a,\mu) + R_0 S(0,T_+)\xi_b \\ &= Q_0\mu + R_0\xi_a + R_0\frac{\partial}{\partial\lambda}x(0,\bar\lambda)(0)\mu + R_0 S(0,T_+)\xi_b && (4.38) \\ &= r_0 - R_0 S(0,T_+)\int_{T_+}^{\infty} S(T_+,s)P_b(s)\Delta(s)\mu\mathrm{d}s && (4.39) \\ &\quad + R_0\int_0^{T_+} S(0,s)P_b(s)w(s)\mathrm{d}s \quad =: \quad R_a(T_+). \end{aligned}$$

¿From $P_a(T_+)v(T_+) = S(T_+,0)\xi_a + s_J(w)(T_+) + P_a(T_+)\frac{\partial}{\partial\lambda}x(0,\bar\lambda)(T_+)\mu$ (see (4.37)) and $P_b(T_+)v(T_+) = \xi_b + P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar\lambda)(T_+)\mu$ (see (4.32) and $P_b(T_+)s_J(w + V(\cdot))(T_+) = 0$)

follows

$$\begin{aligned}
&\hat{M}(T_+,\bar{\lambda})\Big(\xi_b + S(T_+,0)\xi_a + s_J(w)(T_+) \\
&\qquad\qquad + P_a(T_+)\frac{\partial}{\partial\lambda}x(0,\bar{\lambda})(T_+)\mu + P_b(T_+)\frac{\partial}{\partial\lambda}y(\bar{\lambda})(T_+)\mu\Big) \\
&= R_+ v(T_+) \quad = \quad r_+ - Q_+\mu \\
&= r_+ + \hat{M}(T_+,\bar{\lambda})\frac{\partial}{\partial\lambda}y(\bar{\lambda})(T_+)\mu - \frac{\partial}{\partial\lambda}\hat{M}(T_+,\bar{\lambda})\mu\big(\bar{x}(T_+) - y(\bar{\lambda})(T_+)\big)
\end{aligned}$$

and

$$\begin{aligned}
\hat{M}(T_+,\bar{\lambda})\xi_b &= r_+ - \frac{\partial}{\partial\lambda}\hat{M}(T_+,\bar{\lambda})\mu\big(\bar{x}(T_+) - y(\bar{\lambda})(T_+)\big) \\
&\quad + \hat{M}(T_+,\bar{\lambda})P_a(T_+)\big(\frac{\partial}{\partial\lambda}y(\bar{\lambda})(T_+) - \frac{\partial}{\partial\lambda}x(0,\bar{\lambda})(T_+)\big)\mu \\
&\quad - \hat{M}(T_+,\bar{\lambda})S(T_+,0)\xi_a - \hat{M}(T_+,\bar{\lambda})s_J(w)(T_+) \\
&=: R_b(T_+).
\end{aligned}$$

If we define $\hat{\xi}_b := \xi_b e^{-\beta T_+}$ we get the linear system

$$\begin{pmatrix} D\circ B & R_0 S(0,T_+)|_{\mathcal{R}(P_b(T_+))}\cdot e^{\beta T_+} \\ 0 & \hat{M}(T_+,\bar{\lambda})|_{\mathcal{R}(P_b(T_+))} \end{pmatrix}\begin{pmatrix} (\xi_a,\mu) \\ \hat{\xi}_b \end{pmatrix} = \begin{pmatrix} R_a(T_+) \\ R_b(T_+)e^{-\beta T_+} \end{pmatrix}. \tag{4.40}$$

Using (2.11), (2.12) and (3.12) we can estimate $||s_{[T_+,\infty)}(\Delta(\cdot))(T_+)|| \le K\frac{1}{\beta-\gamma}L_f^1(1+M_y^1)C_{\bar{x}}e^{\gamma T_+}$ and $||s_J(\Delta(\cdot))(T_+)|| \le K\frac{1}{\gamma-\alpha}L_f^1(1+M_y^1)C_{\bar{x}}e^{\gamma T_+}$. Thus

$$\begin{aligned}
|R_a(T_+)| &\le |r_0| + ||R_0||\int_0^{T_+}\underbrace{||S(0,s)P_b(s)||}_{\le Ke^{-\beta s}}|w(s)|\mathrm{ds} \\
&\quad + ||R_0||\cdot\underbrace{||S(0,T_+)P_b(T_+)||}_{\le Ke^{-\beta T_+}}\cdot\underbrace{||s_{[T_+,\infty)}(\Delta(\cdot))(T_+)||}_{\le K\frac{1}{\beta-\gamma}L_f^1(1+M_y^1)C_{\bar{x}}e^{\gamma T_+}}|\mu| \\
&\le |r_0| + ||R_0||K||w||_{\mathcal{Z}}^* + ||R_0||K^2\frac{1}{\beta-\gamma}L_f^1(1+M_y^1)C_{\bar{x}}e^{(\gamma-\beta)T_+}|\mu| \\
&\le C_a^R(|r_0| + ||w||_{\mathcal{Z}}^*) + c_a^R e^{(\gamma-\beta)T_+}|\mu|
\end{aligned}$$

for some constant $C_a^R > 0$ and $c_a^R \ge 0$ and

$$\begin{aligned}
&|R_b(T_+)|e^{-\beta T_+} \le |r_+|e^{-\beta T_+} + M_{\hat{M}}^1 C_{\bar{x}}e^{(\gamma-\beta)T_+}|\mu| \\
&\quad + M_{\hat{M}}\underbrace{||S(T_+,0)P_a(0)||}_{\le Ke^{\alpha T_+}}M_y^1|\mu|e^{-\beta T_+} + M_{\hat{M}}\underbrace{||s_J(\Delta(\cdot))(T_+)||}_{\le K\frac{1}{\gamma-\alpha}L_f^1(1+M_y^1)C_{\bar{x}}e^{\gamma T_+}}|\mu|e^{-\beta T_+} \\
&\quad + M_{\hat{M}}||S(T_+,0)P_a(0)||e^{-\beta T_+}|\xi_a| + M_{\hat{M}}e^{-\beta T_+}\int_0^{T_+}\underbrace{||S(T_+,s)P_a(s)||}_{\le Ke^{\alpha(T_+-s)}\le Ke^{\beta(T_+-s)}}|w(s)|\mathrm{ds} \\
&\le |r_+|e^{-\beta T_+} + M_{\hat{M}}K||w||_{\mathcal{Z}}^* + M_{\hat{M}}Ke^{(\alpha-\beta)T_+}|\xi_a| \\
&\quad + \big(M_{\hat{M}}^1 C_{\bar{x}} + M_{\hat{M}}KM_y^1 e^{(\alpha-\gamma)T_+} + KM_{\hat{M}}C_\gamma L_f^1(1+M_y^1)C_{\bar{x}}\big)e^{(\gamma-\beta)T_+}|\mu| \\
&\le C_b^R(|r_+|e^{-\beta T_+} + ||w||_{\mathcal{Z}}^*) + c_b^R e^{(\gamma-\beta)T_+}(|\xi_a| + |\mu|).
\end{aligned}$$

for some constant $C_b^R > 0$ and $c_b^R \geq 0$. Thus there exists some constants $C_{\mu,\xi}$ and $\bar{T}$ such that for all $T_+ \geq \bar{T}$

$$|\xi_a| + |\mu| + |\xi_b| e^{-\beta T_+} \leq C_{\mu,\xi} ||(w, r_0, r_+)||_{\mathcal{Z}}. \tag{4.41}$$

We estimate $v(t)$ by using (4.36) and Theorem 2.3 as follows:

$$\begin{aligned}
|v(t)| \leq\ & K e^{\alpha t} |\xi_a| + K e^{-\beta(T_+ - t)} |\xi_b| + K ||w||_{\mathcal{Z}}^* e^{\beta t} \\
& + \big(\big| s_J(\Delta(\cdot))(t) \big| + \big| \frac{\partial}{\partial\lambda} y(\bar\lambda)(t) \big| + \big| S(t,0) P_a(0) \frac{\partial}{\partial \lambda} y(\bar\lambda)(0) \big| \big) |\mu| \\
\leq\ & \max \big(K e^{\alpha t}, K C_\gamma L_f^1 (1 + M_y^1) C_{\bar x} e^{\gamma t} + (K e^{\alpha t} + 1) M_y^1 \big) \big(|\xi_a| + |\mu| \big) \\
& + K e^{-\beta T_+} |\xi_b| e^{\beta t} + K ||w||_{\mathcal{Z}}^* e^{\beta t} \\
\leq\ & (C_v^1 e^{\beta t} + C_v^2) ||(w, r_0, r_+)||_{\mathcal{Z}} \\
\leq\ & C_v ||(w, r_0, r_+)||_{\mathcal{Z}} \cdot \begin{cases} e^{\beta t} & : \beta \geq 0 \\ 1 & : \beta < 0 \end{cases} = C_v ||(w, r_0, r_+)||_{\mathcal{Z}} q(t)^{-1}
\end{aligned}$$

and therefore $||v||_{\mathcal{Y}}^* = \sup_{t\in J} \big(||v(t)|| q(t) \big) \leq C_v ||w, r_0, r_+||_{\mathcal{Z}}$.

Thus **S1** $||(v, \mu)||_{\mathcal{Y}} \leq C_{lin} ||(w, r_0, r_+)||_{\mathcal{Z}}$ holds with $C_{lin} = C_{\mu,\xi} + C_v$.

S2 Assumption **A4** implies that $G_J'(\bar{x}|_J, \bar\lambda)(v, \mu) = (w, r_0, r_+)$ is a linear boundary value problem of dimension $m_a + p + m_b = m + p$ for which the Fredholm alternative holds. Thus $\sigma := \frac{1}{C_{lin}} \leq ||G_J'(\bar{x}|_J, \bar\lambda)^{-1}||^{-1}$.

S3 For any $(z, \nu), (v, \mu) \in \mathcal{Y}$ we can estimate

$$\begin{aligned}
& \left\| \Big(G_J'(z,\nu) - G_J'(\bar x|_J, \bar\lambda) \Big) \begin{pmatrix} v \\ \mu \end{pmatrix} \right\|_{\mathcal{Z}} \\
=\ & \big\| \big(f_x(z,\nu) - f_x(\bar x|_J, \bar\lambda) \big) v + \big(f_\lambda(z,\nu) - f_\lambda(\bar x|_J, \bar\lambda) \big) \mu \big\|_{\mathcal{Z}}^* \\
& + \big| \big(g_x(z(0),\nu) - g_x(\bar x(0), \bar\lambda) \big) v(0) + \big(g_\lambda(z(0),\nu) - g_\lambda(\bar x(0), \bar\lambda) \big) \mu \big| \\
& + \big| \big(\hat M(T_+, \nu) - \hat M(T_+, \bar\lambda) \big) v(T_+) \\
& \quad + \frac{\partial}{\partial \lambda} \hat M(T_+, \nu) \mu \big(z(T_+) - y(\nu)(T_+) \big) \\
& \quad - \frac{\partial}{\partial \lambda} \hat M(T_+, \bar\lambda) \mu \big(\bar x(T_+) - y(\bar\lambda)(T_+) \big) \\
& \quad - \hat M(T_+, \nu) \frac{\partial}{\partial \lambda} y(\nu)(T_+) \mu + \hat M(T_+, \bar\lambda) \frac{\partial}{\partial\lambda} y(\bar\lambda)(T_+) \mu \big| e^{-\beta T_+} \\
\leq\ & C_{Lip} ||(z - \bar x|_J, \nu - \bar\lambda)||_{\mathcal{Y}} \cdot ||(v,\mu)||_{\mathcal{Y}} \cdot r(\beta, T_+)
\end{aligned} \tag{4.42}$$

with some constant C_{Lip} and therefore with $\kappa := \frac{\sigma}{2}$, $\delta \leq \frac{\sigma}{2C_{Lip}}$ and $\tilde\delta = \frac{\delta}{r(\beta,T_+)}$ we see for any $(z,\nu) \in \mathcal{B}_{\tilde\delta}(\bar x|_J, \bar\lambda)$

$$||G_J'(z,\nu) - G_J'(\bar x|_J, \bar\lambda)||_{\mathcal{Y}\mathcal{Z}} \leq \frac{\sigma}{2} = \kappa < \sigma \leq ||G_J'(x|_J, \bar\lambda)^{-1}||_{\mathcal{Y}\mathcal{Z}}^{-1}. \tag{4.43}$$

For details of this estimate see [19], as example we estimate (4.42) with the functions $p(t) = \frac{e^{-\beta t}}{q(t)}$, $\rho(t) = \frac{p(t)}{q(t)} = e^{|\beta| t}$ and

$$\sup_{t\in J} (\rho(t)) + \int_0^{T_+} \rho(t) \mathrm{d}t \leq r(\beta, T_+) := \begin{cases} e^{|\beta| T_+} + \frac{1}{|\beta|} \left(e^{|\beta| T_+} - 1 \right) & : \beta \neq 0 \\ 1 + T_+ & : \beta = 0. \end{cases}$$

as follows:

$$\begin{aligned}
&||\big(f_x(z,\nu)-f_x(\bar{x}|_J,\bar{\lambda})\big)v+\big(f_\lambda(z,\nu)-f_\lambda(\bar{x}|_J,\bar{\lambda})\big)\mu||^*_{\mathcal{Z}} \\
\leq\ & \sup_{t\in J}\big(\big|f_x\big(z(t),\nu\big)-f_x\big(\bar{x}(t),\bar{\lambda}\big)\big|\cdot p(t)\cdot|v(t)|q(t)\big) \\
&+\int_0^{T_+}|(f_x(z(t),\nu)-f_x(\bar{x}(t),\bar{\lambda}))v(t)|e^{-\beta t}\mathrm{d}t \\
&+\sup_{t\in J}(|(f_\lambda(z(t),\nu)-f_\lambda(\bar{x}(t),\bar{\lambda}))|e^{-\beta t}|\mu|) \\
&+\int_0^{T_+}|(f_\lambda(z(t),\nu)-f_\lambda(\bar{x}(t),\bar{\lambda}))|e^{-\beta t}|\mu|\mathrm{d}t \\
\leq\ & L_f^1\sup_{t\in J}\big(|z(t)-\bar{x}(t)|q(t)\cdot\rho(t)+|\nu-\bar{\lambda}|\cdot p(t)\big)\cdot||v||^*_{\mathcal{Y}} \\
&+L_f^1\left(\int_0^{T_+}|z(t)-\bar{x}(t)|q(t)\cdot\rho(t)\mathrm{d}t+\int_0^{T_+}p(t)\mathrm{d}t|\nu-\bar{\lambda}|\right)\cdot||v||^*_{\mathcal{Y}} \\
&+L_f^1\sup_{t\in J}\left(|z(t)-\bar{x}(t)|q(t)\cdot p(t)+|\nu-\bar{\lambda}|e^{-\beta t}\right)|\mu| \\
&+L_f^1\left(\int_0^{T_+}|z(t)-\bar{x}(t)|q(t)\cdot p(t)\mathrm{d}t+\int_0^{T_+}e^{-\beta t}\mathrm{d}t|\nu-\bar{\lambda}|\right)|\mu| \\
\leq\ & L_f^1 r(\beta,T_+)(||z-\bar{x}|_J||^*_{\mathcal{Y}}+|\nu-\bar{\lambda}|)\cdot(||v||^*_{\mathcal{Y}}+|\mu|)
\end{aligned}$$

On the other hand the truncation error is

$$\begin{aligned}
||G_J(\bar{x}|_J,\bar{\lambda})||_{\mathcal{Z}} &= \left\|\begin{pmatrix}0\\0\\ \hat{M}(T_+)\big(\bar{\lambda}\big)(\bar{x}(T_+)-y(\bar{\lambda})(T_+))\end{pmatrix}\right\|_{\mathcal{Z}} \\
&= \big|\hat{M}(T_+,\bar{\lambda})\big(\bar{x}(T_+)-y(\bar{\lambda})(T_+)\big)\big|e^{-\beta T_+} \\
&\leq \tilde{C_{tr}}||\bar{x}(T_+)-y(\bar{\lambda})(T_+)||^d e^{-\beta T_+} \\
&\leq C_{tr}e^{(d\gamma-\beta)T_+}
\end{aligned}$$

for some $\tilde{C_{tr}}>0$ and $C_{tr}:=\tilde{C_{tr}}C^d_{\bar{x}}$. In the cases $(d=1,\ \gamma<2\beta)$ and $(d=2,\ \gamma<\beta)$ we achieve that $C_{tr}e^{(d\gamma-\beta)T_+}\leq(\sigma-\kappa)\frac{\delta}{r(\beta,T_+)}$ for $T_+\in\mathcal{D}$ sufficiently large.
Now we can apply Lemma 4.2 and there exists a unique solution (z_J,ν_J) in $K^{\mathcal{Y}}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J,\bar{\lambda})$ with

$$||(z_J,\nu_J)-(\bar{x}|_J,\bar{\lambda})||_{\mathcal{Y}}\leq Ce^{(d\gamma-\beta)T_+}$$

where $C=2C_{lin}C_{tr}$. This implies the estimates in Theorem 4.1 ■

As direct result of the estimates in Theorem 4.1 a super-convergence phenomenon in sup–norm as in the connecting orbit case [4], [5], [21] occurs for $\beta>0$.

Corollary 4.3 *Under the assumptions of Theorem 4.1 we have the* sup*–norm estimates*

$$\begin{aligned}
|\nu_J-\bar{\lambda}| &\leq Ce^{(d\gamma-\beta)T_+} & \\
||z_J-\bar{x}|_J||_\infty &\leq Ce^{d\gamma T_+} & :\beta>0 \\
||z_J-\bar{x}|_J||_\infty &\leq Ce^{(d\gamma-\beta)T_+} & :\beta\leq 0.
\end{aligned}$$

4.3 Boundary corrector method

The asymptotic boundary conditions with (4.6) yields “quadratic” estimates (see Theorem 4.1). On the other hand, e. g. in the periodic case, it can be cumbersome to compute $\bar{M}(T_+, \lambda)$ at $\bar{\lambda}$ since $\bar{\lambda}$ is unknown (see section 4.1). But it is realistic to assume

A7 There exists a neighborhood Λ_0 of $\bar{\lambda}$ and a matrix valued function $\bar{M}(T_+, \cdot) \in C^1\big(\Lambda_0, \mathbb{R}^{m_b,m}\big)$ such that **A6** and (4.6) holds for each $\lambda \in \Lambda_0$.

Using the boundary corrector method developed in this section we have to compute $\bar{M}(T_+, \lambda)$ only twice (case $\beta \geq 0$) or three times (case $\beta < 0$).

Boundary Corrector Method

1. Start with some $\mu_0 \in \Lambda_0$, $i = 0$,

2. Calculate (z_{i+1}, μ_{i+1}), the solution of

$$G_J^{\mu_i}(z, \nu) = \begin{pmatrix} \dot{z} - f(z, \nu) \\ g\big(z(0), \nu\big) \\ \bar{M}(T_+, \mu_i)\big(z(T_+) - y(\nu)(T_+)\big) \end{pmatrix} = 0, \tag{4.44}$$

3. Repeat with **2.** once for $\beta \geq 0$ or twice for $\beta < 0$ with “$i = i + 1$”.

The following proposition states that the error of the solution of the boundary corrector method has the same exponential rate $(2\gamma - \beta)$ as the solution of Theorem 4.1 with $d = 2$.

PROPOSITION 4.4 *Let the assumptions of Theorem 4.1 and* **A7** *hold and* $\bar{T}_+$ *sufficiently large. Then there exists some* $C_{bcm} > 0$ *such that for* $T_+ \in \mathcal{D}$, $T_+ \geq \bar{T}_+$ *and* $\mu_0 \in \Lambda_0$ *the result* (z_J, ν_J) *of the* **boundary corrector method** *can be estimated by*

$$||(z_J, \nu_J) - (\bar{x}|_J, \bar{\lambda})||_{\mathcal{Y}} \leq C_{bcm} e^{(2\gamma - \beta)T_+}. \tag{4.45}$$

Proof: For each $\mu_i \in \Lambda_0$ we see that (4.44) is the same as (4.19) with the constant matrix $\hat{M}(T_+, \nu) := \bar{M}(T_+, \mu_i) \quad \forall \nu \in \Lambda_0$.

By inspection of the proof of Theorem 4.1 and the uniformity assumptions in **A6** we see that we can choose $\delta > 0$ and $\bar{T}_+$ independent of ν such that G_J^ν has a unique solution (z_J^ν, μ_J^ν) in $K^{\mathcal{Y}}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J, \bar{\lambda})$ for each $\nu \in \Lambda_0$. Furthermore the estimates (4.20)–(4.22) hold with some constant C independent of ν. In particular the estimates hold with $d = 2$ if $(\bar{z}_J, \bar{\mu}_J) \in K^{\mathcal{Y}}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J, \bar{\lambda})$ is the (unknown) solution of $G_J^{\bar{\lambda}} = 0$. Using this and (4.27) we estimate for each $\mu \in \Lambda_0$ the solution $(z_J, \mu_J) \in K^{\mathcal{Y}}_{\frac{\delta}{r(\beta,T_+)}}(\bar{x}|_J, \bar{\lambda})$ of G_J^μ

by

$$\begin{aligned}
&||(z_J,\mu_J)-(\bar z_J,\bar\mu_J)||_{\mathcal Y}\\
\le\ & \frac{2}{\sigma}||G^\mu_J(z_J,\mu_J)-G^\mu_J(\bar z_J,\bar\mu_J)||_{\mathcal Z}\\
=\ & |\underbrace{\bar M(T_+,\mu)(z_J(T_+)-y(\mu_J)(T_+))}_{=0}-\bar M(T_+,\mu)(\bar z_J(T_+)-y(\bar\mu_J)(T_+))|\frac{2}{\sigma}e^{-\beta T_+}\\
=\ & |\underbrace{\bar M(T_+,\bar\lambda)(\bar z_J(T_+)-y(\bar\mu_J)(T_+))}_{=0}-\bar M(T_+,\mu)(\bar z_J(T_+)-y(\bar\mu_J)(T_+))|\frac{2}{\sigma}e^{-\beta T_+}\\
=\ & |\big(\bar M(T_+,\bar\lambda)-\bar M(T_+,\mu)\big)\big(\bar z_J(T_+)-y(\bar\mu_J)(T_+)\big)|2C_{lin}e^{-\beta T_+}\\
\le\ & ||\bar M(T_+,\bar\lambda)-\bar M(T_+,\mu)||\cdot\underbrace{\big(|\bar z_J(T_+)-\bar x|_J(T_+)|}_{\le Ce^{(2\gamma-\beta)T_+}}\\
&\quad+\underbrace{|\bar x|_J(T_+))-y(\bar\lambda)(T_+)|}_{\le C_{\bar x}e^{\gamma T_+}}+\underbrace{|y(\bar\lambda)(T_+)-y(\bar\mu_J)(T_+)|}_{\le L_yCe^{(2\gamma-\beta)T_+}}\big)2C_{lin}e^{-\beta T_+}\\
\le\ & ||\bar M(T_+,\bar\lambda)-\bar M(T_+,\mu)||2C_{lin}(C+CL_y+C_{\bar x})e^{(\gamma-\beta)T_+}\\
\le\ & |\mu-\bar\lambda|C_be^{(\gamma-\beta)T_+}
\end{aligned}$$

with $C_b=L_{\bar M}2C_{lin}(C+CL_y+C_{\bar x})$.

If we apply this estimate to the boundary corrector method we get

$$\begin{aligned}
||(z_2,\mu_2)-(\bar x|_J,\bar\lambda)||_{\mathcal Y} &\le ||(z_2,\mu_2)-(\bar z_J,\bar\mu_J)||_{\mathcal Y}+||(\bar z_J,\bar\mu_J)-(\bar x|_J,\bar\lambda)||_{\mathcal Y}\\
&\le |\mu_1-\bar\lambda|C_be^{(\gamma-\beta)T_+}+Ce^{(2\gamma-\beta)T_+}\\
&\le CC_be^{2(\gamma-\beta)T_+}+Ce^{(2\gamma-\beta)T_+}.
\end{aligned}$$

Here we used $|\mu_1-\bar\lambda|\le Ce^{(\gamma-\beta)T_+}$ from (4.20). In case $\beta\ge 0$ this implies (4.45) with $C_{bcm}=C(C_b+1)$. Furthermore we see

$$\begin{aligned}
||(z_3,\mu_3)-(\bar x|_J,\bar\lambda)||_{\mathcal Y} &\le ||(z_3,\mu_3)-(\bar z_J,\bar\mu_J)||_{\mathcal Y}+||(\bar z_J,\bar\mu_J)-(\bar x|_J,\bar\lambda)||_{\mathcal Y}\\
&\le |\mu_2-\bar\lambda|C_be^{(\gamma-\beta)T_+}+Ce^{(2\gamma-\beta)T_+}\\
&\le CC_b^2e^{3(\gamma-\beta)T_+}+(C_be^{(\gamma-\beta)T_+}+1)Ce^{(2\gamma-\beta)T_+}.
\end{aligned}$$

Thus $\gamma<2\beta$ yields (4.45) with $C_{bcm}=CC_b^2+C(1+C_b)$ in case $\beta<0$. ■

By the boundary corrector method we avoid the simultaneous computation of $\bar M(\lambda)$ and get "nearly optimal" estimation results.

5 Examples

In the examples we solve the boundary value problems of type (4.9) and (4.15). We use the collocation boundary value solver COLCON and its version for parameter continuation COLPAR from Bader and Kunkel [3].

5.1 A first example

This example can be studied analytically and it will be used to examine the behavior of the truncation error.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_1(\mu - ar^2) - x_2(1+\lambda r^2) \\ x_1(1+\lambda r^2) + x_2(\mu - ar^2) \\ cx_3 + d\mu - \nu r + \frac{\nu}{c} r(\mu - ar^2) \end{pmatrix} =: \hat{f}(x; \lambda, \mu, a, c, d, \nu) \tag{5.1}$$

with $x \in \mathbb{R}^3$, $\mu, a, c, d, \nu > 0$, $\lambda > -\frac{a}{\mu}$ and $r = \sqrt{x_1^2 + x_2^2}$. In polar coordinates with $\varphi = \arccos(\frac{x_1}{r})$ we get $\dot{r} = r(\mu - ar^2)$, $\dot{\varphi} = (1 + \lambda r^2)$.

We use $\chi(x, \tilde{\varphi}_0) := -\sin(\tilde{\varphi}_0)x_1 + \cos(\tilde{\varphi}_0)x_2 = 0$ as phase condition for periodic orbits and $\tilde{\varphi}_0$ as the extra parameter to fix the phase of the periodic orbit.

For given $\tilde{\varphi}_0$ there exists a periodic solution with $\chi(\tilde{x}(0), \tilde{\varphi}_0) = 0$ namely

$$\tilde{x}(t) = \left(\sqrt{\frac{\mu}{a}} \cos\left(\tilde{\varphi}(t)\right), \sqrt{\frac{\mu}{a}} \sin\left(\tilde{\varphi}(t)\right), -\frac{d\mu}{c} + \frac{\nu}{c}\sqrt{\frac{\mu}{a}}\right)$$

with $\tilde{r}(t) = \sqrt{\frac{\mu}{a}}$, $\tilde{\varphi}(t) = t(1 + \lambda\sqrt{\frac{\mu}{a}}) + \tilde{\varphi}_0$. The period is $T = T(\mu, a, \lambda) := \frac{2\pi}{1+\lambda\frac{\mu}{a}}$. Scaling $\hat{f}$ by $f(x; \lambda, \mu, a, c, d, \nu) = T(\lambda, \mu, a)\hat{f}(x; \lambda, \mu, a, c, d, \nu)$ we see that

$$y(t) = \left(\sqrt{\frac{\mu}{a}} \cos(2\pi t + \tilde{\varphi}_0), \sqrt{\frac{\mu}{a}} \sin(2\pi t + \tilde{\varphi}_0), -\frac{d\mu}{c} + \frac{\nu}{c}\sqrt{\frac{\mu}{a}}\right)$$

is a 1–periodic solution of $\dot{x} = f(x; \lambda, \mu, a, c, d, \nu)$ with $\chi\big(x(0), \tilde{\varphi}_0\big) = 0$.

The linearization at the periodic orbit y implies that

$$\zeta(t) = \begin{pmatrix} y_1(t) + \frac{\lambda}{a} y_2(t) & -y_2(t) & 0 \\ y_2(t) - \frac{\lambda}{a} y_1(t) & y_1(t) & 0 \\ \frac{\nu}{c}\sqrt{\frac{\mu}{a}} & 0 & -\frac{\nu}{c}\sqrt{\frac{a}{\mu}} \end{pmatrix} \begin{pmatrix} e^{-2\mu T t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{cTt} \end{pmatrix}$$

solves $\dot{\zeta}(t) = \frac{\partial}{\partial x} f(y(t), \lambda)\zeta(t)$ and

$$\chi^T(t) = \begin{pmatrix} e^{2\mu T t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-cTt} \end{pmatrix} \begin{pmatrix} \frac{a}{\mu} y_1(t) & \frac{a}{\mu} y_2(t) & 0 \\ \frac{1}{\mu}(\lambda y_1(t) - a y_2(t)) & \frac{1}{\mu}(\lambda y_2(t) + a y_1(t)) & 0 \\ y_1(t) & y_2(t) & -\frac{c}{\nu}\sqrt{\frac{\mu}{a}} \end{pmatrix}$$

solves $\dot{\chi}^T(t) = -\chi^T(t)\frac{\partial}{\partial x} f(y(t), \lambda)$ with $\chi^T(t)\zeta(t) = \mathrm{Id}_{\mathbb{R}^3}$.

Thus there are Floquet exponents $-2\mu T < 0 < cT$. Using the periodic solutions y depending on $\tilde{\varphi}_0$ we can define

$$\bar{M}(\tilde{\varphi}_0)(t) = \begin{pmatrix} \frac{1}{\mu}(\lambda y_1(t) - a y_2(t)) & \frac{1}{\mu}(\lambda y_2(t) + a y_1(t)) & 0 \\ y_1(t) & y_2(t) & -\frac{c}{\nu}\sqrt{\frac{\mu}{a}} \end{pmatrix}. \tag{5.2}$$

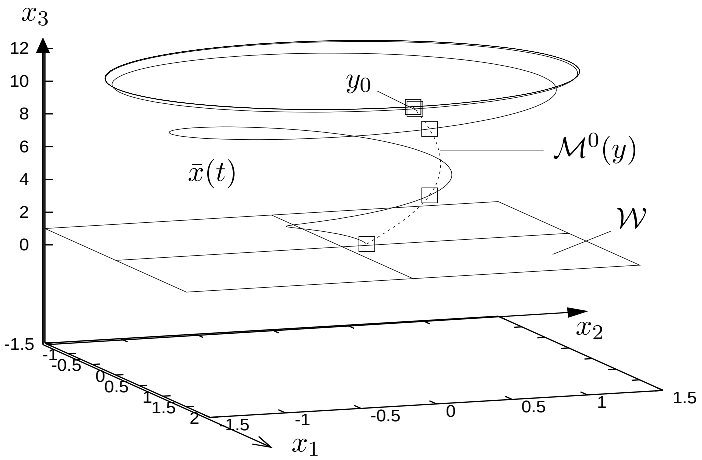

Figure 5.2: The phase portrait of the scaled system (5.1) with parameters $\mu = 0.25$, $a = \frac{1}{9}$, $\lambda = 0.1$, $c = 0.5$, $d = 1$, $\nu = 4$ and $\tilde{\varphi}_0 = 0$.

Thus we can apply Theorem 4.1 with $d = 2$. On the other hand the exact solution of $\dot{x} = f(x; \lambda, \mu, a, c, d, \nu)$ with initial value $x_0 \in \mathbb{R}^3$ and $r_0 := \sqrt{x_{0_1}^2 + x_{0_2}^2}$, $\varphi_0 := \arccos(\frac{x_{0_1}}{r_0})$ is

$$
\begin{aligned}
x(t) &= \Big(r(t)\cos\big(\varphi(t)\big), r(t)\sin\big(\varphi(t)\big), e^{cTt}\big(x_{0_3} + \frac{d\mu}{c} - \frac{\nu}{c}r_0\big) - \frac{d\mu}{c} + \frac{\nu}{c}r(t)\Big),\\
r(t) &= \frac{r_0}{\sqrt{e^{-2\mu Tt}(1 - \frac{ar_0^2}{\mu}) + \frac{ar_0^2}{\mu}}},\\
\varphi(t) &= Tt + \frac{\lambda}{2a}\ln\left(1 + \frac{ar_0^2}{\mu}(e^{2\mu Tt} - 1)\right) + \varphi_0.
\end{aligned}
$$

The solution $\bar{x}$ starting in $\mathcal{W} = \{z \in \mathbb{R}^3 | z_3 = \bar{z}_3\}$, $\bar{z}_3 > 0$ which is $-2\mu T$–asymptotic with y has an initial value $\bar{x}_0 = (\bar{r}_0\cos(\bar{\varphi}_0), \bar{r}_0\sin(\bar{\varphi}_0), \bar{z}_3)$ where $\bar{r}_0 = \frac{c}{\nu}\left(\bar{z}_3 + \frac{d\mu}{c}\right)$ and $\bar{\varphi}_0 = \tilde{\varphi}_0 - \frac{\lambda}{2a}\ln\left(\frac{a\bar{r}_0^2}{\mu}\right)$.

This yields the phase portrait in Figure 5.2 which is shown in the Introduction (see Figure 1.1). The connecting orbit $\bar{x}$ starts in a plane $\mathcal{W}$ and converges to the periodic orbit y. It is indicated by (—). The stable manifold of y which is represented by a dashed line (- -), is approximated by using $\bar{z}_3$ as continuation parameter in $[1, 11.5]$. The intersection of the connecting orbit $\bar{x}$ with the stable manifold is $\bar{x}(N), N = 0, 1, \ldots$ marked by the symbol $\square$. The plane $\mathcal{W}$ is defined by $\{z \in \mathbb{R}^3 \mid \bar{z}_3 = 1.0\}$ and the periodic orbit has an initial value $y_0 = (1.5, 0, 11.5)$.

In typical applications the phase is not fixed, thus we will use $\tilde{\varphi}_0$ as free parameter. In this case the 2–dimensional manifold $\mathcal{W}$ in $\mathbb{R}^4$ is given by $\{(z, \tilde{\varphi}_0) \in \mathbb{R}^4 | z_1 = \bar{z}_1, z_3 = \bar{z}_3\}$

with el$\bar{z}_3 > 0$. The solution has initial values $\bar{x}_0 = (\bar{z}_1, \bar{r}_0 \sin(\bar{\varphi}_0), \bar{z}_3)$ where

$$\bar{r}_0 = \frac{c}{\nu}\left(\bar{z}_3 + \frac{d\mu}{c}\right), \bar{\varphi}_0 = \arccos\left(\frac{\bar{z}_1}{\bar{r}_0}\right) \text{ and } \tilde{\varphi}_0 = \bar{\varphi}_0 + \frac{\lambda}{2a}\ln\left(\frac{a\bar{r}_0^2}{\mu}\right).$$

For an approximation (x_J, μ_J) we define the approximation errors by

$$e^J_{tr}(t) := |x_J(t) - \bar{x}(t)|, e_{max}(T_+) := \max_{t\in[0,T_+]}\left(e^J_{tr}(t)\right) \text{ and } e^J_{par}(T_+) := |\mu_J - \bar{\lambda}|.$$

Each curve in Figure 5.3 shows the natural logarithm of the error $e^J_{tr}(t)$ for $J = [0, T_+]$ with different T_+, when we use the boundary corrector method.

In Figure 5.4 we compare the natural logarithm of the errors in the first and the second step of the boundary corrector method.

- -	first step	$\square$	e^J_{par}
—	second step	$\times$	$e_{max}(T_+)$

In this example the period of the periodic orbit is $T = \frac{2\pi}{1+\lambda\frac{\mu}{a}} = 5.129$ and we apply Theorem 4.1 with $\alpha = -2\mu T$, $\beta = 0$. Hence the theoretical exponential rate for the error estimates is $d\alpha - \beta = -d \cdot 2\mu T = -d \cdot 2.565$ (see Corollary 4.3). The slopes in Figure 5.4 taken from $T_+ = 4$ to $T_+ = 6$ are approximatively 2.563 in the first step and 5.15 in the second step (nearly the same for $e_{max}(T_+)$ and e^J_{par}).

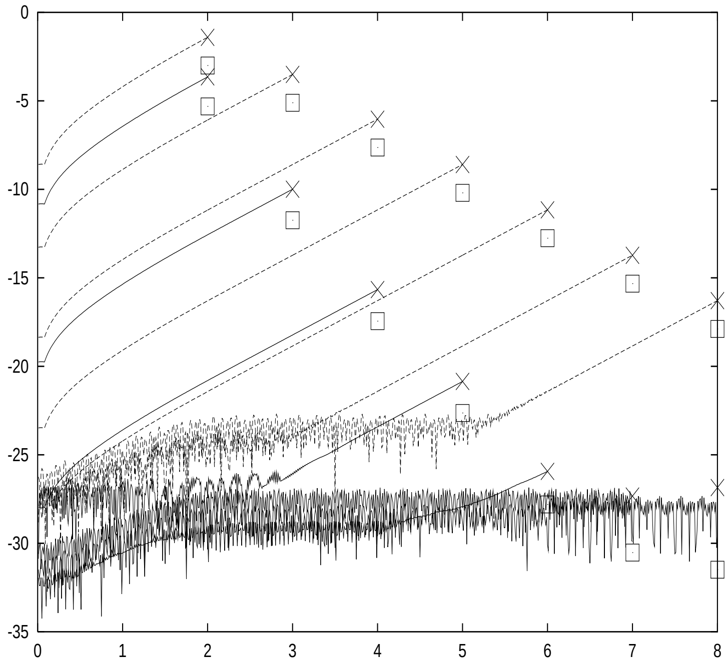

Figure 5.3: The natural logarithm of the error $e^J_{tr}(t)$, $J = [0, T_+]$, $T_+ = 1, \ldots, 8$, for the boundary corrector method. The first approximation with starting parameter $\tilde{\varphi}_0 = 0$ (to compute (5.2)) is represented by (- -). The result of the first step yields the parameter $\tilde{\varphi}_1 = -0.3007$. Using $\bar{M}(\tilde{\varphi}_1)$ from (5.2) we get the second approximation (—).

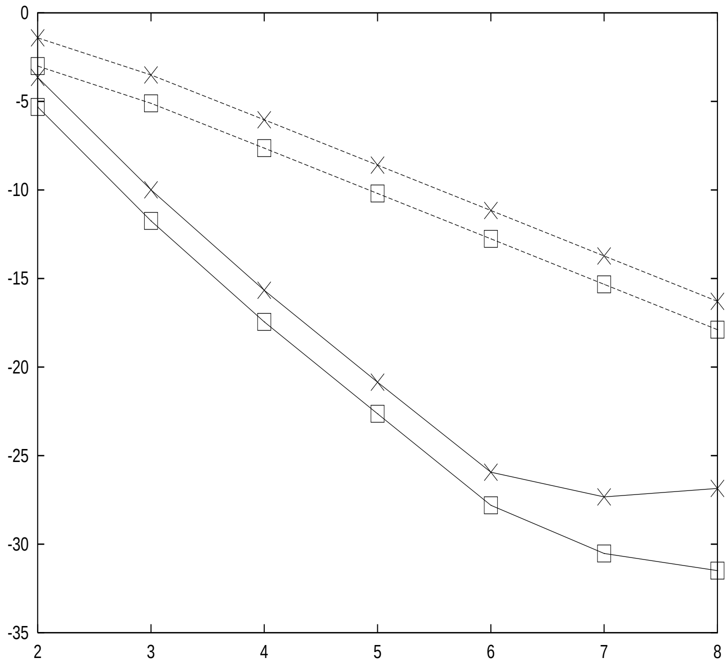

Figure 5.4: The natural logarithm of the errors $e_{max}(T_+)$ ($\times$) and e^J_{par} ($\Box$) for $T_+ = 1, \ldots, 8$ in the first (- - -) and the second (—) step of the boundary corrector method.

5.2 Lorenz system

For the Lorenz system we use our techniques to approximate the stable manifold of the periodic orbit and the strong stable manifold of the non-trivial equilibrium. Moreover, we set up a procedure to find an initial approximation of a point to periodic connecting orbit (see [6], [19]).

For further details about the Lorenz system see [25].

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1)\\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3\\ \dot{x}_3 &= x_1 x_2 - b x_3\end{aligned}$$

We use the parameter values $\sigma = 10$, $b = \frac{8}{3}$ and $r = 24$.

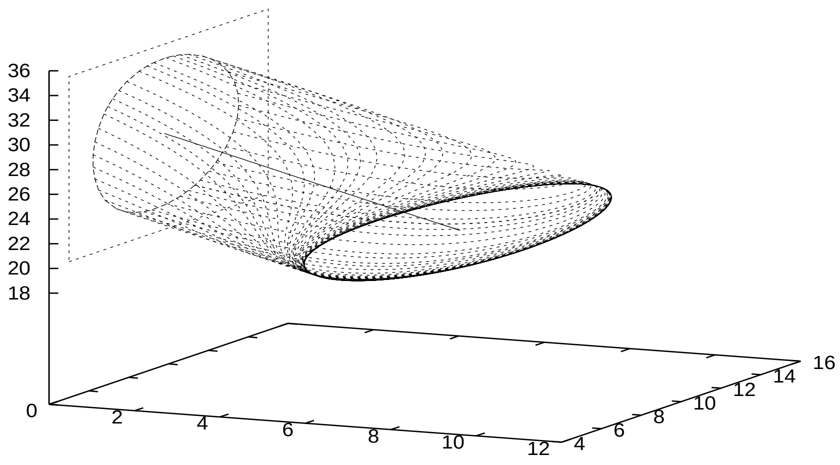

Figure 5.5: Stable manifold of a periodic (- -) orbit and strong-stable manifold of a non-trivial equilibrium (—). The parameters are $\sigma = 10$, $b = \frac{8}{3}$ and $r = 24$. All solutions have initial values with $x_1(0) = 0$.

First we approximate connecting orbits starting in the plane with $x_1(0) = 0$, which are either in the stable manifold of a periodic orbit or in the strong-stable manifold of a non-trivial equilibrium. This yields Figure 5.5.

To approximate the strong-stable manifold we apply Theorem 4.1 in an obvious way. The strong-stable manifold has dimension $m_a = 1$, the plane has dimension $m_b = 2$ and $\sigma x_2 > 0$ for $x_2 > 0$ implies nondegeneracy. The approximation is the line (—) in the interior of the cylinder in Figure 5.5.

For the approximation of the stable manifold of a periodic orbit we use the phase as free parameter and fix $x_3(0)$ in addition to $x_1(0)$. As a result we get one of the dashed lines (- - -) in Figure 5.5. Using $x_3(0)$ as continuation parameter yields the other solutions and hence the whole figure.

Remark: To get a "nice picture", we do not use the phase-variable as continuation parameter, because the resulting curves diverge exponentially and do not give a good graphical representation of the surface.

To find an initial approximation of a point to periodic connecting orbit we use the following procedure:

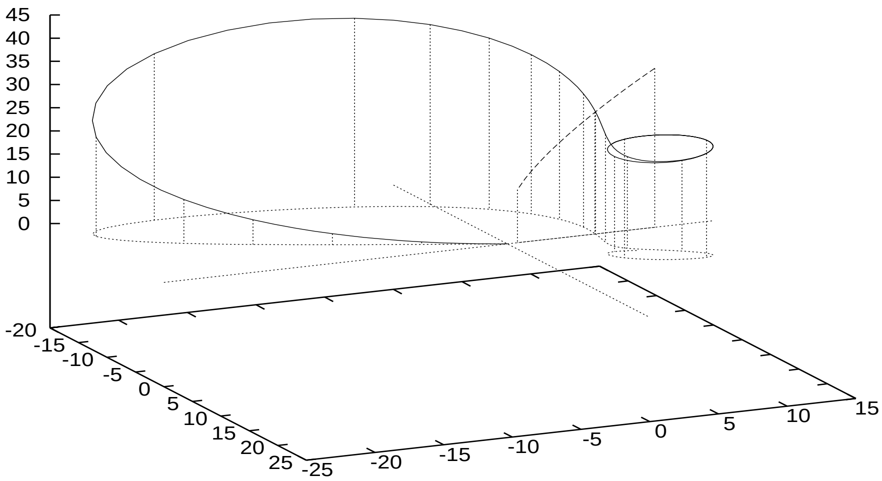

Figure 5.6: Approximation of a point to periodic connecting orbit

First we approximate the 1–dimensional unstable manifold of the origin for $r = 30$. It intersects the x_2–x_3 plane at $(0, 10.7, 34.3)$. By continuation of the parameter r to $r = 15$ we get a curve $\big(0, x_2(r), x_3(r), r\big)$ which is a 1–dimensional manifold in $\mathbb{R}^4$. Its projection onto $\mathbb{R}^3$ is plotted in Figure 5.6 by (- - -). Then we choose a linear approximation $\big(0, a_2 r, a_3 r, r\big)$ of $\big(0, x_2(r), x_3(r), r\big)$.

The center unstable manifold of the periodic orbit has dimension two, thus the manifold $\mathcal{W}$ must be 2–dimensional to get a solution converging to the periodic orbit. Therefore we use the phase fixing parameter φ as additional parameter and define a 2–dimensional manifold $\mathcal{W}$ by $\big(0, a_2 r, a_3 r, r, \varphi\big)$ in $\mathbb{R}^5$. We approximate a solution with initial value on this plane and converging to the periodic orbit to achieve the orbit plotted by (—) in Figure 5.6. The resulting parameter $r = 24.05803$ is in agreement with the value from [6].

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